## Research article

# Two overdetermined problems for anisotropic $p$-Laplacian ${ }^{\dagger}$ 

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#### Abstract

In this paper, we consider two overdetermined problems for the anisotropic $p$-Laplacian $(1<p \leq n)$ in the exterior domains and the bounded punctured domains, respectively, and prove the corresponding Wulff shape characterizations, by using Weinberger type approach.


Keywords: anisotropic p-Laplacian; capacity; overdetermined problems; Wulff shape

## 1. Introduction

The study of symmetry in overdetermined boundary value problems has become an important field of research in the theory of PDEs. The pioneering symmetry result obtained by Serrin [20] are now classical but still influential. The main technique to tackle such problems are the celebrated method of moving planes developed by Alexandrov [2,3] and Serrin [20] as well as Weinberger's approach [25] which is based on maximum principle for so-called $P$-function and Rellich-Pohozaev's integral identity.

There are plenty of considerations for different kinds of overdetermined boundary value problems. For our purpose, we recall a result of Reichel [17], who considered an overdetermined problem for capacity in an exterior domain. The capacity of a smooth bounded domain $\Omega \subset \mathbb{R}^{n}(n \geq 2)$ is defined as

$$
\operatorname{Cap}(\Omega)=\inf \left\{\int_{\mathbb{R}^{n}}|\nabla v|^{2} d x \mid v \in C_{c}^{\infty}\left(\mathbb{R}^{n}\right), v \geq 1 \text { on } \Omega\right\} .
$$

The minimizer for $\operatorname{Cap}(\Omega)$ is characterized by the capacitary potential $u$ satisfying

$$
\left\{\begin{align*}
\Delta u & =0 \text { in } \mathbb{R}^{n} \backslash \bar{\Omega}  \tag{1.1}\\
u & =1 \text { on } \partial \Omega \\
u & \rightarrow 0 \text { as }|x| \rightarrow \infty,
\end{align*}\right.
$$

Reichel [17] considered the overdetermined problem, (1.1) with an extra boundary condition

$$
\begin{equation*}
|\nabla u|=c \text { on } \partial \Omega, \tag{1.2}
\end{equation*}
$$

and proved that (1.1) and (1.2) admits a solution if and only if $\Omega$ is a ball. Reichel's proof is again based on the method of moving planes and he also extended in [18] such result to more general quasilinear equations including $p$-Laplacian equations in exterior domains. Garofalo-Sartori [10] and Poggesi [14] reproved Reichel's result for $p$-capacity by using Weinberger type approach, which was first used by Payne-Philippin [16] for the exterior problem.

The anisotropic PDE problems involving the anisotropic Laplacian attract lots of attention in recent decades. Regarding the overdetermined problem, Cianchi-Salani [6] and Wang-Xia [22] independently extends Serrin's classical result in the anisotropic setting. Due to the anisotropy, the method of moving planes does not work but Weinberger type approach works in general. The correponding overdetermined problem for anisotropic $p$-capacity in an exterior domain considered by Reichel [17,18] has been extended by Bianchini-Ciraolo [4] and Bianchini-Ciraolo-Salani [5]. They proved the symmetric result when the domain is assumed to be convex, by using a totally integral method. In this paper, we remove the convexity assumption in Bianchini-Ciraolo-Salani's result by using Weinberger type approach.

In order to state our result, we introduce the anisotropic $p$-capacity. Let $F \in C^{\infty}\left(\mathbb{R}^{n} \backslash\{0\}\right)$ be a Minkowski norm in $\mathbb{R}^{n}$, see Section 2.1 for the definition. For $p \in(1, n)$, the anisotropic $p$-capacity of $\Omega$ is defined as

$$
\begin{equation*}
\operatorname{Cap}_{F, p}(\Omega)=\inf \left\{\int_{\mathbb{R}^{n}} F^{p}(\nabla v) d x \mid v \in C_{c}^{\infty}\left(\mathbb{R}^{n}\right), v \geq 1 \text { on } \Omega\right\} . \tag{1.3}
\end{equation*}
$$

The associated anisotropic $p$-capacitary potential is namely the unique weak solution $u$ to the following problem

$$
\left\{\begin{align*}
\Delta_{F, p} u & =0 \text { in } \mathbb{R}^{n} \backslash \bar{\Omega}  \tag{1.4}\\
u & =1 \text { on } \partial \Omega \\
u(x) & \rightarrow 0 \text { as }|x| \rightarrow \infty,
\end{align*}\right.
$$

where $\Delta_{F, p}$ is the anisotropic (Finsler-) $p$-Laplacian,

$$
\Delta_{F, p} u=\operatorname{div}\left(F^{p-1}(\nabla u) F_{\xi}(\nabla u)\right), \text { when } \nabla u \neq 0 .
$$

A function $u \in W_{l o c}^{1, p}\left(\mathbb{R}^{n} \backslash \bar{\Omega}\right)$ is called a weak solution of $\Delta_{F, p} u=0$ in $\mathbb{R}^{n} \backslash \bar{\Omega}$ if

$$
\int_{\mathbb{R}^{n} \backslash \bar{\Omega}}\left\langle F^{p-1}(\nabla u) F_{\xi}(\nabla u), \nabla \psi\right\rangle d v=0 .
$$

for any $\psi \in C_{c}^{\infty}\left(\mathbb{R}^{n} \backslash \bar{\Omega}\right)$. It is not hard to see that

$$
\operatorname{Cap}_{F, p}(\Omega)=\int_{\mathbb{R}^{n} \backslash \bar{\Omega}} F^{p}(\nabla u) d x=\int_{\partial \Omega} F^{p-1}(\nabla u) F(v) d \sigma,
$$

where $v=-\frac{\nabla u}{|\nabla u|}$ is a unit normal of $\partial \Omega$ pointing towards $\mathbb{R}^{n} \backslash \bar{\Omega}$.
We will study the problem (1.4) with the overdetermined condition

$$
\begin{equation*}
F(\nabla u)=c \text { on } \partial \Omega, \tag{1.5}
\end{equation*}
$$

for some constant $c>0$. The first main result in this paper is the following
Theorem 1.1. Let $1<p<n$ and $\Omega \subset \mathbb{R}^{n}$ be a bounded domain with boundary of class $C^{2, \alpha}$. Then (1.4) and (1.5) admits a weak solution if and only if $\Omega$ is a Wulff ball.

In the theorem, a Wulff ball means a translation and rescaling of $\mathcal{W}_{F}=\left\{x \in \mathbb{R}^{n}: F^{o}(x)<1\right\}$, where $F^{o}$ is the dual norm of $F$ given by (2.1).

Next, we prove a similar result corresponding to Theorem 1.1 in the special case $p=n$. Let $u \in W^{1, n}\left(\mathbb{R}^{n} \backslash \bar{\Omega}\right)$ be a weak solution to

$$
\left\{\begin{align*}
\Delta_{F, n} u & =0, \text { in } \mathbb{R}^{n} \backslash \bar{\Omega},  \tag{1.6}\\
u & =1 \text { on } \partial \Omega, \\
u(x) & \sim-\ln F^{o}(x) \text { as }|x| \rightarrow \infty,
\end{align*}\right.
$$

where $\sim$ means that

$$
\begin{equation*}
c_{1} \leq \frac{u(x)}{-\ln F^{o}(x)} \leq c_{2} \text { as }|x| \rightarrow \infty, \tag{1.7}
\end{equation*}
$$

for some positive constant $c_{1}, c_{2}$. For this case, we prove the following result with analogous tools.
Theorem 1.2. Let $\Omega \subset \mathbb{R}^{n}$ be a bounded domain with boundary of class $C^{2, \alpha}$. Then (1.6) and (1.5) admit a weak solution if and only if $\Omega$ is a Wulff ball.

We adapt the arguments in Garofalo-Sartori [10] and Poggesi [14] to prove Theorems 1.1 and 1.2. The main ingredients are a strong maximum principle on a well-behaved $P$-function and a Rellich-Pohozaev-type identity.

In the second part of this paper, we consider a similar overdetermined problem for the anisotropic $p$-Laplacian in a bounded punctured domain. More precisely, we are concerned with the following equation in $\Omega \backslash\{0\}$, where 0 is contained in $\Omega$ : for $p \in(1, n]$,

$$
\left\{\begin{align*}
\Delta_{F, p} u & =0 \text { in } \Omega \backslash\{0\},  \tag{1.8}\\
u & =1 \text { on } \partial \Omega, \\
\lim _{|x| \rightarrow 0} u & =+\infty .
\end{align*}\right.
$$

under Serrin's overdetermined condition

$$
\begin{equation*}
F(\nabla u)=c \text { on } \partial \Omega . \tag{1.9}
\end{equation*}
$$

We say a function $u \in W_{l o c}^{1, p}(\Omega \backslash\{0\})$ is a weak solution of $\Delta_{F, p} u=0$ in $\Omega \backslash\{0\}$ if

$$
\int_{\Omega \backslash\{0\}}\left\langle F^{p-1}(\nabla u) F_{\xi}(\nabla u), \nabla \psi\right\rangle d x=0 .
$$

for any $\psi \in C_{c}^{\infty}(\Omega \backslash\{0\})$. $\lim _{|x| \rightarrow 0} u(x)=+\infty$ means $u$ has a non-removable singularity at 0 . A classical result of Serrin [19] says that for $p \in(1, n]$, when $u$ has a non-removable singularity at 0 , then $\frac{u}{\Gamma_{F_{p}, p}}$ is bounded near 0 , where $\Gamma_{F, p}$ is the fundamental solution to the anisotropic $p$-Laplacian given in (3.1).

Theorem 1.3. Let $1<p \leq n$ and $\Omega \subset \mathbb{R}^{n}$ be a bounded domain with boundary of class $C^{2, \alpha}$. Then (1.8) and (1.9) admits a weak solution if and only if $\Omega$ is a Wulff ball centered at 0 .

When $F$ is the Euclidean norm, such symmetric result has been proved by Alessandrini-Rosset [1], and Enciso-Peralta-Salas [9] via the method of moving planes and Weinberger type approach, respectively. We shall adapt Enciso-Peralta-Salas's method [9] which is based on the $P$-function to prove Theorem 1.3.

Throughout this paper, we assume that $F \in C^{\infty}\left(\mathbb{R}^{n} \backslash\{0\}\right)$ is a Minkowski norm on $\mathbb{R}^{n}$, and $\Omega$ is a bounded domain with boundary of class $C^{2, \alpha}$. We will always use Einstein summation convention.

## 2. Preliminaries

### 2.1. Minkowski norm, Wulff shape, anisotropic area

Let $F \in C^{\infty}\left(\mathbb{R}^{n} \backslash\{0\}\right)$ be a Minkowski norm on $\mathbb{R}^{n}$, in the sense that
(i) $F$ is a norm in $\mathbb{R}^{n}$, i.e., $F$ is a convex, 1-homogeneous function satisfying $F(x)>0$ when $x \neq 0$ and $F(0)=0$;
(ii) $F$ satisfies a uniformly elliptic condition: $\nabla^{2}\left(\frac{1}{2} F^{2}\right)$ is positive definite in $\mathbb{R}^{n} \backslash\{0\}$.

The dual norm $F^{o}: \mathbb{R}^{n} \rightarrow[0,+\infty[$ of $F$ is defined as

$$
\begin{equation*}
F^{o}(x)=\sup _{\xi \neq 0} \frac{\langle\xi, x\rangle}{F(\xi)} \tag{2.1}
\end{equation*}
$$

$F^{o}$ is also a Minkowski norm on $\mathbb{R}^{n}$. Furthermore,

$$
F(\xi)=\sup _{x \neq 0} \frac{\langle\xi, x\rangle}{F^{o}(x)} .
$$

We remark that, throughout this paper, we use conventionally $\xi$ as the variable for $F$ and $x$ as the variable for $F^{o}$.

Denote

$$
\mathcal{W}_{F}=\left\{x \in \mathbb{R}^{n}: F^{o}(x)<1\right\} .
$$

For the simplicity of notations, we will denote by $\mathcal{W}_{F}=\mathcal{W}$. We call $\mathcal{W}$ the unit Wulff ball centered at the origin, and $\partial \mathcal{W}$ the Wulff shape.

More generally, we denote

$$
\mathcal{W}_{r}\left(x_{0}\right)=r \mathcal{W}+x_{0},
$$

and call it the Wulff ball of radius $r$ centered at $x_{0}$. We simply denote $\mathcal{W}_{r}=\mathcal{W}_{r}(0)$.
The following properties of $F$ and $F^{o}$ hold true and will be frequently used in this paper (see e.g., $[7,24]$ ).

Proposition 2.1. Let $F: \mathbb{R}^{n} \rightarrow[0, \infty)$ be a Minkowski norm. Then for any $x, \xi \in \mathbb{R}^{n} \backslash\{0\}$, the following hold:
1). $\left\langle F_{\xi}(\xi), \xi\right\rangle=F(\xi), \quad\left\langle F_{x}^{o}(x), x\right\rangle=F^{o}(x)$.
2). $\sum_{j} F_{\xi_{i} \xi_{j}}(\xi) \xi_{j}=0$ for any $i=1, \ldots, n$.
3). $F\left(F_{x}^{o}(x)\right)=F^{o}\left(F_{\xi}(\xi)\right)=1$.
4). $F^{o}(x) F_{\xi}\left(F_{x}^{o}(x)\right)=x, \quad F(\xi) F_{x}^{o}\left(F_{\xi}(\xi)\right)=\xi$.

Let $\Omega \subset \mathbb{R}^{n}$ be a bounded open set with smooth boundary $\partial \Omega$ and $v$ be its unit outward normal of $\partial \Omega$. The anisotropic area $|\partial \Omega|_{F}$ of $\Omega$ is defined by

$$
\begin{equation*}
|\partial \Omega|_{F}=\int_{\partial \Omega} F(v) d \sigma \tag{2.2}
\end{equation*}
$$

The well-known Wulff theorem (see e.g., Theorem 20.8 in [15]) says that Wulff balls are the only minimizers for the anisotropic isoperimetric problem. Equivalently, the Wulff inequality holds true:

$$
\begin{equation*}
|\partial \Omega|_{F} \geq n\left|\mathcal{W}_{F}\right|^{\frac{1}{n}}|\Omega|^{1-\frac{1}{n}} . \tag{2.3}
\end{equation*}
$$

Equality in (2.3) holds if and only if $\Omega$ is a Wulff ball.
Note that when $\Omega=\mathcal{W}$, the unit Wulff ball, one can check by the divergence theorem that

$$
\begin{equation*}
|\partial \mathcal{W}|_{F}=\int_{\partial \mathscr{W}} \frac{1}{\left|\nabla F^{o}\right|} d \sigma=\int_{\mathcal{W}} \operatorname{div}(x) d x=n|\mathcal{W}| . \tag{2.4}
\end{equation*}
$$

For notation simplicity, we denote

$$
\kappa_{n-1}=|\partial \mathcal{W}|_{F}=n|\mathcal{W}| .
$$

### 2.2. Anisotropic p-Laplacian

Let $u$ be twice continuous differentiable at $x \in \mathbb{R}^{n}$. We denote by $F_{i}, F_{i j}, \ldots$ the partial derivatives of $F$ and by $u_{i}, u_{i j}, \ldots$ the partial derivatives of $u$,

$$
F_{i}=\frac{\partial F}{\partial \xi_{i}}, F_{i j}=\frac{\partial^{2} F}{\partial \xi_{i} \partial \xi_{j}}, u_{i}=\frac{\partial u}{\partial x_{i}}, u_{i j}=\frac{\partial^{2} u}{\partial x_{i} \partial x_{j}} .
$$

For $x$ such that $\nabla u(x) \neq 0$, denote

$$
\begin{align*}
a_{i j}(\nabla u)(x) & :=\frac{\partial^{2}}{\partial \xi_{i} \partial \xi_{j}}\left(\frac{1}{2} F^{2}\right)(\nabla u(x))=\left(F_{i} F_{j}+F F_{i j}\right)(\nabla u(x)),  \tag{2.5}\\
a_{i j, p}(\nabla u)(x) & :=\frac{\partial^{2}}{\partial \xi_{i} \partial \xi_{j}}\left(\frac{1}{p} F^{p}\right)(\nabla u(x))=F^{p-2}\left(a_{i j}+(p-2) F_{i} F_{j}\right)(\nabla u(x)) .
\end{align*}
$$

The anisotropic Laplacian and p-Laplacian of $u$ is given by

$$
\begin{align*}
\Delta_{F} u & :=a_{i j}(\nabla u) u_{i j},  \tag{2.6}\\
\Delta_{F, p} u & :=a_{i j, p}(\nabla u) u_{i j}=F^{p-2}\left(\Delta_{F} u+(p-2) F_{i} F_{j} u_{i j}\right) .
\end{align*}
$$

### 2.3. Anisotropic curvature for level sets

We recall the concept of anisotropic curvature for a hypersurface in $\mathbb{R}^{n}$. See e.g., [22, 26].
Let $M$ be a smooth embedded hypersurface in $\mathbb{R}^{n}$ and $v$ be one unit normal of $M$. The corresponding anisotropic normal of $M$ is defined by

$$
v_{F}=F_{\xi}(v) .
$$

The anisotropic principal curvatures $\kappa_{F}=\left(\kappa_{1}^{F}, \ldots, \kappa_{n-1}^{F}\right) \in \mathbb{R}^{n-1}$ are defined as the eigenvalues of the map

$$
d \nu_{F}: T_{x} M \rightarrow T_{\nu_{F}(x)} \mathcal{W} .
$$

The mean curvature (with respect to $v$ ) is defined to be

$$
H_{F}=\sum_{i} \kappa_{i}^{F} .
$$

In this paper we are interested in the case when $M$ is given by a regular level set of a smooth function $u$, that is $M=\{u=t\}$ for some regular value $t$. For our purpose, we choose the unit normal $v=-\frac{\nabla u}{|\nabla u|}$ and

$$
v_{F}=-F_{\xi}(\nabla u), \quad H_{F}=-\operatorname{div}\left(F_{\xi}(\nabla u)\right) .
$$

In this case, we have that

$$
\begin{equation*}
H_{F}=-\operatorname{div}\left(F_{\xi}(\nabla u)\right)=-F_{i j} u_{i j}, \tag{2.7}
\end{equation*}
$$

Here div is the Euclidean divergence. See e.g., [8].
Next we give the formula for the anisotropic mean curvature of regular level sets.
Proposition 2.2 ( [27]). Let $u$ satisfy $\Delta_{F, p} u=0$. Then the anisotropic mean curvature of regular level set of $u$ is given by

$$
\begin{equation*}
H_{F}=(p-1) F^{-1} F_{i} F_{j} u_{i j} \text { in }\{x: \nabla u(x) \neq 0\} . \tag{2.8}
\end{equation*}
$$

Finally, we will give the anisotropic Heintze-Karcher inequality for later use.
Proposition 2.3 ([12,28]). Let $\Omega \subset \mathbb{R}^{n}$ be an open bounded domain with $C^{2}$ boundary $\partial \Omega$ satisfying $H_{F}>0$. Then,

$$
\begin{equation*}
\frac{n-1}{n} \int_{\partial \Omega} \frac{F(v)}{H_{F}} d \sigma \geq|\Omega| . \tag{2.9}
\end{equation*}
$$

and equality holds if and only if $\Omega$ is a Wulff ball.

## 3. Overdetermined problem in an exterior domain

### 3.1. Regularity and asymptotic behavior

Let $u$ be a weak solution to (1.4) (case $1<p<n$ ) or (1.5) (case $p=n$ ). The following regularity result is nowadays standard by the regularity theory for degenerate elliptic PDEs [21] and Schauder theory for uniformly elliptic PDEs [11].

Proposition 3.1 (Regularity). Let $1<p \leq n$ and $u$ be a weak solution to (1.4). Then $u \in C^{1, \beta}\left(\mathbb{R}^{n} \backslash \Omega\right)$ for some $\beta<1$. Moreover, $u \in C^{\infty}\left(\left(\mathbb{R}^{n} \backslash \bar{\Omega}\right) \backslash \operatorname{Crit}(u)\right) \cap C^{2, \alpha}\left(\left(\mathbb{R}^{n} \backslash \Omega\right) \backslash \operatorname{Crit}(u)\right)$, where $\operatorname{Crit}(u)=\left\{x \in \mathbb{R}^{n} \backslash\right.$ $\Omega \mid \nabla u(x)=0\}$. Moreover, $|\nabla u| \neq 0$ in some neighborhood $\mathcal{N}(\partial \Omega)$ of $\partial \Omega$ and $u$ is $C^{2}\left(\mathcal{N}(\partial \Omega) \cap\left(\mathbb{R}^{n} \backslash \Omega\right)\right.$.

For $1<p \leq n$, let

$$
\Gamma_{F, p}(x)= \begin{cases}\frac{p-1}{n-p}\left(\frac{1}{\kappa_{n-1}}\right)^{\frac{1}{p-1}} F^{o}(x)^{\frac{p-n}{p-1}}, & 1<p<n  \tag{3.1}\\ -\kappa_{n-1}^{-\frac{1}{n-1}} \ln F^{o}(x), & p=n .\end{cases}
$$

One can check that

$$
\Delta_{F, p} \Gamma_{F, p}(x)=\delta_{0} \text { in } \mathbb{R}^{n}
$$

where $\delta_{0}$ is the Dirac Delta function about the origin. We call $\Gamma_{F, p}$ the fundamental solution to $\Delta_{F, p} u=0$ in $\mathbb{R}^{n}$. See [23].

Proposition 3.2 (Asymptotic behavior, $1<p<n$ [27]).
Let $1<p<n$ and $u$ is a weak solution to $\Delta_{F, p} u=0$ in $\mathbb{R}^{n} \backslash \bar{\Omega}$. Then
1). $\lim _{|x| \rightarrow+\infty} \frac{u(x)}{\Gamma_{F, p}(x)}=\operatorname{Cap}_{F, p}(\Omega)^{\frac{1}{p-1}}$,
2). $\nabla u(x)=\operatorname{Cap}_{F, p}(\Omega)^{\frac{1}{p-1}} \nabla \Gamma_{F, p}(x)+o\left(|x|^{-\frac{n-1}{p-1}}\right)$. where $\operatorname{Cap}_{F, p}(\Omega)$ is the anisotropic p-capacity of $\Omega$ given in (1.3).

Proposition 3.3 (Asymptotic behavior, $p=n$ ).
Let $p=n$ and $u$ be a weak solution to (1.6) and (1.5). Then
1). $\lim _{|x| \rightarrow+\infty} \frac{u(x)}{\Gamma_{F, n}(x)}=c|\partial \Omega|_{F}^{\frac{1}{n-1}}$,
2). $\nabla u(x)=c|\partial \Omega|_{F}^{\frac{1}{n-1}} \nabla \Gamma_{F, n}(x)+o\left(|x|^{-1}\right)$.

Proof. If $u$ is solution of (1.6), it is a standard argument by using comparison theorem to show that there exists two positive constants $c_{1}, c_{2}$ such that

$$
c_{1} \Gamma_{F, n} \leq u \leq c_{2} \Gamma_{F, n}!‘ £
$$

Following the argument of [13], Theorem 1.1 and Remark 1.5, (see [23], Theorem 4.1 and Remark 4.1 for anisotropic case), we conclude that there exists $\gamma \in \mathbb{R}$ such that

$$
\begin{align*}
& \lim _{|x| \rightarrow+\infty} \frac{u(x)}{\Gamma_{F, n}(x)}=\gamma,  \tag{3.2}\\
& \lim _{|x| \rightarrow+\infty} F^{o}(x)\left(\nabla u-\gamma \nabla \Gamma_{F, n}\right)=0 \tag{3.3}
\end{align*}
$$

By integration by parts for (1.5), we have

$$
\begin{equation*}
\int_{\partial \Omega} F^{n-1}(\nabla u) F(v) d \sigma=-\lim _{R \rightarrow \infty} \int_{\partial \mathcal{W}_{R}} F^{n-1}(\nabla u)\left\langle F_{\xi}(\nabla u), v_{\partial \mathcal{W}_{R}}\right\rangle d \sigma \tag{3.4}
\end{equation*}
$$

where $v=-\frac{\nabla u}{|\nabla u|}$ and $v_{\partial \mathcal{W}_{R}}$ is outward normal vector of $\mathcal{W}_{R}$. From (3.2), we have

$$
F(\nabla u)=\gamma F\left(\nabla \Gamma_{F, n}\right)+o\left(|x|^{-1}\right)=\gamma\left(\kappa_{n-1}\right)^{-\frac{1}{n-1}} \frac{1}{F^{o}}+o\left(|x|^{-1}\right)
$$

On $\partial \mathcal{W}_{R}$,

$$
v_{\partial W_{R}}=\frac{\nabla F^{o}}{\left|\nabla F^{o}\right|}=-\left(\kappa_{n-1}\right)^{\frac{1}{n-1}} \gamma^{-1} F^{o} \frac{\nabla u}{\left|\nabla F^{o}\right|}+o(1) .
$$

Hence

$$
F^{n-1}(\nabla u)\left\langle F_{\xi}(\nabla u), v_{\partial W_{R}}\right\rangle=-\gamma^{n-1} \frac{1}{\kappa_{n-1}} \frac{\left(F^{o}(x)\right)^{1-n}}{\left|\nabla F^{o}\right|}+o\left(|x|^{1-n}\right) .
$$

Combining the fact that

$$
\int_{\partial W_{R}} \frac{\left(F^{o}(x)\right)^{1-n}}{\left|\nabla F^{o}\right|} d \sigma=\kappa_{n-1}
$$

we deduce that

$$
\lim _{R \rightarrow \infty} \int_{\partial \mathcal{W}_{R}} F^{n-1}(\nabla u)\left\langle F_{\xi}(\nabla u), v\right\rangle d \sigma=-\gamma^{n-1}
$$

It follows from (1.5) and (3.4) that

$$
c^{n-1}|\partial \Omega|_{F}=\gamma^{n-1} .
$$

The assertion follows.

Proposition 3.4 ( [27]). Let $1<p<n$ and $u$ be a weak solution to (1.4). Then

$$
\begin{equation*}
\operatorname{Cap}_{F, p}(\Omega)=\int_{\mathbb{R}^{n} \backslash \bar{\Omega}} F^{p}(\nabla u) d x=\int_{\partial \Omega} F^{p-1}(\nabla u) F(v) d \sigma \tag{3.5}
\end{equation*}
$$

### 3.2. Rellich-Pohozaev-type identity

Firstly, we prove the following Rellich-Pohozaev-type identity.
Proposition 3.5. Let $1<p<n$ and $u$ be a weak solution to (1.4). Then

$$
\begin{equation*}
(n-p) \int_{\mathbb{R}^{n} \mid \bar{\Omega}} F^{p}(\nabla u) d x=(p-1) \int_{\partial \Omega} F^{p}(\nabla u)\langle x, v\rangle d \sigma, \tag{3.6}
\end{equation*}
$$

where $v=-\frac{\nabla u}{|\nabla u|}$ is a unit normal of $\partial \Omega$ pointing towards $\mathbb{R}^{n} \backslash \bar{\Omega}$.

Proof. By directing computations, we get, for $R$ large,

$$
\begin{aligned}
\int_{\partial \Omega} F^{p}(\nabla u)\langle x, v\rangle d \sigma= & -\int_{\mathcal{W}_{R} \backslash \bar{\Omega}} \operatorname{div}\left(x F^{p}(\nabla u)\right) d x+\int_{\partial W_{R}} F^{p}(\nabla u)\left\langle x, v_{\partial W_{R}}\right\rangle d \sigma \\
= & -\int_{\mathcal{W}_{R} \backslash \bar{\Omega}} n F^{p}-x_{i} \frac{\partial}{\partial x_{i}}\left(F^{p}(\nabla u)\right) d x+\int_{\partial W_{R}} F^{p}\left\langle x, v_{\partial W_{R}}\right\rangle d \sigma \\
= & -\int_{\mathcal{W}_{R} \backslash \bar{\Omega}} n F^{p}(\nabla u) d x-p \int_{\mathcal{W}_{R} \backslash \bar{\Omega}} x_{i} \partial_{x_{j}}\left(F^{p-1} F_{j} u_{i}\right) \\
& +p \int_{\mathcal{W}_{R} \backslash \bar{\Omega}} x_{i} u_{i} \partial_{x_{j}}\left(F^{p-1} F_{j}\right) d x+\int_{\partial W_{R}} F^{p}(\nabla u)\left\langle x, v_{\partial W_{R}}\right\rangle d \sigma \\
= & -\int_{\mathcal{W}_{R} \backslash \bar{\Omega}} n F^{p}(\nabla u) d x-p \int_{\partial \Omega} x_{i} u_{i} \frac{F^{p}}{|\nabla u|} d \sigma+p \int_{\mathcal{W}_{R} \backslash \bar{\Omega}} F^{p} d x \\
& +\int_{\partial W_{R}} F^{p}\left\langle x, v_{\partial W_{R}}\right\rangle d \sigma-p \int_{\partial W_{R}} x_{i} u_{i} F^{p-1}\left\langle F_{\xi}(\nabla u), v_{\partial W_{R}}\right\rangle d \sigma .
\end{aligned}
$$

Then, by taking the limit for $R \rightarrow+\infty$ and noting that the integrals on $\partial \mathcal{W}_{R}$ converge to zero due to the asymptotic behavior of $u$ at infinity given by Proposition 3.2. Thus, we obtain the assertion.

Proposition 3.6. Let $p=n$ and $u$ be a weak solution to (1.6). Then we have

$$
\begin{equation*}
\int_{\partial \Omega}\langle X, v\rangle d \sigma=\lim _{R \rightarrow \infty} \int_{\partial W_{R}}\left\langle X, v_{\partial W_{R}}\right\rangle d \sigma, \tag{3.7}
\end{equation*}
$$

where $X$ is the vector field given by

$$
\begin{equation*}
X=n\langle x, \nabla u\rangle F^{n-1}(\nabla u) \nabla_{\xi} F(\nabla u)-F^{n}(\nabla u) x . \tag{3.8}
\end{equation*}
$$

Proof. The proof is the same as that of Proposition 3.5 by letting $p=n$. We omit it here.

### 3.3. Case $1<p<n$

First we can compute the value $c$ of $F(\nabla u)$ on $\partial \Omega$ with the overdetermined condition (1.5).
Proposition 3.7. Let $1<p<n$ and $u$ be a weak solution to (1.4) and (1.5). The constant c appearing in (1.5) equals

$$
\begin{equation*}
c=\frac{n-p}{p-1} \frac{|\partial \Omega|_{F}}{n|\Omega|} . \tag{3.9}
\end{equation*}
$$

Moreover, the following explicit expression of the anisotropic p-capacity of $\Omega$ holds:

$$
\begin{equation*}
\operatorname{Cap}_{F, p}(\Omega)=\left(\frac{n-p}{p-1}\right)^{p-1} \frac{|\partial \Omega|_{F}^{p}}{n|\Omega|^{p-1}} . \tag{3.10}
\end{equation*}
$$

Proof. By using (3.5) and (3.6), we obtain that

$$
\operatorname{Cap}_{F, p}(\Omega)=c^{p-1}|\partial \Omega|_{F} \text { and } \operatorname{Cap}_{F, p}(\Omega)=\frac{n(p-1)}{n-p} c^{p}|\Omega|,
$$

which implies (3.9) and (3.10).

Next, we introduce the $P$-function

$$
\begin{equation*}
P=u^{-\frac{p(n-1)}{n-p}} F^{p}(\nabla u) . \tag{3.11}
\end{equation*}
$$

We show that $P$ satisfies a strong maximum principle.
Proposition 3.8. Let $1<p<n$ and $u$ be a weak solution to (1.4). Then, at $\{\nabla u \neq 0\}$,

$$
a_{i j, p} P_{i j}+L_{i} P_{i} \geq 0
$$

where

$$
a_{i j, p}=F^{p-2}\left(F F_{i j}+(p-1) F_{i} F_{j}\right)
$$

and $L_{i} P_{i}$ is lower order term of $P_{i}$.
Moreover, $P$ cannot attain a local maximum at an interior point of $\mathbb{R}^{n} \backslash \bar{\Omega}$, unless $P$ is a constant.
Proof. Set $\operatorname{Crit}(u)=\left\{x \in \mathbb{R}^{n} \backslash \bar{\Omega} \mid \nabla u=0\right\}$. The following calculations are all taken in $\left(\mathbb{R}^{n} \backslash \bar{\Omega}\right) \backslash \operatorname{Crit}(u)$.
First we calculate the first and second derivatives of the $P$-function.
The first and the second derivatives of $P$ are

$$
\begin{align*}
& P_{i}=u^{-\frac{p(n-1)}{n-p}} F^{p}(\nabla u)\left(p \frac{F_{k} u_{i k}}{F}-\frac{p(n-1)}{n-p} \frac{u_{i}}{u}\right),  \tag{3.12}\\
P_{i j}= & u^{-\frac{p(n-1)}{n-p}} F^{p}(\nabla u)\left(p(p-1) \frac{F_{l} F_{k} u_{l j} u_{i k}}{F^{2}}+\frac{p}{F} F_{k m} u_{m j} u_{k i}\right. \\
& -\frac{p^{2}(n-1)}{n-p}\left(\frac{F_{k} u_{k i} u_{j}}{u F}+\frac{F_{k} u_{k j} u_{i}}{u F}\right)+\frac{p}{F} F_{k} u_{k i j}  \tag{3.13}\\
& \left.+\frac{p(n-1)}{n-p}\left(\frac{p(n-1)}{n-p}+1\right) \frac{u_{i} u_{j}}{u^{2}}-\frac{p(n-1)}{n-p} \frac{u_{i j}}{u}\right) .
\end{align*}
$$

It follows from (3.12) and Proposition 2.1 (1) that

$$
\begin{align*}
F_{k} u_{k i} & =p^{-1} u^{\frac{p(n-1)}{n-p}} F^{1-p} P_{i}+\frac{n-1}{n-p} \frac{F u_{i}}{u},  \tag{3.14}\\
F_{i} F_{k} u_{k i} & =p^{-1} u^{\frac{p(n-1)}{n-p}} F^{1-p} P_{i} F_{i}+\frac{n-1}{n-p} \frac{F^{2}}{u} . \tag{3.15}
\end{align*}
$$

The first equation of (1.4) implies that

$$
\begin{equation*}
\left(a_{i j}+(p-2) F_{i} F_{j}\right) u_{i j}=0 . \tag{3.16}
\end{equation*}
$$

(3.15) and (3.16) give us

$$
\begin{equation*}
F F_{i j} u_{i j}=-(p-1)\left[p^{-1} u^{\frac{p(n-1)}{n-p}} F^{1-p} P_{i} F_{i}+\frac{n-1}{n-p} \frac{F^{2}}{u}\right] . \tag{3.17}
\end{equation*}
$$

By taking derivative of (3.16), we obtain

$$
\begin{equation*}
0=F_{i j} F_{l} u_{i j} u_{l k}+F F_{i j l} u_{l k} u_{i j}+2(p-1) F_{i l} F_{j} u_{l k} u_{i j}+F^{2-p} a_{i j, p} u_{i j k} \tag{3.18}
\end{equation*}
$$

From Proposition 2.1 (2), we have

$$
\begin{equation*}
F_{i j} u_{j}=0, \text { for } i=1, \ldots, n . \tag{3.19}
\end{equation*}
$$

Taking derivative of (3.19) w.r.t. $x_{i}$ and summing, we obtain

$$
\begin{equation*}
F_{i j} u_{i j}+F_{i j l} u_{i i} u_{j}=0 . \tag{3.20}
\end{equation*}
$$

From Proposition 2.1, (3.12)-(3.20) and (2.5), we have the following computation

$$
\begin{align*}
a_{i j, p} P_{i j}= & u^{-\frac{p(n-1)}{n-p}} F^{p}(\nabla u)\left(p(p-1) \frac{F_{l} F_{k} u_{l j} u_{i k} a_{i j, p}}{F^{2}}+\frac{p}{F} F_{k m} u_{m j} u_{k i} a_{i j, p}\right. \\
& -2 \frac{p^{2}(n-1)}{n-p}\left(\frac{(p-1) F_{k} u_{k i} F_{i}}{u}\right) F^{p-2}+\frac{p}{F} F_{k} u_{k i j} a_{i j, p}  \tag{3.21}\\
& \left.+(p-1) \frac{p(n-1)}{n-p}\left(\frac{p(n-1)}{n-p}+1\right) \frac{F^{p}}{u^{2}}\right),
\end{align*}
$$

in particular,

$$
\begin{align*}
\frac{F_{l} F_{k} u_{l j} u_{i k} a_{i j, p}}{F^{2}} & =\frac{F_{l} F_{k} u_{l j} u_{i k}\left(F F_{i j}+(p-1) F_{i} F_{j}\right)}{F^{2}} F^{p-2}  \tag{3.22}\\
\frac{p}{F} F_{k m} u_{m j} u_{k i} a_{i j, p} & =\frac{p}{F} F_{k m} u_{m j} u_{k i}\left(F F_{i j}+(p-1) F_{i} F_{j}\right) F^{p-2} \tag{3.23}
\end{align*}
$$

and, by using (3.20)

$$
\begin{align*}
\frac{p}{F} F_{k} u_{k i j} a_{i j, p}= & -\frac{p}{F} F_{k}\left(F_{i j} F_{l} u_{i j} u_{l k}+F F_{i j l} u_{l k} u_{i j}+2(p-1) F_{i l} F_{j} u_{l k} u_{i j}\right) \\
= & \frac{p}{F^{2}}(p-1)\left[p^{-1} u^{-\frac{p^{2}(n-1)}{n-p}} F^{1-p} P_{i} F_{i}+\frac{n-1}{n-p} \frac{F^{2}}{u}\right]^{2} \\
& -p F_{i j l}\left(p^{-1} u^{-\frac{p^{2}(n-1)}{n-p}} F^{1-p} P_{l}+\frac{n-1}{n-p} \frac{F u_{l}}{u}\right) u_{i j} \\
& -\frac{2(p-1) p}{F} F_{i l} F_{j} F_{k} u_{l k} u_{i j}  \tag{3.24}\\
= & \frac{p}{F^{2}}(p-1)\left(\frac{n-1}{n-p} \frac{F^{2}}{u}\right)^{2}-\frac{p}{F^{2}}(p-1)\left(\frac{n-1}{n-p} \frac{F^{2}}{u}\right)^{2} \\
& -\frac{2(p-1) p}{F} F_{i l} F_{j} F_{k} u_{l k} u_{i j}+\text { term of } P_{i} \\
= & -\frac{2(p-1) p}{F} F_{i l} F_{j} F_{k} u_{l k} u_{i j}+\text { term of } P_{i} .
\end{align*}
$$

From Proposition 2.3 in [27], we have

$$
F_{i j} F_{k l} u_{i k} u_{j l}-\frac{1}{n-1}\left(F_{i j} u_{i j}\right)^{2} \geq 0
$$

Substituting (3.22)-(3.24) into (3.21), we obtain

$$
\begin{align*}
a_{i j, p} P_{i j}= & u^{-\frac{p(n-1)}{n-p}} F^{2 p-2}(\nabla u)\left(p(p-1)^{2} \frac{\left(F_{i} F_{j} u_{i j}\right)^{2}}{F^{2}}+p F_{i j} F_{k l} u_{l j} u_{k i}\right. \\
& -2 \frac{p^{2}(n-1)}{n-p}\left(\frac{(p-1) F_{k} u_{k i} F_{i}}{u}\right)+\text { term of } P_{i} \\
& \left.+(p-1) \frac{p(n-1)}{n-p}\left(\frac{p(n-1)}{n-p}+1\right) \frac{F^{2}}{u^{2}}\right) \\
= & u^{-\frac{p(n-1)}{n-p} F^{2 p-2}(\nabla u)\left(\frac{n p(p-1)^{2}}{n-1} \frac{\left(F_{i} F_{j} u_{i j}\right)^{2}}{F^{2}}+\text { term of } P_{i}\right.}  \tag{3.25}\\
& -2 \frac{p^{2}(n-1)}{n-p}\left(\frac{(p-1) F_{k} u_{k i} F_{i}}{u}\right)+F_{i j} F_{k l} u_{i k} u_{j l}-\frac{1}{n-1}\left(F_{i j} u_{i j}\right)^{2} \\
& \left.+(p-1) \frac{p(n-1)}{n-p}\left(\frac{p(n-1)}{n-p}+1\right) \frac{F^{2}}{u^{2}}\right)
\end{align*}
$$

This combing with (3.15) yields to

$$
\begin{align*}
a_{i j, p} P_{i j} \geq & u^{-\frac{p(n-1)}{n-p}-2} F^{2 p}(\nabla u)\left(\frac{n p(p-1)^{2}}{n-1}\left(\frac{n-1}{n-p}\right)^{2}\right. \\
& -2 \frac{p^{2}(n-1)^{2}(p-1)}{(n-p)^{2}}  \tag{3.26}\\
& \left.+(p-1) \frac{p(n-1)}{n-p}\left(\frac{p(n-1)}{n-p}+1\right)\right)+ \text { term of } P_{i}
\end{align*}
$$

Let $-L_{i} P_{i}$ denote the term with $P_{i}$ in (3.26). We have

$$
\begin{equation*}
a_{i j, p} P_{i j}+L_{i} P_{i} \geq 0 . \tag{3.27}
\end{equation*}
$$

If $P$ attains a local maximum at some interior point $x_{0} \in \mathbb{R}^{n} \backslash \bar{\Omega}$, then $x_{0} \in\left(\mathbb{R}^{n} \backslash \bar{\Omega}\right) \backslash \operatorname{Crit}(u)$, or $P \equiv 0$ which is impossible. By using the strong maximum principle for (3.27) on a neighborhood $\mathcal{N}$ of $x_{0}$ in which $\nabla u \neq 0$, one sees $P$ must be a constant on $\mathcal{N}$. It follows that the set where $P$ is a constant is both open and closed. Thus $P$ is a constant in $\mathbb{R}^{n} \backslash \Omega$.

Proof of Theorem 1.1. By using Proposition 3.2 we can check that

$$
\lim _{|x| \rightarrow \infty} P(x)=\left(\frac{n-p}{p-1}\right)^{\frac{p(n-1)}{n-p}}\left(\frac{\kappa_{n-1}}{\operatorname{Cap}_{F, p}(\Omega)}\right)^{\frac{p}{n-p}} .
$$

This combining with (3.10) yields to

$$
\begin{equation*}
\lim _{|x| \rightarrow \infty} P(x)=\left(\frac{n-p}{p-1}\right)^{p}\left(\frac{\kappa_{n-1}(n|\Omega|)^{p-1}}{|\partial \Omega|_{F}}\right)^{\frac{p}{n-p}} . \tag{3.28}
\end{equation*}
$$

On the other hand, by using the boundary conditions (1.4), (1.5) and (3.9) we get

$$
\begin{equation*}
P_{\mid \partial \Omega}=\left(\frac{n-p}{p-1}\right)^{p}\left(\frac{|\partial \Omega|_{F}}{n|\Omega|}\right)^{p} . \tag{3.29}
\end{equation*}
$$

According to the Wulff inequality (2.3), by using (3.28) and (3.29), we check that

$$
\lim _{|x| \rightarrow \infty} P(x) \leq P_{\mid \partial \Omega} .
$$

From Proposition 3.8 we see that either $P$ is a constant or $P$ attains its maximum on $\partial \Omega$. In both cases, we have

$$
\begin{equation*}
\left\langle\nabla P, v_{F}\right\rangle \leq 0 . \tag{3.30}
\end{equation*}
$$

We remark that here $v_{F}=F_{\xi}(v)$ with $v=-\frac{\nabla u}{|\nabla u|}$ which points towards $\mathbb{R}^{n} \backslash \bar{\Omega}$. By direct computation we see

$$
\begin{align*}
\left\langle\nabla P, v_{F}\right\rangle & =-p F^{p-1} u^{-\frac{p(n-1)}{n-p}}\left(\frac{F_{i} F_{j} u_{i j}}{F}-\frac{(n-1)}{n-p} \frac{F(\nabla u)}{u}\right) \\
& =-\frac{p}{p-1} F^{p-1} u^{-\frac{p(n-1)}{n-p}}\left(H_{F}-\frac{(p-1)(n-1)}{n-p} \frac{F(\nabla u)}{u}\right) . \tag{3.31}
\end{align*}
$$

In the last equality we have used (2.8). By using fact that $u=1$ on $\partial \Omega$, (1.5) and (3.9), it follows from (3.30) and (3.31) that

$$
\begin{equation*}
H_{F} \geq \frac{(p-1)(n-1)}{n-p} c=(n-1) \frac{|\partial \Omega|_{F}}{n|\Omega|} . \tag{3.32}
\end{equation*}
$$

It follows from (3.32) that $H_{F}$ is positive and

$$
\begin{equation*}
\int_{\partial \Omega} \frac{F(v)}{H_{F}} \leq \frac{n}{n-1}|\Omega| . \tag{3.33}
\end{equation*}
$$

Combining (3.33) with Proposition 2.3, we conclude $\Omega$ is a Wulff ball. This completes the proof of Theorem 1.1.

### 3.4. Case $p=n$

Proof of Theorem 1.2. We look at the identity (3.7). For the left side of (3.7), by using (1.5), we deduce that

$$
\int_{\partial \Omega}\langle X, v\rangle d \sigma=(n-1) c^{n} n|\Omega| .
$$

For the right side of (3.7), by the asymptotic behavior in Proposition 3.3, we can easy compute that

$$
\begin{gathered}
F(\nabla u)=c^{\frac{1}{n-1}}\left(\frac{|\partial \Omega|_{F}}{\kappa_{n-1}}\right)^{\frac{1}{n-1}} \frac{1}{F^{o}(x)}+o\left(|x|^{-1}\right), \\
\lim _{R \rightarrow \infty} \int_{\partial W_{R}}\left\langle X, v_{\partial W_{R}}\right\rangle d \sigma
\end{gathered}=\lim _{R \rightarrow \infty} \int_{\partial W_{R}}\left\langle X, \frac{\nabla F^{o}(x)}{\left|\nabla F^{o}(x)\right|}\right\rangle d \sigma, \begin{gathered}
\\
\\
=(n-1) c^{n}\left(\kappa_{n-1}\right)^{-\frac{1}{n-1}|\partial \Omega|_{F}^{\frac{n}{n-1}}} .
\end{gathered}
$$

It follows that

$$
n|\Omega|=\left(\kappa_{n-1}\right)^{-\frac{1}{n-1}}|\partial \Omega|_{F}^{\frac{n}{n-1}} .
$$

That means the equality in the Wulff inequality (2.3) holds. Thus $\Omega$ is a Wulff ball. This completes the proof of Theorem 1.2.

## 4. Overdertermined problem in a punctured domain

In this section, we consider the overdetermined problem (1.8) and (1.9) in a bounded punctured domain.

Since 0 is non-removable singular point, by using a result of Serrin [19], we can see that $u / \Gamma_{F, p}$ is bounded in some neighborhood of 0 . Moreover we can see that if $\Delta_{F, p} u=0$ in $\Omega \backslash\{0\}$, then $u$ satisfies (see [13] or Theorem 4.1 in [23])

$$
\begin{equation*}
-\Delta_{F, p} u=K \delta_{0}, \text { in } \Omega, \tag{4.1}
\end{equation*}
$$

where $\delta_{0}$ is the Dirac measure in the origin and $K$ is some non-zero constant.
Proposition 4.1 (Regularity and asymptotic behavior). Let $1<p \leq n$ and $u$ be a weak solution to (1.8) and (1.9). Then
1). $u \in C^{1, \beta}(\bar{\Omega} \backslash\{0\})$ for some $\beta<1$. Moreover, $u \in C^{\infty}(\Omega \backslash(\{0\} \cup \operatorname{Crit}(u))) \cap C^{2, \alpha}(\bar{\Omega} \backslash(\{0\} \cup \operatorname{Crit}(u)))$, where $\operatorname{Crit}(u)=\{x \in \bar{\Omega} \mid \nabla u(x)=0\}$. Moreover, $|\nabla u| \neq 0$ in some neighborhood $\mathcal{N}(\partial \Omega)$ of $\partial \Omega$ and $u$ is $C^{2}(\mathcal{N}(\partial \Omega) \cap \bar{\Omega})$.
2). The constant $K$ that appears in (4.1) satisfies $K=c^{p-1}|\partial \Omega|_{F}$.
3). Asymptotic behavior of the solution $u$ of (1.8) near the origin is given by
(a) $\lim _{|x| \rightarrow+\infty} \frac{u(x)}{\Gamma_{F, p}(x)}=c|\partial \Omega|_{F}^{\frac{1}{n-1}}$,
(b) $\nabla u(x)=c|\partial \Omega|_{F}^{\frac{1}{p-1}} \nabla \Gamma_{F, p}(x)+o\left(|x|^{-\frac{n-1}{p-1}}\right)$.

Proof. We prove (2). Let $U$ and $V$ be tubular neighborhoods of $\partial \Omega$ such that $\nabla u \neq 0$ in $U \cap \Omega$ and $V \subset \subset U$. Hence $u \in C^{2}(U \cap \Omega)$, and

$$
\begin{equation*}
\Delta_{F, p} u=0 \text { in classical sense in } U \cap \Omega . \tag{4.2}
\end{equation*}
$$

We choose a function $\phi_{+}$with $\phi_{+}=1$ on $\Omega \backslash U$ and $\operatorname{supp} \phi_{+} \subset \Omega \backslash \bar{V}$, and define $\phi_{-}:=1-\phi_{+}$. From (4.2), (1.9) and (4.1), we have

$$
\begin{aligned}
0 & =\int_{\Omega} F^{p-1}(\nabla u)\left\langle F_{\xi}(\nabla u), \nabla 1\right\rangle d x \\
& =\int_{\Omega} F^{p-1}(\nabla u)\left\langle F_{\xi}(\nabla u), \nabla\left(\phi_{+}+\phi_{-}\right)\right\rangle d x \\
& =K-\int_{\partial \Omega} F^{p-1} F(v) d \sigma=K-c^{p-1}|\partial \Omega|_{F} .
\end{aligned}
$$

For (3), the proof is similar with that of that of Propositions 3.2 and 3.3. We omit it here.
Proof of Theorem 1.3: $1<p<n$. Firstly, note that the actual value of the function $F(\nabla u)$ on the boundary is irrelevant, because for any arbitrary constant $c_{0} \neq c$, there exists another solution $\tilde{u}=$ $\frac{c_{0}}{c}(u-1)+1$ satisfying $\tilde{u}=1 \Delta_{F, p} \tilde{u}=0$ and $F(\nabla \tilde{u})=c_{0}$. Hence, without loss of generality one can set

$$
\begin{equation*}
c:=\frac{n-p}{p-1}\left(\frac{|\partial \Omega|_{F}}{\kappa_{n-1}}\right)^{-\frac{1}{n-1}} . \tag{4.3}
\end{equation*}
$$

By (4.3) and Proposition 4.1 (3), the maximum principle for the anisotropic $p$-Laplacian yields that $u>0$ in $\Omega \backslash\{0\}$.

Next, we define the $P$-function exactly as in the exterior case, that is,

$$
P:=u^{-\frac{p(n-1)}{n-p}} F^{p}(\nabla u) .
$$

Then we get as in Proposition 3.8 that

$$
a_{i j, p} P_{i j}+L_{i} P_{i} \geq 0 \text { at }\{\nabla u \neq 0\} .
$$

Hence, $P$ cannot attain a local maximum at an interior point of $\Omega \backslash\{0\}$, unless $P$ is a constant. On the other hand, by Proposition 4.1 (3), we obtain

$$
\lim _{x \rightarrow 0} P(x)=\left(\frac{p-1}{n-p}\right)^{-p}\left(\frac{|\partial \Omega|_{F}}{\kappa_{n-1}}\right)^{-\frac{p}{n-1}} .
$$

Moreover, from (4.3) and the boundary condition, one can check that

$$
\left.P\right|_{\partial \Omega}=\lim _{x \rightarrow 0} P(x) .
$$

It follows that

$$
\sup _{\Omega} P=\lim _{x \rightarrow 0} P(x) .
$$

Now, we shall show that $P$ is actually a constant. It follows from (4.1) that

$$
\begin{equation*}
\int_{\Omega} F^{p-1}(\nabla u)\left\langle F_{\xi}(\nabla u), \nabla \tilde{\phi}\right\rangle d x=K \tilde{\phi}(0) \tag{4.4}
\end{equation*}
$$

for $\tilde{\phi} \in C_{c}^{1}(\Omega)$. Let $\tilde{\phi}=\phi u^{\alpha}$ for $\phi \in C_{c}^{\infty}(\Omega)$ and some $\alpha<0$ to be fixed. From the asymptotic behavior of $u$ near 0 and $u \in C^{1}(\bar{\Omega} \backslash\{0\})$, we see $\tilde{\phi} \in C_{c}^{1}(\Omega)$. Using $\tilde{\phi}$ in (4.4) and let $\alpha=-n \frac{p-1}{n-p}$, we see

$$
\begin{equation*}
\int_{\Omega}\left(F^{p-1}(\nabla u)\left\langle F_{\xi}(\nabla u), \nabla \phi\right\rangle u^{-n \frac{p-1}{n-p}}-n \frac{p-1}{n-p} F^{p}(\nabla u) u^{-\frac{p(n-1)}{n-p}} \phi\right) d x=0 . \tag{4.5}
\end{equation*}
$$

Also, near $\partial \Omega$, we have $\Delta_{F, p} u=0$ which yields to

$$
\begin{equation*}
\left(-\frac{n-p}{p-1} u^{-\frac{p-1}{n-p}}\right) \operatorname{div}\left(u^{-\frac{(p-1)(n-1)}{n-p}} F^{p-1}(\nabla u) F_{\xi}(\nabla u)\right)=(n-1) u^{-\frac{p(n-1)}{n-p}} F^{p} . \tag{4.6}
\end{equation*}
$$

in the classical sense. By using the same $\phi_{+}$and $\phi_{-}$stated in the proof of Proposition 4.1, we have

$$
\begin{aligned}
0 & =\int_{\Omega}\left(\left(-\frac{n-p}{p-1} u^{-\frac{p-1}{n-p}}\right) u^{-\frac{(p-1)(n-1)}{n-p}} F^{p-1}\left\langle F_{\xi}(\nabla u), \nabla 1\right\rangle d x\right. \\
& =\int_{\Omega}\left(\left(-\frac{n-p}{p-1} u^{-\frac{p-1}{n-p}}\right) u^{-\frac{(p-1)(n-1)}{n-p}} F^{p-1}\left\langle F_{\xi}(\nabla u), \nabla\left(\phi_{+}+\phi_{-}\right)\right\rangle d x\right. \\
& =-\int_{\Omega} n u^{-\frac{p(n-1)}{n-p}} F^{p}(\nabla u) d x+\int_{\partial \Omega} \frac{n-p}{p-1} u^{-\frac{n(p-1)}{n-p}} F^{p-1}(\nabla u) F(v) d \sigma .
\end{aligned}
$$

Furthermore, by using (1.8), (1.9) and (4.3), we obtain

$$
\begin{align*}
\int_{\Omega} P d x=\int_{\Omega} u^{-\frac{p(n-1)}{n-p}} F^{p}(\nabla u) d x & =\frac{1}{n} \int_{\partial \Omega} \frac{n-p}{p-1} u^{-\frac{n(p-1)}{n-p}} F^{p-1}(\nabla u) F(v) d \sigma \\
& =\frac{|\partial \Omega|_{F}^{\frac{n}{n-1}}}{n \kappa_{n-1}^{\frac{1}{n-1}}} \sup _{\Omega} P \geq|\Omega| \sup _{\Omega} P, \tag{4.7}
\end{align*}
$$

where the last inequality follows directly from the Wulff inequality (2.3). One sees that the equality holds in (4.7) which implies that $\Omega$ is Wulff ball by the equality characterization in the Wulff inequality.
Proof of Theorem 1.3: $p=n$. In this case, we use similar method as the proof of Theorem 1.2. We also have a similar integral equality as (3.7),

$$
\begin{equation*}
\int_{\partial \Omega}\langle X, v\rangle d \sigma=\lim _{r \rightarrow 0} \int_{\partial W_{r}}\left\langle X, v_{\partial W_{r}}\right\rangle d \sigma, \tag{4.8}
\end{equation*}
$$

where $v=-\frac{\nabla u}{|\nabla u|}$ is the unit outward normal of $\Omega$ and $X$ is given by (3.8).
For the left hand side of (4.8), by using (1.9), we have

$$
\int_{\partial \Omega}\langle X, v\rangle d \sigma=(n-1) c^{n} \int_{\partial \Omega}\langle x, v\rangle d \sigma=(n-1) n|\Omega| c^{n}
$$

For the right side of (4.8), by Proposition 4.1 (3), we can compute that

$$
\lim _{r \rightarrow 0} \int_{\partial \mathcal{W}_{r}}\left\langle X, v_{\partial W_{r}}\right\rangle d \sigma=(n-1) c^{n}\left(\kappa_{n-1}\right)^{-\frac{1}{n-1}}|\partial \Omega|_{F}^{\frac{n}{n-1}} .
$$

It follows that

$$
n|\Omega|=\left(\kappa_{n-1}\right)^{-\frac{1}{n-1}}|\partial \Omega|_{F}^{\frac{n}{n-1}},
$$

which is the equality in the Wulff inequality (2.3). It follows that $\Omega$ is a Wulff ball. This completes the proof of Theorem 1.3.

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## Conflict of interest

The authors declare no conflict of interest.

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