Research article

Nonlocal diffusion of smooth sets[†]

Anoumou Attiogbe[1], Mouhamed Moustapha Fall[1] and El Hadji Abdoulaye Thiam[2,*]

[1] African Institute for Mathematical Sciences in Senegal, KM 2, Route de Joal, B.P. 14 18. Mbour, Senegal

[2] Université Iba Der Thiam de Thies, UFR des Sciences et Techniques, département de mathématiques, Thies

* **Correspondence:** Email: elhadjiabdoulaye.thiam@univ-thies.sn.

Abstract: We consider normal velocity of smooth sets evolving by the $s-$fractional diffusion. We prove that for small time, the normal velocity of such sets is nearly proportional to the mean curvature of the boundary of the initial set for $s \in [\frac{1}{2}, 1)$ while, for $s \in (0, \frac{1}{2})$, it is nearly proportional to the fractional mean curvature of the initial set. Our results show that the motion by (fractional) mean curvature flow can be approximated by fractional heat diffusion and by a diffusion by means of harmonic extension of smooth sets.

Keywords: motion by fractional mean curvature flow; fractional heat equation; fractional mean curvature; harmonic extension

1. Introduction

For $N \geq 2$, we let Ω_0 be a bounded open set of $\mathbb{R}^N$ with boundary Γ_0. Consider the heat equation with initial data the indicator function of the set Ω_0:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = 0 & \text{in } \mathbb{R}^N \times (0, t_1] \\[2mm] u(x, 0) = \mathbb{1}_{\Omega_0}(x) & \text{on } \mathbb{R}^N. \end{cases} \tag{1.1}$$

for some time $t_1 > 0$. In 1992, Bence-Merriman-Osher [5] provided a computational algorithm for tracking the evolution in time of the set Ω_0 whose boundary Γ_0 moves with normal velocity proportional to its classical mean curvature. At time $t_1 > 0$, they considered

$$\Omega_1 = \{x \in \mathbb{R}^N : \quad u(x, t_1) \geq 1/2\}.$$

Bence-Merriman-Osher [5] applied iteratively this procedure to generate a sequence of sets $(\Omega_j)_{j \geq 0}$ and conjectured in [5] that their boundaries Γ_j evolved by mean curvature flow. Later Evans [16] provided a rigorous proof for the Bence-Merriman-Osher algorithm by means of the level-set approach to mean curvature flow developed by Osher-Sethian [32], Evans-Spruck [17–20] and Chen-Giga-Goto [10]. For related works in this direction, we refer the reader to [4, 25, 28, 29, 31, 34, 36] and references therein.

Recently Caffarelli and Souganadis considered in [9] nonlocal diffusion of open sets $E \subset \mathbb{R}^N$ given by

$$\begin{cases} \dfrac{\partial u}{\partial t} + (-\Delta)^s u = 0 & \text{in } \mathbb{R}^N \times (0, \infty) \\ u(x, 0) = \tau_E(x) & \text{in } \mathbb{R}^N \times \{t = 0\}, \end{cases} \tag{1.2}$$

where

$$\tau_E(x) = \mathbb{1}_E(x) - \mathbb{1}_{\mathbb{R}^N \setminus \overline{E}}(x).$$

We consider the fractional heat kernel K_s with Fourier transform given by $\widehat{K}(\xi, t) = e^{-t|\xi|^{2s}}$. It satisfies

$$\begin{cases} \dfrac{\partial K_s}{\partial t} + (-\Delta)^s K_s = 0 & \text{in } \mathbb{R}^N \times (0, \infty) \\ K_s = \delta_0 & \text{on } \mathbb{R}^N \times \{t = 0\}. \end{cases}$$

It follows that the unique bounded solution to (1.2) is given by

$$u(x, t) = K_s(\cdot, t) \star \tau_E(x) = \int_{\mathbb{R}^N} K_s(x - y, t)\tau_E(y)dy. \tag{1.3}$$

By solving a finite number of times (1.2) for a small fixed time step $\sigma_s(h)$, the authors in [9] find a discrete family of sets

$$E_0^h = E, \qquad E_{nh}^h = \{x \in \mathbb{R}^N : \quad K_s(\cdot, \sigma_s(h)) \star \tau_{E_{(n-1)h}^h}(x) > 0\},$$

for a suitable scaling function σ_s to be defined below. It is proved in [9] that as $nh \to t$, ∂E_{nh}^h converges, in a suitable sense, to Γ_t. Here, the family of hypersurface $\{\Gamma_t\}_{t>0}$, with $\Gamma_0 = \partial E$, evolves under generalized mean curvature flow for $s \in [\frac{1}{2}, 1)$ and under generalized fractional mean curvature flow for $s \in (0, \frac{1}{2})$. We refer the reader to [9, 16, 27] for the notion generalized (nonlocal) mean curvature flow which considers the level sets of viscosity solutions to quasilinear parabolic integro-differential equations.

In the present paper, we are interested in the normal velocity of the sets

$$E_t := \left\{x \in \mathbb{R}^N : \mathcal{K}_s(\cdot, \sigma_s(t)) \star \tau_E(x) > 0\right\} \tag{1.4}$$

as they depart from a sufficiently smooth initial set $E_0 := E$. We consider here and in the following

$$\mathcal{K}_s(x,t) = t^{-\frac{N}{2s}} P_s(t^{-\frac{1}{2s}} x), \qquad \text{for some radially symmetric function } P_s \in C^1(\mathbb{R}^N). \tag{1.5}$$

We make the following assumptions:

$$\frac{C_{N,s}^{-1}}{1 + |y|^{N+2s}} \le P_s(y) \le \frac{C_{N,s}}{1 + |y|^{N+2s}}, \qquad |\nabla P_s(y)| \le \frac{C_{N,s}}{1 + |y|^{N+2s+1}}. \tag{1.6}$$

and

$$\lim_{t \to 0} t^{-1} \mathcal{K}_s(y,t) = \frac{C_{N,s}}{|y|^{N+2s}} \qquad \text{locally uniformly in } \mathbb{R}^N \setminus \{0\}, \tag{1.7}$$

for some constants $C_{N,s}, C_{N,s} > 0$. In the Section 1.1 below, we provide examples of valuable kernels $\mathcal{K}_s$ satisfying the above properties.

Now, as we shall see below (Lemma 2.1), for $t > 0$ small, $\nabla_x u(x, \sigma_s(t)) \ne 0$ for all $x \in B(y, \sigma_s(t)^{\frac{1}{2s}})$ and $y \in \partial E$. Hence ∂E_t is a C^1 hypersurface, for small $t > 0$. For $t > 0$ and $y \in \partial E$, we let $v = v(t, y)$ be such that

$$y + t v \nu(y) \in \partial E_t \cap B(y, \sigma_s(t)^{\frac{1}{2s}}), \tag{1.8}$$

where $\nu(y)$ is the unit exterior normal of E at y. In the spirit of the work of Evans [16] on diffusion of smooth sets, we provide in this paper an expansion of $v(t, y)$ as $t \to 0$. It turns out that $v(0, y)$ is proportional to the fractional mean curvature of ∂E at y for $s \in (0, 1/2)$ and $v(0, y)$ is proportional to the classical mean curvature of ∂E at y for $s \in [1/2, 1)$.

We notice that it is not a priori clear from (1.8), that v remains finite as $t \to 0$. This is where the (unique) appropriate choice of $\sigma_s(t)$ enters during our estimates. Here and in the following, we define

$$\sigma_s(t) = \begin{cases} t^{\frac{2s}{1+2s}} & \text{for } s \in (0, 1/2), \\ t^s & \text{for } s \in (1/2, 1) \end{cases} \tag{1.9}$$

and for $s = 1/2$, $\sigma_s(t)$ is the unique positive solution to

$$t = \sigma_{1/2}^2(t) |\log(\sigma_{1/2}(t))|. \tag{1.10}$$

Before stating our main result, we recall that for $s \in (0, \frac{1}{2})$ and ∂E is of class $C^{1,\beta}$ for some $\beta > 2s$, the fractional mean curvature of ∂E is defined for $x \in \partial E$ as

$$H_s(x) := P.V. \int_{\mathbb{R}^N} \frac{\tau_E(y)}{|x-y|^{N+2s}} dy = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{\tau_E(y)}{|x-y|^{N+2s}} dy.$$

On the other hand, if ∂E is of class C^2 then the normalized mean curvature of ∂E is given, for $x \in \partial E$, by

$$H(x) := \frac{N-1}{N+1} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon |B_\varepsilon(x)|} \int_{B_\varepsilon(x)} \tau_E(y) \, dy,$$

see also (2.4) and [21]. Having fixed the above definitions, we now state our main result.

Theorem 1.1. *We let $s \in (0,1)$ and $E \subset \mathbb{R}^N$, $N \geq 2$. We assume, for $s \in (0,1/2)$, that ∂E is of class $C^{1,\beta}$ for some $\beta > 2s$ and that ∂E is of class C^3, for $s \in [1/2,1)$. Then, as $t \to 0$, the expansion of $v(t,y)$, defined in (1.8), is given, locally uniformly in $y \in \partial E$, by*

$$
v(t,y) = \begin{cases}
a_{N,s} H_s(y) + o_t(1) & \text{for } s \in (0,1/2) \\[2ex]
b_N(t) H(y) + O\left(\frac{1}{\log(\sigma_{1/2}(t))}\right) & \text{for } s = 1/2 \\[2ex]
c_{N,s} H(y) + O\left(t^{\frac{2s-1}{2}}\right) & \text{for } s \in (1/2,1),
\end{cases}
$$

where H_s and H are respectively the fractional and the classical mean curvatures of ∂E and the positive constants $a_{N,s}$, $b_{N,1/2}$ and $c_{N,s}$ are given by

$$
a_{N,s} = \frac{C_{N,s}}{2 \displaystyle\int_{\mathbb{R}^{N-1}} P_s(y',0)\,dy'}, \quad
b_N(t) = \frac{\displaystyle\int_{B^{N-1}_{\sigma_{\frac{1}{2}}(t)^{-1}}} |y'|^2 P_{1/2}(y',0)\,dy'}{2|\log(\sigma_{\frac{1}{2}}(t))| \displaystyle\int_{\mathbb{R}^{N-1}} P_{1/2}(y',0)\,dy'}, \quad
c_{N,s} = \frac{\displaystyle\int_{\mathbb{R}^{N-1}} |y'|^2 P_s(y',0)\,dy'}{2 \displaystyle\int_{\mathbb{R}^{N-1}} P_s(y',0)\,dy'}
$$

and $P_s(y) := \mathcal{K}_s(y,1)$.

Some remarks are in order. The assumption of E being of class C^3 in Theorem 1.1 is motivated by the result of Evans in [16], where in the case $s = 1$ and $\mathcal{K}_1$, the heat kernel, he obtained $v = (N-1)H(0) + O(t^{\frac{1}{2}})$. We notice that from our argument below, we cannot improve the error term $o_t(1)$ in the case $s \in (0,1/2)$ even if E is of class C^∞. This is due to the definition of the fractional mean curvature H_s as a principal value integral. Using polar coordinates and the estimates in (1.6), it easy follows that

$$
\frac{|S^{N-2}|C_{N,1/2}^{-1}}{2\displaystyle\int_{\mathbb{R}^{N-1}} P_{1/2}(y',0)dy'} \leq \lim_{t \to 0} b_N(t) \leq \frac{|S^{N-2}|C_{N,1/2}}{2\displaystyle\int_{\mathbb{R}^{N-1}} P_{1/2}(y',0)dy'}.
$$

This then justify the choice of the scaling when $s = \frac{1}{2}$. In the particular case, that $\mathcal{K}_{1/2}(y,t) = C_N \frac{t}{(t^2+|y|^2)^{\frac{N+1}{2}}}$, we have that

$$
b_N(t) = \frac{|S^{N-2}|C_N}{2\displaystyle\int_{\mathbb{R}^{N-1}} P_{1/2}(y',0)dy'} + O\left(\frac{1}{\log(\sigma_{1/2}(t))}\right).
$$

For the general case, we get the following

1.1. Some applications of Theorem 1.1

We next put emphasis on two valuable examples where Theorem 1.1 applies.

1) *Fractional heat diffusion of smooth sets.* We recall, see e.g., [6,35], that the fractional heat kernel K_s satisfies (1.5), (1.6) and (1.7) with

$$
C_{N,s} = s2^{2s} \frac{\sin(s\pi)\Gamma\left(\frac{N}{2}+s\right)\Gamma(s)}{\pi^{1+\frac{N}{2}}}. \tag{1.11}
$$

We recall that K_s is known explicitly only in the case $s = 1/2$, where $K_{1/2}(y, t) = C_{N,1/2} \frac{t}{(t^2+|y|^2)^{\frac{N+1}{2}}}$. In this case Theorem 1.1 provides an approximation of the (fractional) mean curvature motion by fractional heat diffusion of smooths sets, thereby extending, in the fractional setting, Evan's result in [16] on heat diffusion of smooth sets.

2) *Diffusion of smooth sets by Harmonic extension.* We consider the Poisson kernel on the half space $\mathbb{R}_+^{N+1} := \mathbb{R}^N \times (0, \infty)$, given by

$$\overline{K}_s(x, t) := t^{-N} P_s(x/t), \qquad P_s(x) = \frac{p_{N,s}}{(1 + |x|^2)^{\frac{N+2s}{2}}}, \tag{1.12}$$

where $p_{N,s} := \frac{1}{\int_{\mathbb{R}^N} (1+|y|^2)^{-\frac{N+2s}{2}} dy}$. Thanks to the result of Caffarelli and Silvestre in [8], the function

$$w(x, t) = \overline{K}_s(\cdot, t) \star \tau_E(x) = p_{N,s} t^{2s} \int_{\mathbb{R}^N} \frac{\tau_E(y)}{(t^2 + |y - x|^2)^{\frac{N+2s}{2}}} dy$$

solves

$$\begin{cases} \operatorname{div}(t^{1-2s} \nabla w) = 0 & \text{in } \mathbb{R}^N \times (0, \infty) \\ w = \tau_E & \text{on } \mathbb{R}^N \times \{t = 0\}. \end{cases}$$

It is clear that $\mathcal{K}_s(x, t) := \overline{K}_s(x, t^{\frac{1}{2s}})$ satisfies (1.5), (1.6) and (1.7) with $C_{N,s} = p_{N,s}$. Hence, Theorem 1.1 provides an expansion of the normal velocities of the boundary of the sets

$$E_t := \left\{ x \in \mathbb{R}^N : \overline{K}_s(\cdot, \sigma_s(t)^{\frac{1}{2s}}) \star \tau_E(x) > 0 \right\},$$

where $\sigma_s(t)$ is given by (1.9) and (1.10). Therefore this Harmonic extension yields an approximation of (fractional) mean curvature motion of smooth sets.

We conclude Section 1 by noting that the notion of nonlocal curvature appeared for the first time in [9]. Later on, the study of geometric problems involving fractional mean curvature has attracted a lot of interest, see [1, 7], the survey paper [21] and the references therein. While the mean curvature flow is well studied, see e.g., [2, 3, 15, 24, 26], its fractional counterpart appeared only recently in the literature, see e.g., [11–14, 27, 30, 33].

We finally remark that the changes of normal velocity of the nonlocal diffused sets as s varies in $(0,1/2)$ and $[1/2,1)$, appeared analogously in phases transition problems, see e.g., [22, 23, 33].

2. Preliminary results and notations

Unless otherwise stated, we assume for the following that E is an open set of class $C^{1,\beta}$, with $0 \in \partial E$ and the unit normal of ∂E at 0 coincides with e_N. We denote by $Q_r = B_r^{N-1} \times (-r, r)$ the cylinder of $\mathbb{R}^N$ centred at the origin with B_r^{N-1} the ball of $\mathbb{R}^{N-1}$ centred at the origin with radius $r > 0$. Decreasing r, if necessary, we may assume that

$$E \cap Q_r = \{(y', y_N) \in B_r^{N-1} \times \mathbb{R} : y_N > \gamma(y')\}, \tag{2.1}$$

with $\gamma \in C^{1,\beta}(B_r^{N-1})$ satisfying

$$\gamma(y') = O\left(|y'|^{1+\beta}\right). \tag{2.2}$$

In the following, for $f, g : \mathbb{R} \to \mathbb{R}$, we write $g(t) := O(f(t))$ if

$$|g(t)| \leq C|f(t)|.$$

We also write $g(t) = o(f(t))$ if $g(t) = O(f(t))$ and moreover when $f(t) \neq 0$, we have

$$\lim_{t \to 0} \frac{|g(t)|}{|f(t)|} = 0.$$

We denote by $o_t(1)$ any function that tends to zero when $t \to 0$. If in addition, ∂E is of class C^3, then for $y' \in B_r^{N-1}$, we have

$$\gamma(y') = \frac{1}{2} D^2 \gamma(0)[y', y'] + O\left(|y'|^3\right) \tag{2.3}$$

and the normalized mean curvature of ∂E at 0 is given by

$$H(0) = \frac{\Delta\gamma(0)}{N-1} = \frac{N-1}{N+1} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon|Q_\varepsilon(0)|} \int_{Q_\varepsilon(0)} \tau_E(y) \, dy. \tag{2.4}$$

Recall that the unit exterior normal $\nu(y') := \nu(y', \gamma(y'))$ of E and the volume element $d\sigma(y')$ on $\partial E \cap Q_r$ are given by

$$\nu(y') = \frac{(-\nabla\gamma(y'), 1)}{\sqrt{1 + |\nabla\gamma(y')|^2}} \quad \text{and} \quad d\sigma(y') = \sqrt{1 + |\nabla\gamma(y')|^2} dy'. \tag{2.5}$$

We finally note, in view of (1.5) and (1.6), that we have

$$0 < \mathcal{K}_s(y, t) \leq C \frac{t}{|y|^{N+2s}} \qquad \text{for all } y \in \mathbb{R}^N \setminus \{0\}, t > 0, \tag{2.6}$$

for some positive constant $C = C(N, s)$. We start with the following result.

Lemma 2.1. *Let $s \in (0, 1)$ and E be a set of class $C^{1,\beta}$ satisfying (2.1). Define*

$$w(z, t) = \int_{\mathbb{R}^N} \mathcal{K}_s(z - y, t)\tau_E(y) dy.$$

Then there exist $t_0, C > 0$, only depending on N, s, β and E, such that for all $t \in (0, t_0)$ and $z \in B_{t^{\frac{1}{2s}}}$,

$$\frac{\partial w}{\partial z_N}(z, t) \geq Ct^{-1/2s}. \tag{2.7}$$

As a consequence, for all $t \in (0, t_0)$, the set

$$\{z \in \mathbb{R}^N : w(z, t) = 0\} \cap B_{t^{\frac{1}{2s}}} \quad \text{is of class } C^1. \tag{2.8}$$

Proof. We fix $t > 0$ small so that $t^{\frac{1}{2s}} < \frac{r}{8}$ and let $z \in B_{t^{\frac{1}{2s}}}$. We write

$$\frac{\partial w}{\partial z_N}(z, t) = \int_{\mathbb{R}^N} \frac{\partial \mathcal{K}_s}{\partial z_N}(z - y, t)\tau_E(y) dy = \int_{B_{\frac{r}{2}}(z)} \frac{\partial \mathcal{K}_s}{\partial y_N}(z - y, t)\tau_E(y) dy + \int_{\mathbb{R}^N \setminus B_{\frac{r}{2}}(z)} \frac{\partial \mathcal{K}_s}{\partial y_N}(z - y, t)\tau_E(y) dy. \tag{2.9}$$

By a change of variable, (1.5) and (1.6), we have

$$
\int_{\mathbb{R}^N \setminus B_{\frac{r}{2}}(z)} \frac{\partial \mathcal{K}_s}{\partial y_N}(z - y, t)dy = t^{-\frac{1}{2s}} \int_{\mathbb{R}^N \setminus t^{-\frac{1}{2s}} B_{\frac{r}{2}}(z)} \frac{\partial P_s}{\partial y_N}(t^{-\frac{1}{2s}}z - y)dy
$$

$$
= O\left(t^{-\frac{1}{2s}} \int_{\mathbb{R}^N \setminus t^{-\frac{1}{2s}} B_{\frac{r}{2}}(z)} |t^{-\frac{1}{2s}}z - y|^{-N-1-2s}dy \right) = O(t). \qquad (2.10)
$$

Integrating by parts, we have

$$
\int_{B_{\frac{r}{2}}(z)} \frac{\partial \mathcal{K}_s}{\partial y_N}(z - y, t)\tau_E(y)dy = \int_{B_{\frac{r}{2}}(z) \cap E} \frac{\partial \mathcal{K}_s}{\partial y_N}(z - y, t)\tau_E(y)dy - \int_{B_{\frac{r}{2}}(z) \cap E^c} \frac{\partial \mathcal{K}_s}{\partial y_N}(z - y, t)dy
$$

$$
= 2 \int_{B_{\frac{r}{2}}(z) \cap \partial E} \mathcal{K}_s(z - y, t)e_N \cdot \nu_E(y)d\sigma(y) + \int_{\partial B_r(z)} \mathcal{K}_s(z - y, t)e_N \cdot \nu_{B_{\frac{r}{2}}}(y)\tau_E(y)d\sigma'(y).
$$

By a change of variable, (1.5), (1.6) and the fact that $Q_{r/8} \subset B_{r/4} \subset B_{\frac{r}{2}}(z) \subset Q_r$, we have

$$
\int_{B_{\frac{r}{2}}(z) \cap \partial E} \mathcal{K}_s(z - y, t)e_N \cdot \nu_E(y)d\sigma(y) \geq C \int_{B_{r/8}^{N-1}} \mathcal{K}_s(z' - y', z_N - \gamma(y'), t)dy'
$$

$$
= Ct^{-\frac{1}{2s}} \int_{B_{\frac{r}{8}t^{-\frac{1}{2s}}}^{N-1}} P_s(t^{-\frac{1}{2s}}z' - y', t^{-\frac{1}{2s}}z_N - t^{-\frac{1}{2s}}\gamma(t^{\frac{1}{2s}}y'))dy'
$$

$$
\geq Ct^{-\frac{1}{2s}} \int_{B_{\frac{r}{8}t^{-\frac{1}{2s}}}^{N-1} \setminus B_2} \frac{dy'}{1 + |y'|^{N+2s}}, \qquad (2.11)
$$

provided $\frac{r}{8} > t^{\frac{1}{2s}}$. Next, using (2.6) and recalling that $z \in B_{t^{\frac{1}{2s}}}$, we then have

$$
\left| \int_{\partial B_{\frac{r}{2}}(z)} \mathcal{K}_s(z - y, t)e_N \cdot \nu_{B_{\frac{r}{2}}(z)}(y)d\sigma'(y) \right| \leq \int_{\partial B_{\frac{r}{2}}(z)} \mathcal{K}_s(z - y, t)d\sigma'(y) \leq tC \int_{\partial B_{\frac{r}{2}}(z)} \frac{1}{|y|^{N+2s}}d\sigma'(y) = O(t).
$$

From this and (2.11), we deduce that

$$
\int_{B_r(z)} \frac{\partial \mathcal{K}_s}{\partial y_N}(z - y, t)\tau_E(y)dy \geq Ct^{-1/2s}.
$$

Combining this with (2.9) and (2.10), we get

$$
\frac{\partial w}{\partial z_N}(z, t) \geq Ct^{-1/2s}.
$$

Therefore (2.7) follows. Finally (2.8) follows from the inverse function theorem and the fact that w is of class C^1 on $\mathbb{R}^N \times (0, \infty)$. $\qquad \square$

In the sequel, we will need the following lemmas to estimate some error terms.

Lemma 2.2. *For $s \in (0, 1)$, we let $E \subset \mathbb{R}^N$ be a set of class $C^{1,\beta}$, for some $\beta > 2s$, as in Section 2. For $r > 0$, we set*

$$J_r(t) := \int_{Q_r} \mathcal{K}_s(y, t)\tau_E(y)dy \qquad \text{and} \qquad I_r(t) = \int_{\mathbb{R}^N \setminus Q_r} \left(t^{-1}\mathcal{K}_s(y, t) - \frac{C_{N,s}}{|y|^{N+2s}}\right)\tau_E(y)dy.$$

Then we have

$$|J_r(t)| \leq Ctr^{\beta-2s} \qquad \text{and} \qquad \lim_{t \to 0} I_r(t) = 0,$$

where C is a positive constant depending only on N, β, s and E.

Proof. Since $\tau_E = \mathbb{1}_E - \mathbb{1}_{\mathbb{R}^N \setminus \overline{E}}$, we get

$$J_r(t) = \int_{Q_r \cap E} \mathcal{K}_s(y, t)dy - \int_{Q_r \cap E^c} \mathcal{K}_s(y, t)dy$$

$$= \int_{B_r^{N-1}} \int_{\gamma(y')}^r \mathcal{K}_s((y', y_N), t)dy_N dy' - \int_{B_r^{N-1}} \int_{-r}^{\gamma(y')} \mathcal{K}_s((y', y_N), t)dy_N dy'$$

$$= \int_{B_r^{N-1}} \left(\int_{\gamma(y')}^r \mathcal{K}_s((y', y_N), t)dy_N - \int_{-r}^{\gamma(y')} \mathcal{K}_s((y', y_N), t)dy_N\right)dy'$$

$$= \int_{B_r^{N-1}} \left(\int_{\gamma(y')}^r \mathcal{K}_s((y', y_N), t)dy_N - \int_{-r}^{-\gamma(y')} \mathcal{K}_s((y', y_N), t)dy_N - \int_{-\gamma(y')}^{\gamma(y')} \mathcal{K}_s((y', y_N), t)dy_N\right)dy'$$

$$= \int_{B_r^{N-1}} \left(\int_{\gamma(y')}^r \mathcal{K}_s((y', y_N), t)dy_N + \int_r^{\gamma(y')} \mathcal{K}_s((y', -y_N), t)dy_N - \int_{-\gamma(y')}^{\gamma(y')} \mathcal{K}_s((y', y_N), t)dy_N\right)dy'.$$

Since the map $y \longmapsto K_s(y, t)$ is radial, we have $\mathcal{K}_s(y', y_N, t) = \mathcal{K}_s(y', -y_N, t)$ so that

$$\int_{\gamma(y')}^r \mathcal{K}_s((y', y_N), t)dy_N + \int_r^{\gamma(y')} \mathcal{K}_s((y', -y_N), t)dy_N = 0.$$

Therefore

$$J_r(t) = -\int_{B_r^{N-1}} \int_{-\gamma(y')}^{\gamma(y')} \mathcal{K}_s((y', y_N), t)dy_N dy' = -2\int_{B_r^{N-1}} \int_0^{\gamma(y')} \mathcal{K}_s((y', y_N), t)dy_N dy'.$$

Then, by (2.6),

$$|J_r(t)| \leq 2C_{N,s}t \left|\int_{B_r^{N-1}} \int_0^{\gamma(y')} \frac{1}{|(y', y_N)|^{N+2s}}dy_N dy'\right| \leq Ctr^{\beta-2s}.$$

where C is a positive constant depending on N, β and s and which may change from a line to another. Next, using (2.6), (1.7) and the dominate convergence theorem, we obtain

$$\lim_{t \to 0} I_r(t) = 0.$$

This then ends the proof. □

Lemma 2.3. *Let $s \in (0, 1)$ and let $x = vte_N \in \partial E_t \cap B_{\sigma_s(t)^{\frac{1}{2s}}}$, with E_t given by (1.4). Then*

$$vt = O\left((\sigma_s(t))^{\frac{1+2s}{2s}}\right) \qquad as\ t \to 0.$$

Proof. Since $x = vte_N \in \partial E_t$, we have that $u(x, \sigma_s(t)) = 0$. By the fundamental theorem of calculus, we have

$$u(x, \sigma_s(t)) = u(0, \sigma_s(t)) + vt \int_0^1 \frac{\partial u}{\partial x_N}(\theta v t e_N, \sigma_s(t))d\theta = 0$$

so that

$$vt \int_0^1 \frac{\partial u}{\partial x_N}(\theta v t e_N, \sigma_s(t))d\theta = -u(0, \sigma_s(t)). \tag{2.12}$$

We write

$$u(0, \sigma_s(t)) = \int_{\mathbb{R}^N} \mathcal{K}_s(y, \sigma_s(t))\tau_E(y)dy = \int_{Q_r} \mathcal{K}_s(y, \sigma_s(t))\tau_E(y)dy + \int_{Q_r^c} \mathcal{K}_s(y, \sigma_s(t))\tau_E(y)dy.$$

Then by Lemma 2.2 and (2.6), we have

$$\left| \int_{Q_r} \mathcal{K}_s(y, \sigma_s(t))\tau_E(y)dy \right| \le C\sigma_s(t) \qquad \text{and} \qquad \int_{Q_r^c} \mathcal{K}_s(y, \sigma_s(t))\tau_E(y)dy = O(\sigma_s(t)) \tag{2.13}$$

for some constant C depending on r. Furthermore by (2.7), we have

$$\frac{\partial u}{\partial x_N}(\theta v t e_N, \sigma_s(t)) \ge C(\sigma_s(t))^{-1/2s}. \tag{2.14}$$

Therefore, by (2.12), (2.13) and (2.14), we obtain

$$|vt| = \frac{u(0, \sigma_s(t))}{\left| \int_0^1 \frac{\partial u}{\partial x_N}(\theta v t e_n, \sigma_s(t))d\theta \right|} \le C \frac{\sigma_s(t)}{(\sigma_s(t))^{-\frac{1}{2s}}} = C(\sigma_s(t))^{\frac{1+2s}{2s}}.$$

This then ends the proof. $\qquad\square$

Lemma 2.4. *Under the assumptions of Lemma 2.3, we have*

$$\int_0^1 \int_{B_r^{N-1}} \int_0^{\gamma(y')-vt} y_N \frac{\partial \mathcal{K}_s}{\partial y_N}(y', \theta y_N, \sigma_s(t))dy d\theta = O(\sigma_s(t)) \qquad as\ t \to 0.$$

Proof. Let $\theta \in [0, 1]$. By (1.5), (1.6) and a change of variable, we have

$$\int_{B_r^{N-1}} \int_0^{\gamma(y')-vt} y_N \frac{\partial \mathcal{K}_s}{\partial y_N}(y', \theta y_N, \sigma_s(t))dy$$

$$= (\sigma_s(t))^{\frac{-N-1}{2s}} \int_{B_r^{N-1}} \int_0^{\gamma(y')-vt} y_N \frac{\partial P_s}{\partial y_N}(y'(\sigma_s(t))^{-\frac{1}{2s}}, \theta y_N(\sigma_s(t))^{-\frac{1}{2}})dy$$

$$\leq C \int_{B^{N-1}_{r(\sigma_s(t))^{-1/2s}}} \int_0^{(\sigma_s(t))^{-1/2s}\left(\gamma((\sigma_s(t))^{1/2s}y')-vt\right)} \frac{y_N}{1+|y'|^{N+2s+1}} dy$$

$$\leq C(\sigma_s(t))^{-1/s} \int_{B^{N-1}_{r(\sigma_s(t))^{-1/2s}}} \frac{\left(\gamma((\sigma_s(t))^{1/2s}y') - vt\right)^2}{1+|y'|^{N+2s+1}} dy$$

$$= O\left(\sigma_s(t)\right) + O\left(v^2 t^2 (\sigma_s(t))^{-1/s}\right) + O\left(vt\right).$$

Applying Lemma 2.3, we get

$$\int_0^{\gamma(y')-vt} y_N \frac{\partial \mathcal{K}_s}{\partial y_N}(y', \theta y_N, \sigma_s(t)) dy d\theta = O\left(\sigma_s(t)\right),$$

as $t \to 0$. This then ends the proof. $\qquad\square$

3. Proof of Theorem 1.1 in the case $s \in (0, \frac{1}{2})$

In this section, we start by the following preliminary result.

Lemma 3.1. *Let $s \in (0, 1/2)$. We assume that E is of class $C^{1,\beta}$ for some $\beta > 2s$ satisfying (2.1). Then, for all $\theta \in [0, 1]$, we have*

$$\int_{\mathbb{R}^N} \frac{\partial \mathcal{K}_s}{\partial y_N}(y', y_N - vt\theta, \sigma_s(t))\tau_E(y) dy = 2(\sigma_s(t))^{-1/2s} \int_{\mathbb{R}^{N-1}} P_s(y', 0) dy' + O\left((\sigma_s(t))^{\frac{2s-1}{2s}}\right). \tag{3.1}$$

Proof. We have

$$\int_{\mathbb{R}^N} \frac{\partial \mathcal{K}_s}{\partial y_N}(y', y_N - vt\theta, \sigma_s(t))\tau_E(y) dy = \int_{B_r} \frac{\partial \mathcal{K}_s}{\partial y_N}(y', y_N - vt\theta, \sigma_s(t))\tau_E(y) dy$$

$$+ \int_{B_r^c} \frac{\partial \mathcal{K}_s}{\partial y_N}(y', y_N - vt\theta, \sigma_s(t))\tau_E(y) dy,$$

where B_r is the ball of $\mathbb{R}^N$ centered at the origin and of radius $r > 0$. By integration by parts, we have

$$\int_{B_r} \frac{\partial \mathcal{K}_s}{\partial y_N}(y', y_N - vt\theta, \sigma_s(t))\tau_E(y) dy = 2 \int_{\partial E \cap B_r} \mathcal{K}_s(y', y_N - vt\theta, \sigma_s(t))\nu_N(y) d\sigma(y)$$

$$+ \int_{\partial B_r} \mathcal{K}_s(y', y_N - vt\theta, \sigma_s(t)) e_N \cdot \nu_{B_r}(y)\tau_E(y) d\sigma'(y).$$

Therefore

$$\int_{\mathbb{R}^N} \frac{\partial \mathcal{K}_s}{\partial y_N}(y', y_N - vt\theta, \sigma_s(t))\tau_E(y) dy = 2 \int_{\partial E \cap B_r} \mathcal{K}_s(y', y_N - vt\theta, \sigma_s(t))\nu_N(y') d\sigma(y')$$

$$+ \int_{B_r^c} \frac{\partial \mathcal{K}_s}{\partial y_N}(y', y_N - vt\theta, \sigma_s(t))\tau_E(y) dy + \int_{\partial B_r} \mathcal{K}_s(y', y_N - vt\theta, \sigma_s(t))\frac{y_N}{r}\tau_E(y) d\sigma'(y). \tag{3.2}$$

Then by a change of variable and (2.5), we have

$$\int_{\partial E \cap B_r} \mathcal{K}_s(y', y_N - vt\theta, \sigma_s(t))v_N(y)d\sigma(y) = \int_{B_r^{N-1}} \mathcal{K}_s(y', \gamma(y') - vt\theta, \sigma_s(t))dy'.$$

By the Fundamental Theorem of calculus, we can write

$$\mathcal{K}_s(y', \gamma(y') - vt\theta, \sigma_s(t)) = \mathcal{K}_s(y', 0, \sigma_s(t)) + (\gamma(y') - vt\theta)\int_0^1 \frac{\partial \mathcal{K}_s}{\partial y_N}(y', \theta'(\gamma(y') - vt\theta)), \sigma_s(t))d\theta'. \quad (3.3)$$

In the following, we let

$$\varepsilon(y') := \gamma(y') - vt\theta. \quad (3.4)$$

Then we have

$$\int_{\partial E \cap B_r} \mathcal{K}_s(y', y_N - vt\theta, \sigma_s(t))v_N(y)d\sigma(y) = \int_{B_r^{N-1}} \mathcal{K}_s(y', \varepsilon(y'), \sigma_s(t))dy'$$

$$= \int_{B_r^{N-1}} \mathcal{K}_s(y', 0, \sigma_s(t))dy' + \int_0^1 \int_{B_r^{N-1}} \varepsilon(y')\frac{\partial \mathcal{K}_s}{\partial y_N}(y', \theta'\varepsilon(y'), \sigma_s(t))dy'd\theta'. \quad (3.5)$$

Therefore By a change of variable, (1.5) and (1.6), we have

$$\int_{B_r^{N-1}} \mathcal{K}_s(y', 0, \sigma_s(t))dy' = \int_{B_r^{N-1}} (\sigma_s(t))^{\frac{-N}{2s}} P_s(y'(\sigma_s(t))^{-1/2s}, 0)dy'$$

$$= (\sigma_s(t))^{-1/2s}\int_{\mathbb{R}^{N-1}} P_s(y', 0)dy' + (\sigma_s(t))^{-1/2s}\int_{\mathbb{R}^{N-1}\setminus B_{r(\sigma_s(t))^{-1/2s}}^{N-1}} P_s(y', 0)dy'$$

$$= (\sigma_s(t))^{-1/2s}\int_{\mathbb{R}^{N-1}} P_s(y', 0)dy' + O\left((\sigma_s(t))^{-1/2s}(\sigma_s(t))^{\frac{1+2s}{2s}}\right). \quad (3.6)$$

By a change of variable, (1.6) and (3.4), we have

$$\int_{B_r^{N-1}} \varepsilon(y')\frac{\partial \mathcal{K}_s}{\partial y_N}(y', \theta'\varepsilon(y'), \sigma_s(t))dy'$$

$$= (\sigma_s(t))^{-1/s}\int_{B_{r(\sigma_s(t))^{-1/2s}}^{N-1}} \varepsilon(y'(\sigma_s(t))^{1/2s})\frac{\partial P_s}{\partial y_N}(y', \theta'(\sigma_s(t))^{-1/2s}\varepsilon(y'(\sigma_s(t))^{1/2s}))dy'.$$

We use (2.2), (3.4) and Lemma 2.3 to get

$$(\sigma_s(t))^{-1/s}\varepsilon(y'(\sigma_s(t))^{1/2s}) = O\left(|y'|^{1+\beta}(\sigma_s(t))^{\frac{\beta-1}{2s}}\right) - vt\theta(\sigma_s(t))^{-1/s}$$

$$= O\left(|y'|^{1+\beta}(\sigma_s(t))^{\frac{2s-1}{2s}}\right) + O\left((\sigma_s(t))^{\frac{2s-1}{2s}}\right) \quad \text{in } B_{r(\sigma_s(t))^{-1/2s}}^{N-1}.$$

Then by (1.6), we have

$$\int_{B_r^{N-1}} \varepsilon(y')\frac{\partial \mathcal{K}_s}{\partial y_N}(y', \theta'\varepsilon(y'), \sigma_s(t))dy'$$

$$= O\left((\sigma_s(t))^{\frac{2s-1}{2s}}\int_{B_{r(\sigma_s(t))^{-1/2s}}^{N-1}} |y'|^{1+\beta}\frac{\partial P_s}{\partial y_N}(y', \theta'(\sigma_s(t))^{-1/2s}\varepsilon(y'(\sigma_s(t))^{1/2s}))dy'\right)$$

$$+ O\left((\sigma_s(t))^{\frac{2s-1}{2s}} \int_{B_{r(\sigma_s(t))^{-1/2s}}^{N-1}} \frac{\partial P_s}{\partial y_N}(y', \theta'(\sigma_s(t))^{-1/2s}\varepsilon(y'(\sigma_s(t))^{1/2s}))dy'\right)$$

$$= O\left((\sigma_s(t))^{\frac{2s-1}{2s}} \int_{\mathbb{R}^{N-1}} \frac{1 + |y'|^{1+\beta}}{1 + |y'|^{N+2s+1}}dy'\right) = O\left((\sigma_s(t))^{\frac{2s-1}{2s}}\right).$$

Hence

$$\int_0^1 \int_{B_r^{N-1}} \varepsilon(y')\frac{\partial \mathcal{K}_s}{\partial y_N}(y', \theta'\varepsilon(y'), \sigma_s(t))dy'd\theta' = O\left((\sigma_s(t))^{\frac{2s-1}{2s}}\right) \qquad \text{as } t \to 0. \qquad (3.7)$$

It follows from (3.5), (3.6) and (3.7) that

$$\int_{\partial E \cap B_r} \mathcal{K}_s(y', y_N - vt\theta, \sigma_s(t))v_N(y)d\sigma(y) = (\sigma_s(t))^{-1/2s} \int_{\mathbb{R}^{N-1}} P_s(y', 0)dy' + O\left((\sigma_s(t))^{\frac{2s-1}{2s}}\right) \qquad \text{as } t \to 0. \tag{3.8}$$

By a change of variable and the fact that $|\tau_E(y)| \le 1$, we have

$$\int_{B_r^c} \frac{\partial \mathcal{K}_s}{\partial y_N}(y', y_N - vt\theta, \sigma_s(t))\tau_E(y)dy = O\left((\sigma_s(t))^{-1/2s} \int_{B_{r(\sigma_s(t))^{-1/2s}}^c} \frac{\partial P_s}{\partial y_N}(y', y_N - vt(\sigma_s(t))^{-1/2s}\theta)dy\right)$$

$$= O\left((\sigma_s(t))^{-1/2s} \int_{B_{r(\sigma_s(t))^{-1/2s}}^c} \frac{1}{1 + |y|^{N+2s+1}}dy\right) = O(\sigma_s(t)). \qquad (3.9)$$

We use (1.7) to get, as $t \to 0$,

$$\left|\int_{\partial B_r} \mathcal{K}_s(y', y_N - vt\theta, \sigma_s(t))\frac{y_N}{r}\tau_E(y)d\sigma'(y)\right| \le \int_{\partial B_r} \mathcal{K}_s(y', y_N - vt\theta, \sigma_s(t))d\sigma'(y)$$

$$\le \sigma_s(t) \int_{\partial B_r} \frac{C_{N,s}}{|(y', y_N - vt\theta)|^{N+2s}}d\sigma'(y) = O(\sigma_s(t)).$$

Therefore, the expansion (3.1) follows immediately from (3.2), (3.8), (3.9) and the above estimate. This ends the proof. □

The following result completes the proof of Theorem 1.1 in the case $s \in (0, 1/2)$.

Proposition 3.2. *Under the assumptions of Lemma 3.1, we have*

$$v = a_{N,s}H_s(0) + o_t(1) \qquad \text{as } t \to 0, \tag{3.10}$$

where $H_s(0)$ is the fractional mean curvature of ∂E at the point 0 and the positive constant $a_{N,s}$ is given by

$$a_{N,s} = \frac{C_{N,s}}{2\int_{\mathbb{R}^{N-1}} P_s(y', 0)\, dy'}.$$

Proof. We put with $x = vte_N$ and we recall that

$$u(x, \sigma_s(t)) = \int_{\mathbb{R}^N} \mathcal{K}_s(y - x, \sigma_s(t))\tau_E(y)dy = 0.$$

By the fundamental theorem of calculus, we have

$$\mathcal{K}_s(y - x, \sigma_s(t)) = \mathcal{K}_s(y, \sigma_s(t)) - vt \int_0^1 \frac{\partial \mathcal{K}_s}{\partial y_N}(y', y_N - vt\theta, \sigma_s(t))d\theta.$$

Then

$$u(x, \sigma_s(t)) = \widetilde{J}_r(t) + \sigma_s(t)\widetilde{I}_r(t) + \sigma_s(t)C_{N,s} \int_{\mathbb{R}^N \backslash Q_r} \frac{\tau_E(y)}{|y|^{N+2s}}dy - vt \int_{\mathbb{R}^N} \int_0^1 \frac{\partial \mathcal{K}_s}{\partial y_N}(y', y_N - vt\theta, \sigma_s(t))\tau_E(y)d\theta dy,$$

(3.11)

where $\widetilde{J}_r(t) = J_r(\sigma_s(t))$, $\widetilde{I}_r(t) = I_r(\sigma_s(t))$, while $I_r(t)$ and $J_r(t)$ are given by Lemma 2.2. Moreover, by Lemma 3.1, we have

$$\int_{\mathbb{R}^N} \frac{\partial \mathcal{K}_s}{\partial y_N}(y', y_N - vt\theta, \sigma_s(t))\tau_E(y)dy = 2(\sigma_s(t))^{-1/2s} \int_{\mathbb{R}^{N-1}} P_s(y', 0)dy' + O\left((\sigma_s(t))^{\frac{2s-1}{2s}}\right).$$

Therefore

$$\int_0^1 \int_{\mathbb{R}^N} \frac{\partial \mathcal{K}_s}{\partial y_N}(y', y_N - vt\theta, \sigma_s(t))\tau_E(y)dyd\theta = 2(\sigma_s(t))^{-1/2s} \int_{\mathbb{R}^{N-1}} P_s(y', 0)dy' + O\left((\sigma_s(t))^{\frac{2s-1}{2s}}\right) \qquad \text{as } t \to 0.$$

(3.12)

Putting (3.12) in (3.11), we obtain that

$$u(x, \sigma_s(t)) = \widetilde{J}_r(t) + \sigma_s(t)\widetilde{I}_r(t) + \sigma_s(t)C_{N,s} \int_{\mathbb{R}^N \backslash Q_r} \frac{\tau_E(y)}{|y|^{N+2s}}dy$$

$$- vt \left(2(\sigma_s(t))^{-1/2s} \int_{\mathbb{R}^{N-1}} P_s(y', 0)dy' + O\left((\sigma_s(t))^{\frac{2s-1}{2s}}\right)\right)$$

$$= \sigma_s(t)\left[(\sigma_s(t))^{-1}\widetilde{J}_r(t) + \widetilde{I}_r(t) + C_{N,s} \int_{\mathbb{R}^N \backslash Q_r} \frac{\tau_E(y)}{|y|^{N+2s}}dy - 2vt(\sigma_s(t))^{\frac{-2s-1}{2s}} \int_{\mathbb{R}^{N-1}} P_s(y', 0)dy' + O(\sigma_s(t))\right].$$

Recalling that $\sigma_s(t) = t^{\frac{2s}{1+2s}}$ and using the fact that $u(x, \sigma_s(t)) = 0$, we have

$$0 = t^{\frac{-2s}{1+2s}}\widetilde{J}_r(t) + \widetilde{I}_r(t) + C_{N,s} \int_{\mathbb{R}^N \backslash Q_r} \frac{\tau_E(y)}{|y|^{N+2s}}dy - 2v \int_{\mathbb{R}^{N-1}} P_s(y', 0)dy' + O(t^{\frac{2s}{1+2s}}) \qquad \text{as } t \to 0.$$

As a consequence,

$$\left|C_{N,s}H_s(0) - 2v \int_{\mathbb{R}^{N-1}} P_s(y', 0)dy'\right| \leq \left|C_{N,s}H_s(0) - C_{N,s} \int_{\mathbb{R}^N \backslash Q_r} \frac{\tau_E(y)}{|y|^{N+2s}}dy\right| + |t^{\frac{-2s}{1+2s}}\widetilde{J}_r(t)| + \widetilde{I}_r(t) + O(t^{\frac{2s}{1+2s}}).$$

Therefore by Lemma 2.2, taking the limsups as $t \to 0$ and as $r \to 0$ respectively, we obtain

$$\limsup_{t \to 0} \left|C_{N,s}H_s(0) - 2v \int_{\mathbb{R}^{N-1}} P_s(y', 0)dy'\right| = 0.$$

Hence

$$v = \frac{C_{N,s}H_s(0)}{2 \int_{\mathbb{R}^{N-1}} P_s(y', 0)dy'} + o_t(1) \qquad \text{as } t \to 0.$$

This then ends the proof. □

4. Proof of Theorem 1.1 in the case $s \in (\frac{1}{2}, 1)$

We have the following result.

Proposition 4.1. *We consider E a set of class C^3 satisfying the condition in Section 2. For $s \in (1/2, 1)$, we have*

$$v = c_{N,s}H(0) + O\left(t^{\frac{2s-1}{2}}\right), \qquad as\ t \to 0. \tag{4.1}$$

where $H(0)$ is the normalized mean curvature of ∂E at 0 and the positive constant $c_{N,s}$ is given by

$$c_{N,s} = \frac{\displaystyle\int_{\mathbb{R}^{N-1}} |y'|^2 P_s(y', 0)dy'}{2\displaystyle\int_{\mathbb{R}^{N-1}} P_s(y', 0)dy'}.$$

Proof. We let $x = vte_N \in \partial E_t$ and we expand

$$u(x, \sigma_s(t)) = \int_{\mathbb{R}^N} \mathcal{K}_s(y - x, \sigma_s(t))\tau_E(y)dy = \int_{Q_r} \mathcal{K}_s(y - x, \sigma_s(t))\tau_E(y)dy + \int_{Q_r^c} \mathcal{K}_s(y - x, \sigma_s(t))\tau_E(y)dy, \tag{4.2}$$

By (2.6) and Lemma 2.3, we have

$$\int_{Q_r^c} \mathcal{K}_s(y - x, \sigma_s(t))\tau_E(y)dy = O(\sigma_s(t)) \qquad as\ t \to 0.$$

Therefore

$$u(x, \sigma_s(t)) = \int_{Q_r} \mathcal{K}_s(y - x, \sigma_s(t))\tau_E(y)dy + O(\sigma_s(t)). \tag{4.3}$$

By a change of variable, the fact that $\tau_E = \mathbb{1}_E(x) - \mathbb{1}_{\mathbb{R}^N \setminus \overline{E}}(x)$ and $x = vte_N$, we have

$$\int_{Q_r} \mathcal{K}_s(y - x, \sigma_s(t))\tau_E(y)dy = \int_{E \cap Q_r} \mathcal{K}_s(y - x, \sigma_s(t))dy - \int_{E^c \cap Q_r} \mathcal{K}_s(y - x, \sigma_s(t))dy$$

$$= \int_{B_r^{N-1}} \int_{-r}^{\gamma(y')} \mathcal{K}_s(y - x, \sigma_s(t))dy - \int_{B_r^{N-1}} \int_{\gamma(y')}^{r} \mathcal{K}_s(y - x, \sigma_s(t))dy$$

$$= \int_{B_r^{N-1}} \int_{-r-vt}^{\gamma(y')-vt} \mathcal{K}_s(y, \sigma_s(t))dy - \int_{B_r^{N-1}} \int_{\gamma(y')-vt}^{r-vt} \mathcal{K}_s(y, \sigma_s(t))dy$$

$$= 2\int_{B_r^{N-1}} \int_{0}^{\gamma(y')-vt} \mathcal{K}_s(y, \sigma_s(t))dy + \int_{B_r^{N-1}} \int_{-r-vt}^{0} \mathcal{K}_s(y, \sigma_s(t))dy - \int_{B_r^{N-1}} \int_{0}^{r-vt} \mathcal{K}_s(y, \sigma_s(t))dy$$

$$= 2\int_{B_r^{N-1}} \int_{0}^{\gamma(y')-vt} \mathcal{K}_s(y, \sigma_s(t))dy + \int_{B_r^{N-1}} \int_{-r-vt}^{-r+vt} \mathcal{K}_s(y, \sigma_s(t))dy.$$

The last line is due to the fact that the map $y_N \to \mathcal{K}_s(y, \sigma_s(t))$ is even so that

$$\int_0^{r-vt} \mathcal{K}_s(y, \sigma_s(t))dy_N = -\int_0^{-r+vt} \mathcal{K}_s(y, \sigma_s(t))dy_N.$$

Therefore we have

$$\int_{Q_r} \mathcal{K}_s(y - x, \sigma_s(t))\tau_E(y)dy = 2\int_{B_r^{N-1}} \int_0^{\gamma(y')-vt} \mathcal{K}_s(y, \sigma_s(t))dy + \int_{B_r^{N-1}} \int_{-r-vt}^{-r+vt} \mathcal{K}_s(y, \sigma_s(t))dy. \quad (4.4)$$

By (2.6) and the fact that $vt = o_t(1)$, we have

$$\int_{B_r^{N-1}} \int_{-r-vt}^{-r+vt} \mathcal{K}_s(y, \sigma_s(t))dy = O(\sigma_s(t)). \quad (4.5)$$

By a change of variable, the Fundamental Theorem of Calculus, (1.5) and (2.3), we have

$$\int_{B_r^{N-1}} \int_0^{\gamma(y')-vt} \mathcal{K}_s(y, \sigma_s(t))dy$$

$$= \int_{B_r^{N-1}} \int_0^{\gamma(y')-vt} \mathcal{K}_s(y', 0, \sigma_s(t))dy + \int_{B_r^{N-1}} \int_0^{\gamma(y')-vt} \int_0^1 y_N \frac{\partial \mathcal{K}_s}{\partial y_N}(y', \theta y_N, \sigma_s(t))dyd\theta$$

$$= \int_{B_r^{N-1}} \mathcal{K}_s(y', 0, \sigma_s(t))\left(\frac{1}{2}\gamma_{y_iy_j}(0)y_iy_j + O(|y'|^3) - vt\right)dy'$$

$$+ \int_{B_r^{N-1}} \int_0^{\gamma(y')-vt} \int_0^1 y_N \frac{\partial \mathcal{K}_s}{\partial y_N}(y', \theta y_N, \sigma_s(t))dyd\theta$$

$$= \frac{\Delta\gamma(0)}{2(N-1)} \int_{B_r^{N-1}} |y'|^2 \mathcal{K}_s(y', 0, \sigma_s(t))dy' - vt \int_{B_r^{N-1}} \mathcal{K}_s(y', 0, \sigma_s(t))dy'$$

$$+ \int_{B_r^{N-1}} \int_0^{\gamma(y')-vt} \int_0^1 y_N \frac{\partial \mathcal{K}_s}{\partial y_N}(y', \theta y_N, \sigma_s(t))dyd\theta + O\left(\int_{B_r^{N-1}} |y'|^3 \mathcal{K}_s(y', 0, \sigma_s(t))dy'\right).$$

Therefore, recalling (2.4),

$$\int_{B_r^{N-1}} \int_0^{\gamma(y')-vt} \mathcal{K}_s(y, \sigma_s(t))dy = \frac{H(0)}{2} \int_{B_r^{N-1}} |y'|^2 \mathcal{K}_s(y', 0, \sigma_s(t))dy' - vt \int_{B_r^{N-1}} \mathcal{K}_s(y', 0, \sigma_s(t))dy'$$

$$+ \int_{B_r^{N-1}} \int_0^{\gamma(y')-vt} \int_0^1 y_N \frac{\partial \mathcal{K}_s}{\partial y_N}(y', \theta y_N, \sigma_s(t))dyd\theta$$

$$+ O\left(\int_{B_r^{N-1}} |y'|^3 \mathcal{K}_s(y', 0, \sigma_s(t))dy'\right). \quad (4.6)$$

By a change of variable and (1.5), we have

$$\int_{B_r^{N-1}} |y'|^2 \mathcal{K}_s(y', 0, \sigma_s(t))dy' = (\sigma_s(t))^{\frac{1}{2s}} \int_{B^{N-1}_{r(\sigma_s(t))^{-\frac{1}{2s}}}} |y'|^2 P_s(y', 0)dy' \quad (4.7)$$

and

$$\int_{B_r^{N-1}} \mathcal{K}_s(y', 0, \sigma_s(t))dy' = (\sigma_s(t))^{-\frac{1}{2s}} \int_{B_{r(\sigma_s(t))^{-\frac{1}{2s}}}^{N-1}} P_s(y', 0)dy'. \tag{4.8}$$

Moreover by (1.6), we get

$$(\sigma_s(t))^{\frac{1}{2s}} \int_{\mathbb{R}^{N-1}\setminus B_{r(\sigma_s(t))^{-\frac{1}{2s}}}^{N-1}} |y'|^2 P_s(y', 0)dy' + (\sigma_s(t))^{\frac{-1}{2s}} \int_{\mathbb{R}^{N-1}\setminus B_{r(\sigma_s(t))^{-\frac{1}{2s}}}^{N-1}} P_s(y', 0)dy' = O(\sigma_s(t)) \qquad \text{as } t \to 0 \tag{4.9}$$

and

$$\int_{B_r^{N-1}} |y'|^3 \mathcal{K}_s(y', 0, \sigma_s(t))dy' = O(\sigma_s(t)). \tag{4.10}$$

By Lemma 2.4, we get

$$\int_{B_r^{N-1}} \int_0^{\gamma(y')-vt} \int_0^1 y_N \frac{\partial \mathcal{K}_s}{\partial y_N}(y', \theta y_N, \sigma_s(t))dyd\theta = O\left(\sigma_s(t)\right). \tag{4.11}$$

Combining (4.4), (4.6), (4.7), (4.8), (4.9) and (4.11), we obtain

$$\int_{Q_r} \mathcal{K}_s(y - x, \sigma_s(t))\tau_E(y)dy = (\sigma_s(t))^{\frac{1}{2s}} \frac{H(0)}{2} \int_{\mathbb{R}^{N-1}} |y'|^2 P_s(y', 0)dy'$$
$$- vt(\sigma_s(t))^{\frac{-1}{2s}} \int_{\mathbb{R}^{N-1}} P_s(y', 0)dy' + O\left(\sigma_s(t)\right). \tag{4.12}$$

By (4.3) and (4.12), we obtain

$$u(x, \sigma_s(t)) = (\sigma_s(t))^{1/2s} H(0) \int_{\mathbb{R}^{N-1}} |y'|^2 P_s(y', 0)dy' - 2vt(\sigma_s(t))^{-1/2s} \int_{\mathbb{R}^{N-1}} P_s(y', 0)dy' + O\left(\sigma_s(t)\right)$$
$$= (\sigma_s(t))^{-1/2s} \left[(\sigma_s(t))^{1/s} H(0) \int_{\mathbb{R}^{N-1}} |y'|^2 P_s(y', 0)dy' - 2vt \int_{\mathbb{R}^{N-1}} P_s(y', 0)dy' + O\left((\sigma_s(t))^{\frac{1+2s}{2s}}\right) \right].$$

Since $x = vt\nu \in \partial E_t$, we have $u(x, \sigma_s(t)) = 0$. Now, from the definition of $\sigma_s(t) = t^s$, we deduce that

$$H(0) \int_{\mathbb{R}^{N-1}} |y'|^2 P_s(y', 0)dy' - 2v \int_{\mathbb{R}^{N-1}} P_s(y', 0)dy' + O\left(t^{\frac{2s-1}{2}}\right) = 0.$$

Thus

$$v = c_{N,s} H(0) + O(t^{\frac{2s-1}{2}}),$$

where

$$c_{N,s} = \frac{\int_{\mathbb{R}^{N-1}} |y'|^2 P_s(y', 0)dy'}{2 \int_{\mathbb{R}^{N-1}} P_s(y', 0)dy'}.$$

This then ends the proof. □

5. Proof of Theorem 1.1 in the case $s = \frac{1}{2}$

As usual, we consider the function

$$u(x, t) = \mathcal{K}_{1/2}(\cdot, t) \star \tau_E(x)$$

and recall that

$$E_t := \{x \in \mathbb{R}^N \quad : \quad u(x, \sigma_{1/2}(t)) \geq 0\}.$$

To alleviate the notations, for the following of this section, we write $\sigma_{1/2}(t) := \sigma(t)$.

Proposition 5.1. *For $s = 1/2$, we have*

$$v = b_N(t)H(0) + O\left(\frac{1}{\log(\sigma_{1/2}(t))}\right) \qquad \text{as } t \to 0, \tag{5.1}$$

where $H(0)$ is the mean curvature of ∂E at 0.

Proof. Recall that $x = vt\nu \to 0$ as $t \to 0$, thanks to Lemma 2.3. We write

$$u(x, \sigma(t)) = \int_{\mathbb{R}^N} \mathcal{K}_{1/2}(y - x, \sigma(t))\tau_E(y)dy = \int_{Q_r} \mathcal{K}_{1/2}(y - x, \sigma(t))\tau_E(y)dy + \int_{Q_r^c} \mathcal{K}_{1/2}(y - x, \sigma(t))\tau_E(y)dy, \tag{5.2}$$

where $Q_r = B_r^{N-1} \times (-r, r)$. By (2.6), we have

$$\int_{Q_r^c} \mathcal{K}_{1/2}(y - x, \sigma(t))\tau_E(y)dy = O(\sigma(t)) \qquad \text{as } t \to 0.$$

Then, we have

$$u(x, \sigma(t)) = \int_{Q_r} \mathcal{K}_{1/2}(y - x, \sigma(t))\tau_E(y)dy + O(\sigma(t)). \tag{5.3}$$

By a change of variable and (2.3), we have

$$\int_{Q_r} \mathcal{K}_{1/2}(y - x, \sigma(t))\tau_E(y)dy = \int_{E \cap Q_r} \mathcal{K}_{1/2}(y - x, \sigma(t))dy - \int_{E^c \cap Q_r} \mathcal{K}_{1/2}(y - x, \sigma(t))dy$$

$$= \int_{B_r^{N-1}} \int_{-r}^{\gamma(y')} \mathcal{K}_{1/2}(y - x, \sigma(t))dy - \int_{B_r^{N-1}} \int_{\gamma(y')}^{r} \mathcal{K}_{1/2}(y - x, \sigma(t))dy$$

$$= \int_{B_r^{N-1}} \int_{-r-vt}^{\gamma(y')-vt} \mathcal{K}_{1/2}(y, \sigma(t))dy - \int_{B_r^{N-1}} \int_{\gamma(y')-vt}^{r-vt} \mathcal{K}_{1/2}(y, \sigma(t))dy$$

$$= 2\int_{B_r^{N-1}} \int_{0}^{\gamma(y')-vt} \mathcal{K}_{1/2}(y, \sigma(t))dy + \int_{B_r^{N-1}} \int_{-r-vt}^{0} \mathcal{K}_{1/2}(y, \sigma(t))dy - \int_{B_r^{N-1}} \int_{0}^{r-vt} \mathcal{K}_{1/2}(y, \sigma(t))dy$$

$$= 2\int_{B_r^{N-1}} \int_{0}^{\gamma(y')-vt} \mathcal{K}_{1/2}(y, \sigma(t))dy + \int_{B_r^{N-1}} \int_{-r-vt}^{-r+vt} \mathcal{K}_{1/2}(y, \sigma(t))dy.$$

The last line is due to the fact that the map $y_N \to \mathcal{K}_{1/2}(y, \sigma(t))$ is even so that

$$\int_{0}^{r-vt} \mathcal{K}_{1/2}(y, \sigma(t))dy_N = -\int_{0}^{-r+vt} \mathcal{K}_{1/2}(y, \sigma(t))dy_N.$$

Therefore we have

$$\int_{Q_r} \mathcal{K}_{1/2}(y-x, \sigma(t))\tau_E(y)dy = 2\int_{B_r^{N-1}} \int_0^{\gamma(y')-vt} \mathcal{K}_{1/2}(y, \sigma(t))dy + \int_{B_r^{N-1}} \int_{-r-vt}^{-r+vt} \mathcal{K}_{1/2}(y, \sigma(t))dy. \quad (5.4)$$

Using (2.6), we find that

$$\int_{B_r^{N-1}} \int_{-r-vt}^{-r+vt} \mathcal{K}_{1/2}(y, \sigma(t))dy = O(\sigma(t)). \quad (5.5)$$

By a change of variable, the fundamental theorem of calculus, (1.5) and (2.3), we have

$$\int_{B_r^{N-1}} \int_0^{\gamma(y')-vt} \mathcal{K}_{1/2}(y, \sigma(t))dy$$

$$= \int_{B_r^{N-1}} \int_0^{\gamma(y')-vt} \mathcal{K}_{1/2}(y', 0, \sigma(t))dy + \int_{B_r^{N-1}} \int_0^{\gamma(y')-vt} \int_0^1 y_N \frac{\partial \mathcal{K}_{1/2}}{\partial y_N}(y', \theta y_N, \sigma(t))dyd\theta$$

$$= \int_{B_r^{N-1}} \mathcal{K}_{1/2}(y', 0, \sigma(t))\left(\frac{1}{2}\gamma_{y_i y_j}(0)y_i y_j + O(|y'|^3) - vt\right)dy'$$

$$+ \int_{B_r^{N-1}} \int_0^{\gamma(y')-vt} \int_0^1 y_N \frac{\partial \mathcal{K}_{1/2}}{\partial y_N}(y', \theta y_N, \sigma(t))dyd\theta$$

$$= \frac{\Delta\gamma(0)}{2(N-1)} \int_{B_r^{N-1}} |y'|^2 \mathcal{K}_{1/2}(y', 0, \sigma(t))dy' - vt \int_{B_r^{N-1}} \mathcal{K}_{1/2}(y', 0, \sigma(t))dy'$$

$$+ \int_{B_r^{N-1}} \int_0^{\gamma(y')-vt} \int_0^1 y_N \frac{\partial \mathcal{K}_{1/2}}{\partial y_N}(y', \theta y_N, \sigma(t))dyd\theta + O\left(\int_{B_r^{N-1}} |y'|^3 \mathcal{K}_{1/2}(y', 0, \sigma(t))dy'\right).$$

Therefore

$$\int_{B_r^{N-1}} \int_0^{\gamma(y')-vt} \mathcal{K}_{1/2}(y, \sigma(t))dy = \frac{H(0)}{2} \int_{B_r^{N-1}} |y'|^2 \mathcal{K}_{1/2}(y', 0, \sigma(t))dy'$$

$$- vt \int_{B_r^{N-1}} \mathcal{K}_{1/2}(y', 0, \sigma(t))dy' + \int_{B_r^{N-1}} \int_0^{\gamma(y')-vt} \int_0^1 y_N \frac{\partial \mathcal{K}_{1/2}}{\partial y_N}(y', \theta y_N, \sigma(t))dyd\theta + O(\sigma(t)). \quad (5.6)$$

By (1.5), (1.6) and a change of variable, we have

$$\int_{B_r^{N-1}} \int_0^{\gamma(y')-vt} y_N \frac{\partial \mathcal{K}_{1/2}}{\partial y_N}(y', \theta y_N, \sigma(t))dy$$

$$= O\left(\int_{B_{r\sigma(t)^{-1}}^{N-1}} \int_0^{\sigma(t)^{-1}(\gamma(y'\sigma(t))-vt)} \frac{y_N}{(1+|(y', \theta y_N)|^2)^{\frac{N+2}{2}}}dy\right) = O\left(\int_{B_{r\sigma(t)^{-1}}^{N-1}} \frac{\sigma(t)^{-2}(\gamma(y'\sigma(t))-vt)^2}{(1+|y'|^2)^{\frac{N+2}{2}}}dy'\right)$$

$$= O\left(\sigma^{-2}(t) \int_{B^{N-1}_{r\sigma(t)^{-1}}} \frac{\left(\sigma(t)^2|y'|^2 - vt\right)^2}{(1 + |y'|^2)^{\frac{N+2}{2}}} dy'\right) = O\left(\sigma(t)\right) + O(vt) + O\left(v^2 t^2 (\sigma(t))^{-2}\right).$$

Now Lemma 2.3 yields $vt = O(\sigma(t)^2)$ and thus

$$\int_{B^{N-1}_r} \int_0^{\gamma(y')-vt} \int_0^1 y_N \frac{\partial \mathcal{K}_{1/2}}{\partial y_N}(y', \theta y_N, \sigma(t)) dy d\theta = O\left(\sigma(t)\right). \tag{5.7}$$

We get from (5.3), (5.4), (5.5), (5.6) and (5.7) that

$$u(x, \sigma(t)) = \sigma(t)H(0) \int_{B^{N-1}_{r\sigma(t)^{-1}}} |y'|^2 P_{1/2}(y', 0) dy' - 2vt \int_{B^{N-1}_{r\sigma(t)^{-1}}} P_{1/2}(y', 0) dy' + O(\sigma(t)) \quad \text{as } t \to 0.$$

Thanks to (1.6), we have

$$\int_{\mathbb{R}^{N-1}\setminus B^{N-1}_{r\sigma(t)^{-1}}} P_{1/2}(y', 0) dy' = O(\sigma(t)^2) \quad \text{and} \quad \int_{B^{N-1}_{\sigma(t)^{-1}}\setminus B^{N-1}_{r\sigma(t)^{-1}}} |y'|^2 P_{1/2}(y', 0) dy' = O(1) \quad \text{as } t \to 0.$$

This implies that

$$u(x, \sigma(t)) = \sigma(t)H(0) \int_{B^{N-1}_{\sigma(t)^{-1}}} |y'|^2 P_{1/2}(y', 0) dy' - 2vt(\sigma(t))^{-1} \int_{\mathbb{R}^{N-1}} P_{1/2}(y', 0) dy' + O(\sigma(t)). \tag{5.8}$$

Using polar coordinates and (1.6), we then have

$$\int_{B^{N-1}_{\sigma(t)^{-1}}} |y'|^2 P_{1/2}(y', 0) dy' \asymp C_{N,1/2} \omega_{N-2} \int_0^{1/\sigma(t)} \frac{m^N}{(1 + m^2)^{\frac{N+1}{2}}} dm, \tag{5.9}$$

where $\omega_{N-2} := |S^{N-2}|$. By the change of variable $\rho = \frac{1}{m}$, we have

$$\int_0^{1/\sigma(t)} \frac{m^N}{(1 + m^2)^{\frac{N+1}{2}}} dm = \int_{\sigma(t)}^{+\infty} \frac{1}{\rho (1 + \rho^2)^{\frac{N+1}{2}}} d\rho$$

$$= \int_{\sigma(t)}^1 \frac{1}{\rho (1 + \rho^2)^{\frac{N+1}{2}}} d\rho + \int_1^{+\infty} \frac{1}{\rho (1 + \rho^2)^{\frac{N+1}{2}}} d\rho$$

$$= \int_{\sigma(t)}^1 \frac{1}{\rho (1 + \rho^2)^{\frac{N+1}{2}}} d\rho + O(1) = -\log(\sigma(t)) + O(1).$$

Letting $b_N(t) := \dfrac{\displaystyle\int_{B^{N-1}_{\sigma(t)^{-1}}} |y'|^2 P_{1/2}(y', 0) dy'}{-2\log(\sigma(t)) \displaystyle\int_{\mathbb{R}^{N-1}} P_{1/2}(y', 0) dy'}$, by (5.8), (5.9) and the above estimate, we obtain, as $t \to 0$,

$$u(x, \sigma(t)) = \sigma(t)H(0) \int_{B^{N-1}_{\sigma(t)^{-1}}} |y'|^2 P_{1/2}(y', 0) dy' - 2vt(\sigma(t))^{-1} \int_{\mathbb{R}^{N-1}} P_{1/2}(y', 0) dy' + O(\sigma(t))$$

$$= (\sigma(t))^{-1}\left[\sigma^2(t)H(0)\int_{B^{N-1}_{\sigma(t)^{-1}}}|y'|^2 P_{1/2}(y',0)dy' - 2vt\int_{\mathbb{R}^{N-1}}P_{1/2}(y',0)dy' + O(\sigma^2(t))\right].$$

Since $x = vt\nu \in \partial E_t$, we have $u(x,\sigma(t)) = 0$. Recalling that $t = \sigma^2(t)|\log(\sigma(t))|$, we finally get

$$0 = \sigma(t)\log(\sigma(t))\left[H(0)\frac{\int_{B^{N-1}_{\sigma(t)^{-1}}}|y'|^2 P_{1/2}(y',0)dy'}{|\log(\sigma(t))|} - 2v\int_{\mathbb{R}^{N-1}}P_{1/2}(y',0)dy' + O\left(\frac{1}{\log(\sigma(t))}\right)\right].$$

Hence

$$v = b_N(t)H(0) + O\left(\frac{1}{\log(\sigma(t))}\right) \qquad \text{as } t \to 0.$$

The proof is then ended. □

Acknowledgments

Conflict of interest

The authors declare no conflict of interest.

References

1. N. Abatangelo, E. Valdinoci, A notion of nonlocal curvature, *Numer. Func. Anal. Opt.*, **35** (2014), 793–815.

2. P. Alexander, H. Gerhard, Geometric evolution equations for hypersurfaces, In: *Calculus of variations and geometric evolution problems*, Springer, 1999, 45–84.

3. S. J. Altschuler, M. A. Grayson, *Shortening space curves and flow through singularities*, Institute for Mathematics and its Applications, USA, 1991.

4. G. Barles, C. Georgelin, A simple proof of convergence for an approximation scheme for computing motions by mean curvature, *SIAM J. Numer. Anal.*, **32** (1995), 484–500.

5. J. Bence, B. Merriman, S. Osher, *Diffusion generated motion by mean curvature*, Computational Crystal Growers Workshop, 1992.

6. R. M. Blumenthal, R. K. Getoor, Some theorems on stable processes, *T. Am. Math. Soc.*, **95** (1960), 263–273.

7. L. Caffarelli, J. M. Roquejoffre, O. Savin, Nonlocal minimal surfaces, *Commun. Pure Appl. Math.*, **63** (2010), 1111–1144.

8. L. Caffarelli, L. Silvestre, An extension problem related to the fractional Laplacian, *Commun. Part. Diff. Eq.*, **32** (2007), 1245–1260.

9. L. A. Caffarelli, P. E. Souganidis, Convergence of nonlocal threshold dynamics approximations to front propagation, *Arch. Ration. Mech. Anal.*, **195** (2010), 1–23.

10. L. Capogna, G. Citti, C. S. G. Magnani, Sub-riemannian heat kernels and mean curvature flow of graphs, *J. Funct. Anal.*, **264** (2013), 1899–1928.

11. A. Chambolle, M. Morini, M. Ponsiglione, A nonlocal mean curvature flow and its semi-implicit time-discrete approximation, *SIAM J. Math. Anal.*, **44** (2012), 4048–4077.

12. A. Chambolle, M. Morini, M. Ponsiglione, Minimizing movements and level set approaches to nonlocal variational geometric flows, In: *Geometric partial differential equations*, Edizioni della Normale, 2013, 93–104.

13. A. Chambolle, M. Morini, M. Ponsiglione, Nonlocal curvature flows, *Arch. Ration. Mech. Anal.*, **218** (2015), 1263–1329.

14. E. Cinti, C. Sinestrari, E.Valdinoci, Neckpinch singularities in fractional mean curvature flows, *P. Am. Math. Soc.*, **146** (2018), 2637–2646.

15. K. Ecker, G. Huisken, Mean curvature evolution of entire graphs, *Ann. Math.*, **130** (1989), 453–471.

16. L. C. Evans, Convergence of an algorithm for mean curvature motion, *Indian U. Math. J.*, **42** (1993), 533–557.

17. L. C. Evans, J. Spruck, Motion of level sets by mean curvature I, *J. Differ. Geom.*, **33** (1991), 635–681.

18. L. C. Evans, J. Spruck, Motion of level sets by mean curvature II, *T. Am. Math. Soc.*, **330** (1992), 321–332.

19. L. C. Evans, J. Spruck, Motion of level sets by mean curvature III, *J. Geom. Anal.*, **2** (1992), 121–150.

20. L. C. Evans, J. Spruck, Motion of level sets by mean curvature IV, *J. Geom. Anal.*, **5** (1995), 77–114.

21. M. M. Fall, Constant nonlocal mean curvatures surfaces and related problems, In: *Proceedings of the International Congress of Mathematicians (ICM 2018)*, 2018, 1631–1656.

22. J. A. Garroni, S. Müller, Γ-limit of a phase-field model for dislocations, *SIAM J. Math. Anal.*, **36** (2005), 1943–1964.

23. J. A. Garroni, S. Müller, A variational model for dislocations in the line tension limit, *Arch. Ration. Mech. Anal.*, **181** (2006), 535–578.

24. Y. Giga, *Surface evolution equations*, Springer, 2006.

25. Y. Goto, K. Ishii, T. Ogawa, *Method of the distance function to the BenceMerriman-Osher algorithm for motion of mean curvature*, Graduate School of Mathematics, Kyushu University, 2002.

26. G. Huisken, *Flow by mean curvature of convex surfaces into spheres*, Australian National University, Centre for Mathematical Analysis, 1984.

27. C. Imbert, Level set approach for fractional mean curvature flows, *Interface. Free Bound.*, **11** (2009), 153–176.

28. H. Ishii, A generalization of the Bence, Merriman and Osher algorithm for motion by mean curvature, *Curvature flows and related topics (Levico, 1994)*, **5**, 111–127. Available from: https://mathscinet.ams.org/mathscinet/.

29. H. Ishii, G. E. Pires, P. E. Souganidis, Threshold dynamics type approximation schemes for propagating fronts, *J. Math. Soc. Jpn.*, **51** (1999), 267–308.

30. H. C. Lara, G. Davila, Regularity for solutions of nonlocal, non symmetric equations, *Ann. Inst. H. Poincaré Anal. Non linéaire*, **29** (2012), 833–859.

31. F. Leoni, Convergence of an approximation scheme for curvature-dependent motions of sets, *SIAM J. Numer. Anal.*, **39** (2001), 1115–1131.

32. S. Osher, J. A. Sethian, Fronts propagating with curvature-dependent speed: algorithms based on Hamilton-Jacobi formulations, *J. Comput. Phys.*, **79** (1988), 12–49.

33. O. Savin, E. Valdinoci, G-convergence for nonlocal phase transitions, *Ann. Inst. H. Poincaré Anal. Non Lineaire*, **29** (2012), 479–500.

34. D. Swartz, N. K. Yip, Convergence of diffusion generated motion to motion by mean curvature, *Commun. Part. Diff. Eq.*, **42** (2017), 1598–1643.

35. J. L. Vázquez, Recent progress in the theory of nonlinear diffusion with fractional Laplacian operators, *Discrete Cont. Dyn S*, **7** (2014), 857–885.

36. L. Vivier, Convergence of an approximation scheme for computing motions with curvature dependent velocities, *Differ. Integral Equ.*, **13** (2000), 1263–1288.