http://www.aimspress.com/journal/mine

Mathematics in Engineering, 3(4): 1–20.
DOI:10.3934/mine.2021029
Received: 20 February 2020
Accepted: 31 May 2020
Published: 03 August 2020

Research article

Low dimensional completely resonant tori in Hamiltonian Lattices and a Theorem of Poincaré†

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† **This contribution is part of the Special Issue:** Modern methods in Hamiltonian perturbation theory
Guest Editors: Marco Sansottera; Ugo Locatelli
Link: www.aimspress.com/mine/article/5514/special-articles

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Abstract: We present an extension of a classical result of Poincaré (1892) about continuation of periodic orbits and breaking of completely resonant tori in a class of nearly integrable Hamiltonian systems, which covers most Hamiltonian Lattice models. The result is based on the fixed point method of the period map and exploits a standard perturbation expansion of the solution with respect to a small parameter. Two different statements are given, about existence and linear stability: a first one, in the so called non-degenerate case, and a second one, in the completely degenerate case. A pair of examples inspired to the existence of localized solutions in the discrete NLS lattice is provided.

Keywords: Hamiltonian lattices; perturbation theory; average methods; resonant tori; periodic orbits; linear stability

Foreword

The present paper is dedicated to Antonio Giorgilli, in occasion of his 70th birthday.

There is no need to stress here his scientific merits: his publications are worth more than a thousand words. And for sure he would deserve a much better paper than the present contribution, to celebrate his career.

We thus prefer to praise the human qualities of Antonio, which are perceived immediately even by those who know him since only a short period of time; and they get absolutely clear for those who had the luck and the honor to know him since a long time. Like one of us, who met Antonio almost 30 years ago as a professor at the second year of the physics degree.

For all of us, to various extent, he has been a master; we could even say a scientific father. And we are proud to claim he is more than a good friend.

1. Introduction

The present paper deals with an extension of a classical result on periodic orbits in nearly integrable Hamiltonian Systems, due to Poincaré at the end of XIX century [28, 29]. The problem considered is the continuation of periodic orbits foliating a completely resonant torus $I = I^*$ of maximal dimension m of an integrable Hamiltonian $H_0(I)$ of m degrees of freedom, once a small perturbation $\epsilon H_1(\theta, I)$ is added to the system. Given $I = I^*$ and $\theta^{(0)}(t) = \omega t + \theta(0)$ the corresponding unperturbed periodic flow (where $\omega := D_I H_0(I^*)$), and assuming the invertibility of $D_I^2 H_0(I^*)$, Poincaré's Theorem ensures the existence of the continuation at small $|\epsilon|$ for those choices $\theta^*(0) \in \mathbb{T}^m$ which are non-degenerate critical points of the time averaged perturbation $\left\langle H_1\Big|_0\right\rangle_T$ defined as

$$\left\langle H_1\Big|_0\right\rangle_T := \frac{1}{T}\int_0^T H_1(\omega t + \theta(0), I^*)dt\,, \tag{1.1}$$

where $\langle\cdot\rangle_T$ and $\Big|_0$ in the rest of the paper will denote respectively the time-average over one period and the evaluation at the unperturbed dynamics on the torus I^*. Since any unperturbed periodic orbit on $I = I^*$ is uniquely identified by the quotient of the resonant torus with respect to the periodic flow, it turns out that the functional $\left\langle H_1\Big|_0\right\rangle_T$ is defined on the quotient manifold $\mathbb{T}^{m-1}$ and can be expressed in terms of $m-1$ suitable resonant angles φ (see for example formula (10) in [19]), namely

$$\left\langle H_1\Big|_0\right\rangle_T(\varphi) : \mathbb{T}^{m-1} \to \mathbb{R}\,.$$

Hence, in terms of these "phase-shift" variables φ, the values φ^* fulfilling

$$\nabla_\varphi \left\langle H_1\Big|_0\right\rangle_T(\varphi^*) = 0 \qquad\qquad \det\left(D^2_\varphi\left\langle H_1\Big|_0\right\rangle_T(\varphi^*)\right) \neq 0\,, \tag{1.2}$$

identify unperturbed periodic solutions that can be continued via Implicit Function Theorem. Even more, Poincaré's result relates linear stability of these continued periodic orbits to the product of the two matrices $\epsilon D^2_\theta\left\langle H_1\Big|_0\right\rangle_T(\varphi^*)$ and $D^2_I H_0(I^*)$, showing that the Floquet multipliers (different from one) are of order $\mathcal{O}(\sqrt{\epsilon})$; in particular, if $D^2_I H_0(I^*)$ is definite (positive or negative), linear stability is encoded in the nature of φ^* as critical point on $\mathbb{T}^{m-1}$. An extension of this result to the completely degenerate case $\left\langle H_1\Big|_0\right\rangle_T \equiv$ constant is included in [20]; with the same original approach of Poincaré, based on the ϵ Taylor-expansion of the solutions, the authors are able to replace the average $\left\langle H_1\Big|_0\right\rangle_T$ with a $\mathcal{O}(\epsilon^2)$ higher order functional F on the quotient torus $\mathbb{T}^{m-1}$, whose non-degenerate critical points φ^* are the new candidate for the continuation. A further and complete generalization of Poincaré's result is instead given by [26], where a KAM-like normal form approach, combined with fixed point methods, has been used to explore any kind of Poincaré degeneracy. Among the various degenerate scenarios, the normal form approach allows to investigate the case of isolated critical points φ^* whose hessian in (1.2) is not invertible (hence non-degeneracy does not hold). In this case the existence of the continuation depends on the higher order corrections, which are needed to be explicitly calculated.

One of the typical mechanism for the complete degeneracy to occur, is the lack of any terms, in the given leading order perturbation H_1, related to all the resonances of the torus; indeed in these cases $\left\langle H_1\Big|_0\right\rangle_T$ turns out to be a constant function on the torus. This kind of degeneracy is naturally met in the study of spatially localized solutions in Hamiltonian Lattices (see for example the Special Issue [21] for a recent collection of results on this topic), like Multibreathers in the Klein-Gordon model

$$H = \sum_{j\in\mathcal{J}}\left(\frac{1}{2}y_j^2 + V(x_j)\right) + \epsilon\sum_{j\in\mathcal{J}}(x_{j+1} - x_j)^2 , \qquad V(x) = \frac{1}{2}x^2 + O(|x|^3) , \tag{1.3}$$

or multi-pulse discrete solitons in standard discrete NLS model

$$H = \sum_{j\in\mathcal{J}}\left(|\psi_j|^2 + \frac{\gamma}{2}|\psi_j|^4\right) + \epsilon\sum_{j\in\mathcal{J}}|\psi_{j+1} - \psi_j|^2 , \tag{1.4}$$

where $\mathcal{J}$ is a finite* set of indexes. Although quite trivial, let us remark that if $\epsilon = 0$ these models reduce to a collection of uncoupled identical nonlinear oscillators. One typically considers an unperturbed ($\epsilon = 0$) periodic solution given by a subset $S = \{j_1, \ldots, j_m\}$ of m of these uncoupled identical oscillators, having the same action $I_{j_l} = I^*$ and hence the same nonlinear frequency ω; in other words, one is considering a low-dimensional completely resonant torus of the unperturbed system, corresponding to the lowest order resonance $(1 : 1 : \ldots : 1)$. Once the perturbation H_1 is added (in the above models it is given by the weak linear interactions among the oscillators), the torus breaks down and only some, typically a finite number, of the unperturbed periodic orbits survive at small ϵ: these solutions are still spatially localized (in terms of amplitude of the oscillations) around the unperturbed oscillators j_l. The mathematical investigation of these solutions naturally represents an extension of the original Poincaré's result to low-dimensional completely resonant tori. The literature on the topic provides two suitable methods to prove existence of such solutions, both based on variational arguments and leading to the study of critical points on the average perturbation $\langle H_1\rangle_T$ as in Poincaré. The first method is formulated and developed in [2, 3, 6, 11, 14, 15, 17] and applies to models like (1.3), while the second exploits the rotation symmetry in (1.4) and is developed and applied in [1, 10, 12, 13]. All these results deal with the non-degenerate case, which occurs for example in (1.3) and (1.4) when, due to the nearest-neighbours linear interaction, consecutive oscillators $S = \{1, 2, \ldots, m\}$ are considered. If instead some of the unperturbed oscillators are not consecutive, degeneracy occurs: in particular total degeneracy happens in models like (1.3) and (1.4) when all the oscillators are not consecutive (the easiest case is $S = \{j_1, j_2\}$ with $|j_1 - j_2| \neq 1$). Some results treating partial or total degeneracy in this context might be found in [24, 25, 27]: the first two works are based on the Lyapunov-Schmidt decomposition, while the last one represents the extension of [26] to the low-dimensional case, hence the perturbation scheme is performed at the level of Hamiltonian normal forms.

The results we here present are the extensions to low dimensional tori of both the original non-degenerate Theorem, due to Poincaré, and of the totally degenerate one of Meletlidou-Stagika [20];

*Part of the literature consider the lattice as infinite; we here prefer to take the finite case, in order to avoid technical difficulties related to the infinite dimensional phase space structure which are not essential with respect to the problem we are tackling.

such extensions are performed in the special case when the unperturbed Hamiltonian H_0 is already decoupled into two non-interacting integrable subsystems

$$H_0(I,\eta,\zeta) = H_0^\sharp(I) + H_0^\flat(\eta,\zeta) , \tag{1.5}$$

as in the Hamiltonian Lattice models (1.3) and (1.4). From a technical point of view, the continuation is obtained looking for fixed points of the period map, and the expansions in ϵ are performed at the level of the (analytic) solutions; the above restriction (1.5) allows to decompose the differential of the period map into a block triangular form, so that the Implicit Function Theorem can be easily applied. The same decomposition also allows to successfully investigate the linear stability of the periodic orbits thus obtained. For integrable Hamiltonian more general than (1.5) and for extensions to partial or higher order degeneracies, normal form techniques are needed, to put the system into a suitable form to investigate the continuation (see [4, 5, 8, 9, 16, 27, 30, 31]).

Theorem 1.1. *Consider a Hamiltonian system of the form*

$$H = H_0^\sharp(I) + H_0^\flat(z) + \epsilon H_1(\theta, I, z) + O(\epsilon^2) , \tag{1.6}$$

where $(\theta, I) \in \mathbb{T}^m \times \mathcal{U}(I^*)$ while $z = (\eta,\zeta) \in \mathbb{C}^{2n}$. Assume $z = 0$ to be an elliptic equilibrium for $H_0^\flat(z)$, with

$$H_0^\flat(z) = \sum_{j=1}^{n} i\Omega_j\zeta_j\eta_j + O(\|z\|^3) , \qquad \Omega_j > 0 , \tag{1.7}$$

and $I^* \in \mathcal{U}$ to be a value of the actions which identifies a completely resonant torus of maximal dimension for $H_0^\sharp(I)$, with frequency vector $\boldsymbol{\omega} = \omega\boldsymbol{k}$. Let φ^* be a critical point of the average $\left\langle H_1\Big|_0\right\rangle_T(\varphi)$. If the following three assumptions hold true:

1).

$$\mathrm{det}\Big(D_I^2 H_0^\sharp(I^*)\Big) \neq 0 , \tag{K-ND}$$

2).

$$\mathrm{det}\left(D_\varphi^2\left\langle H_1\Big|_0\right\rangle_T(\varphi^*)\right) \neq 0 , \tag{P-ND}$$

3).

$$l\omega \pm \Omega_j \neq 0 , \qquad \forall l \in \mathbb{Z} ,\ j = 1, \ldots, n \tag{M1}$$

then, there exists ϵ^* such that, for $|\epsilon| < \epsilon^*$, there exists a periodic orbit with initial datum $(\varphi_{po}(\epsilon), I_{po}(\epsilon), z_{po}(\epsilon))$ analytic in ϵ and $O(\epsilon)$-close to $(\varphi^*, I^*, 0)$.

Moreover, if the second Melnikov non-resonance condition holds true

$$l\omega \pm \Omega_j \pm \Omega_i \neq 0 , \qquad \forall l \in \mathbb{Z} , \qquad \forall i, j = 1, \ldots, n , \tag{M2}$$

then the spectrum Σ of the monodromy matrix splits into two different subsets: Σ_1 given by multipliers which have characteristic exponents $\mu_j(\epsilon)$ becoming zero in the limit $\epsilon \to 0$, which have the asymptotic behaviour $\mu_j(\epsilon) \sim \sigma_j\sqrt{\epsilon}$, with σ_j^2 eigenvalues of $-T^2 D_\theta^2\left\langle H_1\Big|_0\right\rangle_T(\varphi^*) D_I^2 H_0^\sharp(I^*)$ and Σ_2 given by multipliers having purely imaginary characteristic exponents $\pm 2\pi i\frac{\Omega_j(\epsilon)}{\omega}$ with $\Omega_j(\epsilon) = \Omega_j + O(\epsilon) \in \mathbb{R}$.

In the sufficient assumptions needed for the existence statement above we recover the so called *Kolmogorov-non-degeneracy* (K-ND), which implies that the resonant torus I^* is isolated in $\mathcal{U}$, and the so called *Poincaré-non-degeneracy* (P-ND), which similarly implies that the critical points φ^* on the torus $\mathbb{T}^{m-1}$, selecting the periodic orbits, are isolated. Moreover, we need a third assumption in order to split the tori variables (θ_1, φ, I) from the transversal ones: this is easily obtained by assuming the so called *First Melnikov condition* (M1), which is a non-resonance assumption between the frequency of the unperturbed periodic orbit ω and the transversal (small) oscillations of frequencies Ω_j. The linear stability statement is instead a consequence of the continuity of the spectrum with respect to ϵ, which is enough in the easy case of distinct eigenvalues, and of the Krein signature theory, which here applies if the frequencies Ω_j all have the same sign. In particular, it is evident that if $D^2_I H_0^\sharp(I^*)$ is either positive or negative definite, the relevant role in the stability is played by the nature of φ^* as critical point of $\left\langle H_1\Big|_0\right\rangle_T$. The second Melnikov condition (M2) is instead needed (see for example [18]) to preserve transversal ellipticity of the periodic orbits, namely that multipliers of Σ_2 do not leave the unitary circle, via Krein signature: indeed, combining (M2) with the assumption (1.7) on the (positive) sign of Ω_j, one can derive the (positive) definite signature for all the unperturbed Floquet multipliers $e^{2\pi i\frac{\Omega_j}{\omega}}$ on the unitary circle. Linear stability then depends on the effect of the $O(\epsilon)$ perturbation on the approximate internal characteristic exponents $\sigma_j\sqrt{\epsilon}$; for example, in the generic case of distinct σ_j, any sufficiently small perturbation would still give distinct $\mu_j(\epsilon)$ and linear stability is then encoded in the spectrum of $D^2_\theta\left\langle H_1\Big|_0\right\rangle_T(\varphi^*)$.

Theorem 1.2. *Consider a Hamiltonian system of the form* (1.6) *which fulfills assumptions* (K-ND) *and* (M1). *Suppose that* $\left\langle H_1\Big|_0\right\rangle_T(\varphi)$ *is identically constant on* $\mathbb{T}^{m-1}$. *Then there exists a function* $F_2(\varphi) : \mathbb{T}^{m-1} \to \mathbb{R}$ *such that if* φ^* *is a nondegenrate critical point of* F_2*, namely*

$$F_2(\varphi^*) = 0\,, \qquad \det\Big(D_\varphi F_2(\varphi^*)\Big) \neq 0\,, \tag{1.8}$$

then there exists ϵ^* *such that, for* $|\epsilon| < \epsilon^*$*, there exists a periodic orbit with initial datum* $(\varphi_{po}(\epsilon), I_{po}(\epsilon), z_{po}(\epsilon))$ *analytic in* ϵ *and* $O(\epsilon)$*-close to* $(\varphi^*, I^*, 0)$. *Moreover, under the same assumption* (M2)*, the same splitting of* Σ *as in Theorem 1.1 holds true, with the only difference that* $\mu_j(\epsilon) \sim \sigma_j\epsilon$*, with* σ_j^2 *eigenvalues of* $-T^2 D^2_\theta F_2(\varphi^*) D^2_I H_0^\sharp(I^*)$.

The precise definition of F_2 in the above statement requires the expansion of the solution up to order $O(\epsilon)$, hence it is deferred to formula (2.19) in Section 2, where we prove the two Theorems.

The two statements claimed above are then applied to show existence of discrete solitons in dNLS models, by studying first a typical non-degenerate vortex-like configuration for the ZigZag model, and then showing the existence of only in/out-of-phase discrete solitons in the easiest totally degenerate case ($S = -1, 1$) for the standard model. This part, which reproduces results already existing in the literature (see for example [22, 23]), is developed in Section 3.

2. Proof

The present section includes the proofs of the two Theorems. A first part shows the formal perturbation scheme here used (which is the same of [20]); the second and third parts provide the continuation argument in the non-degenerate and totally degenerate case; the last part is dedicated to linear stability, hence giving the splitting of the spectrum and asymptotic behaviour of Floquet exponents in the limit $\epsilon \to 0$.

2.1. Formal expansions and leading order approximations

Due to the near integrability of the model, it is known that any T-periodic solution of (1.6) $(\theta, I, z)(t; \epsilon)$ is analytic in ϵ; we can thus Taylor-expand all the variables in ϵ and write

$$\begin{cases} \theta(t) = & \theta^{(0)} + \epsilon\theta^{(1)} + O(\epsilon^2) \\ I(t) = & I^{(0)} + \epsilon I^{(1)} + O(\epsilon^2) \\ z(t) = & z^{(0)} + \epsilon z^{(1)} + O(\epsilon^2) \end{cases} .$$

According to the same splitting of the integrable part H_0 given in (1.5), we can decompose the perturbation H_1 as

$$H_1 := H_1^\sharp(\theta, I) + H_1^\flat(z) + H_1^\natural(\theta, I, z) , \tag{2.1}$$

where $H_1^\sharp$ depends only on the action-angle variables, $H_1^\flat$ depends only on the complex variables $z = (\eta, \zeta)$ while $H_1^\natural$ properly depends on all the variables, and provide the interaction among the two subsystems. Notice that the term $H_1^\natural$ has necessarily to be at least linear in z, so that it has to vanish when evaluated at the unperturbed flow, namely $H_1^\natural\big|_0 = 0$. Hamilton equations read

$$\begin{cases} \dot\theta = & \nabla_I H_0^\sharp + \epsilon\nabla_I H_1^\sharp + \epsilon\nabla_I H_1^\natural \\ \dot I = & -\epsilon\nabla_\theta H_1^\sharp - \epsilon\nabla_\theta H_1^\natural \\ \dot z = & J\nabla_z H_0^\flat + \epsilon J\nabla_z H_1^\flat + \epsilon J\nabla_z H_1^\natural \end{cases} . \tag{2.2}$$

If we expand w.r.t. ϵ both the Hamiltonian vector field and the time derivatives we get the two systems:

$$\begin{cases} \dot\theta = & \dot\theta^{(0)} + \epsilon\dot\theta^{(1)} + O(\epsilon^2) \\ \dot I = & \dot I^{(0)} + \epsilon\dot I^{(1)} + O(\epsilon^2) \\ \dot z = & \dot z^{(0)} + \epsilon\dot z^{(1)} + O(\epsilon^2) \end{cases} , \tag{2.3}$$

and

$$\begin{cases} \dot\theta = & \nabla_I H_0^\sharp\left(I^{(0)}\right) + \epsilon D_I^2 H_0^\sharp\left(I^{(0)}\right) I^{(1)} + \epsilon\nabla_I H_1^\sharp\left(I^{(0)}, \theta^{(0)}\right) + \epsilon\nabla_I H_1^\natural\left(I^{(0)}, \theta^{(0)}, z^{(0)}\right) + O(\epsilon^2) \\ \dot I = & -\epsilon\nabla_\theta H_1^\sharp\left(I^{(0)}, \theta^{(0)}\right) - \epsilon\nabla_\theta H_1^\natural\left(I^{(0)}, \theta^{(0)}, z^{(0)}\right) + O(\epsilon^2) \\ \dot z = & J\nabla_z H_0^\flat\left(z^{(0)}\right) + \epsilon D_z^2 H_0^\flat\left(z^{(0)}\right) z^{(1)} + \epsilon J\nabla_z H_1^\flat\left(z^{(0)}\right) + \epsilon J\nabla_z H_1^\natural\left(I^{(0)}, \theta^{(0)}, z^{(0)}\right) + O(\epsilon^2) \end{cases} ; \tag{2.4}$$

these necessarily have to coincide at any order in ϵ. Equating terms of order 0 we get

$$\begin{cases} \dot\theta^{(0)} & = \nabla_I H_0^\sharp(I^{(0)}) \\ \dot I^{(0)} & = 0 \\ \dot z^{(0)} & = J\nabla_z H_0^\flat(z^{(0)}) = JDz^{(0)} + O\left(\left\|z^{(0)}\right\|^2\right) \end{cases} ,$$

where $D := D_z^2 H_0^\flat\big|_0$ is the diagonal matrix

$$JD = \mathrm{diag}\{\mathrm{diag}\{i\Omega_j\}, \mathrm{diag}\{-i\Omega_j\}\}\ . \tag{2.5}$$

We deduce immediately that actions keep their unperturbed initial values $I^{(0)}(0)$, the angles rotate on the torus $\mathbb{T}^m$ with frequencies $\nabla_I H_0^\sharp(I^{(0)})$, while the external variables z evolve according to their unperturbed nonlinear dynamics

$$I^{(0)}(t) = I^{(0)}(0)\ , \qquad \theta^{(0)}(t) = \theta^{(0)}(0) + \nabla_I H_0^\sharp(I^{(0)})t\ , \qquad z^{(0)}(t) = e^{JDt} z^{(0)}(0) + \text{h.o.t.}\ .$$

At order $O(\epsilon)$ we have the time-dependent system

$$\begin{cases} \dot\theta^{(1)} = & D_I^2 H_0^\sharp(I^{(0)}) I^{(1)} + \nabla_I H_1^\sharp\left(\theta^{(0)}, I^{(0)}\right) + \nabla_I H_1^\natural\left(\theta^{(0)}, I^{(0)}, z^{(0)}\right) \\ \dot I^{(1)} = & -\nabla_\theta H_1^\sharp\left(\theta^{(0)}, I^{(0)}\right) - \nabla_\theta H_1^\natural\left(\theta^{(0)}, I^{(0)}, z^{(0)}\right) \\ \dot z^{(1)} = & JD_z^2 H_0^\flat\left(z^{(0)}\right) z^{(1)} + J\nabla_z H_1^\flat\left(z^{(0)}\right) + J\nabla_z H_1^\natural\left(\theta^{(0)}, I^{(0)}, z^{(0)}\right) \end{cases};$$

by inserting order 0 approximation of the periodic solutions on the unperturbed torus I^*

$$I^{(0)}(t) = I^*\ , \qquad \theta^{(0)}(t) = \theta^{(0)}(0) + \omega t\ , \qquad z^{(0)}(t) = 0\ , \qquad\qquad \omega := \nabla_I H_0^\sharp(I^*)\ ,$$

we get the system

$$\begin{cases} \dot\theta^{(1)} = & D_I^2 H_0^\sharp(I^*) I^{(1)} + \nabla_I H_1^\sharp\left(\theta^{(0)}(t), I^*\right) \\ \dot I^{(1)} = & -\nabla_\theta H_1^\sharp\left(\theta^{(0)}(t), I^*\right) \\ \dot z^{(1)} = & JD_z^2 H_0^\flat\left(z^{(0)}\right) z^{(1)} + J\nabla_z H_1^\flat\left(z^{(0)}\right) + J\nabla_z H_1^\natural\left(\theta^{(0)}(t), I^*, z^{(0)}\right) \end{cases},$$

where equation for $\dot I^{(1)}$ and $\dot\theta^{(1)}$ are uncoupled from $\dot z^{(1)}$. Notice that equation for $z^{(1)}$ simplifies in those models with $H_1^\flat(z) = O(z^2)$; in fact in such cases one has $\nabla_z H_1^\flat(z) = O(z)$, which implies a vanishing contribute of $\nabla_z H_1^\flat$ in the third equation, if at $\epsilon = 0$ the variables $z^{(0)}$ stay at rest

$$\nabla_z H_1^\flat\left(z^{(0)} = 0\right) = 0\,.$$

Solving the equation for $\dot I$, we get the leading order corrections to I^* of the actions

$$I^{(1)}(t) = I^{(1)}(0) - \int_0^t \nabla_\theta H_1^\sharp(\theta^{(0)}(\tau), I^*) d\tau\,. \tag{2.6}$$

Since $I^{(1)}(t)$ have to be periodic of period T, namely $I^{(1)}(T) = I^{(1)}(0)$, we have to impose

$$\int_0^T \nabla_\theta H_1^\sharp(\theta^{(0)}(\tau), I^*) d\tau = 0\,. \tag{2.7}$$

Let us set for the sake of simplicity $k_1 = 1$ in the resonant frequency vector ωk, with $\omega := \frac{2\pi}{T}$. We introduce the $m-1$ resonant angles $\varphi = \{\varphi_j\}_{j=1}^{m-1} = k_j\theta_1 - \theta_j$, which are the natural coordinates of the quotient torus $\mathbb{T}^{m-1}$ of the resonant torus over the periodic flow. Hence we get (by setting $\theta_1^{(0)}(0) = 0$)

$$\frac{1}{T}\int_0^T \nabla_\varphi H_1^\sharp(\omega t + \theta_1^{(0)}(0), \varphi, I^*) dt = \nabla_\varphi \frac{1}{2\pi}\int_0^{2\pi} H_1^\sharp(\theta_1, \varphi, I^*) d\theta_1\ .$$

Indeed, in the previous average, we can exchange the time-variable with θ_1, so that the first element of the gradient $\nabla_\theta H_1^\sharp$ has zero average and in the $m-1$ remaining components, only the dependence on the angles φ is left in the average

$$\left\langle H_1^\sharp\Big|_0\right\rangle_T(\varphi) = \frac{1}{2\pi}\int_0^{2\pi} H_1^\sharp(\theta_1,\varphi,I^*)d\theta_1\ . \tag{2.8}$$

Thus, the periodicity condition (2.7) easily reads

$$\nabla_\varphi\left\langle H_1^\sharp\Big|_0\right\rangle_T = 0\ ; \tag{2.9}$$

solutions φ^* of (2.9) represent critical points of the functional (2.8) defined on the torus $\mathbb{T}^{m-1}$.

Le us define the Hessian matrix

$$C := D_I^2 H_0^\sharp(I^*)\ , \tag{2.10}$$

thus the equation for $\dot\theta$ becomes

$$\theta^{(1)}(t) = \theta^{(1)}(0) + C\int_0^t I^{(1)}(\tau)d\tau + \int_0^t \nabla_I H_1^\sharp\left(\theta^{(0)}(\tau),I^*\right)d\tau\ ;$$

since also the angles $\theta^{(1)}$ have to be T-periodic (uniformly in ϵ), we have to ask

$$\theta^{(1)}(T) = \theta^{(1)}(0)\ , \qquad \Leftrightarrow \qquad C\left\langle I^{(1)}\right\rangle_T + \left\langle \nabla_I H_1^\sharp(\theta^{(0)}(\tau),I^*)\right\rangle_T = 0\ ,$$

where the first term is the average of $I^{(1)}(t)$ while the second represents the average of $\nabla_I H_1^\sharp$ restricted (as usual) to the unperturbed flow. Recalling (2.6) one has

$$I^{(1)}(0) - \left\langle\int_0^t \nabla_\theta H_1^\sharp(\theta^{(0)}(\tau),I^*)d\tau\right\rangle_T + C^{-1}\left\langle\nabla_I H_1^\sharp(\theta^{(0)}(\tau),I^*)\right\rangle_T = 0\ , \tag{2.11}$$

which provides the correction at order $\mathcal{O}(\epsilon)$ of the initial values of the actions

$$I(0) = I^* + \epsilon I^{(1)}(0) + O(\epsilon^2)\ .$$

It is clear from (2.11) that $I^{(1)}(0)$ depends on the unperturbed initial data I^* and φ_0.

Let us now move to the equation for $\dot z$

$$\dot z^{(1)} = JDz^{(1)} + J\nabla_z H_1^\flat(0) + J\nabla_z H_1^\natural(\theta^{(0)},I^*,0)\ . \tag{2.12}$$

In the Taylor expansions of $H_1^\natural$ and $H_1^\flat$ with respect to the external variable z, the important terms are linear in z; indeed all those terms which are $O(z^2)$ produce contributions, in the equation, that vanish when evaluated at $z^{(0)} = 0$. Hence

$$h_1(t) := J\Big[\nabla_z H_1^\flat(0) + \nabla_z H_1^\natural(\theta^{(0)}(t),I^*,0)\Big]$$

represents a periodically forcing term in the inhomogeneous linear equation (2.12)

$$\dot z^{(1)} = JDz^{(1)} + h_1(t)\ ;$$

if we assume the non-resonance (M1) condition between Ω_j and the frequency ω, then h_1 is non-resonant with respect to JD and the solution of (2.12) is given by the convolution integral

$$z^{(1)}(t) = e^{JDt} z^{(1)}(0) + \int_0^t h_1(s) e^{JD(t-s)} ds \, .$$

The T-periodicity of $z^{(1)}$ provides, as for $I^{(1)}$, the correction at order $O(\epsilon)$ of the initial value $z_0 = \epsilon z^{(1)}(0) + O(\epsilon^2)$; also in this case, $z^{(1)}(0)$ has to depend on I^* and φ_0. Indeed, from

$$z^{(1)}(T) = z^{(1)}(0) \qquad \Longleftrightarrow \qquad e^{JDT} z^{(1)}(0) + \int_0^T h_1(s) e^{JD(t-s)} ds = z^{(1)}(0)$$

one gets the correction

$$z^{(1)}(0) = \left(\mathbb{I} - e^{JDT}\right)^{-1} \int_0^T h_1(s) e^{JD(T-s)} ds \, ,$$

with $\left(\mathbb{I} - e^{JDT}\right)$ invertible because of (M1).

2.2. Continuation in the non-degenerate case

Given a generic initial datum P_0 of the unperturbed system ($\epsilon = 0$), we look for a correction $P(\epsilon)$ at $\epsilon \neq 0$ such that the variation of the Hamiltonian flow after one period T vanishes, namely $\Phi^T(P(\epsilon), \epsilon) - P(\epsilon) = 0$. The condition we have to impose is then the T-periodicity of the flow, assuming its validity at $\epsilon = 0$, and exploiting the analyticity of the flow $\Phi^T(P(\epsilon), \epsilon)$, and of its initial datum $P(\epsilon)$, with respect to ϵ.

In order to identify the periodic orbit, we can ignore its phase θ_1 and consider as unknowns only the other initial $2n + 2m - 1$ "transversal" variables $\{\varphi_0(\epsilon), I_0(\epsilon), z_0(\epsilon)\}$. Moreover, since the Hamiltonian is kept constant along the orbit, the $2(n + m)$ equation $\Phi^T(P(\epsilon), \epsilon) - P(\epsilon) = 0$ are not independent, and we can get rid of one of them; we choose to ignore the action I_1 associated to the angle θ_1. In this way, the periodicity condition reduces to a set of $2n + 2m - 1$ equations in $2n + 2m - 1$ variables of the form

$$X(\varphi_0, I_0, z_0, \epsilon) = 0 \, . \tag{2.13}$$

Since the velocity of variation of the actions $I_{l=2,\dots,m}$ (which are approximately constant in time) are of order $O(\epsilon)$, we can rescale them by ϵ; in this way the leading terms of (2.13), with respect to ϵ, read

$$X(\varphi_0, I_0, z_0, 0) = \begin{cases} F_1\,(I_0) \\ F_2\,(\varphi_0, I_0) \\ F_3\,(z_0) \end{cases} ,$$

where now $\{\varphi_0, I_0, z_0\}$ are the variables which identify the generic unperturbed initial datum P_0, modulo a shift of θ_1, and the components F_j read

$$\begin{cases} F_1\,(I_0) & = \nabla_I H_0^\sharp\,(I_0) - \omega \boldsymbol{k} \\ F_2\,(\varphi_0, I_0) & = \nabla_\varphi \left\langle H_1^\sharp(\tau) \right\rangle_T (\varphi_0, I_0) + \nabla_\varphi \left\langle H_1^\natural(\tau) \right\rangle_T (\varphi_0, I_0, z_0) \\ F_3\,(z_0) & = \left(e^{JDT} - \mathbb{I}\right) z_0 + O\left(\|z_0\|^2\right) \end{cases} ,$$

where $H_1(\tau)$ is the perturbation restricted to the unperturbed flow corresponding to the initial datum (φ_0, I_0, z_0). By redefining X coherently with the previous scaling of the actions J, we get the compact form

$$X(\varphi_0, I_0, z_0, \epsilon) = F(\varphi_0, I_0, z_0) + O(\epsilon) . \tag{2.14}$$

Solving the above equation (2.14) at $\epsilon = 0$ is equivalent to solve the system $F(\varphi_0, I_0, z_0) = 0$. Hence, if $I_0 = I^*$ corresponds to a completely resonant torus with frequency vector $\omega \boldsymbol{k}$, then $F_1(I^*) = 0$; given such I^* and setting at rest the complex variables $z_0 = 0$, if $\varphi_0 = \varphi^*$ is a critical point of $\left\langle H_1^\sharp\big|_0\right\rangle_T$

$$\nabla_\varphi \left\langle H_1^\sharp\big|_0\right\rangle_T (\varphi^*) = 0 , \tag{2.15}$$

then also $F_2(\varphi^*, I^*) = 0$. Thus, the point $(\varphi^*, I^*, 0)$ solves

$$X(\varphi^*, I^*, 0, 0) = F(\varphi^*, I^*, 0) = 0 .$$

In order to apply the Implicit Function Theorem, we need invertibility of X', which is the differential of X evaluated on the approximate solution $(\varphi^*, I^*, 0)$ at $\epsilon = 0$; this coincides[†] with the differential of F at its zero and is block triangular. Indeed

$$X' := D_{(\varphi_0, I_0, z_0)} F\Big|_{(\varphi^*, I^*, 0)} = \begin{pmatrix} X'_{11} & X'_{12} \\ 0 & X'_{22} \end{pmatrix} ,$$

where

$$\begin{aligned}
X'_{11} &= \begin{pmatrix} 0 & C \\ D^2_{\varphi_0} \langle H_1(\tau) \rangle_T (\varphi^*, I^*, 0) & D^2_{\varphi_0, I_0} \langle H_1(\tau) \rangle_T (\varphi^*, I^*, 0) \end{pmatrix} \\
X'_{12} &= \begin{pmatrix} 0 & 0 \\ D^2_{\varphi_0, \xi_0} \left\langle H_1^\natural(\tau) \right\rangle_T (\varphi^*, I^*, 0) & D^2_{\varphi_0, \eta_0} \left\langle H_1^\natural(\tau) \right\rangle_T (\varphi^*, I^*, 0) \end{pmatrix} \\
X'_{22} &= e^{JDT} - \mathbb{I} , \qquad JDT = \operatorname{diag}\Big\{ \operatorname{diag}\{2\pi i \left(\frac{\Omega_j}{\omega}\right)\}, \operatorname{diag}\{-2\pi i \left(\frac{\Omega_j}{\omega}\right)\} \Big\} .
\end{aligned}$$

From the first assumption (K-ND) the matrix C is invertible, and the same holds true for $D^2_{\varphi_0} \langle H_1(\tau) \rangle_T (\varphi^*, I^*, 0) = D^2_\varphi \left\langle H_1\big|_0\right\rangle_T (\varphi^*)$ due to the second one (P-ND); in particular notice that

$$D^2_{\varphi_0} \langle H_1(\tau) \rangle_T (\varphi^*, I^*, 0) = D^2_\varphi \left\langle H_1^\sharp\big|_0\right\rangle_T (\varphi^*) ,$$

because $D^2_{\varphi_0} \left\langle H_1^\natural(\tau) \right\rangle_T$ is at least linear in z_0. The invertibility of block $e^{JDT} - \mathbb{I}$ of X' is due to the first Melnikov condition (M1), which ensures that $\frac{\Omega_j}{\omega} \notin \mathbb{Z}$. Due to the triangular structure of X', the block X'_{12} does not play any role, even if different from zero.

[†]Recall that to compute X' we can first restrict to $\epsilon = 0$ and then differentiate w.r.t. the unperturbed initial datum z_0.

2.3. Continuation in the totally degenerate case

The totally degenerate case corresponds to $\langle H_1\rangle_T \equiv$ const; what follows represents the extension of what has been already developed in [20] in the case of maximal tori.

In the degenerate scenario, the variation of the actions I_l on the approximate periodic orbit is much slower than what happens in the non-degenerate case, being $\dot{I}_l = O(\epsilon^2)$; as a consequence, an expansion up to order ϵ^2 of the actions I_l is required. Such an expansion provides the following equation for $I^{(2)}$

$$\dot{I}^{(2)} = -\{D^2_\theta H^\sharp_1(t)\theta^{(1)} + D^2_{I\theta}H^\sharp_1(t)I^{(1)} + D^2_\theta H^\natural_1(t)\theta^{(1)} + D^2_{I\theta}H^\natural_1(t)I^{(1)} + D^2_{z\theta}H^\natural_1(t)z^{(1)}\} . \tag{2.16}$$

Since the term $H^\natural_1(\theta, I, z)$ is at least linear in z, then two of the above terms will vanish when evaluated at $z = 0$

$$D^2_\theta H^\natural_1\Big|_0 = D^2_{I\theta}H^\natural_1\Big|_0 = 0 ,$$

hence we can reduce (2.16) to the simplified form

$$\dot{I}^{(2)} = -\{D^2_\theta H^\sharp_1(t)\theta^{(1)} + D^2_{I\theta}H^\sharp_1(t)I^{(1)} + D^2_{z\theta}H^\natural_1(t)z^{(1)}\} , \tag{2.17}$$

where $D^2_{z\theta}H^\natural_1(t)z^{(1)}(t)$ is given only by linear terms in z, and the first order corrections $\theta^{(1)}(t)$, $I^{(1)}(t)$ and $z^{(1)}(t)$ have been determined by the periodicity conditions at order $O(\epsilon)$. By integrating on the period interval $[0, T]$, the periodicity condition $I^{(2)}(T) - I^{(2)}(0) = 0$ reads

$$\Big\langle D^2_\theta H^\sharp_1(\tau)(\theta^{(1)} - \theta^{(1)}(0)) + D^2_{I\theta}H^\sharp_1(\tau)I^{(1)} + D^2_{z\theta}H^\natural_1(\tau)z^{(1)}\Big\rangle_T = 0 , \tag{2.18}$$

where in the first addendum the initial datum $\theta^{(1)}(0)$ can be "added" without affecting the computation of the average; indeed one gets $\Big\langle D^2_\theta H^\sharp_1(\tau)\theta^{(1)}(0)\Big\rangle_T \equiv 0$, due to the degeneracy assumptions. Moreover, thanks to the conservation of the energy along the periodic orbit, it is possible to prove that only $m - 1$ of the above m equations are independent. Hence, as in the non-degenerate case, we can ignore the evolution of the first action I_1 and consider the periodicity condition only for the remaining $m-1$ ones. Due to the complete degeneracy, the Taylor expansion of the $m - 1$ actions I_l starts with term of order $O(\epsilon^2)$; thus we rescale $I_l(T) - I_l(0)$ by ϵ^2 and define the components of F_2 as

$$F_{2,l}(\varphi_0, I_0) := \Big\langle D_\theta\partial_{\theta_l}H^\sharp_1(\tau)(\theta^{(1)} - \theta^{(1)}(0)) + D_I\partial_{\theta_l}H^\sharp_1(\tau)I^{(1)} + D_z\partial_{\theta_l}H^\natural_1(\tau)z^{(1)}\Big\rangle_T , \tag{2.19}$$

where the initial values $I^{(1)}_0$ and $z^{(1)}_0$ explicitly depend on the unperturbed initial values φ_0 and I_0. We want to apply the IFT to the periodicity condition (2.14), where now φ^* has to solve

$$F_2(\varphi^*, I^*) = 0 . \tag{2.20}$$

The above Eq (2.20) selects those $\varphi_0 = \varphi^*$ on the torus $\mathbb{T}^{m-1}$ which can be continued, and that provide the solution of (2.14) at $\epsilon = 0$

$$X(\varphi^*, I^*, 0) = 0 .$$

In order to apply again the IFT we need such φ^* to be non-degenerate critical points for $F_2(\varphi_0, I^*)$, namely

$$\det\Big(D_\varphi F_2(\varphi^*, I^*)\Big) \neq 0 ,$$

which is indeed hypothesis (1.8).

2.4. Linear stability

In order to investigate the linear stability of the periodic orbits, we have to exploit the fact that the monodromy matrix $\Lambda(\epsilon)$ coincides with the differential of the Hamiltonian flow at time T with respect to the initial datum of the periodic orbit. We develop the non-degenerate case, being the totally degenerate one a minor variation of the forthcoming arguments.

With standard calculations, which resemble those for the differential X' of the period map, one obtains the following structure for $\Lambda(\epsilon)$

$$d_{(\theta_0,I_0,z_0)}\Phi^T(I_{po},\varphi_{po},z_{po}) = \Lambda(\epsilon) = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix} , \tag{2.21}$$

with:

- the Λ_{11} block reads

$$\Lambda_{11} = \begin{pmatrix} \mathbb{I} + O(\epsilon) & CT + O(\epsilon) \\ -\epsilon T D^2_{\theta_0}\left\langle H_1^\sharp\Big|_0\right\rangle_T(\varphi^*, I^*) + O(\epsilon^2) & \mathbb{I} + O(\epsilon) \end{pmatrix} ,$$

  where we have included in the $O(\epsilon)$ also the term $-\epsilon T D^2_{\theta_0 I_0}\left\langle H_1^\sharp\Big|_0\right\rangle_T(\varphi^*, I^*)$;
- the out-of-diagonal blocks are perturbations of the same order $O(\epsilon)$

$$\Lambda_{12} = O(\epsilon) , \qquad \Lambda_{21} = O(\epsilon) ,$$

  since the interaction between internal variables (θ, I) and the external ones (ξ, η) is only due to the perturbation H_1;
- the block Λ_{22} reads

$$\Lambda_{22} = e^{JDT} + O(\epsilon) .$$

By continuity with respect to ϵ, the multipliers of $\Lambda(\epsilon)$ have to converge to those of $\Lambda(0)$ as $\epsilon \to 0$, the last ones being collected into two different sets (due to the block diagonal shape of $\Lambda(0)$): $2n$ multipliers of $\Lambda_{22}(0)$ are on the unitary circle, with purely imaginary exponents $\pm 2\pi i\frac{\Omega_j}{\omega}$, and $2m$ multipliers of $\Lambda_{11}(0)$ equal to 1 (due to the complete resonance of the n-dimensional torus). Moreover, due to the two Melnikov conditions[‡], the $2n$ multipliers on the unitary circle are different from ± 1. By continuity in ϵ, the two sets stay disjoint and provide the two different sets Σ_1 and Σ_2 of the statements: Σ_1 is made of multipliers $\lambda(\epsilon)$ which all bifurcate from 1, hence having vanishing characteristic exponents, while Σ_2 is made of multipliers which are deformations of $e^{\pm 2\pi i\frac{\Omega_j}{\omega}} \neq \pm 1$. If some (or all) of the unperturbed Floquet multiplier of $\Lambda_{22}(0)$ coincide, then having the same definite Krein signature (see [7, 18, 32]) is a sufficient condition for their perturbations not to leave the unitary circle. Due to the Melnikov conditions, and since the frequencies Ω_j share the same sign (rotation direction), the unperturbed Floquet multipliers $e^{\pm 2\pi i\frac{\Omega_j}{\omega}}$ are all positive definite, so any sufficiently small perturbation $\lambda_j(\epsilon)$ cannot leave the unitary circle, still being different from ± 1. If all Ω_j are

[‡]Notice that (M1) implies $\pm\frac{\Omega_j}{\omega} \neq l \in \mathbb{Z}$, hence the multipliers have to be different from 1. The second condition (M2) implies in particular $\pm\frac{\Omega_j}{\omega} \neq \frac{l}{2} \in \mathbb{Z}$ hence the multipliers have to be different also from -1. We recommend reference [18].

distinct, then the same holds true for the corresponding unperturbed Floquet multipliers (which are all simple eigenvalues of $\Lambda(\epsilon)$) and for any sufficiently small perturbations, without invoking their Krein signatures.

It remains to show the prescribed asymptotic behaviour of the Floquet multipliers of Σ_1. These are solutions of

$$\det(M(\lambda,\epsilon)) = 0 \,, \qquad M(\epsilon) := \Lambda(\epsilon) - \lambda\mathbb{I} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} ,$$

with vanishing characteristic exponents, hence $\lambda_j(\epsilon) \to 1$. We thus assume for these part of the spectrum the ansatz $\lambda_j(\epsilon) = e^{\sqrt{\epsilon}\sigma_j + O(\epsilon)}$, where the $\sqrt{\epsilon}$ scaling is expected (as in the original Poincaré result, but see also [30]) and will be clarified in a while. Since $M_{22}(\epsilon) = e^{JDT} - \lambda\mathbb{I} + O(\epsilon)$ is invertible for sufficiently small ϵ, because of the first Melnikov assumption (M1), we can apply the Schur complement to rewrite the characteristic equation as

$$\det(M(\epsilon)) = \det(M_{22}(\epsilon))\det\Big(M_{11} - M_{12}M_{22}^{-1}M_{21}\Big) = 0 \,.$$

Since $M_{22}^{-1} = O(1)$ and $M_{21} = O(\epsilon) = M_{12}$, it turns out that $M_{21}M_{22}^{-1}M_{12} = O(\epsilon^2)$ is a perturbation of M_{11}, thus it is enough to study $\det(M_{11}(\sigma,\epsilon)) = 0$, where

$$M_{11}(\epsilon) = \begin{pmatrix} -\mu\mathbb{I} + O(\epsilon) & CT + O(\epsilon) \\ & \\ -\epsilon T D^2_{\theta_0}\Big\langle H_1^\sharp\Big|_0\Big\rangle_T(\varphi^*, I^*) + O(\epsilon^2) & -\mu\mathbb{I} + O(\epsilon) \end{pmatrix} , \qquad \mu := \lambda - 1 \approx \sqrt{\epsilon}\sigma \,.$$

Here we can apply a second time the Schur complement, being the diagonal blocks invertible for $\epsilon \neq 0$ small enough, so that $\det(M_{11}(\sigma,\epsilon)) = 0$ reads

$$\det\Big(\sigma^2\mathbb{I} + T^2 D^2_{\theta_0}\Big\langle H_1^\sharp\Big|_0\Big\rangle_T(\varphi^*, I^*)C + R(\varepsilon)\Big) = 0 \,, \qquad \lim_{\epsilon\to 0} R(\epsilon) = 0 \,.$$

Taking the limit $\epsilon \to 0$, we obtain the asymptotic expansion claimed in the statement of Theorem 1.1; this concludes the non-degenerate case, since by adding the smaller perturbation term $M_{12}M_{22}^{-1}M_{21}$ the same arguments hold true.

The totally degenerate case can be faced following the same steps above, with a minor but crucial variation which involves the perturbation $M_{12}M_{22}^{-1}M_{21}$. Indeed, since the magnitude of $\dot{I}$ is of order $O(\epsilon^2)$, it follows that M_{12} has the special structure

$$M_{12}(\epsilon) = \begin{pmatrix} O(\epsilon) & O(\epsilon) \\ O(\epsilon^2) & O(\epsilon^2) \end{pmatrix} ,$$

which provides

$$M_{12}M_{22}^{-1}M_{21} = \begin{pmatrix} O(\epsilon^2) & O(\epsilon^2) \\ O(\epsilon^3) & O(\epsilon^3) \end{pmatrix} ;$$

hence it is component wise a perturbation of the leading term M_{11}

$$M_{11} = \begin{pmatrix} -\mu\mathbb{I} + O(\epsilon) & CT + O(\epsilon) \\ & \\ -\epsilon^2 T D^2_{\theta_0} F_2(\varphi^*, I^*) + O(\epsilon^3) & -\mu\mathbb{I} + O(\epsilon^2) \end{pmatrix} , \qquad \mu := \lambda - 1 \,,$$

where F_2 is taken with all its m components.

3. Spatially localized solutions in discrete NLS models

The aim of this section is to exploit Theorem 1.1 to investigate the existence of a special class of spatially localized time-periodic solutions in a dNLS model of the form

$$i\dot{\psi}_j = \psi_j - \epsilon(L\psi)_j + \gamma\psi_j|\psi_j|^2 , \tag{3.1}$$

where the (weak) linear term L is beyond nearest-neighbours

$$L\psi = \sum_{l=1}^{r} \kappa_l(\Delta_l\psi) , \qquad (\Delta_l\psi)_j := \psi_{j+l} - 2\psi_j + \psi_{j-l} \qquad \forall j \in \mathcal{J} , \tag{3.2}$$

the coupling parameter ϵ has to be considered small enough (so called anti-continuum limit) and the boundary conditions are of Dirichlet type in the case of $\mathcal{J}$ finite. The equations can be written in Hamiltonian form $i\dot{\psi}_j = \frac{\partial H}{\partial\bar{\psi}_j}$ with

$$H = H_0 + \epsilon H_1 , \tag{3.3}$$

where

$$\begin{aligned} H_0 &:= \sum_{j\in\mathcal{J}} |\psi_j|^2 + \frac{\gamma}{2}\sum_{j\in\mathcal{J}} |\psi_j|^4 \\ H_1 &:= \sum_{l=1}^{r} \kappa_l \sum_{j\in\mathcal{J}} |\psi_{j+l} - \psi_j|^2 = \sum_{l=1}^{r} \kappa_l\langle -\Delta_l\psi, \bar{\psi}\rangle . \end{aligned} \tag{3.4}$$

Being interested in solutions which are spatially localized on many-sites, we introduce the set of excited sites

$$S = \{j_1, \ldots, j_m\} ,$$

where we stress that the indexes do not have to be necessarily consecutive; in other words, we are including also configurations where the localization of the amplitude (hence of the energy), is clustered, with "holes" separating the different clusters along the chain.

The easiest and most popular spatially localized solutions in the considered dNLS model correspond to the so-called *discrete solitons* (or also *multi-pulse solutions*). These are periodic solutions $\psi(t)$ of (3.1) which at $\epsilon = 0$ take the form

$$\psi^{(0)}(t) = e^{-i\omega t}\phi^{(0)} , \qquad t \in [0, T := 2\pi/\omega] , \tag{3.5}$$

where $\phi^{(0)}$ is independent of time. Since the unperturbed excited oscillators $\{\psi_j^{(0)}\}_{j\in\mathcal{J}}$ in (3.5) are required to keep the same frequency ω, thus being in $1:1$ resonance, a common amplitude $R > 0$ is necessary, so that the unperturbed spatial profile $\phi^{(0)}$ reads

$$\phi_j^{(0)} = \begin{cases} Re^{i\theta_j} , & j \in S \\ 0 , & j \in \mathcal{J} \setminus S \end{cases} , \tag{3.6}$$

where

$$\omega(R) = 1 + \gamma R^2 . \tag{3.7}$$

All these orbits are uniquely defined except for a phase shift, which corresponds to a change of the initial configuration in the ansatz (3.5). According to this geometrical interpretation, all the unperturbed periodic orbits foliate a completely resonant m-dimensional torus $\mathbb{T}^m$ of the phase space, and any orbit is uniquely identified by a point in the quotient space $\mathbb{T}^{m-1} = \mathbb{T}^m \setminus \mathbb{T}$; such a point can be well represented by introducing a set of $m-1$ new "phase shifts" variables

$$\varphi_j := \theta_{j_{l+1}} - \theta_{j_l} \,, \qquad\qquad l = 1, \ldots, m \,. \tag{3.8}$$

In order to put the dNLS model in the form (1.6) of Theorem 1.1, we introduce action-angle variables for the sites $j \in S$

$$\psi_j = \sqrt{I_j} e^{-i\theta_j} \qquad \Rightarrow \qquad |\psi_j|^2 = I_j \,,$$

and Birkhoff complex variables $z = (\eta, \zeta)$ (replacing the complex variable ψ) for the remaining sites

$$\zeta_j = \psi_j \,, \qquad\qquad i\eta_j = \bar{\psi}_j = \bar{\zeta}_j \,;$$

in this way the symplectic form reads $d\eta \wedge d\zeta + d\theta \wedge dI$ and the integrable part of the Hamiltonian takes the form

$$\begin{aligned} H_0^\sharp &= \sum_{j\in S} \left(I_j + \frac{\gamma}{2} I_j^2 \right) \qquad \Rightarrow \qquad C = D_I^2 H_0^\sharp = \gamma \mathbb{I} \,, \\ H_0^\flat &= \sum_{j\notin S} i\zeta_j\eta_j - \frac{\gamma}{2}\zeta_j^2\eta_j^2 \qquad \Rightarrow \qquad \Omega_j = 1 \,, \end{aligned}$$

while the perturbation H_1 can be split in the three parts $H_1^\sharp$, $H_1^\flat$, $H_1^\natural$ depending on the shape of L in (3.2) and of the choice of S in the considered example. Moreover, once fixed at I^* the common action of the excited nonlinear oscillators ψ_j, $j \in S$, the frequency (3.7) reads

$$\omega(I^*) = 1 + \gamma I^* \,.$$

We will benefit of rewriting the generic linear interaction term as

$$|\psi_{j+l} - \psi_j|^2 = |\psi_{j+l}|^2 + |\psi_j|^2 - \left(\psi_{j+l}\bar{\psi}_j + c.c \right) .$$

In this way, a term of $H_1^\sharp$ would read

$$I_{j+l} + I_j - 2\sqrt{I_{j+l} I_j} \cos(\theta_{j+l} - \theta_j) \,,$$

one of $H_1^\flat$ would read

$$i\zeta_{j+l}\eta_{j+l} + i\zeta_j\eta_j - i\left(\zeta_{j+l}\eta_j + \eta_{j+l}\zeta_j \right) ,$$

and one of $H_1^\natural$ (assuming $j \in S$ and $j + l \in \mathcal{J} \setminus S$)

$$-\sqrt{I_j}\zeta_{j+l} e^{i\theta_j} - i\sqrt{I_j}\eta_{j+l} e^{-i\theta_j} \,.$$

3.1. Non-degeneracy and consecutive excited sites

Let us consider, instead of the generic model (3.4), the dNLS Hamiltonian with nearest neighbor and next to nearest neighbor linear interactions, on a lattice $\mathcal{J} = \{-N, \ldots, N\}$

$$H = \sum_{j\in\mathcal{J}} \left[|\psi_j|^2 + \frac{\gamma}{2}|\psi_j|^4 \right] + \epsilon \sum_{j\in\mathcal{J}} \left[k_1|\psi_{j+1} - \psi_j|^2 + k_2|\psi_{j+2} - \psi_j|^2 \right],$$

and take $m \ll N$ excited consecutive sites in the set $S = \{1, \ldots, m\}$. In order to find candidate solutions, we need to solve (2.15), i.e., to find critical points of $\left\langle H_1^\sharp\big|_0 \right\rangle_T$, which explicitly reads

$$\left\langle H_1^\sharp\big|_0 \right\rangle_T = \frac{1}{T}\int_0^T H_1^\sharp\left(\omega t + \theta^{(0)}, I^*\right) dt\,,$$

where

$$\begin{aligned} H_1^\sharp = {} & 2k_1 \sum_{i=1}^{m-1} \sqrt{I_{i+1}I_i}\cos\left(\theta_{i+1} - \theta_i\right) + \\ & + 2k_2 \sum_{i=1}^{m-2} \sqrt{I_{i+2}I_i}\cos\left(\theta_{i+2} - \theta_i\right) + \\ & + k_1(I_1 + I_m) + k_2(I_1 + I_2 + I_{m-1} + I_m) + 2k_1\sum_{i=2}^{m-1} I_i + 2k_2 \sum_{i=3}^{m-2} I_i \end{aligned}$$

while both $H_1^\flat$ and $H_1^\natural$ have to vanish, since $z^{(0)} \equiv 0$. Explicit calculations provide

$$\left\langle H_1^\sharp\big|_0 \right\rangle_T = \frac{2I^*}{T}\int_0^T \left[k_1 \sum_{i=1}^{m-1} \cos\left(\theta_{i+1}^{(0)} - \theta_i^{(0)}\right) + k_2 \sum_{i=1}^{m-2} \cos\left(\theta_{i+2}^{(0)} - \theta_i^{(0)}\right) \right] dt + \text{constant terms;}$$

by introducing the coordinates $\varphi_i = \theta_{i+1}^{(0)} - \theta_i^{(0)}$ one gets the (ϕ-dependent part of the) functional on the torus $\mathbb{T}^{m-1}$

$$\left\langle H_1^\sharp\big|_0 \right\rangle_T = 2I^* \left[k_1 \sum_{i=1}^{m-1} \cos\left(\varphi_i\right) + k_2 \sum_{i=1}^{m-2} \cos\left(\varphi_{i+1} + \varphi_i\right) \right].$$

whose critical points provide the solutions φ^* we were looking for.

For examples, let us consider three consecutive sites $m = 3$ with $S = \{1, 2, 3\}$, in the so-called ZigZag model $k_1 = k_2 = 1$. The average $\left\langle H_1^\sharp\big|_0 \right\rangle_T$ defined on the torus $\mathbb{T}^2$ is

$$\left\langle H_1^\sharp\big|_0 \right\rangle_T = \cos(\varphi_1) + \cos(\varphi_2) + \cos(\varphi_1 + \varphi_2)\,,$$

so that Eq (2.15) reads

$$\begin{cases} \sin\left(\varphi_1\right) + \sin\left(\varphi_1 + \varphi_2\right) = 0 \\ \sin\left(\varphi_2\right) + \sin\left(\varphi_1 + \varphi_2\right) = 0 \end{cases},$$

which provides six different solutions: four of them correspond to in/out-of-phase configurations $\varphi^* \in \{(0,0),(0,\pi),(\pi,0),(\pi,\pi)\}$ and the other two represent the so-called vortex solutions (see [24]) with $\varphi^* = \pm(\frac{2}{3}\pi, \frac{2}{3}\pi)$. All the six solutions are isolated, hence we expect non-degeneracy to hold: this is confirmed by an explicit computation of the differential in (P-ND).

Since $C = \gamma\mathbb{I}$, approximate linear stability is completely encoded by the nature of the critical points φ^*. Indeed the nonzero eigenvalues of $D^2_\theta\left\langle H_1^\sharp\big|_0\right\rangle_T$ are given by the eigenvalues of $-\gamma T^2 D^2_\varphi\left\langle H_1^\sharp\big|_0\right\rangle_T$, with

$$D^2_\varphi\left\langle H_1^\sharp\big|_0\right\rangle_T = -\begin{pmatrix} \cos(\varphi_1)+\cos(\varphi_1+\varphi_2) & \cos(\varphi_1+\varphi_2) \\ \cos(\varphi_1+\varphi_2) & \cos(\varphi_2)+\cos(\varphi_1+\varphi_2) \end{pmatrix} .$$

In particular we recover the property, already illustrated in the literature, that for consecutive excited sites the center and saddle directions are exchanged when changing the sign of the product $\epsilon\gamma$. If such a product is positive, we found one maximum $(0,0)$, two minima $\pm(\frac{2}{3}\pi, \frac{2}{3}\pi)$ and the remaining are saddles of $\left\langle H_1^\sharp\big|_0\right\rangle_T$.

3.2. Total degeneracy and non consecutive sites

Let us now consider the standard dNLS model on a lattice $\mathcal{J} = \{-N,\dots,N\}$

$$H = \sum_{j\in\mathcal{J}}\left[|\psi_j|^2 + \frac{\gamma}{2}|\psi_j|^4\right] + \epsilon\sum_{j\in\mathcal{J}}|\psi_{j+1}-\psi_j|^2 ,$$

and take two excited but not consecutive sites, namely $S = \{-1,1\}$. The perturbation H_1 can be split in the three parts

$$\begin{aligned} H_1^\sharp &= 2I_{-1} + 2I_1 , \\ H_1^\flat &= 2\sum_{j\notin S} i\zeta_j\eta_j - \sum_{j\le -3,\, j\ge 2} i\big(\zeta_{j+1}\eta_j + \eta_{j+1}\zeta_j\big) , \\ H_1^\natural &= -\sqrt{I_{-1}}e^{i\theta_{-1}}(\zeta_{-2}+\zeta_0) - \sqrt{I_{-1}}e^{-i\theta_{-1}}(i\eta_{-2}+i\eta_0) + \\ &\quad -\sqrt{I_1}e^{i\theta_1}(\zeta_2+\zeta_0) - \sqrt{I_1}e^{-i\theta_1}(i\eta_2+i\eta_0) . \end{aligned}$$

The complete degeneracy of the continuation appears in the independence of $H_1^\sharp$ with respect to the angles

$$\nabla_\theta H_1^\sharp \equiv 0 .$$

In this case, condition (2.18) further simplifies, being

$$D^2_\theta H_1^\sharp \equiv D^2_{I\theta} H_1^\sharp \equiv 0 ,$$

hence the persistence condition reduces to

$$\left\langle D_z\partial_{\theta_l} H_1^\natural\big|_0 z^{(1)}\right\rangle = 0 , \tag{3.9}$$

with l chosen as either $l = -1$ or $l = 1$, since the two conditions are dependent. Due to $i\eta_j = \zeta_j$, it is evident that $H_1^\natural$ is a real function, and the same holds also for its gradient. We choose $l = -1$, so that

$$D_z\partial_{\theta_{-1}} H_1^\natural\big|_0 z^{(1)}(t) = -i\sqrt{I^*}e^{i\theta_{-1}}\Big(\zeta^{(1)}_{-2} + \zeta^{(1)}_0\Big) + i\sqrt{I^*}e^{-i\theta_{-1}}\Big(i\eta^{(1)}_{-2} + i\eta^{(1)}_0\Big) ,$$

which is a real function since the second addendum is the complex conjugate of the first; in the above and in what follows, we have to keep in mind that

$$\theta_{-1}(t) = \omega t\ , \qquad \theta_1(t) = \omega t + \phi\ . \tag{3.10}$$

We can also write explicitly

$$D_z \partial_{\theta_l} H_1^\natural \Big|_0 z^{(1)}(t) = -2\sqrt{I^*}\,\Re\Big[ie^{i\omega t}\Big(\zeta_{-2}^{(1)} + \zeta_0^{(1)}\Big)\Big]\ , \tag{3.11}$$

which tells us that only evolutions of the complex configurations $\zeta_0^{(1)}$ and $\zeta_{-2}^{(1)}$ are needed. Their equations are (remember that ζ are the momenta of the canonical variables)

$$\begin{aligned}\dot\zeta_{-2}^{(1)} &= -i\zeta_{-2}^{(1)} + i\sqrt{I^*}e^{-i\omega t}\\ \dot\zeta_{0}^{(1)} &= -i\zeta_{0}^{(1)} + i\sqrt{I^*}\Big(e^{-i(\omega t+\phi)} + e^{-i\omega t}\Big)\end{aligned}$$

where the different forcing terms depend on the fact that the 0-site is excited by two neighbours, while the -2-site only by one. The solutions, because of the non-resonance condition between the linear frequency $\Omega = 1$ and ω, are then given by

$$\begin{aligned}\zeta_{-2}^{(1)}(t) &= e^{-it}\zeta_{-2}^{(1)}(0) + \frac{\sqrt{I^*}}{(1-\omega)}\Big(e^{-i\omega t} - e^{-it}\Big)\\ \zeta_{0}^{(1)}(t) &= e^{-it}\zeta_{0}^{(1)}(0) + \frac{\sqrt{I^*}}{(1-\omega)}\Big(e^{-i(\omega t+\phi)} + e^{-i\omega t} - e^{-it} - e^{-i(t+\phi)}\Big)\end{aligned}$$

where the Cauchy problems are given by the periodicity conditions $\zeta_l^{(1)}(T) = \zeta_l^{(1)}(0)$, thus

$$\zeta_{-2}^{(1)}(0) = \frac{\sqrt{I^*}}{(1-\omega)}\ , \qquad \zeta_{-2}^{(1)}(0) = \frac{\sqrt{I^*}}{(1-\omega)}\Big(e^{-i\phi} + 1\Big)\ .$$

By inserting the above initial data in the general solutions, one obtains

$$\begin{aligned}\zeta_{-2}^{(1)}(t) &= \frac{\sqrt{I^*}}{(1-\omega)}e^{-i\omega t}\\ \zeta_{0}^{(1)}(t) &= \frac{\sqrt{I^*}}{(1-\omega)}\Big(e^{-i\omega t} + e^{-i(\omega t+\phi)}\Big)\end{aligned} \tag{3.12}$$

which provides

$$D_z \partial_{\theta_{-1}} H_1^\natural \Big|_0 z^{(1)}(t) = \frac{2I^*}{\omega - 1}\Re\Big(2i + ie^{-i\phi}\Big) = \frac{2I^*}{1-\omega}\sin(\varphi)\ .$$

Hence, being the above term constant in time, the average provides exactly

$$F_2(\varphi, I^*) = -\frac{2}{\gamma}\sin(\varphi)\ ,$$

where we have used the explicit dependence of ω with respect to I^*; critical points of F_2 are only $\varphi^* \in \{0, \pi\}$, which are non-degenerate, and the IFT applies.

Linear stability immediately follows from $-\epsilon F_2'(\varphi^*)$: we stress that here γ does not play any role, and stability is exchanged only by switching from positive to negative values of ϵ, hence from attractive to repulsive linear interaction.

Acknowledgments

T. P. thanks his former Master student Sarah Dal Lago and the group of Studitorium of Mechanics in the Department of Physics of the Aristotle University of Thessaloniki, for useful and friendly discussions which at the end of 2018 have motivated the present contribution. T. P. and V. D. have been supported by the GNFM - Progetto Giovani funding “Low-dimensional Invariant Tori in FPU-like Lattices via Normal Forms” and by the MIUR-PRIN 20178CJA2B “New Frontiers of Celestial Mechanics: theory and Applications”.

Conflict of interest

The authors declare no conflict of interest.

References

1. Kapitula T, Kevrekidis P (2001) Stability of waves in discrete systems. *Nonlinearity* 14: 533–566.
2. Ahn T, Mackay RS, Sepulchre JA (2001) Dynamics of relative phases: Generalised multibreathers. *Nonlinear Dynam* 25: 157–182.
3. Aubry S (1997) Breathers in nonlinear lattices: Existence, linear stability and quantization. *Phys D* 103: 201–250.
4. Bruno AD (2020) Normalization of a periodic Hamiltonian system. *Program Comput Soft* 46: 76–83.
5. Cheng CQ, Wang S (1999) The surviving of lower dimensional tori from a resonant torus of Hamiltonian Systems. *J Differ Equations* 155: 311–326.
6. Cuevas J, Koukouloyannis V, Kevrekidis PG, et al. (2011) Multibreather and vortex breather stability in Klein-Gordon lattices: Equivalence between two different approaches. *Int J Bifurcat Chaos* 21: 2161–2177.
7. Ekeland I (1990) *Convexity Methods in Hamiltonian Mechanics*, Berlin: Springer-Verlag.
8. Graff SM (1974) On the conservation of hyperbolic invariant for Hamiltonian systems. *J Differ Equations* 15: 1-69.
9. Han Y, Li Y, Yi Y (2006) Degenerate lower dimensional tori in Hamiltonian Systems. *J Differ Equations* 227: 670–691.
10. Kapitula T (2001) Stability of waves in perturbed Hamiltonian systems. *Phys D* 156: 186–200.
11. Koukouloyannis V, Kevrekidis PG (2009) On the stability of multibreathers in Klein-Gordon chains. *Nonlinearity* 22: 2269–2285.
12. Kevrekidis PG (2009) *The Discrete Nonlinear Schrödinger Equation*, Berlin: Springer-Verlag.
13. Kevrekidis PG (2009) Non-nearest-neighbor interactions in nonlinear dynamical lattices. *Phys Lett A* 373: 3688–3693.
14. Koukouloyannis V, Kevrekidis PG, Cuevas J, et al. (2013) Multibreathers in Klein-Gordon chains with interactions beyond nearest neighbors. *Phys D* 242: 16–29.

15. Koukouloyannis V (2013) Non-existence of phase-shift breathers in one-dimensional Klein-Gordon lattices with nearest-neighbor interactions. *Phys Lett A* 377: 2022–2026.

16. Li Y, Yi Y (2003) A quasi-periodic Poincaré's theorem. *Math Ann* 326: 649–690.

17. MacKay RS (1996) Dynamics of networks: Features which persist from the uncoupled limit, In: *Stochastic and Spatial Structures of Dynamical Systems*, Amsterdam: North-Holland, 81–104.

18. MacKay RS, Sepulchre JS (1998) Stability of discrete breathers. *Phys D* 119: 148-162.

19. Meletlidou E, Ichtiaroglou S (1994) On the number of isolating integrals in perturbed Hamiltonian systems with $n \geq 3$ degrees of freedom. *J Phys A Math Gen* 27: 3919-3926.

20. Meletlidou E, Stagika G (2006) On the continuation of degenerate periodic orbits in Hamiltonian systems. *Regul Chaotic Dyn* 11: 131–138.

21. Paleari S, Penati T (2019) Hamiltonian lattice dynamics. *Mathematics in Engineering* 1: 881–887.

22. Pelinovsky DE, Kevrekidis PG, Frantzeskakis DJ (2005) Persistence and stability of discrete vortices in nonlinear Schrödinger lattices. *Phys D* 212: 20–53.

23. Pelinovsky DE, Kevrekidis PG, Frantzeskakis DJ (2005) Stability of discrete solitons in nonlinear Schrödinger lattices. *Phys D* 212: 1–19.

24. Penati T, Sansottera M, Paleari S, et al. (2018) On the nonexistence of degenerate phase-shift discrete solitons in a dNLS nonlocal lattice. *Phys D* 370: 1–13.

25. Penati T, Koukouloyannis V, Sansottera M, et al. (2019) On the nonexistence of degenerate phase-shift multibreathers in Klein-Gordon models with interactions beyond nearest neighbors. *Phys D* 398: 92-114.

26. Penati T, Sansottera M, Danesi V (2018) On the continuation of degenerate periodic orbits via normal form: Full dimensional resonant tori. *Commun Nonlinear Sci* 61: 198-224.

27. Sansottera M, Danesi V, Penati T, et al. (2020) On the continuation of degenerate periodic orbits via normal form: Lower dimensional resonant tori. *Commun Nonlinear Sci* 90: 105360.

28. Poincaré H (1957) *Les méthodes nouvelles de la mécanique céleste. Tome I. Solutions périodiques. Non-existence des intégrales uniformes. Solutions asymptotiques*. New York: Dover Publications, Inc.

29. Poincaré H (1996) *Œuvres. Tome VII*. Les Grands Classiques Gauthier-Villars, Sceaux: Jacques Gabay.

30. Treshchev DV (1991) The mechanism of destruction of resonant tori of Hamiltonian systems. *Math USSR Sb* 68: 181–203.

31. Voyatzis G, Ichtiaroglou S (1999) Degenerate bifurcations of resonant tori in Hamiltonian systems. *Int J Bifurcat Chaos* 9: 849–863.

32. Yakubovich VA, Starzhinskii VM (1975) *Linear differential equations with periodic coefficients. 1 and 2*. New York-Toronto: John Wiley & Sons.