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Research article

Stability of the standing waves of the concentrated NLSE in dimension two†

Riccardo Adami[1,*], Raffaele Carlone[2], Michele Correggi[3] and Lorenzo Tentarelli[1]

[1] Politecnico di Torino, Dipartimento di Scienze Matematiche "G.L. Lagrange", Corso Duca degli Abruzzi, 24, 10129, Torino, Italy

[2] Università degli Studi di Napoli "Federico II", Dipartimento di Matematica e Applicazioni "R. Caccioppoli", MSA, via Cinthia, I-80126, Napoli, Italy

[3] Politecnico di Milano, Dipartimento di Matematica, Piazza Leonardo da Vinci, 32, 20133, Milano, Italy

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* **Correspondence:** Email: riccardo.adami@polito.it.

Abstract: In this paper we will continue the analysis of two dimensional Schrödinger equation with a fixed, pointwise, nonlinearity started in [2, 13]. In this model, the occurrence of a blow-up phenomenon has two peculiar features: the energy threshold under which all solutions blow up is strictly negative and coincides with the infimum of the energy of the standing waves; there is no critical power nonlinearity, i.e., for every power there exist blow-up solutions. Here we study the stability properties of stationary states to verify whether the anomalies mentioned before have any counterpart on the stability features.

Keywords: nonlinear Schrödinger equation; point interactions; standing waves; orbital stability
Mathematics Subject Classification: 35Q40, 35Q55

1. Introduction

The Nonlinear Schrödinger Equation (NLSE) with concentrated nonlinearity in $d = 2$ is the subject of several recent papers, finalizing a research program developed over the last twenty years (see [3, 4, 8, 14] for the NLSE with concentrated nonlinearity and also [15] and [12] for the fractional case and the Dirac equation, respectively).

Such a research line was originally motivated by some mesoscopic physical models. For instance, in semiconductor theory the effect of electronic charge accumulation in a resonant tunneling in a double barrier heterostructure [20] is typically studied using a concentrated NLSE. More recently, other applications have been suggested: the spontaneous formation of quantum coherent non-dissipative patterns in semiconductor heterostructures with nonlinear properties [11]; the dynamics of the mixed states of statistical physics [23]; the appearance of quantum turbulence in the probability density [9]; the scattering in nuclear physics models for the disexcitation of isomeric states and also the production of weakly bounded states in heavy nuclei close to the instability; the analysis of resonant tunneling diodes, which exhibits intrinsic instability [31] or the fabrication of semiconductor superlattices, for the estimate of the time decay rates for the solutions to the Schrödinger-Poisson equations in the repulsive case [10, 25].

In [13] and [16] the local well-posedness is established, i.e., the problem of existence and uniqueness of the solution for short times, as well as the mass and energy conservation. Global existence is also proven in the defocusing case irrespective of the power of the nonlinearity. In [2] it is studied the occurrence of a blow-up phenomenon for a focusing nonlinearity, with two peculiar features: first, the energy threshold under which all solutions blow up is strictly negative and coincides with the infimum of the energy of standing waves; second, there is no critical power nonlinearity, i.e., for every power there exist blow up solutions. We remark that such a behavior is anomalous compared to the conventional NLSE, also because such anomalies are not a direct consequence of the dimension, or of the concentrated nonlinearity. In fact, there is a critical power for standard nonlinearities in dimension two [21], and there is also a critical power for concentrated nonlinearities in dimension one and three [3, 8]. In the present paper we investigate further whether such peculiarities also show up in the stability of stationary states.

Let us preliminarily recall the results on the standard NLSE [29]: Consider the Cauchy problem for a focusing NLSE, where the word focusing refers to the attractive character of the nonlinearity, with initial data in the energy space

$$\imath\partial_t\psi(t,x) + \triangle\psi + |\psi|^{2\sigma}\psi = 0, \qquad \psi(0,x) = \psi_0(x) \in H^1(\mathbb{R}^d).$$

In [17], using a variational characterization, it was established the orbital stability of the ground-states in the subcritical case, i.e., for $\sigma < 2\backslash d$. On the other hand, [19, 26, 30] presents an alternative proof of orbital stability for the subcritical solitary waves and shows the orbital instability in the critical and supercritical case ($\sigma d \geqslant 2$).

It turns out that there is a strict relation between blow-up and orbital stability of standing waves [28]. The NLSE admits blow-up solutions if and only if its solitary waves are orbitally unstable. This behavior has some relevant exceptions as in the case of NLSE in bounded domains or in [27] where the key feature of all this models is always the absence of translational invariance in space.

The analysis of stationary states stability for concentrated nonlinearities traces back to [5–7]. For the concentrated NLSE in dimension 2 the scenario is different and, in some sense, surprising. As it will be illustrated in Section 2 there are, at any fixed value of the mass, two branches of stationary states, distinguished by the value of the frequency ω, with opposite orbital stability behavior. To the best of our knowledge, there is no similar behavior for a standard Schrödinger equation on $\mathbb{R}^d$, but some analogy exists with the 1d NLSE in the presence of a point interaction [18] and with NLSE on compact domains [22, 24]. In all these cases, the nonlinearity is supercritical.

As for the case of concentrated NLSE in the defocusing case, the scenario is really puzzling. In

Section 3 the analysis of stationary waves reveals that they are stable and moreover that they are ground states.

1.1. Setting and known results

The problem under investigation can be formally written as

$$\begin{cases} \imath \dfrac{\partial \psi}{\partial t} = (-\Delta + \beta |\psi|^{2\sigma} \delta_{\mathbf{0}})\psi, & \text{in } \mathbb{R}^+ \times \mathbb{R}^2, \\ \psi(0) = \psi_0, & \text{in } \mathbb{R}^2, \end{cases} \tag{1.1}$$

where $\sigma > 0, \beta \in \mathbb{R}$ and $\delta_{\mathbf{0}}$ is a Dirac delta function centered at the origin of $\mathbb{R}^2$.

As extensively explained in [2, 13], the Cauchy problem in (1.1) can be rigorously formulated in a weak form. To this aim, one has to first introduce the so-called energy space

$$V := \left\{ \chi \in L^2(\mathbb{R}^2) : \chi = \chi_\lambda + q\mathcal{G}_\lambda,\, \chi_\lambda \in H^1(\mathbb{R}^2),\ q \in \mathbb{C} \right\}, \tag{1.2}$$

with $\lambda > 0$ and $\mathcal{G}_\lambda$ denoting the Green's function of $-\Delta + \lambda$ in $\mathbb{R}^2$, i.e.,

$$\mathcal{G}_\lambda(\mathbf{x}) := \frac{K_0(\sqrt{\lambda}\mathbf{x})}{2\pi} = \frac{1}{2\pi}\mathcal{F}^{-1}[(|\cdot|^2 + \lambda)^{-1}](\mathbf{x}), \tag{1.3}$$

(recall that K_0 is the Macdonald function of order zero given, e.g., in [1] and that $\mathcal{F}$ is the unitary Fourier transform of $\mathbb{R}^2$). Note that the parameter λ does not affect the definition of V. Indeed, it is possible to rewrite the space V without the parameter λ, as

$$V = \left\{ \chi \in L^2(\mathbb{R}^2),\, \chi = \chi_0 - q\frac{\log|x|}{2\pi},\, \chi_0 \in \dot{H}^1(\mathbb{R}^2),\ q \in \mathbb{C} \right\} \tag{1.4}$$

where $\dot{H}^1(\mathbb{R}^2)$ is the homogeneous Sobolev space. However, as (1.4) is not easily implemented in the expression of the energy, we shall keep using (1.2) throughout.

Hence, we can define a weak solution of (1.1) as a function ψ such that

$$\psi(t) = \phi_\lambda(t) + q(t)\, G_\lambda \in V, \qquad \forall t \geqslant 0, \tag{1.5}$$

and such that, for every $\chi = \chi_\lambda + q_\chi G_\lambda \in V$,

$$\begin{cases} \imath \dfrac{d}{dt}\langle \chi, \psi(t)\rangle = \langle \nabla\chi_\lambda, \nabla\psi(t)\rangle + \lambda(\langle \chi_\lambda, \phi_\lambda(t)\rangle - \langle \chi, \psi(t)\rangle) + \theta_\lambda(|q(t)|)q_\chi^* q(t), & \forall t > 0, \\ \psi(0) = \psi_0, \end{cases} \tag{1.6}$$

where $\langle \cdot, \cdot \rangle$ is the usual scalar product in $L^2(\mathbb{R}^2)$, and $\theta_\lambda : \mathbb{R}^+ \to \mathbb{R}$ is defined as

$$\theta_\lambda(s) := \frac{\log(\sqrt{\lambda}/2) + \gamma}{2\pi} + \beta s^{2\sigma},$$

with γ the Euler-Mascheroni constant (note that the parameter λ does not affect (1.6) too). According to (1.5) we will usually refer to $\phi_\lambda(t)$ as the regular part of $\psi(t)$, to $q(t)G_\lambda$ as the singular part and to $q(t)$ as the charge.

It has been proven in [2, 13] that, for $\sigma \geqslant 1/2$, (1.6) is locally well-posed in V (with the additional assumption $\phi_\lambda(0) \in H^{1+\eta}$, $\eta > 0$) and that the mass

$$\mathrm{M}(t) = \mathrm{M}(\psi(t)) := \|\psi(t)\|^2,$$

$\|\cdot\|$ denoting the usual norm in $L^2(\mathbb{R}^2)$, and the energy

$$\mathrm{E}(t) \;=\; \mathrm{E}(\psi(t)) \;:=\; \|\nabla\phi_\lambda(t)\|^2 + \lambda(\|\phi_\lambda(t)\|^2 - \|\psi(t)\|^2) + \left(\frac{\beta|q(t)|^{2\sigma}}{\sigma+1} + \frac{\log(\sqrt{\lambda}/2)+\gamma}{2\pi}\right)|q(t)|^2, \quad (1.7)$$

which is independent of λ as well, are preserved along the flow. In addition, when $\beta > 0$, i.e., in the defocusing case, the solution is global in time, whereas when $\beta < 0$, i.e., in the focusing case, the solution blows up in a finite time. In order to prove these results, one has to require [2] that $\phi_\lambda(0)$ belongs to the Schwartz space, which is only a technical hypothesis, and, more important, its energy satisfies

$$\mathrm{E}(\psi_0) < \Lambda = \Lambda(\sigma,\beta) := -\frac{\sigma}{4\pi(\sigma+1)(-4\pi\sigma\beta)^{1/\sigma}}.$$

In the following sections we study the problem of the stability of stationary states separately in the focusing and defocusing case.

2. Focusing case

In the focusing case, i.e., for $\beta < 0$, (1.6) admits (see [2]) a unique family of standing waves of the form

$$\psi_\omega(t,\mathbf{x}) := \mathrm{e}^{\imath\omega t}\,\mathrm{e}^{\imath\eta}\,u_\omega(\mathbf{x}), \qquad \eta \in \mathbb{R}, \quad \omega \in (\widetilde{\omega}, +\infty), \quad \widetilde{\omega} := 4\mathrm{e}^{-2\gamma}, \quad (2.1)$$

where

$$u_\omega(\mathbf{x}) := q(\omega)\mathcal{G}_\omega(\mathbf{x}), \qquad q(\omega) := \left(-\frac{\log(\sqrt{\omega}/2)+\gamma}{2\pi\beta}\right)^{1/2\sigma}. \quad (2.2)$$

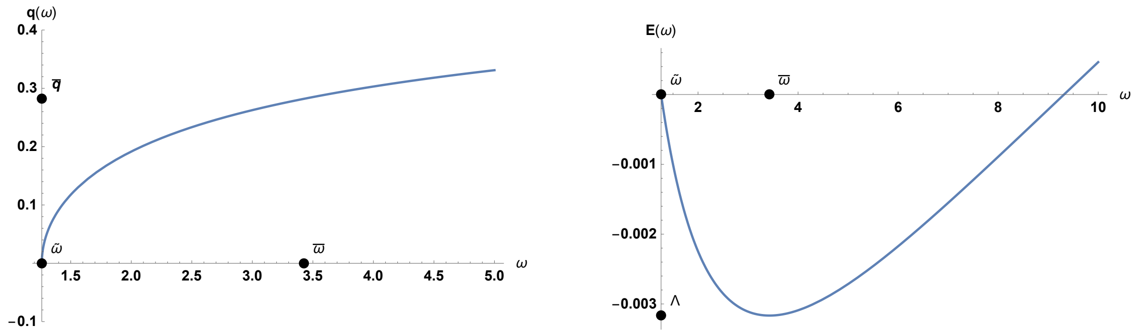
The behavior of $q(\omega)$ is depicted in Figure 1a.

Now, plugging (2.2) into (1.7), one finds that the energy of the standing waves as a function of the frequency ω reads

$$\mathrm{E}(\omega) := \mathrm{E}(u_\omega) = \left(\frac{\sigma\log(\sqrt{\omega}/2)+\gamma\sigma}{2\pi(\sigma+1)} - \frac{1}{4\pi}\right)\left(-\frac{\log(\sqrt{\omega}/2)+\gamma}{2\pi\beta}\right)^{1/\sigma}, \qquad \forall\omega \in (\widetilde{\omega}, +\infty). \quad (2.3)$$

The behavior of $E(\omega)$ is represented in Figure 1b. In addition,

$$\min_{\omega\in(\widetilde{\omega},+\infty)} \mathrm{E}(\omega) = \mathrm{E}(\overline{\omega}) = \Lambda, \qquad \text{where} \quad \overline{\omega} := 4\mathrm{e}^{-2\gamma+1/\sigma}. \quad (2.4)$$



(a) Behavior of $q(\omega)$. **(b)** Behavior of $\mathrm{E}(\omega)$.

Figure 1. Plots of $q(\omega)$ and $\mathrm{E}(\omega)$ for $\omega \in (\widetilde{\omega}, +\infty)$, when $\sigma = 1$ and $\beta = -1$. Here $\widetilde{\omega} \approx 1.26$, $\overline{\omega} \approx 3.43$, $\overline{q} \approx 0.2$ and $\Lambda \approx -0.0016$.

On the other hand, noting that $q(\widetilde{\omega}) = 0$ and that $q(\cdot)$ is smooth and strictly increasing on $(\widetilde{\omega}, +\infty)$, one can take the inverse $q(\omega)$ of the function

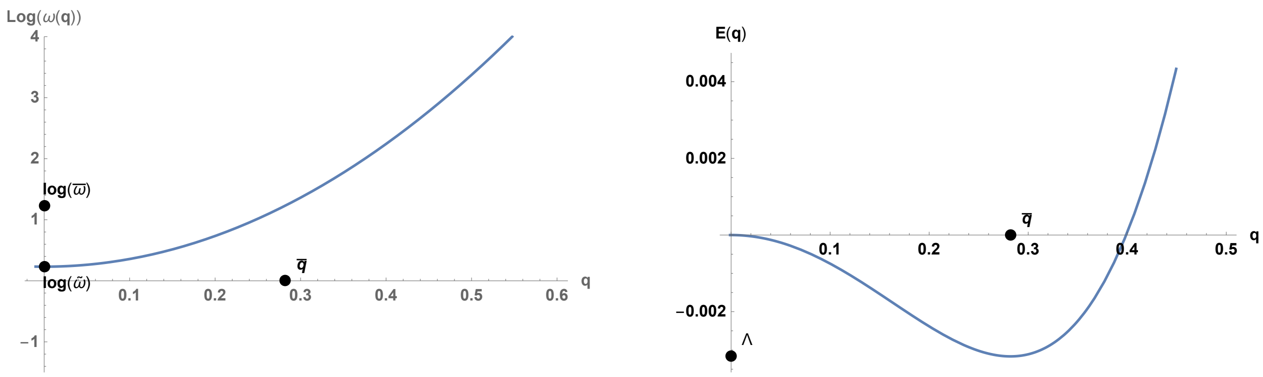
$$\omega(q) := 4\,\mathrm{e}^{-2\gamma - 4\pi\beta q^{2\sigma}}, \qquad q > 0, \tag{2.5}$$

and plug it into (2.3), to obtain the energy as a function of q, i.e.,

$$\mathrm{E}(q) = -\frac{q^2}{4\pi} - \frac{\sigma\beta q^{2\sigma+2}}{\sigma + 1}. \tag{2.6}$$

The behaviors of $\omega(q)$ and $\mathrm{E}(q)$ are depicted in Figure 2a and b, respectively. This alternative form can be useful in computation since (2.6) is more manageable than (2.3). Furthermore,

$$\inf_{q>0} \mathrm{E}(q) = \mathrm{E}(\overline{q}) = \Lambda < 0, \qquad \text{where} \quad \overline{q} := q(\overline{\omega}) = (-4\pi\sigma\beta)^{-1/2\sigma}.$$



(a) Behavior of $\log(\omega(q))$. **(b)** Behavior of $\mathrm{E}(q)$.

Figure 2. Plots of $\omega(q)$ and $\mathrm{E}(q)$ for $q \in \mathbb{R}^+$, when $\sigma = 1$ and $\beta = -1$.

The natural question arising at this point is about the stability of the standing waves. In view of the application of Grillakis-Shatah-Strauss theory [19], it is first necessary to compute the mass M as a function of ω and q. Exploiting (1.3), one has that

$$\mathrm{M}(\omega) := \mathrm{M}(u_\omega) = \frac{q^2(\omega)}{4\pi\omega} = \frac{1}{4\pi\omega}\left(-\frac{\log(\sqrt{\omega}/2) + \gamma}{2\pi\beta}\right)^{1/\sigma}. \tag{2.7}$$

On the other hand, one can easily check that

$$\mathrm{M}'(\omega) = \frac{q^2(\omega)}{4\pi\omega^2}\underbrace{\left[\frac{\left(\log(\sqrt{\omega}/2)+\gamma\right)^{-1}}{2\sigma} - 1\right]}_{=:h(\omega)},$$

whence

$$\begin{aligned}\mathrm{M}'(\omega) > 0 \quad (\text{resp. } \mathrm{M}'(\omega) < 0) \qquad &\Longleftrightarrow \qquad h(\omega) > 0 \quad (\text{resp. } h(\omega) < 0)\\ &\Longleftrightarrow \qquad \widetilde{\omega} < \omega < \overline{\omega} \quad (\text{resp. } \omega > \overline{\omega}).\end{aligned} \tag{2.8}$$

In addition, as $\lim_{\omega\to\widetilde{\omega}} \mathrm{M}(\omega) = \lim_{\omega\to+\infty} \mathrm{M}(\omega) = 0$, there results

$$\sup_{\omega\in(\widetilde{\omega},+\infty)} \mathrm{M}(\omega) = \mathrm{M}(\overline{\omega}) = \frac{\mathrm{e}^{2\gamma-1/\sigma}}{16\pi(-4\pi\sigma\beta)^{1/\sigma}} =: \overline{\mu}.$$

As a consequence, for every value of the mass $\mu \in (0, \overline{\mu})$ (or, alternatively, of the energy $\mathrm{E} \in (\Lambda, 0)$) there exists two distinct families of standing waves u_{ω_1}, u_{ω_2}, such that $\mathrm{M}(\omega_1) = \mathrm{M}(\omega_2) = \mu$, with $\omega_1 \in (\widetilde{\omega}, \overline{\omega})$ and $\omega_2 \in (\overline{\omega}, +\infty)$.

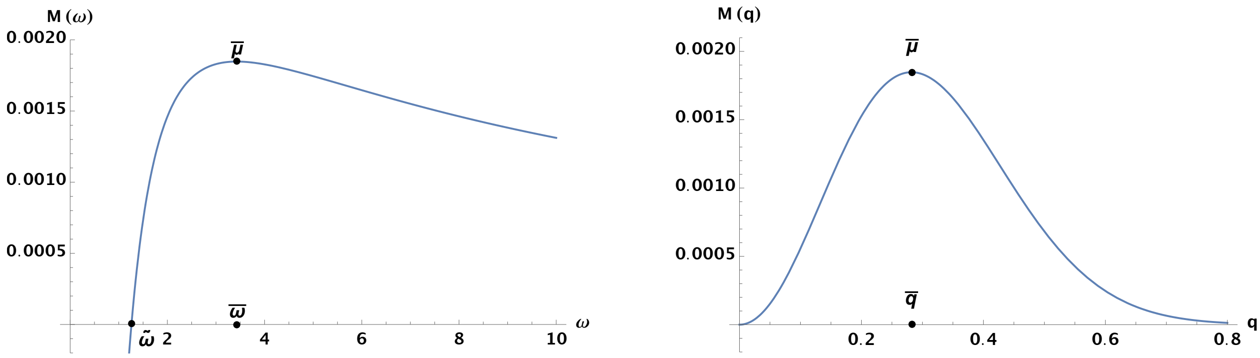
Analogous results can be obtained writing the mass of the standing waves in terms of q in place of ω, so that

$$\mathrm{M}(q) = \frac{q^2 \mathrm{e}^{2\gamma+4\pi\beta q^{2\sigma}}}{16\pi} \tag{2.9}$$

and

$$\sup_{q>0} \mathrm{M}(q) = \mathrm{M}(\overline{q}) = \overline{\mu}.$$

The qualitative behavior of $\mathrm{M}(\omega)$ and $\mathrm{M}(q)$ is depicted in Figure 3.



(a) Behavior of $\mathrm{M}(\omega)$. **(b)** Behavior of $\mathrm{M}(q)$.

Figure 3. Plots of $\mathrm{M}(\omega)$ and $\mathrm{M}(q)$ when $\sigma = 1$ and $\beta = -1$.

For any $\mu > 0$ there is no ground state of mass μ, i.e., no global minimizer of the energy constrained on

$$V_\mu := \{\psi \in V : \mathrm{M}(\psi) = \mu\}.$$

This can be easily seen if one defines a sequence $\{u_n\}_{n\in\mathbb{N}}$ such that

$$u_n(x) := 2\sqrt{\pi\mu\, n}\,\mathcal{G}_1(\sqrt{n}\,x), \qquad \mathrm{M}(u_n) = \mu.$$

Indeed, $\{u_n\} \subset V_\mu$ and

$$\mathrm{E}(u_n) = -\sqrt{n}\mu + \left(\frac{\beta(4\pi\mu\, n)^\sigma}{\sigma+1} + \frac{\log(\sqrt{n}/2)+\gamma}{2\pi}\right)(\pi\mu\, n)^{1/2} \xrightarrow[n\to+\infty]{} -\infty,$$

since $\beta < 0$. Hence, the stability analysis requires the use of the techniques developed in [19], as shown in Theorem 2.1.

First, we recall the definition of *orbitally* stable standing wave. To this aim, preliminarily, we endow the energy space V with a norm. Due to the several possible decompositions of a function $\chi \in V$ for different values of the spectral parameter $\lambda > 0$ (see (1.2)), in order to obtain a suitable norm, one has to fix a value $\lambda = \overline{\lambda} > 0$ and then set

$$\|\chi\|_{\overline{\lambda}}^2 := \|\chi_{\overline{\lambda}}\|_{H^1(\mathbb{R}^2)}^2 + \frac{|q|^2}{4\pi\overline{\lambda}}.$$

Clearly, any other choice of λ gives rise to an equivalent norm. In this section we will set $\overline{\lambda} = 1$ for the sake of simplicity.

Definition 2.1. The standing wave u_ω is said to be *orbitally stable* whenever for every $\varepsilon > 0$ there exists $\delta > 0$ such that: if $\|\psi_0 - e^{\imath\eta}u_\omega\|_1 < \delta$, for some $\eta \in \mathbb{R}$, and $\psi(t)$ is a solution of (1.6) on $[0, T^*)$ with initial condition ψ_0, then $\psi(t)$ can be continued to a solution on $[0, +\infty)$ and

$$\sup_{t\in\mathbb{R}^+}\inf_{\eta\in\mathbb{R}} \|\psi(t) - \mathrm{e}^{\imath\eta}u_\omega\|_1 < \varepsilon.$$

Otherwise the standing wave is called *unstable*.

Theorem 2.1 (Stability in the focusing case)**.** *Let $\sigma \geqslant 1/2$ and $\beta < 0$. The standing waves defined in* (2.1) *and* (2.2) *are orbitally stable if $\omega \in (\widetilde{\omega}, \overline{\omega})$ and unstable if $\omega > \overline{\omega}$ (where $\overline{\omega}$ is given in* (2.4)*).*

Note that the previous theorem entails that, for every mass $\mu \in (0, \overline{\mu})$, there is a pair of standing waves of mass μ, where the one with low frequency is stable, while the one with high frequency is unstable.

Remark 2.1. The assumption $\sigma \geqslant 1/2$ is only related to the local well-posedness of (1.6) proved in [13]. It is likely that it could be dropped by means of a more refined analysis of the local well-posedness and hence is not actually relevant in the stability analysis.

The proof of Theorem 2.1 is based on [19, Theorem 3]. For the sake of completeness, we recall here the statement suitably adapted to (1.6).

Theorem 2.2. *Assume that:*

(A1) *there exists a local solution of* (1.6)*, which preserves mass and energy along the flow;*

(A2) *there exist $\omega_2 > \omega_1 > 0$ and a family $(u_\omega)_\omega$ of standing waves of* (1.6) *such that the a mapping $(\omega_1, \omega_2) \ni \omega \mapsto u_\omega \in V$ is of class C^1;*

(A3) *letting*

$$\mathrm{S}_\omega : V \to \mathbb{R}, \qquad \mathrm{S}_\omega(u) := \mathrm{E}(u) + \omega \mathrm{M}(u),$$

be the action functional associated with (1.6) *and defining the operator*

$$H_\omega : V \to V^*, \qquad H_\omega := d^2\mathrm{S}_\omega(u_\omega) \tag{2.10}$$

($d^2\mathrm{S}_\omega$ denoting the second Fréchet differential), suppose that, for every $\omega \in (\omega_1, \omega_2)$,

(i) H_ω has exactly one negative simple eigenvalue,
(ii) the kernel of H_ω coincides with the span of u_ω,
(iii) the rest of $\sigma(H_\omega)$ is positive and bounded away from zero.

If the function

$$D : (\omega_1, \omega_2) \to \mathbb{R}, \qquad D(\omega) := \mathrm{S}_\omega(u_\omega),$$

is strictly convex, then u_ω is orbitally stable. If, on the contrary, D is strictly concave, then u_ω is unstable.

Proof of Theorem 2.1. **(A1)** of Theorem 2.2 has been proven by [13, Theorem 1.1 & Theorem 1.2], while the fulfillment of **(A2)** is a direct consequence of the form of the standing waves given by (2.1) and (2.2), with $\omega_1 = \widetilde{\omega}$ and $\omega_2 = +\infty$. Concerning **(A3)** we argue as follows.

As $d^2\mathrm{M}(u_\omega) = 2 \times \mathbb{I}$, it is sufficient to compute only $d^2\mathrm{E}(u_\omega)$. Since E is a functional of class C^2 we can compute the Gâteaux second differential in place of the Fréchet second differential, i.e.,

$$d^2\mathrm{E}(u_\omega)[h,k] = \left.\frac{\partial^2 \mathrm{E}(u_\omega + \nu h + \tau k)}{\partial\nu\partial\tau}\right|_{\nu=\tau=0}.$$

In addition, for the sake of simplicity, we can set $\lambda = \omega$ in the definition of E. Therefore, standard computations yields

$$\frac{\partial \mathrm{E}(u_\omega + \nu h + \tau k)}{\partial\tau} = 2\mathrm{Re}\Big\{\langle \nu\nabla h_\omega + \tau\nabla k_\omega, \nabla k_\omega\rangle + \omega(\langle \nu h_\omega + \tau k_\omega, k_\omega\rangle - \langle u_\omega + \nu h + \tau k, k\rangle) +$$
$$+ q_k(q^*(\omega) + \nu q_h^* + \tau q_k^*)\frac{\log(\sqrt{\omega}/2) + \gamma}{2\pi} + \beta q_k(q(\omega) + \nu q_h + \tau q_k)^\sigma (q^*(\omega) + \nu q_h^* + \tau q_k^*)^{\sigma+1}\Big\},$$

so that

$$\frac{\partial^2 \mathrm{E}(u_\omega + \nu h + \tau k)}{\partial\nu\partial\tau} = 2\mathrm{Re}\Big\{\langle \nabla h_\omega, \nabla k_\omega\rangle + \omega(\langle h_\omega, k_\omega\rangle - \langle h, k\rangle) +$$
$$+ q_k q_h^* \frac{\log(\sqrt{\omega}/2) + \gamma}{2\pi} + \sigma\beta q_k q_h (q(\omega) + \nu q_h + \tau q_k)^{\sigma-1}(q^*(\omega) + \nu q_h^* + \tau q_k^*)^{\sigma+1} +$$
$$+ (\sigma+1)\beta q_k q_h^* (q(\omega) + \nu q_h + \tau q_k)^\sigma (q^*(\omega) + \nu q_h^* + \tau q_k^*)^\sigma\Big\}$$

and hence

$$\left.\frac{\partial^2 \mathrm{E}(u_\omega + \nu h + \tau k)}{\partial\nu\partial\tau}\right|_{\nu=\tau=0} = 2\mathrm{Re}\{\langle\nabla h_\omega, \nabla k_\omega\rangle + \omega(\langle h_\omega, k_\omega\rangle - \langle h, k\rangle)\} + $$

$$+ \frac{\log(\sqrt{\omega}/2)+\gamma}{\pi}\mathrm{Re}\{q_k q_h^*\} + 2\beta q^{2\sigma}(\omega)\mathrm{Re}\{\sigma q_k^* q_h^* + (\sigma+1) q_k^* q_h\}. \quad (2.11)$$

Now, if we split each quantity as real and imaginary part, i.e.,

$$h = h^r + \imath h^i, \quad k = k^r + \imath k^i,$$
$$h_\omega = h_\omega^r + \imath h_\omega^i, \quad k_\omega = k_\omega^r + \imath k_\omega^i,$$
$$q_h = q_h^r + \imath q_h^i, \quad q_k = q_k^r + \imath q_k^i,$$

then (2.11) reads

$$\left.\frac{\partial^2 \mathrm{E}(u_\omega + \nu h + \tau k)}{\partial\nu\partial\tau}\right|_{\nu=\tau=0} = B_1[h^r, k^r] + B_1[h^i, k^i],$$

where B_1, B_2 are two sesquilinear forms given by

$$B_1[h^r, k^r] := 2(\langle\nabla h_\omega^r, \nabla k_\omega^r\rangle + \omega(\langle h_\omega^r, k_\omega^r\rangle - \langle h^r, k^r\rangle)) + \left(\frac{\log(\sqrt{\omega}/2)+\gamma}{\pi} + 2\beta(2\sigma+1)q^{2\sigma}(\omega)\right) q_k^r q_h^r$$

and

$$B_2[h^i, k^i] := 2(\langle\nabla h_\omega^i, \nabla k_\omega^i\rangle + \omega(\langle h_\omega^i, k_\omega^i\rangle - \langle h^i, k^i\rangle)) + \left(\frac{\log(\sqrt{\omega}/2)+\gamma}{\pi} + 2\beta q^{2\sigma}(\omega)\right) q_k^i q_h^i.$$

Furthermore, one notes that B_1, B_2 are the sesquilinear form (restricted to real-valued functions) associated with the operators H_{α_1}, $H_{\alpha_2} : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ with domains

$$\mathrm{dom}(H_{\alpha_j}) := \Big\{\psi \in L^2(\mathbb{R}^2) : \psi = \phi_\lambda + q\mathcal{G}_\lambda,\ \phi_\lambda \in H^2(\mathbb{R}^2),\ q \in \mathbb{C},$$

$$\phi_\lambda(\mathbf{0}) = \left(\alpha_i + \frac{\log(\sqrt{\lambda}/2)+\gamma}{2\pi}\right)q\Big\}, \qquad i = 1, 2, \quad (2.12)$$

with $\lambda > 0$, and action

$$(H_{\alpha_i} + \lambda)\psi := (-\Delta + \lambda)\phi_\lambda, \qquad \forall\psi \in \mathrm{dom}(H_{\alpha_i}), \qquad i = 1, 2, \quad (2.13)$$

where

$$\alpha_1 = (2\sigma+1)\beta q^{2\sigma}(\omega), \qquad \alpha_2 = \beta q^{2\sigma}(\omega). \quad (2.14)$$

Summing up,

$$\left.\frac{\partial^2 \mathrm{E}(u_\omega + \nu h + \tau k)}{\partial\nu\partial\tau}\right|_{\nu=\tau=0} = 2(h^r, h^i)\begin{pmatrix} H_{\alpha_1} & 0 \\ 0 & H_{\alpha_2}\end{pmatrix}\begin{pmatrix} k^r \\ k^i\end{pmatrix},$$

whence

$$d^2\mathrm{E}(u_\omega) = 2\begin{pmatrix} H_{\alpha_1} & 0 \\ 0 & H_{\alpha_2} \end{pmatrix}$$

and, consequently,

$$H_\omega = 2\begin{pmatrix} H_{\alpha_1} + \omega & 0 \\ 0 & H_{\alpha_2} + \omega \end{pmatrix}.$$

In order to verify (i), (ii) and (iii) of **(A3)**, it suffices to observe that

$$\sigma(H_{\alpha_1}) = \left\{-\omega\, \mathrm{e}^{-8\pi\beta\sigma q^{2\sigma}}\right\} \cup [0, +\infty), \qquad \sigma(H_{\alpha_2}) = \{-\omega\} \cup [0, +\infty),$$

with $-\omega\, \mathrm{e}^{-8\pi\beta\sigma q^{2\sigma}}$ and $-\omega$ simple eigenvalues, and that u_ω is the eigenfunction associated with $-\omega$. Indeed, this entails

$$\sigma(H_\omega) = \left\{\omega(1 - \mathrm{e}^{-8\pi\beta\sigma q^{2\sigma}})\right\} \cup \{0\} \cup [\omega, +\infty), \tag{2.15}$$

which proves that H_ω possesses one simple negative eigenvalue (since $1 - \mathrm{e}^{-8\pi\beta\sigma q^{2\sigma}} < 0$, as $\beta < 0$), that the kernel of H_ω is the span of u_ω and that the rest of the spectrum is positive and bounded away from zero.

Finally, it is sufficient to detect for which values of ω the scalar function $D(\omega)$ is strictly convex and for which values of ω it is strictly concave. However, as u_ω is a standing wave, $d\mathrm{S}_\omega(u_\omega) = 0$ ($d\mathrm{S}_\omega$ denoting the Fréchet differential) so that $D''(\omega) = \mathrm{M}'(\omega)$. Therefore, recalling (2.8), one can conclude the proof. □

3. Defocusing case

One can easily check that there exists a family of standing waves in the defocusing case $\beta > 0$ as well:

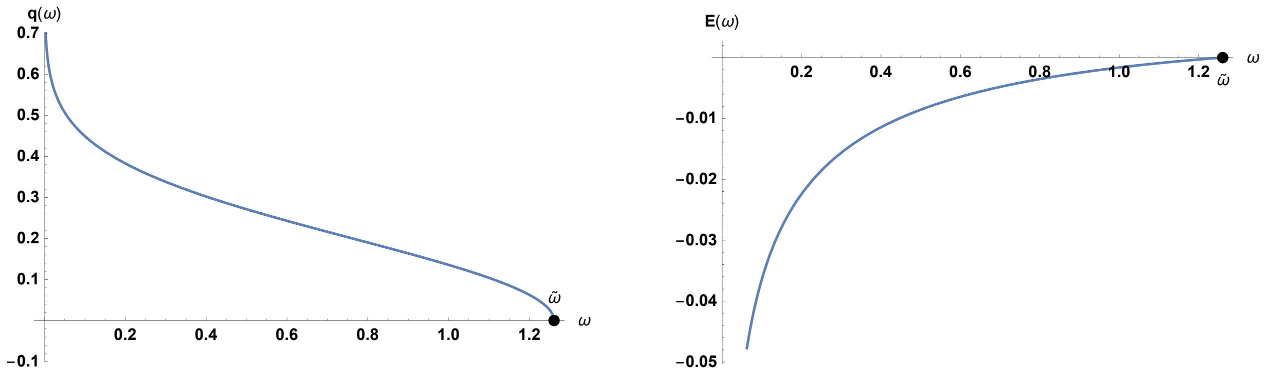
$$\psi_\omega(t, \mathbf{x}) := \mathrm{e}^{\imath\omega t}\, \mathrm{e}^{\imath\eta}\, u_\omega(\mathbf{x}), \qquad u_\omega(\mathbf{x}) := q(\omega)\mathcal{G}_\omega(\mathbf{x}), \qquad q(\omega) := \left(-\frac{\log\left(\sqrt{\omega}/2\right) + \gamma}{2\pi\beta}\right)^{1/2\sigma} \tag{3.1}$$

with $\eta \in \mathbb{R}$, defined for

$$\omega \in (0, \widetilde{\omega}), \qquad \text{where} \quad \widetilde{\omega} := 4\mathrm{e}^{-2\gamma}.$$

The behavior of $q(\omega)$ is shown in Figure 4a.

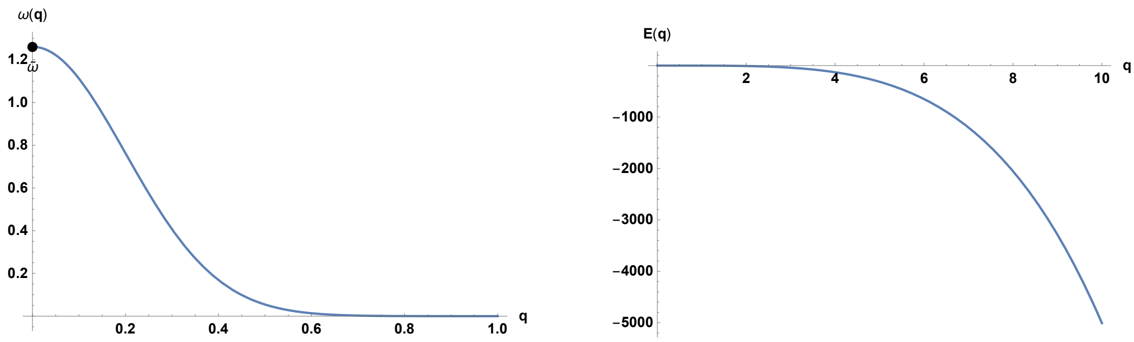
In addition, simple computations show that the form of $\mathrm{E}(\omega)$ is still given by (2.3), but in this case the function $\mathrm{E}(\omega)$ is unbounded from below, due to the fact that $\beta > 0$. The behavior of $E(\omega)$ is depicted in Figure 4b.



(a) Behavior of $q(\omega)$. **(b)** Behavior of $\mathrm{E}(\omega)$.

Figure 4. Plots of $q(\omega)$ and $\mathrm{E}(\omega)$ for $\omega \in (0, \widetilde{\omega})$, when $\sigma = 1$ and $\beta = 1$.

From (3.1) one has that the function $q(\omega)$ is invertible. Again we get that $\omega(q)$ reads as (2.5) and, plugging (2.5) into (2.3), one obtains (2.6) for $E(q)$. The behavior of $\omega(q)$ and $\mathrm{E}(q)$ is depicted in Figure 5a and b.



(a) Behavior of $\omega(q)$. **(b)** Behavior of $\mathrm{E}(q)$.

Figure 5. Plots of $\omega(q)$ and $\mathrm{E}(q)$ for $q \in \mathbb{R}^+$, when $\sigma = 1$ and $\beta = 1$.

Remark 3.1. Let us point out a relevant difference between the focusing and the defocusing case: $\mathrm{M}(\omega)$ and $\mathrm{M}(q)$, given by (2.7) and (2.9), respectively, are strictly monotone on their domain with range $\mathbb{R}^+$. In particular, this means that, in the defocusing case, for every $\mu \in \mathbb{R}^+$, there exists a unique (up to a phase factor) standing wave u_{ω_μ} of mass μ.

Concerning the stability of these standing waves, one can prove the following

Theorem 3.1 (Stability in the defocusing case)**.** *Let* $\sigma \geqslant 1/2$ *and* $\beta > 0$. *The standing waves defined by* (3.1) *are orbitally stable for every* $\omega \in (0, \widetilde{\omega})$.

The proof of Theorem 3.1 is analogous to that of Theorem 2.1. The main difference is that the key tool now is [19, Theorem 1], instead of [19, Theorem 3]. For the sake of completeness, we recall also this statement (again, suitably adapted to (1.6)).

Theorem 3.2. *Assume that* **(A1)** *and* **(A2)** *of Theorem 2.2 are satisfied. If, in addition, the operator* H_ω*, defined by* (2.10)*, satisfies (ii) and (iii) of* **(A3)** *in Theorem 2.1, then* u_ω *is orbitally stable.*

Proof of Theorem 3.1. Arguing as in the proof of Theorem 2.1 one immediately sees that **(A1)** and **(A2)** are fulfilled, with $\omega_1 = 0$ and $\omega_2 = \widetilde{\omega}$.

In addition, following again the proof of Theorem 2.1, one obtains that

$$H_\omega = 2\begin{pmatrix} H_{\alpha_1} + \omega & 0 \\ 0 & H_{\alpha_2} + \omega \end{pmatrix}$$

with H_{α_1}, H_{α_2} defined in (2.12) and (2.13) and α_1, α_2 given by (2.14). Hence, the spectrum of H_ω is given again by (2.15), but now, as $\beta > 0$ and $\omega \in (0, \widetilde{\omega})$, there is no negative eigenvalue so that (ii) and (iii) are satisfied and the proof is complete. □

Moreover, in the defocusing case it is possible to give a further characterization of the standing waves, given by the following

Theorem 3.3 (Ground states in the defocusing case)**.** *Let $\beta > 0$ and $\mu > 0$. Then, the energy functional* E *restricted to the manifold V_μ has a unique (up to a phase factor) global minimizer, which is of the form* (3.1) *with $\omega = \omega_\mu$, where ω_μ is the unique solution of*

$$\log(2/\sqrt{\omega}) - \gamma = 2\pi\beta(4\pi\omega\mu)^\sigma. \tag{3.2}$$

Proof. Preliminarily, one can see that (3.2) is equivalent to $\mathrm{M}(\omega) = \mu$ with $\mathrm{M}(\omega)$ defined by (2.7). Hence, by Remark 3.1, there is a unique solution ω_μ for any value of $\mu > 0$. It is thus clear that, if a minimizer does exist, then it has to be equal to u_{ω_μ} up to phase factor.

First, let us fix $\lambda = \omega_\mu$ in (1.7) and in the definition of the norm of V_μ, which is the same of V. Consider, therefore, a minimizing sequence $\{\psi_n\} = \{\phi_{\omega_\mu,n} + q_n\mathcal{G}_{\omega_\mu}\} \subset V_\mu$ for E. As $\|\psi_n\|^2 = \mu$ and $\beta > 0$, E is coercive on V_μ and hence $\|\psi_n\|_{\omega_\mu} \leqslant C$ for every n. As a consequence there exists $\psi = \phi_{\omega_\mu} + q\,\mathcal{G}_{\omega_\mu} \in V$ such that, up to subsequences,

$$\begin{aligned} \psi_n &\xrightarrow[n\to\infty]{w} \psi, && \text{in } L^2(\mathbb{R}^2), \\ \phi_{\omega_\mu,n} &\xrightarrow[n\to\infty]{w} \phi_{\omega_\mu}, && \text{in } H^1(\mathbb{R}^2), \\ q_n &\xrightarrow[n\to\infty]{} q, && \text{in } \mathbb{C}. \end{aligned}$$

Furthermore, by the weak lower semicontinuity of E

$$\mathrm{E}(\psi) \leqslant \liminf_{n\to+\infty} \mathrm{E}(\psi_n),$$

and, by the weak lower semicontinuity of the norms, $\|\psi\|^2 \leqslant \mu$. Hence, if one can prove that $\|\psi\|^2 = \mu$, the proof is complete.

To this aim, first note that

$$\begin{aligned} \mathrm{E}(\psi) &\geqslant -\omega_\mu\|\psi\|^2 + \left(\frac{\beta|q|^{2\sigma}}{\sigma+1} + \frac{\log(\sqrt{\omega_\mu}/2) + \gamma}{2\pi}\right)|q|^2 \\ &\geqslant -\omega_\mu\mu + \left(\frac{\beta|q|^{2\sigma}}{\sigma+1} + \frac{\log(\sqrt{\omega_\mu}/2) + \gamma}{2\pi}\right)|q|^2 =: f(|q|). \end{aligned}$$

Assuming that q is real-valued (which is not restrictive), one can check that f is minimized for $q = q(\omega_\mu)$ and that

$$f(q(\omega_\mu)) = -\frac{q^2(\omega_\mu)}{4\pi} - \frac{\sigma\beta q^{2\sigma+2}(\omega_\mu)}{\sigma+1} = \mathrm{E}(u_{\omega_\mu}).$$

Therefore,

$$\mathrm{E}(\psi) \geqslant \mathrm{E}(u_{\omega_\mu})$$

and, since $\mathrm{M}(u_{\omega_\mu}) = \mu$, this implies that u_{ω_μ} is the minimizer of E on V_μ up to a phase factor. □

Conflict of interest

The authors declare no conflict of interest.

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