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**Research article**

## **Lipschitz continuity of minimizers in a problem with nonstandard growth<sup>†</sup>**

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**Abstract:** In this paper we obtain the Lipschitz continuity of nonnegative local minimizers of the functional  $J(v) = \int_{\Omega} (F(x, v, \nabla v) + \lambda(x)\chi_{\{v>0\}}) dx$ , under nonstandard growth conditions of the energy function  $F(x, s, \eta)$  and  $0 < \lambda_{\min} \leq \lambda(x) \leq \lambda_{\max} < \infty$ . This is the optimal regularity for the problem. Our results generalize the ones we obtained in the case of the inhomogeneous  $p(x)$ -Laplacian in our previous work. Nonnegative local minimizers  $u$  satisfy in their positivity set a general nonlinear degenerate/singular equation  $\operatorname{div} A(x, u, \nabla u) = B(x, u, \nabla u)$  of nonstandard growth type. As a by-product of our study, we obtain several results for this equation that are of independent interest.

**Keywords:** minimization problem; Lipschitz continuity; variable exponent spaces; nonstandard growth; singular and degenerate elliptic equation; nonlinear elliptic operator;  $p(x)$ -Laplacian; free boundary problem

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*To our dear friend Sandro Salsa on the occasion of his 70th birthday.*

### **1. Introduction**

In this paper we study the regularity properties of nonnegative, local minimizers of the functional

$$J(v) = \int_{\Omega} (F(x, v, \nabla v) + \lambda(x)\chi_{\{v>0\}}) dx, \quad (1.1)$$

under nonstandard growth conditions of the energy function  $F(x, s, \eta)$  and  $0 < \lambda_{\min} \leq \lambda(x) \leq \lambda_{\max} < \infty$ .

There has been a great deal of interest in these type of problems. Their study started with the seminal paper of Alt and Caffarelli [2] where the case  $F(x, s, \eta) = \frac{1}{2}|\eta|^2$  was considered. Later on, [3] considered the case  $F(x, s, \eta) = G(|\eta|^2)$  under uniform ellipticity assumptions. The general power case  $F(x, s, \eta) = \frac{1}{p}|\eta|^p$  with  $1 < p < \infty$  was studied in [8], and  $F(x, s, \eta) = G(|\eta|)$  with  $G$  convex under the assumption that  $G'$  satisfies Lieberman's condition namely,  $G''(t) \sim G'(t)/t$ , was analyzed in [19]. The linear inhomogeneous case  $F(x, s, \eta) = \frac{1}{2}|\eta|^2 + f(x)s$  was addressed in [12] and [15].

The minimization problem for the functional (1.1) with  $F(x, s, \eta) = \frac{1}{p(x)}|\eta|^{p(x)}$  was first considered in [6] for  $p(x) \geq 2$  and then, in [16] and [17] in the inhomogeneous case  $F(x, s, \eta) = \frac{1}{p(x)}|\eta|^{p(x)} + f(x)s$ , for  $1 < p(x) < \infty$  and  $f \in L^\infty(\Omega)$ . In [17], among other results, we proved that nonnegative local minimizers  $u$  are locally Lipschitz continuous and satisfy

$$\Delta_{p(x)}u := \operatorname{div}(|\nabla u(x)|^{p(x)-2}\nabla u) = f \quad \text{in } \{u > 0\}.$$

The operator  $\Delta_{p(x)}$ , called the  $p(x)$ -Laplacian, extends the Laplacian, where  $p(x) \equiv 2$  and the  $p$ -Laplacian, where  $p(x) \equiv p$ . This is a prototype operator with nonstandard growth. The functional setting for the study of this type of operators are the variable exponent Lebesgue and Sobolev spaces  $L^{p(\cdot)}$  and  $W^{1,p(\cdot)}$ .

Functionals and PDEs with nonstandard growth have a wide range of applications, such as the modelling of non-Newtonian fluids, as for instance, electrorheological [21] or thermorheological fluids [4]. Other areas of application include non-linear elasticity [24], image reconstruction [1, 7], the modelling of electric conductors [25], as well as processes of filtration of gases in non-homogeneous porous media [5].

As far as we know, no result on the minimization of (1.1) with  $F(x, s, \eta)$  a general function with nonstandard growth has been obtained.

The main purpose of our work is to prove the local Lipschitz continuity of nonnegative local minimizers of such an energy. We stress that this is the optimal regularity since it is known from the particular cases referred to above that the gradient of a minimizer  $u$  jumps across  $\Omega \cap \partial\{u > 0\}$ .

We prove that nonnegative minimizers of (1.1) are solutions to the associated equation in their positivity sets. That is, a local minimizer  $u \geq 0$  satisfies

$$\operatorname{div}A(x, u, \nabla u) = B(x, u, \nabla u) \tag{1.2}$$

in  $\{u > 0\}$ , where

$$A(x, s, \eta) = \nabla_\eta F(x, s, \eta), \quad B(x, s, \eta) = F_s(x, s, \eta).$$

Under our assumptions, the governing equation (1.2) is given by  $A(x, s, \eta)$  satisfying

$$\lambda_0|\eta|^{p(x)-2}|\xi|^2 \leq \sum_{i,j} \frac{\partial A_i}{\partial \eta_j}(x, s, \eta)\xi_i\xi_j \leq \Lambda_0|\eta|^{p(x)-2}|\xi|^2,$$

and has a right hand side given by  $B(x, s, \eta) \not\equiv 0$  of  $p(x)$ -type growth in  $\eta$ . This equation is singular in the regions where  $1 < p(x) < 2$  and degenerate in the ones where  $p(x) > 2$ .

Our study thus presents new features, needed in order to overcome the deep technical difficulties arising due to the nonlinear degenerate/singular nature and the  $x$  and  $s$  dependence of this general operator associated to our energy functional (1.1).

The first part of the paper is devoted to the study of equation (1.2) in a domain  $\Omega$ , under nonstandard growth conditions of  $p(x)$ -type. We prove existence results, a comparison principle, a uniqueness result, a maximum principle and other local  $L^\infty$  bounds of solutions of this equation. These delicate results are of independent interest.

Some of these results are obtained under the growth assumption (3.14). We remark that this hypothesis on the functions  $A$  and  $B$  allows to consider very general equations. This condition not only enables us to get the inequality in Proposition 3.3 that is a main tool for all the proofs in the paper, but also it is invariant under rescalings. All these results are included in Section 3.

In the second part of the paper we deal with the minimization problem for the functional (1.1). In fact, in Section 4 we first get an existence result for minimizers. We also prove nonnegativity and boundedness, under suitable assumptions. Then, we prove the local Hölder and Lipschitz continuity of nonnegative local minimizers (Theorems 4.3 and 4.5).

The proofs in Section 4 involve delicate rescalings. One of the main difficulties this problem presents is that it is not invariant under the rescaling  $u(x) \mapsto \frac{u(tx)}{t}$ , if  $t \neq k$  —rescaling that is a crucial tool in dealing with this type of problems. The rescaled functionals lose the uniform properties and nontrivial modifications are needed to get through the proofs. Even after these modifications, there is in general no limit equation for the rescaled problems due to the growth we are allowing to the function  $B(x, s, \eta)$ . Novel arguments are used to complete the proof of Theorem 4.4. In fact, we are able to show that, although there is in general no limit equation for the rescaled problems, there is a limit function and it satisfies Harnack's inequality (see (4.58)).

A thorough follow up of the dependence of the bounds found in Section 3 with respect to the structural conditions on  $F, A$  and  $B$  is of most importance as well.

Let us point out that the results in the paper are new even in the case  $p(x) \equiv p$  constant.

Finally, in Section 5 we present some examples of functionals (1.1) where our results can be used.

Our examples include functionals (1.1) involving energy functions of the form

$$F(x, s, \eta) = a(x, s) \frac{|\eta|^{p(x)}}{p(x)} + f(x, s).$$

A possible example of admissible functions  $a(x, s), f(x, s)$  is given by

$$a(x, s) = a_0(x)(1 + s)^{-q(x)}, \quad a_0(x) > 0, \quad 0 < q(x) \leq q_0(x),$$

for  $s$  in the range where the nonnegative local minimizer takes values,  $q_0(x)$  a function depending on  $p(x)$  and

$$f(x, s) = b(x)|s|^{\tau(x)}, \quad b(x) \geq 0, \quad \tau(x) \geq 2,$$

with  $\tau(x)$  satisfying (2.7).

Our results also apply to functionals (1.1) involving energy functions of the form

$$F(x, s, \eta) = G(x, \eta) + f(x, s).$$

Some admissible  $G(x, \eta), f(x, s)$  are

$$G(x, \eta) = a(x)\tilde{G}(|\eta|^{p(x)}) \quad a(x) > 0, \quad \tilde{G}'' \geq 0,$$

$$G(x, \eta) = \tilde{A}(x)\eta \cdot \eta |\eta|^{p(x)-2} \quad \tilde{A}(x) \in \mathbb{R}^{N \times N} \text{ uniformly elliptic},$$

$$f(x, s) = g(x)s.$$

Also,

$$F(x, s, \eta) = a_1(x)F_1(x, s, \eta) + a_2(x)F_2(x, s, \eta), \quad a_i(x) > 0,$$

is an admissible function if both  $F_1(x, s, \eta)$  and  $F_2(x, s, \eta)$  are admissible.

We begin our paper with a section where we state the hypotheses on  $F, A, B, \lambda$  and  $p(x)$  that will be used through the article. And we end it with an Appendix where we state some properties of the function spaces  $L^{p(\cdot)}$  and  $W^{1,p(\cdot)}$  where the problem is well posed.

### 1.1. Preliminaries on Lebesgue and Sobolev spaces with variable exponent

Let  $p : \Omega \rightarrow [1, \infty)$  be a measurable bounded function, called a variable exponent on  $\Omega$  and denote  $p_{\max} = \text{esssup } p(x)$  and  $p_{\min} = \text{essinf } p(x)$ . We define the variable exponent Lebesgue space  $L^{p(\cdot)}(\Omega)$  to consist of all measurable functions  $u : \Omega \rightarrow \mathbb{R}$  for which the modular  $\varrho_{p(\cdot)}(u) = \int_{\Omega} |u(x)|^{p(x)} dx$  is finite. We define the Luxemburg norm on this space by

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \|u\|_{p(\cdot)} = \inf\{\lambda > 0 : \varrho_{p(\cdot)}(u/\lambda) \leq 1\}.$$

This norm makes  $L^{p(\cdot)}(\Omega)$  a Banach space.

There holds the following relation between  $\varrho_{p(\cdot)}(u)$  and  $\|u\|_{L^{p(\cdot)}(\Omega)}$ :

$$\begin{aligned} \min\left\{\left(\int_{\Omega} |u|^{p(x)} dx\right)^{1/p_{\min}}, \left(\int_{\Omega} |u|^{p(x)} dx\right)^{1/p_{\max}}\right\} &\leq \|u\|_{L^{p(\cdot)}(\Omega)} \\ &\leq \max\left\{\left(\int_{\Omega} |u|^{p(x)} dx\right)^{1/p_{\min}}, \left(\int_{\Omega} |u|^{p(x)} dx\right)^{1/p_{\max}}\right\}. \end{aligned}$$

Moreover, the dual of  $L^{p(\cdot)}(\Omega)$  is  $L^{p'(\cdot)}(\Omega)$  with  $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ .

Let  $W^{1,p(\cdot)}(\Omega)$  denote the space of measurable functions  $u$  such that  $u$  and the distributional derivative  $\nabla u$  are in  $L^{p(\cdot)}(\Omega)$ . The norm

$$\|u\|_{1,p(\cdot)} := \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}$$

makes  $W^{1,p(\cdot)}(\Omega)$  a Banach space.

The space  $W_0^{1,p(\cdot)}(\Omega)$  is defined as the closure of the  $C_0^\infty(\Omega)$  in  $W^{1,p(\cdot)}(\Omega)$ .

For the sake of completeness we include in an Appendix at the end of the paper some additional results on these spaces that are used throughout the paper.

### 1.2. Notation

- $N$  spatial dimension
- $|S|$   $N$ -dimensional Lebesgue measure of the set  $S$
- $B_r(x_0)$  open ball of radius  $r$  and center  $x_0$
- $B_r$  open ball of radius  $r$  and center  $0$
- $\chi_S$  characteristic function of the set  $S$
- $u^+ = \max(u, 0)$ ,  $u^- = \max(-u, 0)$
- $\langle \xi, \eta \rangle$  and  $\xi \cdot \eta$  both denote scalar product in  $\mathbb{R}^N$

## 2. Assumptions

In this section we collect all the assumptions that will be made along the paper.

Throughout the paper  $\Omega$  will denote a  $C^1$  bounded domain in  $\mathbb{R}^N$ . In addition, the following assumptions will be made:

### 2.1. Assumptions on $p(x)$

We assume that the function  $p(x)$  is measurable in  $\Omega$  and verifies

$$1 < p_{\min} \leq p(x) \leq p_{\max} < \infty, \quad x \in \Omega.$$

We assume further that  $p(x)$  is Lipschitz continuous in  $\Omega$  and we denote by  $L$  the Lipschitz constant of  $p(x)$ , namely,  $\|\nabla p\|_{L^\infty(\Omega)} \leq L$ .

When we are restricted to a ball  $B_r$  we use  $p_r^-$  and  $p_r^+$  to denote the infimum and the supremum of  $p(x)$  over  $B_r$ .

### 2.2. Assumptions on $\lambda(x)$

We assume that the function  $\lambda(x)$  is measurable in  $\Omega$  and verifies

$$0 < \lambda_{\min} \leq \lambda(x) \leq \lambda_{\max} < \infty, \quad x \in \Omega.$$

### 2.3. Assumptions on $F$

We assume that  $F$  is measurable in  $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N$ , and for every  $x \in \overline{\Omega}$ ,  $F(x, \cdot, \cdot) \in C^1(\mathbb{R} \times \mathbb{R}^N) \cap C^2(\mathbb{R} \times \mathbb{R}^N \setminus \{0\})$ .

We denote  $A(x, s, \eta) = \nabla_\eta F(x, s, \eta)$  and  $B(x, s, \eta) = F_s(x, s, \eta)$ .

### 2.4. Assumptions on $A$

We assume that  $A \in C(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N, \mathbb{R}^N)$  and for every  $x \in \overline{\Omega}$ ,  $A(x, \cdot, \cdot) \in C^1(\mathbb{R} \times \mathbb{R}^N \setminus \{0\}, \mathbb{R}^N)$ . Moreover, there exist positive constants  $\lambda_0$  and  $\Lambda_0$ , and  $\beta \in (0, 1)$  such that for every  $x, x_1, x_2 \in \overline{\Omega}$ ,  $s, s_1, s_2 \in \mathbb{R}$ ,  $\eta \in \mathbb{R}^N \setminus \{0\}$  and  $\xi \in \mathbb{R}^N$ , the following conditions are satisfied:

$$A(x, s, 0) = 0, \tag{2.1}$$

$$\sum_{i,j} \frac{\partial A_i}{\partial \eta_j}(x, s, \eta) \xi_i \xi_j \geq \lambda_0 |\eta|^{p(x)-2} |\xi|^2, \tag{2.2}$$

$$\sum_{i,j} \left| \frac{\partial A_i}{\partial \eta_j}(x, s, \eta) \right| \leq \Lambda_0 |\eta|^{p(x)-2}, \tag{2.3}$$

$$|A(x_1, s, \eta) - A(x_2, s, \eta)| \leq \Lambda_0 |x_1 - x_2|^\beta (|\eta|^{p(x_1)-1} + |\eta|^{p(x_2)-1}) (1 + |\log |\eta||), \tag{2.4}$$

$$|A(x, s_1, \eta) - A(x, s_2, \eta)| \leq \Lambda_0 |s_1 - s_2| |\eta|^{p(x)-1}. \tag{2.5}$$

### 2.5. Assumptions on $B$

We assume that  $B$  is measurable in  $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N$  and for every  $x \in \overline{\Omega}$ ,  $B(x, \cdot, \cdot) \in C^1(\mathbb{R} \times \mathbb{R}^N)$ , and for every  $(x, s, \eta) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N$ ,

$$|B(x, s, \eta)| \leq \Lambda_0(1 + |\eta|^{p(x)} + |s|^{\tau(x)}), \quad (2.6)$$

where  $\Lambda_0$  is as in the assumptions on  $A$  and

$$\begin{aligned} \tau(x) &\geq p(x) \quad \text{and} \quad \tau \in C(\overline{\Omega}), \\ \tau(x) &\leq p^*(x) = \frac{Np(x)}{N - p(x)} \quad \text{if } p_{\max} < N, \\ \tau(x) &\text{arbitrary if } p_{\min} > N, \\ \tau(x) &= p(x) \quad \text{if } p_{\min} \leq N \leq p_{\max}. \end{aligned} \quad (2.7)$$

**Remark 2.1.** From (2.1) and (2.3) we get

$$|A_i(x, s, \eta)| = |A_i(x, s, \eta) - A_i(x, s, 0)| = \left| \int_0^1 \sum_j \frac{\partial A_i}{\partial \eta_j}(x, s, t\eta) \eta_j dt \right| \leq \bar{\alpha}(p_{\min}) \Lambda_0 |\eta|^{p(x)-1},$$

so that

$$|A(x, s, \eta)| \leq \bar{\alpha}(p_{\min}) N \Lambda_0 |\eta|^{p(x)-1}. \quad (2.8)$$

From (2.1) and (2.2) we have

$$A(x, s, \eta) \cdot \eta = (A(x, s, \eta) - A(x, s, 0)) \cdot \eta = \int_0^1 \sum_{ij} \frac{\partial A_i}{\partial \eta_j}(x, s, t\eta) \eta_j \eta_i dt,$$

so that

$$A(x, s, \eta) \cdot \eta \geq \alpha(p_{\max}) \lambda_0 |\eta|^{p(x)}. \quad (2.9)$$

### 3. Existence, uniqueness and bounds of solutions to equation (1.2)

In this section we consider  $A$  and  $B$  as in Section 2 and we prove results for solutions of the equation

$$\operatorname{div} A(x, u, \nabla u) = B(x, u, \nabla u) \quad \text{in } \Omega. \quad (3.1)$$

Namely, existence, comparison principle, uniqueness, maximum principle and bounds of solutions.

Our first result is Proposition 3.1, where we prove existence of a solution to (3.1) with given boundary data. In order to prove the existence of a solution to (3.1) we show that, given  $u \in W^{1,p(\cdot)}(\Omega)$ , there exists a minimizer of the functional

$$\mathcal{J}_{\Omega}(v) = \int_{\Omega} F(x, v, \nabla v) dx \quad (3.2)$$

in  $u + W_0^{1,p(\cdot)}(\Omega)$ , where  $F$  is as in Section 2,  $A(x, s, \eta) = \nabla_{\eta} F(x, s, \eta)$  and  $B(x, s, \eta) = F_s(x, s, \eta)$ .

Then, in Proposition 3.2 we get an existence result under a growth assumption on the function  $F$  stronger than (3.3) in Proposition 3.1, but without the small oscillation hypothesis there.

In Proposition 3.4 and Corollary 3.2 we prove comparison and uniqueness for this problem, assuming that condition (3.14) below holds. In Proposition 3.5 we prove that solutions to (3.1) with bounded boundary data are bounded and in Proposition 3.6 we prove a maximum principle for this problem, under suitable assumptions. In Proposition 3.7 we give another existence result of a bounded solution.

We start with the definition of solution to (3.1).

**Definition 3.1.** Let  $p$ ,  $A$  and  $B$  be as in Section 2. We say that  $u$  is a solution to (3.1) if  $u \in W^{1,p(\cdot)}(\Omega)$  and, for every  $\varphi \in C_0^\infty(\Omega)$ , there holds that

$$-\int_{\Omega} A(x, u, \nabla u) \cdot \nabla \varphi \, dx = \int_{\Omega} B(x, u, \nabla u) \varphi \, dx.$$

We are using that, under the conditions in (2.7), the embedding theorem (see Theorem A.5) applies.

Our first existence result is

**Proposition 3.1.** Let  $p, F, A, B$  as in Section 2 and let  $\Omega' \subset \Omega$  be a  $C^1$  domain. Let  $u \in W^{1,p(\cdot)}(\Omega')$  and let us call  $p^+ = \sup_{\Omega'} p(x)$ ,  $p^- = \inf_{\Omega'} p(x)$ . Assume that there exist  $\mu, c_1 \in \mathbb{R}_+$ ,  $p_{\min} > \delta > 0$  and  $g \in L^1(\Omega)$  such that

$$F(x, s, \eta) \geq \mu |\eta|^{p(x)} - c_1 |s|^{p(x)-\delta} - g(x) \quad \text{in } \Omega. \quad (3.3)$$

Assume, moreover that  $\delta > p^+ - p^-$  and that

$$F(x, s, \eta) \leq \mu^{-1} |\eta|^{p(x)} + c_1 |s|^{\tau(x)} + g(x) \quad \text{in } \Omega, \quad (3.4)$$

with  $\tau$  satisfying (2.7).

Then, there exists a solution  $v \in u + W_0^{1,p(\cdot)}(\Omega')$  to (3.1) in  $\Omega'$ .

Moreover,  $\|v\|_{W^{1,p(\cdot)}(\Omega')} \leq C$ , for a constant  $C$  depending only  $\|u\|_{W^{1,p(\cdot)}(\Omega')}$ ,  $\|g\|_{L^1(\Omega')}$ ,  $|\Omega'|$ ,  $\text{diam}(\Omega')$ ,  $N$ ,  $p^-$ ,  $p^+$ ,  $\delta$ ,  $L$ ,  $\mu$ ,  $c_1$ ,  $\|\tau\|_{L^\infty(\Omega')}$  and the  $C^1$  norm of  $\partial\Omega'$ .

*Proof.* We will show that there is a minimizer of  $\mathcal{J}_{\Omega'}$  in  $u + W_0^{1,p(\cdot)}(\Omega')$  where

$$\mathcal{J}_{\Omega'}(v) = \int_{\Omega'} F(x, v, \nabla v) \, dx.$$

This minimizer is a solution to the associated Euler-Lagrange equation (3.1) in  $\Omega'$ .

We will use the embedding theorem (see Theorem A.5) that states that, under the conditions in (2.7),  $W^{1,p(\cdot)}(\Omega') \hookrightarrow L^{\tau(\cdot)}(\Omega')$  continuously.

So, let  $v_n$  be a minimizing sequence. That is,  $v_n \in u + W_0^{1,p(\cdot)}(\Omega')$  and

$$I = \lim_{n \rightarrow \infty} \mathcal{J}_{\Omega'}(v_n) = \inf_{u + W_0^{1,p(\cdot)}(\Omega')} \mathcal{J}_{\Omega'}(v) \leq \int_{\Omega'} F(x, u, \nabla u) \, dx.$$

Let us show that there is a constant  $\kappa > 0$  such that  $\|v_n\|_{L^{p(\cdot)}(\Omega')} \leq \kappa$ . In fact, by (3.3), for  $n$  large,

$$\int_{\Omega'} |\nabla v_n|^{p(x)} dx \leq 1 + \int_{\Omega'} F(x, u, \nabla u) dx + \frac{c_1}{\mu} \int_{\Omega'} |v_n|^{p(x)-\delta} dx + \frac{1}{\mu} \int_{\Omega'} g(x) dx.$$

By Poincaré's inequality (Theorem A.4)

$$\|v_n - u\|_{L^{p(\cdot)}(\Omega')} \leq C_{\Omega'} \|\nabla(v_n - u)\|_{L^{p(\cdot)}(\Omega')}.$$

Hence, recalling Proposition A.1,

$$\begin{aligned} \|v_n\|_{L^{p(\cdot)}(\Omega')} &\leq \|u\|_{L^{p(\cdot)}(\Omega')} + C_{\Omega'} [\|\nabla v_n\|_{L^{p(\cdot)}(\Omega')} + \|\nabla u\|_{L^{p(\cdot)}(\Omega')}] \\ &\leq C [\|u\|_{W^{1,p(\cdot)}(\Omega')} + \max \{ (\int_{\Omega'} |\nabla v_n|^{p(x)} dx)^{1/p^-}, (\int_{\Omega'} |\nabla v_n|^{p(x)} dx)^{1/p^+} \}] \\ &\leq \bar{C} [1 + \max \{ (\int_{\Omega'} |v_n|^{p(x)-\delta} dx)^{1/p^-}, (\int_{\Omega'} |v_n|^{p(x)-\delta} dx)^{1/p^+} \}] \end{aligned}$$

with  $\bar{C}$  depending on  $\|u\|_{W^{1,p(\cdot)}(\Omega')}$ ,  $\|g\|_{L^1(\Omega')}$ ,  $N$ ,  $p^-$ ,  $p^+$ ,  $\delta$ ,  $|\Omega'|$ ,  $\text{diam}(\Omega')$ ,  $L$ ,  $\|\tau\|_{L^\infty(\Omega')}$ , the  $C^1$  norm of  $\partial\Omega'$ , and the constants in (3.3).

Observe that in case  $u \equiv M$ , there holds that  $\int_{\Omega'} F(x, u, \nabla u) dx$  is bounded by a constant that depends only on  $M$ ,  $\|\tau\|_{L^\infty(\Omega')}$  and  $|\Omega'|$ . Hence, in that case  $\bar{C}$  is independent of the regularity of  $\Omega'$ .

Since we want to find a uniform bound of  $\|v_n\|_{L^{p(\cdot)}(\Omega')}$ , we may assume that this norm is larger than 1. Let  $q$  be the middle point of the interval  $[p^+ - \delta, p^-]$ . By Young's inequality with  $r(x) = \frac{q}{p(x)-\delta}$ ,

$$\int_{\Omega'} |v_n|^{p(x)-\delta} dx \leq C_\varepsilon + \varepsilon \int_{\Omega'} |v_n|^q dx,$$

for  $0 < \varepsilon < 1$  with  $C_\varepsilon$  depending only on  $|\Omega'|$ ,  $\varepsilon$ ,  $p^-$ ,  $p^+$  and  $\delta$ . On the other hand, since  $\|v_n\|_{L^q(\Omega')} \leq C\|v_n\|_{L^{p(\cdot)}(\Omega')}$  with  $C$  depending only on  $|\Omega'|$ ,  $p^-$ ,  $p^+$  and  $\delta$ ,

$$\int_{\Omega'} |v_n|^q dx \leq (C\|v_n\|_{L^{p(\cdot)}(\Omega')})^q.$$

So that

$$\begin{aligned} \|v_n\|_{L^{p(\cdot)}(\Omega')} &\leq C[\bar{C}_\varepsilon + \varepsilon^{\frac{1}{p^+}} (\|v_n\|_{L^{p(\cdot)}(\Omega')})^{\frac{q}{p^+}}] \\ &\leq C[\bar{C}_\varepsilon + \varepsilon^{\frac{1}{p^+}} \|v_n\|_{L^{p(\cdot)}(\Omega')}] \end{aligned}$$

By choosing  $\varepsilon$  small enough, we find that

$$\|v_n\|_{L^{p(\cdot)}(\Omega')} \leq C \tag{3.5}$$

with  $C$  depending on  $|\Omega'|$ ,  $\text{diam}(\Omega')$ ,  $\|u\|_{W^{1,p(\cdot)}(\Omega')}$ ,  $p^-$ ,  $p^+$ ,  $N$ ,  $\delta$ ,  $\|g\|_{L^1(\Omega')}$ ,  $L$ ,  $\|\tau\|_{L^\infty(\Omega')}$ , the  $C^1$  norm of  $\partial\Omega'$ ,  $\mu$  and  $c_1$ .

From the computations above we find that  $\int_{\Omega'} |v_n|^{p(x)-\delta} dx \leq C_1$ . So that we have that  $I > -\infty$  and

$$\|\nabla v_n\|_{L^{p(\cdot)}(\Omega')} \leq C_2, \tag{3.6}$$

with  $C_2$  depending on  $|\Omega'|$ ,  $\text{diam}(\Omega')$ ,  $\|u\|_{W^{1,p(\cdot)}(\Omega')}$ ,  $p^-$ ,  $p^+$ ,  $N$ ,  $\delta$ ,  $\|g\|_{L^1(\Omega')}$ ,  $L$ ,  $\|\tau\|_{L^\infty(\Omega')}$ , the  $C^1$  norm of  $\partial\Omega'$ ,  $\mu$  and  $c_1$ .



From our comment above, we have that in case  $u \equiv M$  in  $\Omega'$ , the constant  $C_2$  is independent of the regularity of  $\partial\Omega'$ .

Let us proceed with the proof of the existence of a minimizer. By the estimates above, for a subsequence that we still call  $v_n$ , there holds that there exists  $v \in u + W_0^{1,p(\cdot)}(\Omega')$ , such that

$$v_n \rightharpoonup v \quad \text{in } W^{1,p(\cdot)}(\Omega'), \quad v_n \rightarrow v \quad \text{in } L^{p^-}(\Omega') \quad \text{and almost everywhere,}$$

and such that the bounds (3.5) and (3.6) also hold for  $v$ .

By Egorov's Theorem, for every  $\varepsilon > 0$  there exists  $\Omega_\varepsilon$  such that  $|\Omega' \setminus \Omega_\varepsilon| < \varepsilon$  and  $v_n \rightarrow v$  uniformly in  $\Omega_\varepsilon$ .

On the other hand, if we set  $\Omega_K = \{x \in \Omega' / |v| + |\nabla v| \leq K\}$ , there holds that  $|\Omega' \setminus \Omega_K| \rightarrow 0$  as  $K \rightarrow \infty$ . Let  $\Omega_{\varepsilon,K} = \Omega_\varepsilon \cap \Omega_K$ . Then,  $|\Omega' \setminus \Omega_{\varepsilon,K}| \rightarrow 0$  as  $\varepsilon \rightarrow 0$  and  $K \rightarrow \infty$ .

There holds

$$\limsup_{n \rightarrow \infty} \int_{\Omega_{\varepsilon,K}} F(x, v_n, \nabla v_n) dx \leq I + c_1 \int_{\Omega' \setminus \Omega_{\varepsilon,K}} |v|^{p(x)-\delta} dx + \int_{\Omega' \setminus \Omega_{\varepsilon,K}} g dx. \quad (3.7)$$

Let us prove that

$$\int_{\Omega_{\varepsilon,K}} F(x, v, \nabla v) dx \leq I + c_1 \int_{\Omega' \setminus \Omega_{\varepsilon,K}} |v|^{p(x)-\delta} dx + \int_{\Omega' \setminus \Omega_{\varepsilon,K}} g dx.$$

In fact,

$$\begin{aligned} \int_{\Omega_{\varepsilon,K}} F(x, v_n, \nabla v_n) dx - \int_{\Omega_{\varepsilon,K}} F(x, v, \nabla v) dx &= \int_{\Omega_{\varepsilon,K}} [F(x, v_n, \nabla v_n) - F(x, v, \nabla v)] dx \\ &+ \int_{\Omega_{\varepsilon,K}} [F(x, v_n, \nabla v) - F(x, v, \nabla v)] dx = \mathcal{A} + \mathcal{B}. \end{aligned}$$

On the one hand,  $\mathcal{B} \rightarrow 0$  since  $F(x, v_n, \nabla v) - F(x, v, \nabla v) \rightarrow 0$  uniformly in  $\Omega_{\varepsilon,K}$  and it is uniformly bounded. On the other hand, by the convexity assumption on  $F(x, s, \eta)$  with respect to  $\eta$ ,

$$\mathcal{A} \geq \int_{\Omega_{\varepsilon,K}} A(x, v_n, \nabla v) \cdot (\nabla v_n - \nabla v) dx \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

since  $A(x, v_n, \nabla v) \rightarrow A(x, v, \nabla v)$  uniformly in  $\Omega_{\varepsilon,K}$ , they are uniformly bounded and  $\nabla v_n \rightharpoonup \nabla v$  weakly in  $L^{p(\cdot)}(\Omega_{\varepsilon,K})$ .

Hence, for every  $\varepsilon, K$ ,

$$\int_{\Omega_{\varepsilon,K}} F(x, v, \nabla v) dx \leq I + c_1 \int_{\Omega' \setminus \Omega_{\varepsilon,K}} |v|^{p(x)-\delta} dx + \int_{\Omega' \setminus \Omega_{\varepsilon,K}} g dx.$$

Now, by letting  $\varepsilon \rightarrow 0$  and  $K \rightarrow \infty$ , we get

$$\int_{\Omega'} F(x, v, \nabla v) dx \leq I,$$

and therefore,  $v$  is a minimizer of  $\mathcal{J}_{\Omega'}$  in  $u + W_0^{1,p(\cdot)}(\Omega')$  and a solution to (3.1).  $\square$

As a corollary of Proposition 3.1 we have the following existence result that will be used in the next section.

**Corollary 3.1.** *Let  $p, F, A, B$  as in Section 2 and let  $\Omega' \subset \Omega$  be a  $C^1$  domain. Let  $u \in W^{1,p(\cdot)}(\Omega')$  and let us call  $p^+ = \sup_{\Omega'} p(x)$ ,  $p^- = \inf_{\Omega'} p(x)$ . Assume that there exist  $\mu, c_1 \in \mathbb{R}_+$  and  $p_{\min} > \delta > 0$  such that*

$$F(x, s, \eta) \geq \mu|\eta|^{p(x)} - c_1(|s|^{p(x)-\delta} + 1) \quad \text{in } \Omega. \quad (3.8)$$

Assume, moreover that  $\delta > p^+ - p^-$  and that

$$F(x, s, \eta) \leq \mu^{-1}|\eta|^{p(x)} + c_1(|s|^{\tau(x)} + 1) \quad \text{in } \Omega, \quad (3.9)$$

with  $\tau(x)$  satisfying (2.7).

Then, there exists a solution  $v \in u + W_0^{1,p(\cdot)}(\Omega')$  to (3.1) in  $\Omega'$  and  $\|v\|_{W^{1,p(\cdot)}(\Omega')} \leq C$ , for a constant  $C$  depending only  $\|u\|_{W^{1,p(\cdot)}(\Omega')}$ ,  $|\Omega'|$ ,  $\text{diam}(\Omega')$ ,  $N$ ,  $p^-$ ,  $p^+$ ,  $\delta$ ,  $L$ ,  $\mu$ ,  $c_1$ ,  $\|\tau\|_{L^\infty(\Omega')}$  and the  $C^1$  norm of  $\partial\Omega'$ .

With a stronger growth assumption on the  $s$  variable for the function  $F(x, s, \eta)$  we get an existence result without the small oscillation assumption of the function  $p$ .

**Proposition 3.2.** *Let  $p, F, A, B$  as in Section 2 and let  $\Omega' \subset \Omega$  be a  $C^1$  domain. Let  $u \in W^{1,p(\cdot)}(\Omega')$ . Assume that there exist  $\mu, c_1 \in \mathbb{R}_+$ ,  $g \in L^1(\Omega)$  and  $1 \leq q < p_{\min}$  such that*

$$F(x, s, \eta) \geq \mu|\eta|^{p(x)} - c_1|s|^q - g(x) \quad \text{in } \Omega. \quad (3.10)$$

Assume, moreover that

$$F(x, s, \eta) \leq \mu^{-1}|\eta|^{p(x)} + c_1|s|^{\tau(x)} + g(x) \quad \text{in } \Omega, \quad (3.11)$$

with  $\tau$  satisfying (2.7).

Then, there exists a solution  $v \in u + W_0^{1,p(\cdot)}(\Omega')$  to (3.1) in  $\Omega'$  and  $\|v\|_{W^{1,p(\cdot)}(\Omega')} \leq C$ , for a constant  $C$  depending only  $\|u\|_{W^{1,p(\cdot)}(\Omega')}$ ,  $\|g\|_{L^1(\Omega')}$ ,  $|\Omega'|$ ,  $\text{diam}(\Omega')$ ,  $N$ ,  $p_{\min}$ ,  $p_{\max}$ ,  $q$ ,  $L$ ,  $\mu$ ,  $c_1$ ,  $\|\tau\|_{L^\infty(\Omega')}$  and the  $C^1$  norm of  $\partial\Omega'$ .

*Proof.* We proceed as in the proof of Proposition 3.1 and we prove that a minimizing sequence  $\{v_n\}$  satisfies

$$\mu \int_{\Omega'} |\nabla v_n|^{p(x)} dx \leq \int_{\Omega'} F(x, u, \nabla u) + 1 + \int_{\Omega'} g(x) dx + c_1 \int_{\Omega'} |v_n|^q dx. \quad (3.12)$$

We want to prove that there is a constant such that  $\int_{\Omega'} |\nabla v_n|^{p(x)} dx \leq C$ . So, we can assume that  $\int_{\Omega'} |\nabla v_n|^{p(x)} dx > 1$ .

Thus,

$$\begin{aligned} \|v_n\|_{L^q(\Omega')} &\leq C\|v_n\|_{L^{p(\cdot)}(\Omega')} \leq C[\|u\|_{W^{1,p(\cdot)}(\Omega')} + \|\nabla v_n\|_{L^{p(\cdot)}(\Omega')}] \\ &\leq C[\|u\|_{W^{1,p(\cdot)}(\Omega')} + (\int_{\Omega'} |\nabla v_n|^{p(x)} dx)^{1/p_{\min}}], \end{aligned}$$

where  $C$  depends on  $q$ ,  $p_{\min}$ ,  $p_{\max}$ ,  $N$ ,  $L$  and  $|\Omega'|$ ,  $\text{diam}(\Omega')$ . Hence, as  $q < p_{\min}$ ,

$$\begin{aligned} \int_{\Omega'} |v_n|^q dx &\leq C(1 + (\int_{\Omega'} |\nabla v_n|^{p(x)} dx)^{q/p_{\min}}) \\ &\leq \tilde{C} + \varepsilon \int_{\Omega'} |\nabla v_n|^{p(x)} dx \end{aligned} \quad (3.13)$$

with  $C$  depending only on  $q, p_{\min}, p_{\max}, N, |\Omega'|, \text{diam}(\Omega'), L, \|u\|_{W^{1,p(\cdot)}(\Omega')}$ , and  $\widetilde{C}$  depending on the same constants and also on  $\varepsilon$ .

Thus, by (3.12) and (3.13),

$$\int_{\Omega'} |\nabla v_n|^{p(x)} dx \leq \widetilde{C}$$

with  $\widetilde{C}$  depending only on  $q, p_{\min}, p_{\max}, N, \mu, |\Omega'|, \text{diam}(\Omega'), L, \int_{\Omega'} g(x) dx, c_1, \|\tau\|_{L^\infty(\Omega')}$ , the  $C^1$  norm of  $\partial\Omega'$  and  $\|u\|_{W^{1,p(\cdot)}(\Omega')}$ .

Now, as in the proof of Proposition 3.1, we get that there exists a subsequence that we still call  $\{v_n\}$  and a function  $v \in u + W_0^{1,p(\cdot)}(\Omega')$  such that

$$v_n \rightarrow v \quad \text{in } L^{p_{\min}}(\Omega'), \quad v_n \rightharpoonup v \quad \text{weakly in } W^{1,p(\cdot)}(\Omega').$$

Now, the proof follows as that of Proposition 3.1.  $\square$

We next prove a result valid for solutions of equation (3.1) that will be of use in the proofs of Hölder and Lipschitz continuity of minimizers of the energy functional (1.1)

**Proposition 3.3.** *Let  $p, F, A$  and  $B$  be as in Section 2. Assume moreover that*

$$2|A_s(x, s, \eta) \cdot \xi w| \leq \frac{1}{2} \sum_{i,j} \frac{\partial A_i}{\partial \eta_j}(x, s, \eta) \xi_i \xi_j + B_s(x, s, \eta) w^2, \quad (3.14)$$

for every  $(x, s, \eta) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \setminus \{0\}$ ,  $\xi \in \mathbb{R}^N$  and  $w \in \mathbb{R}$ .

Let  $u \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$  and let  $v \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$  be such that

$$\begin{cases} \text{div} A(x, v, \nabla v) = B(x, v, \nabla v) & \text{in } \Omega, \\ v = u & \text{on } \partial\Omega. \end{cases} \quad (3.15)$$

Then,

$$\begin{aligned} \int_{\Omega} (F(x, u, \nabla u) - F(x, v, \nabla v)) dx &\geq \\ \frac{1}{2} \alpha \lambda_0 &\left( \int_{\Omega \cap \{p(x) \geq 2\}} |\nabla u - \nabla v|^{p(x)} dx + \int_{\Omega \cap \{p(x) < 2\}} (|\nabla u| + |\nabla v|)^{p(x)-2} |\nabla u - \nabla v|^2 dx \right), \end{aligned} \quad (3.16)$$

where  $\alpha = \alpha(p_{\min}, p_{\max})$  and  $\lambda_0$  is as in (2.2).

*Proof.* For  $0 \leq \sigma \leq 1$ , let  $u^\sigma = v + \sigma(u - v)$ . Then, denoting  $\nabla_\eta F = A$  and  $F_s = B$ , we obtain

$$\begin{aligned} \int_{\Omega} (F(x, u, \nabla u) - F(x, v, \nabla v)) dx &= \int_0^1 \int_{\Omega} A(x, u^\sigma, \nabla u^\sigma) \cdot \nabla(u^\sigma - v) \frac{1}{\sigma} dx d\sigma \\ &+ \int_0^1 \int_{\Omega} B(x, u^\sigma, \nabla u^\sigma)(u^\sigma - v) \frac{1}{\sigma} dx d\sigma = \int_0^1 \int_{\Omega} (A(x, u^\sigma, \nabla u^\sigma) - A(x, v, \nabla v)) \cdot \nabla(u^\sigma - v) \frac{1}{\sigma} dx d\sigma \\ &+ \int_0^1 \int_{\Omega} (B(x, u^\sigma, \nabla u^\sigma) - B(x, v, \nabla v))(u^\sigma - v) \frac{1}{\sigma} dx d\sigma = I + II, \end{aligned} \quad (3.17)$$

where we have used (3.15). Moreover,

$$\begin{aligned}
I &= \int_0^1 \int_0^1 \int_{|\nabla v| \geq |\nabla u^\sigma|} A_s(x, u^{\sigma\tau}, \nabla u^{\sigma\tau}) \cdot \nabla(u^\sigma - v)(u^\sigma - v) \frac{1}{\sigma} dx d\sigma d\tau \\
&+ \int_0^1 \int_0^1 \int_{|\nabla v| < |\nabla u^\sigma|} A_s(x, u^{\sigma(1-\tau)}, \nabla u^{\sigma(1-\tau)}) \cdot \nabla(u^\sigma - v)(u^\sigma - v) \frac{1}{\sigma} dx d\sigma d\tau \\
&+ \int_0^1 \int_0^1 \int_{|\nabla v| \geq |\nabla u^\sigma|} \sum_{i,j} \frac{\partial A_i}{\partial \eta_j}(x, u^{\sigma\tau}, \nabla u^{\sigma\tau})(u^\sigma - v)_{x_i}(u^\sigma - v)_{x_j} \frac{1}{\sigma} dx d\sigma d\tau \\
&+ \int_0^1 \int_0^1 \int_{|\nabla v| < |\nabla u^\sigma|} \sum_{i,j} \frac{\partial A_i}{\partial \eta_j}(x, u^{\sigma(1-\tau)}, \nabla u^{\sigma(1-\tau)})(u^\sigma - v)_{x_i}(u^\sigma - v)_{x_j} \frac{1}{\sigma} dx d\sigma d\tau \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{3.18}$$

Now, using (2.2), and the inequality

$$|\eta' + t(\eta - \eta')| \geq \frac{1}{4}|\eta - \eta'|, \quad \text{for } |\eta'| \geq |\eta|, \quad 0 \leq t \leq \frac{1}{4}, \tag{3.19}$$

we get

$$\begin{aligned}
I_3 + I_4 &\geq \int_0^1 \int_0^1 \int_{|\nabla v| \geq |\nabla u^\sigma|} \lambda_0 |\nabla u^{\sigma\tau}|^{p(x)-2} |\nabla(u^\sigma - v)|^2 \frac{1}{\sigma} dx d\sigma d\tau \\
&+ \int_0^1 \int_0^1 \int_{|\nabla v| < |\nabla u^\sigma|} \lambda_0 |\nabla u^{\sigma(1-\tau)}|^{p(x)-2} |\nabla(u^\sigma - v)|^2 \frac{1}{\sigma} dx d\sigma d\tau \\
&\geq \alpha \lambda_0 \left( \int_{\{p(x) \geq 2\}} |\nabla u - \nabla v|^{p(x)} dx + \int_{\{p(x) < 2\}} (|\nabla u| + |\nabla v|)^{p(x)-2} |\nabla u - \nabla v|^2 dx \right),
\end{aligned} \tag{3.20}$$

where  $\alpha = \alpha(p_{\min}, p_{\max})$  and  $\lambda_0$  is as in (2.2). On the other hand,

$$\begin{aligned}
II &= \int_0^1 \int_0^1 \int_{|\nabla v| \geq |\nabla u^\sigma|} B_s(x, u^{\sigma\tau}, \nabla u^{\sigma\tau})(u^\sigma - v)^2 \frac{1}{\sigma} dx d\sigma d\tau \\
&+ \int_0^1 \int_0^1 \int_{|\nabla v| < |\nabla u^\sigma|} B_s(x, u^{\sigma(1-\tau)}, \nabla u^{\sigma(1-\tau)})(u^\sigma - v)^2 \frac{1}{\sigma} dx d\sigma d\tau \\
&+ \int_0^1 \int_0^1 \int_{|\nabla v| \geq |\nabla u^\sigma|} \nabla_\eta B(x, u^{\sigma\tau}, \nabla u^{\sigma\tau}) \cdot \nabla(u^\sigma - v)(u^\sigma - v) \frac{1}{\sigma} dx d\sigma d\tau \\
&+ \int_0^1 \int_0^1 \int_{|\nabla v| < |\nabla u^\sigma|} \nabla_\eta B(x, u^{\sigma(1-\tau)}, \nabla u^{\sigma(1-\tau)}) \cdot \nabla(u^\sigma - v)(u^\sigma - v) \frac{1}{\sigma} dx d\sigma d\tau.
\end{aligned} \tag{3.21}$$

Finally, using that  $A_s(x, s, \eta) = \nabla_\eta B(x, s, \eta)$ , the assumption (3.14) and estimates (3.17), (3.18), (3.20) and (3.21), we get (3.16).  $\square$

We now prove a comparison principle for equation (3.1), which holds under assumption (3.14).

**Proposition 3.4.** *Let  $p, A$  and  $B$  be as in Section 2. Assume moreover that condition (3.14) holds. Let  $u, v \in W^{1,p(\cdot)}(\Omega)$  be such that*

$$\begin{aligned}
\operatorname{div} A(x, u, \nabla u) &\geq B(x, u, \nabla u) \quad \text{in } \Omega, \\
\operatorname{div} A(x, v, \nabla v) &\leq B(x, v, \nabla v) \quad \text{in } \Omega, \\
u &\leq v \quad \text{on } \partial\Omega.
\end{aligned} \tag{3.22}$$

Then,

$$u \leq v \quad \text{in } \Omega. \quad (3.23)$$

*Proof.* We will use arguments similar to those in Proposition 3.3. In fact, for  $R > 0$  we consider the nonnegative function  $w_R \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$  given by

$$w_R = \begin{cases} 0 & \text{if } u - v \leq 0, \\ u - v & \text{if } 0 < u - v < R, \\ R & \text{if } u - v \geq R, \end{cases} \quad (3.24)$$

and by (3.22) we have

$$0 \geq \int_{\Omega} (A(x, u, \nabla u) - A(x, v, \nabla v)) \cdot \nabla w_R \, dx + \int_{\Omega} (B(x, u, \nabla u) - B(x, v, \nabla v)) w_R \, dx = I + II. \quad (3.25)$$

Then, denoting  $\Omega_R = \Omega \cap \{0 < u - v < R\}$  and, for  $0 \leq \tau \leq 1$ ,  $u^\tau = v + \tau(u - v)$ , we get

$$\begin{aligned} I &= \int_0^1 \int_{\Omega_R \cap \{|\nabla v| \geq |\nabla u|\}} A_s(x, u^\tau, \nabla u^\tau) \cdot \nabla(u - v)(u - v) \, dx \, d\tau \\ &\quad + \int_0^1 \int_{\Omega_R \cap \{|\nabla v| < |\nabla u|\}} A_s(x, u^{(1-\tau)}, \nabla u^{(1-\tau)}) \cdot \nabla(u - v)(u - v) \, dx \, d\tau \\ &\quad + \int_0^1 \int_{\Omega_R \cap \{|\nabla v| \geq |\nabla u|\}} \sum_{i,j} \frac{\partial A_i}{\partial \eta_j}(x, u^\tau, \nabla u^\tau)(u - v)_{x_i}(u - v)_{x_j} \, dx \, d\tau \\ &\quad + \int_0^1 \int_{\Omega_R \cap \{|\nabla v| < |\nabla u|\}} \sum_{i,j} \frac{\partial A_i}{\partial \eta_j}(x, u^{(1-\tau)}, \nabla u^{(1-\tau)})(u - v)_{x_i}(u - v)_{x_j} \, dx \, d\tau \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned} \quad (3.26)$$

Now, proceeding as in Proposition 3.3, we obtain

$$\begin{aligned} I_3 + I_4 &\geq \int_0^1 \int_{\Omega_R \cap \{|\nabla v| \geq |\nabla u|\}} \lambda_0 |\nabla u^\tau|^{p(x)-2} |\nabla(u - v)|^2 \, dx \, d\tau \\ &\quad + \int_0^1 \int_{\Omega_R \cap \{|\nabla v| < |\nabla u|\}} \lambda_0 |\nabla u^{(1-\tau)}|^{p(x)-2} |\nabla(u - v)|^2 \, dx \, d\tau \\ &\geq \tilde{\alpha} \lambda_0 \left( \int_{\Omega_R \cap \{p(x) \geq 2\}} |\nabla u - \nabla v|^{p(x)} \, dx + \int_{\Omega_R \cap \{p(x) < 2\}} (|\nabla u| + |\nabla v|)^{p(x)-2} |\nabla u - \nabla v|^2 \, dx \right), \end{aligned} \quad (3.27)$$

where  $\tilde{\alpha} = \tilde{\alpha}(p_{\min}, p_{\max})$  and  $\lambda_0$  is as in (2.2).

On the other hand, we observe that the evaluation of (3.14) in  $\xi = 0$  implies that  $B_s(x, s, \eta) \geq 0$ .

Then, we get

$$\begin{aligned}
II \geq & \int_0^1 \int_{\Omega_R \cap \{|\nabla v| \geq |\nabla u|\}} B_s(x, u^\tau, \nabla u^\tau)(u-v)^2 dx d\tau \\
& + \int_0^1 \int_{\Omega_R \cap \{|\nabla v| < |\nabla u|\}} B_s(x, u^{(1-\tau)}, \nabla u^{(1-\tau)})(u-v)^2 dx d\tau \\
& + \int_0^1 \int_{\Omega_R \cap \{|\nabla v| \geq |\nabla u|\}} \nabla_\eta B(x, u^\tau, \nabla u^\tau) \cdot \nabla(u-v)(u-v) dx d\tau \\
& + \int_0^1 \int_{\Omega_R \cap \{|\nabla v| < |\nabla u|\}} \nabla_\eta B(x, u^{(1-\tau)}, \nabla u^{(1-\tau)}) \cdot \nabla(u-v)(u-v) dx d\tau \\
& + \int_0^1 \int_{\{u-v>R\} \cap \{|\nabla v| \geq |\nabla u|\}} B_s(x, u^\tau, \nabla u^\tau) w_R^2 dx d\tau \\
& + \int_0^1 \int_{\{u-v>R\} \cap \{|\nabla v| < |\nabla u|\}} B_s(x, u^{(1-\tau)}, \nabla u^{(1-\tau)}) w_R^2 dx d\tau \\
& + \int_0^1 \int_{\{u-v>R\} \cap \{|\nabla v| \geq |\nabla u|\}} \nabla_\eta B(x, u^\tau, \nabla u^\tau) \cdot \nabla(u-v) w_R dx d\tau \\
& + \int_0^1 \int_{\{u-v>R\} \cap \{|\nabla v| < |\nabla u|\}} \nabla_\eta B(x, u^{(1-\tau)}, \nabla u^{(1-\tau)}) \cdot \nabla(u-v) w_R dx d\tau.
\end{aligned} \tag{3.28}$$

Now, using that  $A_s(x, s, \eta) = \nabla_\eta B(x, s, \eta)$ , (2.3), (3.19), assumption (3.14) and estimates (3.25), (3.26), (3.27) and (3.28), we get

$$\begin{aligned}
0 \geq & \frac{1}{2} \tilde{\alpha} \lambda_0 \left( \int_{\Omega_R \cap \{p(x) \geq 2\}} |\nabla u - \nabla v|^{p(x)} dx + \int_{\Omega_R \cap \{p(x) < 2\}} (|\nabla u| + |\nabla v|)^{p(x)-2} |\nabla u - \nabla v|^2 dx \right) \\
& - \hat{\alpha} \Lambda_0 \left( \int_{\{u-v>R\} \cap \{p(x) \geq 2\}} (|\nabla u| + |\nabla v|)^{p(x)} dx + \int_{\{u-v>R\} \cap \{p(x) < 2\}} |\nabla u - \nabla v|^{p(x)} dx \right),
\end{aligned} \tag{3.29}$$

where  $\hat{\alpha} = \hat{\alpha}(p_{\min}, p_{\max})$  and  $\Lambda_0$  is as in (2.3). Since  $R > 0$  is arbitrary, we can use that  $u, v \in W^{1,p(\cdot)}(\Omega)$  and let  $R \rightarrow \infty$  and we obtain

$$0 \geq \frac{1}{2} \tilde{\alpha} \lambda_0 \left( \int_{\Omega \cap \{p(x) \geq 2\}} |\nabla(u-v)^+|^{p(x)} dx + \int_{\Omega \cap \{p(x) < 2\}} (|\nabla u| + |\nabla v|)^{p(x)-2} |\nabla(u-v)^+|^2 dx \right), \tag{3.30}$$

which implies that  $\nabla(u-v)^+ = 0$  in  $\Omega$ . Since  $(u-v)^+ \in W_0^{1,p(\cdot)}(\Omega)$ , Poincaré's inequality (Theorem A.4) gives  $(u-v)^+ = 0$  in  $\Omega$ . That is, (3.23) holds.  $\square$

As a corollary of Proposition 3.4 we obtain the following uniqueness result

**Corollary 3.2.** *Let  $p, A$  and  $B$  be as in Section 2. Assume moreover that condition (3.14) holds. Let  $\varphi \in W^{1,p(\cdot)}(\Omega)$  and let  $u_1, u_2 \in W^{1,p(\cdot)}(\Omega)$  be such that*

$$\begin{cases} \operatorname{div} A(x, u_i, \nabla u_i) = B(x, u_i, \nabla u_i) & \text{in } \Omega, \\ u_i = \varphi & \text{on } \partial\Omega, \end{cases} \tag{3.31}$$

for  $i = 1, 2$ . Then,  $u_1 = u_2$  in  $\Omega$ .

We next prove that solutions to (3.1) with bounded boundary data are bounded, under the assumptions of Proposition 3.1.

**Proposition 3.5.** *Let  $p, A$  and  $B$  be as in Section 2 and let  $\Omega' \subset \Omega$  be a  $C^1$  domain. Assume moreover, that conditions (3.3), (3.4) and (3.14) hold in  $\Omega'$  for some  $p^+ - p^- < \delta < p_{\min}$  where  $p^+ = \sup_{\Omega'} p$  and  $p^- = \inf_{\Omega'} p$  and with  $\tau$  satisfying (2.7). Let us also assume that there exists a positive constant  $\Lambda_0$  such that the following condition holds:*

$$|B(x, s, \eta)| \leq \Lambda_0(1 + |s|^{p(x)-1} + |\eta|^{p(x)-1}), \quad (3.32)$$

for every  $(x, s, \eta) \in \overline{\Omega'} \times \mathbb{R} \times \mathbb{R}^N$ . Let  $u \in W^{1,p(\cdot)}(\Omega')$  be such that

$$\begin{cases} \operatorname{div} A(x, u, \nabla u) = B(x, u, \nabla u) & \text{in } \Omega', \\ |u| \leq M & \text{on } \partial\Omega', \end{cases} \quad (3.33)$$

for some positive constant  $M$ . Then, there exists  $C$  such that  $|u| \leq C$  in  $\Omega'$ , where  $C$  depends only on  $M, |\Omega'|, \operatorname{diam}(\Omega'), N, \lambda_0, \Lambda_0, L, p^-, p^+, \delta, \|g\|_{L^1(\Omega')}, \|\tau\|_{L^\infty(\Omega')}, \mu$  and  $c_1$ .

*Proof.* Let  $v^+$  be the solution to (3.1) with boundary data  $M$ . Then, from the proof of Proposition 3.1 it follows that  $\|v^+\|_{W^{1,p(\cdot)}(\Omega')}$  depends only on the constants in the structural conditions, on  $|\Omega'|$ ,  $\operatorname{diam}(\Omega')$  and  $M$ . Since (recall Remark 2.1) we are under the assumptions of Theorem 4.1 in [11], then  $v^+ \in L^\infty(\Omega')$  with bounds depending only on the constants in the structural conditions, on  $|\Omega'|$ ,  $\operatorname{diam}(\Omega')$  and  $M$ . Now, the comparison principle (Proposition 3.4) implies that  $u \leq v^+$  in  $\Omega'$  and the upper bound follows. Proceeding in an analogous way with  $v^-$  the solution to (3.1) with boundary data  $-M$ , we obtain the lower bound, thus concluding the proof.  $\square$

As a corollary of Propositions 3.1 and 3.5 we get

**Corollary 3.3.** *Let  $p, F, A$  and  $B$  as in Section 2 and let  $\Omega' \subset \Omega$  be a  $C^1$  domain. Assume, moreover that  $F$  satisfies (3.8) and (3.9) with  $\tau$  satisfying (2.7) and  $A$  and  $B$  satisfy (3.14) and (3.32) in  $\Omega'$  for some  $p^+ - p^- < \delta < p_{\min}$  where  $p^+ = \sup_{\Omega'} p$  and  $p^- = \inf_{\Omega'} p$ .*

*Let  $u \in W^{1,p(\cdot)}(\Omega') \cap L^\infty(\Omega')$ . Then, there exists  $v \in u + W_0^{1,p(\cdot)}(\Omega')$  a solution to*

$$\operatorname{div} A(x, v, \nabla v) = B(x, v, \nabla v) \quad \text{in } \Omega'.$$

*Moreover,  $v \in L^\infty(\Omega')$  and  $\|v\|_{L^\infty(\Omega')}$  is bounded by a constant  $C$  that depends only on  $\|u\|_{L^\infty(\Omega')}, |\Omega'|, \operatorname{diam}(\Omega'), N, \lambda_0, \Lambda_0, L, p^-, p^+, \delta, \|\tau\|_{L^\infty(\Omega')}, \mu$  and  $c_1$ .*

We also prove the following maximum principle

**Proposition 3.6.** *Let  $p, A$  and  $B$  be as in Section 2. Assume moreover that condition (3.14) holds. We also assume that  $B(x, 0, 0) \equiv 0$  for every  $x \in \overline{\Omega}$ . Let  $u \in W^{1,p(\cdot)}(\Omega)$  be such that*

$$\begin{cases} \operatorname{div} A(x, u, \nabla u) = B(x, u, \nabla u) & \text{in } \Omega, \\ -M_1 \leq u \leq M_2 & \text{on } \partial\Omega, \end{cases} \quad (3.34)$$

for some nonnegative constants  $M_1, M_2$ . Then,  $-M_1 \leq u \leq M_2$  in  $\Omega$ .

*Proof.* Since condition (3.14) implies that  $B_s(x, s, \eta) \geq 0$  in  $\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N \setminus \{0\}$ , we have  $B(x, M_2, 0) \geq 0$  and also  $B(x, -M_1, 0) \leq 0$ , for every  $x \in \Omega$ . Recalling (2.1), we take  $v^+ \equiv M_2$  and  $v^- \equiv -M_1$  and observe that  $\operatorname{div} A(x, v^+, \nabla v^+) \leq B(x, v^+, \nabla v^+)$  and  $\operatorname{div} A(x, v^-, \nabla v^-) \geq B(x, v^-, \nabla v^-)$  in  $\Omega$ . Then, we can apply the comparison principle (Proposition 3.4) and obtain  $-M_1 \equiv v^- \leq u \leq v^+ \equiv M_2$  in  $\Omega$  and the conclusion follows.  $\square$

As a corollary of Propositions 3.1 and 3.6 we get

**Corollary 3.4.** *Let  $p, F, A$  and  $B$  as in Section 2 and let  $\Omega' \subset \Omega$  be a  $C^1$  domain. Assume, moreover that  $F$  satisfies (3.8) and (3.9) with  $\tau$  satisfying (2.7) and  $A$  and  $B$  satisfy (3.14) in  $\Omega'$  for some  $p^+ - p^- < \delta < p_{\min}$  where  $p^+ = \sup_{\Omega'} p$  and  $p^- = \inf_{\Omega'} p$ . We also assume that  $B(x, 0, 0) \equiv 0$  for every  $x \in \bar{\Omega}'$ .*

*Let  $u \in W^{1,p(\cdot)}(\Omega') \cap L^\infty(\Omega')$ . Then, there exists  $v \in u + W_0^{1,p(\cdot)}(\Omega')$  a solution to*

$$\operatorname{div} A(x, v, \nabla v) = B(x, v, \nabla v) \quad \text{in } \Omega'.$$

*Moreover,  $v \in L^\infty(\Omega')$  and  $\|v\|_{L^\infty(\Omega')} \leq \|u\|_{L^\infty(\Omega')}.$*

We also have the following existence result of a bounded solution

**Proposition 3.7.** *Let  $p$  as in Section 2. Assume that  $F(x, \cdot, \cdot)$  is locally Lipschitz in  $\mathbb{R} \times \mathbb{R}^N$  for almost every  $x \in \Omega$  and that  $F(x, s, \cdot) \in C^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N \setminus \{0\})$  for  $s \in \mathbb{R}$  and almost every  $x \in \Omega$ . Let  $A = \nabla_\eta F$ ,  $B = F_s$ . Assume that  $A$  satisfies (2.2) and (2.5),*

$$|A(x, s, \eta)|, |B(x, s, \eta)| \leq \Lambda_0(1 + |s|^{\tau(x)} + |\eta|^{p(x)}) \quad \text{a.e. in } \Omega \times \mathbb{R} \times \mathbb{R}^N,$$

*and  $F$  satisfies (3.3) and (3.4), where  $\tau$  satisfies (2.7). Assume moreover that*

$$F(x, s, \eta) = G(x, s, \eta) + f(x, s) \quad \text{with } G, f \text{ measurable functions} \quad (3.35)$$

*and,*

$$G \geq 0 \text{ in } \Omega \times \mathbb{R} \times \mathbb{R}^N, \quad G(x, s, \eta) = 0 \iff \eta = 0, \quad (3.36)$$

$$f(x, \cdot) \text{ monotone decreasing in } (-\infty, 0] \text{ and monotone increasing in } [0, +\infty). \quad (3.37)$$

*Then, for every  $\Omega' \subset \Omega$  of class  $C^1$  there holds that, if  $p^+ - p^- < \delta < p_{\min}$  where  $p^+ = \sup_{\Omega'} p$  and  $p^- = \inf_{\Omega'} p$  for  $\delta$  in (3.3), given  $u \in W^{1,p(\cdot)}(\Omega')$  such that  $0 \leq u \leq M$  in  $\Omega'$  there exists  $v$  that minimizes the functional  $\mathcal{J}_{\Omega'}(v)$  in  $u + W_0^{1,p(\cdot)}(\Omega')$ . Moreover,  $0 \leq v \leq M$  in  $\Omega'$ .*

*In addition, if there exists  $\varepsilon_0 > 0$  such that for almost every  $x \in \Omega$ ,  $F(x, \cdot, \cdot) \in C^1((-\varepsilon_0, M + \varepsilon_0) \times \mathbb{R}^N)$ , then there holds that  $v$  is a solution to*

$$\begin{cases} \operatorname{div} A(x, v, \nabla v) = B(x, v, \nabla v) & \text{in } \Omega', \\ v = u & \text{on } \partial\Omega'. \end{cases} \quad (3.38)$$

*Proof.* To begin with, the existence of a minimizer  $v$  follows proceeding as in Proposition 3.1. Let us prove that a minimizer satisfies  $0 \leq v \leq M$ . In fact, both  $w_1 = v - (v - M)^+$  and  $w_2 = v + v^-$  are admissible functions. So that on the one hand,

$$\begin{aligned} 0 &\leq \int_{\Omega'} F(x, w_1, \nabla w_1) - F(x, v, \nabla v) = \int_{v > M} F(x, M, 0) - F(x, v, \nabla v) \\ &= \int_{v > M} f(x, M) - f(x, v) - \int_{v > M} G(x, v, \nabla v) \\ &\leq - \int_{v > M} G(x, v, \nabla v) \leq 0. \end{aligned}$$



Hence,  $G(x, v, \nabla v) = 0$  in  $\{v > M\}$ . So that,  $\nabla(v - M)^+ = 0$  in  $\Omega'$ . As  $(v - M)^+ = 0$  on  $\partial\Omega'$ , we deduce that  $v \leq M$  in  $\Omega'$ .

On the other hand, proceeding in a similar way with  $w_2$ ,

$$\begin{aligned} 0 &\leq \int_{\Omega'} F(x, w_2, \nabla w_2) - F(x, v, \nabla v) = \int_{v < 0} F(x, 0, 0) - F(x, v, \nabla v) \\ &= \int_{v < 0} f(x, 0) - f(x, v) - \int_{v < 0} G(x, v, \nabla v) \\ &\leq - \int_{v < 0} G(x, v, \nabla v) \leq 0, \end{aligned}$$

and we deduce as before that  $v^- = 0$ . This is,  $v \geq 0$  in  $\Omega'$ .

Now, in order to proceed with the proof we assume further regularity of  $F$  for  $-\varepsilon_0 \leq s \leq M + \varepsilon_0$ . Let  $0 \leq \varphi \in C_0^\infty(\Omega')$  and  $0 < \varepsilon < \varepsilon_0 / \|\varphi\|_{L^\infty}$ . Then,  $w = v + \varepsilon\varphi$  is an admissible function,  $-\varepsilon_0 < w < M + \varepsilon_0$  and we deduce that

$$\operatorname{div} A(x, v, \nabla v) \leq B(x, v, \nabla v) \quad \text{in } \Omega'.$$

Replacing  $\varphi$  by  $-\varphi$  we reverse the inequality. So that,  $v$  is a solution to (3.38).  $\square$

#### 4. Energy minimizers of energy functional (1.1)

In this section we prove properties of nonnegative local minimizers of the energy functional (1.1). We prove that nonnegative local minimizers are locally Hölder continuous (Theorem 4.3) and are solutions to

$$\operatorname{div} A(x, u, \nabla u) = B(x, u, \nabla u) \quad \text{in } \{u > 0\},$$

where  $A(x, s, \eta) = \nabla_\eta F(x, s, \eta)$  and  $B(x, s, \eta) = F_s(x, s, \eta)$ . In particular we prove our main result which is the local Lipschitz continuity on nonnegative local minimizers (Theorem 4.5).

We start with a definition, some related remarks and an existence result of a minimizer. We also prove nonnegativity and boundedness, under suitable assumptions.

**Definition 4.1.** Let  $p, F$  and  $\lambda$  be as in Section 2. Assume that  $F$  satisfies (3.3) and (3.4) with  $\tau$  satisfying (2.7). We say that  $u \in W^{1,p(\cdot)}(\Omega)$  is a local minimizer in  $\Omega$  of

$$J(v) = J_\Omega(v) = \int_{\Omega} (F(x, v, \nabla v) + \lambda(x)\chi_{\{v>0\}}) dx$$

if for every  $\Omega' \subset\subset \Omega$  and for every  $v \in W^{1,p(\cdot)}(\Omega)$  such that  $v = u$  in  $\Omega \setminus \Omega'$  there holds that  $J(v) \geq J(u)$ .

We point out that the energy  $J$  is well defined in  $W^{1,p(\cdot)}(\Omega)$  since, under the conditions in (2.7), the embedding theorem (see Theorem A.5) applies.

**Remark 4.1.** Let  $u$  be as in Definition 4.1. Let  $\Omega' \subset\subset \Omega$  and  $w - u \in W_0^{1,p(\cdot)}(\Omega')$ . If we define

$$\bar{w} = \begin{cases} w & \text{in } \Omega', \\ u & \text{in } \Omega \setminus \Omega', \end{cases}$$

then  $\bar{w} \in W^{1,p(\cdot)}(\Omega)$  and therefore  $J(\bar{w}) \geq J(u)$ . If we now let

$$J_{\Omega'}(v) = \int_{\Omega'} (F(x, v, \nabla v) + \lambda(x)\chi_{\{v>0\}}) dx$$

it follows that  $J_{\Omega'}(w) \geq J_{\Omega'}(u)$ .

**Remark 4.2.** Let  $J$  be as in Definition 4.1. If  $u \in W^{1,p(\cdot)}(\Omega)$  is a minimizer of  $J$  among the functions  $v \in u + W_0^{1,p(\cdot)}(\Omega)$ , then  $u$  is a local minimizer of  $J$  in  $\Omega$ .

We start with an existence result of a minimizer to (1.1).

**Theorem 4.1.** Let  $p, F, A, B$  and  $\lambda$  be as in Section 2. Let  $\phi \in W^{1,p(\cdot)}(\Omega)$  and assume moreover that  $F$  satisfies (3.10) and (3.11) with  $\tau$  satisfying (2.7).

Then, there exists a minimizer  $u \in \phi + W_0^{1,p(\cdot)}(\Omega)$  to (1.1) and there holds that  $\|u\|_{W^{1,p(\cdot)}(\Omega)} \leq C$ , for a constant  $C$  depending only on  $\|\phi\|_{W^{1,p(\cdot)}(\Omega)}$ ,  $\|g\|_{L^1(\Omega)}$ ,  $\lambda_{\max}$ ,  $|\Omega|$ ,  $\text{diam}(\Omega)$ ,  $N$ ,  $p_{\min}$ ,  $p_{\max}$ ,  $q$ ,  $L$ ,  $\mu$ ,  $c_1$ ,  $\|\tau\|_{L^\infty(\Omega)}$  and the  $C^1$  norm of  $\partial\Omega$ .

*Proof.* The proof is immediate from the computations in the proof of Proposition 3.2.  $\square$

We also have,

**Theorem 4.2.** Let  $p$  and  $\lambda$  be as in Section 2. Let  $F, A$  and  $B$  be as in Proposition 3.7, except for the fact that we require that  $F$  satisfies (3.10) and (3.11) with  $\tau$  satisfying (2.7), instead of (3.3) and (3.4), and with no oscillation assumption on  $p$ . Let  $\phi \in W^{1,p(\cdot)}(\Omega)$  such that  $0 \leq \phi \leq M$ , for some  $M > 0$ .

Then, there exists a minimizer  $u \in \phi + W_0^{1,p(\cdot)}(\Omega)$  to (1.1) and  $0 \leq u \leq M$  in  $\Omega$ .

*Proof.* Proceeding as in the proof of Proposition 3.2 we obtain that there exists a minimizer  $u \in \phi + W_0^{1,p(\cdot)}(\Omega)$  to (1.1). The proof that  $0 \leq u \leq M$  is similar to that of Proposition 3.7. We only have to observe that

$$\{u - (u - M)^+ > 0\} = \{u > 0\} \quad \text{and} \quad \{u + u^- > 0\} = \{u > 0\}.$$

$\square$

For local minimizers of (1.1) we first have

**Lemma 4.1.** Let  $p, F, A, B$  and  $\lambda$  be as in Section 2. Assume that  $F$  satisfies (3.3) and (3.4) with  $\tau$  satisfying (2.7). Let  $u \in W^{1,p(\cdot)}(\Omega)$  be a local minimizer of

$$J(v) = \int_{\Omega} (F(x, v, \nabla v) + \lambda(x)\chi_{\{v>0\}}) dx.$$

Then

$$\text{div}A(x, u, \nabla u) \geq B(x, u, \nabla u) \quad \text{in } \Omega, \quad (4.1)$$

where  $A(x, s, \eta) = \nabla_\eta F(x, s, \eta)$  and  $B(x, s, \eta) = F_s(x, s, \eta)$ .

*Proof.* In fact, let  $t > 0$  and  $0 \leq \xi \in C_0^\infty(\Omega)$ . Using the minimality of  $u$  we have

$$0 \leq \frac{1}{t}(J(u - t\xi) - J(u)) \leq \frac{1}{t} \int_{\Omega} (F(x, u - t\xi, \nabla u - t\nabla \xi) - F(x, u, \nabla u)) dx$$

and if we take  $t \rightarrow 0$ , we obtain

$$0 \leq - \int_{\Omega} \nabla_{\eta} F(x, u, \nabla u) \cdot \nabla \xi dx - \int_{\Omega} F_s(x, u, \nabla u) \xi dx, \quad (4.2)$$

which gives (4.1).  $\square$

From now on we will deal with nonnegative, bounded, local minimizers of (1.1). Next we will prove that they are locally Lipschitz continuous.

We first prove that nonnegative, bounded, local minimizers are locally Hölder continuous.

**Theorem 4.3.** *Let  $p, F, A, B$  and  $\lambda$  be as in Section 2. Assume that  $F$  satisfies (3.3) and (3.4) with  $\tau$  satisfying (2.7). Let  $x_0 \in \Omega$ ,  $\hat{r}_0 > 0$  such that  $B_{\hat{r}_0}(x_0) \subset\subset \Omega$ . Assume that  $A, B$  satisfy condition (3.14) in  $B_{\hat{r}_0}(x_0)$  and either  $B(x, 0, 0) \equiv 0$  for  $x \in B_{\hat{r}_0}(x_0)$  or  $B$  satisfies (3.32) for  $x \in B_{\hat{r}_0}(x_0)$ . Let  $u \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$  be a nonnegative local minimizer of (1.1). Then, there exist  $0 < \gamma < 1$ ,  $\gamma = \gamma(N, p_{\min})$  and  $0 < \hat{\rho}_0 < \hat{r}_0$ , such that  $u \in C^\gamma(\overline{B_{\hat{\rho}_0}(x_0)})$ . Moreover,  $\|u\|_{C^\gamma(\overline{B_{\hat{\rho}_0}(x_0)})} \leq C$  with  $\hat{\rho}_0$  and  $C$  depending only on  $\beta, p_{\max}, p_{\min}, N, L, \hat{r}_0, \lambda_0, \Lambda_0, \|g\|_{L^1(B_{\hat{r}_0}(x_0))}, \mu, c_1, \lambda_{\max}, \|u\|_{L^\infty(B_{\hat{r}_0}(x_0))}, \|\tau\|_{L^\infty(B_{\hat{r}_0}(x_0))}$  and  $\delta$ .*

*Proof.* We will prove that there exist  $0 < \gamma < 1$  and  $0 < \rho_0 < r_0 < \hat{r}_0$  such that, if  $B_{r_0}(y) \subset B_{\hat{r}_0}(x_0)$  and  $\rho \leq \rho_0$ , then

$$\left( \int_{B_\rho(y)} |\nabla u|^{p_-} dx \right)^{1/p_-} \leq C \rho^{\gamma-1}, \quad (4.3)$$

where  $p_- = \inf\{p(x), x \in B_{r_0}(y)\}$ . Without loss of generality we will assume that  $y = 0$ .

In fact, let  $0 < r_0 \leq \min\{\frac{\hat{r}_0}{2}, 1\}$ ,  $0 < r \leq r_0$  and  $v$  the solution of

$$\operatorname{div} A(x, v, \nabla v) = B(x, v, \nabla v) \quad \text{in } B_r, \quad v - u \in W_0^{1,p(\cdot)}(B_r). \quad (4.4)$$

Observe that, under our assumptions we can apply either Proposition 3.1 and Proposition 3.5 or Proposition 3.6 and deduce that such a solution exists and it is bounded in  $\overline{B_r}$  if  $r_0$  is small enough depending on  $\delta$  and  $L = \|\nabla p\|_{L^\infty(\Omega)}$ . Hence, by Proposition 3.3, we have

$$\begin{aligned} \int_{B_r} (F(x, u, \nabla u) - F(x, v, \nabla v)) dx \geq \\ \frac{1}{2} \alpha \lambda_0 \left( \int_{B_r \cap \{p(x) \geq 2\}} |\nabla u - \nabla v|^{p(x)} dx + \int_{B_r \cap \{p(x) < 2\}} (|\nabla u| + |\nabla v|)^{p(x)-2} |\nabla u - \nabla v|^2 dx \right), \end{aligned} \quad (4.5)$$

where  $\alpha = \alpha(p_{\min}, p_{\max})$  and  $\lambda_0$  is as in (2.2).

By the minimality of  $u$ , we have (if  $A_1 = B_r \cap \{p(x) < 2\}$  and  $A_2 = B_r \cap \{p(x) \geq 2\}$ )

$$\int_{A_2} |\nabla u - \nabla v|^{p(x)} dx \leq C r^N, \quad (4.6)$$

$$\int_{A_1} |\nabla u - \nabla v|^2 (|\nabla u| + |\nabla v|)^{p(x)-2} dx \leq Cr^N, \quad (4.7)$$

where  $C = C(p_{\min}, p_{\max}, N, \lambda_{\max}, \lambda_0)$ .

Let  $\varepsilon > 0$ . Take  $\rho = r^{1+\varepsilon}$  and suppose that  $r^\varepsilon \leq 1/2$ . Take  $0 < \eta < 1$  to be chosen later. Then, by Young's inequality, the definition of  $A_1$  and (4.7), we obtain

$$\begin{aligned} \int_{A_1 \cap B_\rho} |\nabla u - \nabla v|^{p(x)} dx &\leq \frac{C}{\eta^{2/p_{\min}}} \int_{A_1 \cap B_r} (|\nabla u| + |\nabla v|)^{p(x)-2} |\nabla u - \nabla v|^2 dx \\ &\quad + C\eta \int_{B_\rho \cap A_1} (|\nabla u| + |\nabla v|)^{p(x)} dx \\ &\leq \frac{C}{\eta^{2/p_{\min}}} r^N + C\eta \int_{B_\rho \cap A_1} (|\nabla u| + |\nabla v|)^{p(x)} dx. \end{aligned} \quad (4.8)$$

Therefore, by (4.6) and (4.8), we get

$$\int_{B_\rho} |\nabla u - \nabla v|^{p(x)} dx \leq \frac{C}{\eta^{2/p_{\min}}} r^N + C\eta \int_{B_\rho \cap A_1} (|\nabla u| + |\nabla v|)^{p(x)} dx, \quad (4.9)$$

where  $C = C(p_{\min}, p_{\max}, N, \lambda_{\max}, \lambda_0)$ .

Since,  $|\nabla u|^q \leq C(|\nabla u - \nabla v|^q + |\nabla v|^q)$ , for any  $q > 1$ , with  $C = C(q)$ , we have, by (4.9), choosing  $\eta$  small, that

$$\int_{B_\rho} |\nabla u|^{p(x)} dx \leq Cr^N + C \int_{B_\rho} |\nabla v|^{p(x)} dx, \quad (4.10)$$

where  $C = C(p_{\min}, p_{\max}, N, \lambda_{\max}, \lambda_0)$ .

Now let  $M \geq 1$  such that  $\|v\|_{L^\infty(B_r)} \leq M$  and define

$$w(x) = \frac{v(rx)}{M} \quad \text{in } B_1.$$

Observe that  $M$  depends only on  $\|u\|_{L^\infty(B_{\hat{r}_0}(x_0))}$  if  $B(x, 0, 0) \equiv 0$  or it depends also on the structural conditions on  $F$ ,  $A$  and  $B$ , on  $\hat{r}_0$  and on the bound  $L$  of  $\|\nabla p\|_{L^\infty}$  if not.

There holds that,

$$\operatorname{div} \bar{A}(x, w, \nabla w) = \bar{B}(x, w, \nabla w) \quad \text{in } B_1$$

where

$$\bar{A}(x, s, \eta) = A(rx, Ms, \frac{M}{r}\eta), \quad \bar{B}(x, s, \eta) = rB(rx, Ms, \frac{M}{r}\eta).$$

Now, let

$$\tilde{A}(x, s, \eta) = \left(\frac{r}{M}\right)^{p_r^- - 1} \bar{A}(x, s, \eta), \quad \tilde{B}(x, s, \eta) = \left(\frac{r}{M}\right)^{p_r^- - 1} \bar{B}(x, s, \eta).$$

Observe that  $w \in W^{1, \tilde{p}(\cdot)}(B_1) \cap L^\infty(B_1)$  satisfies

$$\operatorname{div} \tilde{A}(x, w, \nabla w) = \tilde{B}(x, w, \nabla w) \quad \text{in } B_1, \quad (4.11)$$

where  $\tilde{p}(x) = p(rx)$ .

Let us see that (4.11) is under the conditions of Theorem 1.1 in [10].

First, we clearly have  $\widetilde{A}(x, s, 0) = 0$ . Moreover, as  $1 \leq r^{p_r^- - p_r^+} \leq C_L < \infty$  if  $r \leq 1$  and we have assumed that  $M \geq 1$ ,

$$\begin{aligned} \sum_{ij} \frac{\partial \widetilde{A}_i}{\partial \eta_j}(x, s, \eta) \xi_i \xi_j &= \left(\frac{r}{M}\right)^{p_r^- - 1} \left(\frac{M}{r}\right) \sum_{ij} \frac{\partial A_i}{\partial \eta_j}(rx, Ms, \frac{M}{r}\eta) \xi_i \xi_j \\ &\geq \lambda_0 \left(\frac{r}{M}\right)^{p_r^- - 1} \left(\frac{M}{r}\right)^{p(rx) - 1} |\eta|^{\bar{p}(x) - 2} |\xi|^2 \geq \lambda_0 |\eta|^{\bar{p}(x) - 2} |\xi|^2. \end{aligned} \quad (4.12)$$

On the other hand,

$$\sum_{ij} \left| \frac{\partial \widetilde{A}_i}{\partial \eta_j}(x, s, \eta) \right| \leq \Lambda_0 \left(\frac{r}{M}\right)^{p_r^- - 1} \left(\frac{M}{r}\right)^{p(rx) - 1} |\eta|^{\bar{p}(x) - 2} \leq \Lambda_0 C_L M^{p_{\max} - p_{\min}} |\eta|^{\bar{p}(x) - 2}. \quad (4.13)$$

Then, assuming without loss of generality that  $p(rx_1) \geq p(rx_2)$ ,

$$\begin{aligned} |\widetilde{A}(x_1, s, \eta) - \widetilde{A}(x_2, s, \eta)| &\leq \left(\frac{r}{M}\right)^{p_r^- - 1} |A(rx_1, Ms, \frac{M}{r}\eta) - A(rx_2, Ms, \frac{M}{r}\eta)| \\ &\leq \left(\frac{r}{M}\right)^{p_r^- - 1} \Lambda_0 \left(\left(\frac{M}{r}\right)^{p(rx_1) - 1} |\eta|^{p(rx_1) - 1} + \left(\frac{M}{r}\right)^{p(rx_2) - 1} |\eta|^{p(rx_2) - 1}\right) \\ &\quad (1 + |\log(\frac{M}{r}|\eta|)|) r^\beta |x_1 - x_2|^\beta \\ &\leq \Lambda_0 C_L M^{p_{\max} - p_{\min}} (|\eta|^{\bar{p}(x_1) - 1} + |\eta|^{\bar{p}(x_2) - 1}) (1 + |\log |\eta||) |x_1 - x_2|^\beta \end{aligned} \quad (4.14)$$

if  $r \leq r_{M, \beta}$ .

Similarly,

$$|\widetilde{A}(x, s_1, \eta) - \widetilde{A}(x, s_2, \eta)| \leq \Lambda_4 |s_1 - s_2| |\eta|^{\bar{p}(x) - 1}$$

with  $\Lambda_4 = \Lambda_0 C_L M^{p_{\max} - p_{\min} + 1}$ .

On the other hand, denoting  $\bar{\tau}(x) = \tau(rx)$ ,

$$\begin{aligned} |\widetilde{B}(x, s, \eta)| &\leq \Lambda_0 r \left(\frac{r}{M}\right)^{p_r^- - 1} + \Lambda_0 C_L M^{p_{\max} - p_{\min} + 1} |\eta|^{\bar{p}(x)} + \Lambda_0 r \left(\frac{r}{M}\right)^{p_r^- - 1} |Ms|^{\bar{\tau}(x)} \\ &\leq \Lambda_5 (1 + |\eta|^{\bar{p}(x)} + |s|^{\bar{\tau}(x)}) \end{aligned}$$

with  $\Lambda_5$  depending on  $\Lambda_0, L, p_{\max}, p_{\min}, M$  and  $\|\tau\|_{L^\infty(B_{\hat{r}_0}(x_0))}$ .

Since  $|w| \leq 1$ , we may assume that

$$\widetilde{B}(x, s, \eta) \leq \Lambda_6 (1 + |\eta|^{\bar{p}(x)}),$$

with  $\Lambda_6$  depending on  $\Lambda_0, L, p_{\min}, p_{\max}, M$  and  $\|\tau\|_{L^\infty(B_{\hat{r}_0}(x_0))}$ .

From Theorem 1.1 in [10], it follows that  $w \in C_{\text{loc}}^{1, \alpha}(B_1)$  for some  $0 < \alpha < 1$  and that

$$\sup_{B_{1/2}} |\nabla w| \leq C,$$

with  $C$  depending only on  $\beta, p_{\max}, p_{\min}, N, L, \lambda_0, \Lambda_0, M$  and  $\|\tau\|_{L^\infty(B_{\hat{r}_0}(x_0))}$ , which implies

$$\sup_{B_{r/2}} |\nabla v| \leq \frac{CM}{r}. \quad (4.15)$$

Therefore, from (4.10) and (4.15), we deduce that if  $r$  is small depending on  $M$  and  $\beta$ ,

$$\int_{B_\rho} |\nabla u|^{p(x)} dx \leq Cr^N + C\rho^N r^{-p_+}, \quad (4.16)$$

with  $p_+ = \sup\{p(x), x \in B_{r_0}\}$  and  $C$  depending on  $\beta, p_{\max}, p_{\min}, N, L, \lambda_0, \Lambda_0, \lambda_{\max}, M$  and  $\|\tau\|_{L^\infty(B_{\hat{r}_0}(x_0))}$ .

Then, if we take  $\varepsilon \leq \frac{p_{\min}}{N}$ , we have by (4.16) and by our election of  $\rho$ , that

$$\begin{aligned} \int_{B_\rho} |\nabla u|^{p_-} dx &\leq \int_{B_\rho} |\nabla u|^{p(x)} dx + \frac{1}{|B_\rho|} \int_{B_\rho \cap \{|\nabla u| < 1\}} |\nabla u|^{p_-} dx \\ &\leq \int_{B_\rho} |\nabla u|^{p(x)} dx + 1 \\ &\leq 1 + C\left(\frac{r}{\rho}\right)^N + Cr^{-p_+} \\ &\leq 1 + Cr^{-\varepsilon N} + Cr^{-p_+} \\ &\leq Cr^{-p_+} = C\rho^{-\frac{p_+}{(1+\varepsilon)}}. \end{aligned}$$

Now let  $r_0 \leq r_0(\varepsilon, p_{\min}, L)$  so that

$$\frac{p_+}{p_-} = \frac{p_+(B_{r_0})}{p_-(B_{r_0})} \leq 1 + \frac{\varepsilon}{2},$$

and small enough so that, in addition,  $r_0^\varepsilon \leq 1/2$ . Then, if  $\rho \leq \rho_0 = r_0^{1+\varepsilon}$  and moreover,  $r_0$  is small depending on  $M$  and  $\beta$ ,

$$\int_{B_\rho} |\nabla u|^{p_-} dx \leq C\rho^{-\frac{(1+\frac{\varepsilon}{2})}{(1+\varepsilon)}p_-} = C\rho^{-(1-\gamma)p_-},$$

where  $\gamma = \frac{\varepsilon}{(1+\varepsilon)} = \gamma(N, p_{\min})$ . That is, if  $\rho \leq \rho_0 = r_0^{1+\varepsilon}$

$$\left(\int_{B_\rho} |\nabla u|^{p_-} dx\right)^{1/p_-} \leq C\rho^{\gamma-1}.$$

Thus (4.3) holds, with  $C$  depending only on  $\beta, p_{\max}, p_{\min}, N, L, \hat{r}_0, \lambda_0, \Lambda_0, \|g\|_{L^1(B_{\hat{r}_0}(x_0))}, \mu, c_1, \lambda_{\max}, \|u\|_{L^\infty(B_{\hat{r}_0}(x_0))}, \|\tau\|_{L^\infty(B_{\hat{r}_0}(x_0))}$  and  $\delta$ .

Applying Morrey's Theorem, see e.g., [18], Theorem 1.53, we conclude that  $u \in C^\gamma(B_{\rho_0}(x_0))$  and  $\|u\|_{C^\gamma(\overline{B_{\rho_0/2}(x_0)})} \leq C$  for  $C$  depending only on  $\beta, p_{\max}, p_{\min}, N, L, \hat{r}_0, \lambda_0, \Lambda_0, \|g\|_{L^1(B_{\hat{r}_0}(x_0))}, \mu, c_1, \lambda_{\max}, \|u\|_{L^\infty(B_{\hat{r}_0}(x_0))}, \|\tau\|_{L^\infty(B_{\hat{r}_0}(x_0))}$  and  $\delta$ .  $\square$

As a corollary we obtain

**Corollary 4.1.** *Let  $p, F, A, B$  and  $\lambda$  be as in Section 2. Assume that  $F$  satisfies (3.8) and (3.9) with  $\tau$  satisfying (2.7). Assume that  $A, B$  satisfy condition (3.14) and either  $B(x, 0, 0) \equiv 0$  for  $x \in \Omega$  or  $B$  satisfies (3.32) for  $x \in \Omega$ . Let  $u \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$  be a nonnegative local minimizer of (1.1). Then, there exists  $0 < \gamma < 1$ ,  $\gamma = \gamma(N, p_{\min})$  such that  $u \in C^\gamma(\Omega)$ . Moreover, if  $\Omega' \subset\subset \Omega$ , then  $\|u\|_{C^\gamma(\overline{\Omega'})} \leq C$  with  $C$  depending only on  $\text{dist}(\Omega', \partial\Omega), \beta, N, p_{\min}, p_{\max}, L, \lambda_{\max}, \lambda_0, \Lambda_0, \mu, c_1, \|u\|_{L^\infty(\Omega)}, \|\tau\|_{L^\infty(\Omega)}$  and  $\delta$ .*

Then, under the assumptions of the previous corollary we have that  $u$  is continuous in  $\Omega$  and therefore,  $\{u > 0\}$  is open. We can now prove the following property for nonnegative local minimizers of (1.1).

**Lemma 4.2.** *Let  $p, F, A, B$  and  $\lambda$  be as in Corollary 4.1. If  $u \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$  is a nonnegative local minimizer of*

$$J(v) = \int_{\Omega} (F(x, v, \nabla v) + \lambda(x)\chi_{\{v>0\}}) dx,$$

*there holds that,*

$$\operatorname{div} A(x, u, \nabla u) = B(x, u, \nabla u) \quad \text{in } \{u > 0\}, \quad (4.17)$$

*where  $A(x, s, \eta) = \nabla_{\eta} F(x, s, \eta)$  and  $B(x, s, \eta) = F_s(x, s, \eta)$ .*

*Proof.* From Lemma 4.1 we already know that (4.1) holds. In order to obtain the opposite inequality in  $\{u > 0\}$ , we let  $0 \leq \xi \in C_0^\infty(\{u > 0\})$  and consider  $u - t\xi$ , for  $t < 0$ , with  $|t|$  small.

Using the minimality of  $u$  we have

$$0 \geq \frac{1}{t}(J(u - t\xi) - J(u)) = \frac{1}{t} \int_{\Omega} (F(x, u - t\xi, \nabla u - t\nabla \xi) - F(x, u, \nabla u)) dx$$

and if we take  $t \rightarrow 0$ , we obtain

$$0 \geq - \int_{\Omega} \nabla_{\eta} F(x, u, \nabla u) \cdot \nabla \xi dx - \int_{\Omega} F_s(x, u, \nabla u) \xi dx,$$

which gives the desired inequality, so (4.17) follows.  $\square$

We will next prove the Lipschitz continuity of nonnegative local minimizers of (1.1).

Before getting the Lipschitz continuity we prove the following result

**Theorem 4.4.** *Let  $p, F, A, B, \lambda$  and  $u$  be as in Corollary 4.1. Let  $\Omega' \subset\subset \Omega$ . There exist constants  $C > 0$ ,  $r_0 > 0$  such that if  $x_0 \in \Omega' \cap \partial\{u > 0\}$  and  $r \leq r_0$  then*

$$\sup_{B_r(x_0)} u \leq Cr.$$

*The constants depend only on  $\operatorname{dist}(\Omega', \partial\Omega)$ ,  $\beta$ ,  $N$ ,  $p_{\min}$ ,  $p_{\max}$ ,  $L$ ,  $\lambda_{\max}$ ,  $\lambda_0$ ,  $\Lambda_0$ ,  $\mu$ ,  $c_1$ ,  $\|u\|_{L^\infty(\Omega)}$ ,  $\|\tau\|_{L^\infty(\Omega)}$  and  $\delta$ .*

*Proof.* Let us suppose by contradiction that there exist a sequence of nonnegative local minimizers  $u_k$  corresponding to functionals  $J_k$  given by

$$J_k(v) = \int_{\Omega} (F_k(x, v, \nabla v) + \lambda_k(x)\chi_{\{v>0\}}) dx,$$

with  $u_k \in W^{1,p_k(\cdot)}(\Omega) \cap L^\infty(\Omega)$ ,  $p_{\min} \leq p_k(x) \leq p_{\max}$ ,  $\|\nabla p_k\|_{L^\infty} \leq L$ ,  $0 \leq \lambda_k(x) \leq \lambda_{\max}$ ,  $\|u_k\|_{L^\infty(\Omega)} \leq M$ , for some  $M \geq 1$ , and points  $\bar{x}_k \in \Omega' \cap \partial\{u_k > 0\}$ , such that

$$\sup_{B_{r_k/4}(\bar{x}_k)} u_k \geq kr_k \quad \text{and} \quad r_k \leq \frac{1}{k}.$$

We denote  $A_k(x, s, \eta) = \nabla_\eta F_k(x, s, \eta)$  and  $B_k(x, s, \eta) = (F_k)_s(x, s, \eta)$  and we also suppose that  $p_k, F_k, A_k, B_k$  and  $\tau_k$  satisfy the assumptions in Section 2 with constants  $\lambda_0, \Lambda_0$  and  $\beta$ , we assume that  $A_k, B_k$  satisfy condition (3.14) and  $F_k$  satisfy (3.8) and (3.9) with  $\tau_k$  satisfying (2.7) and either and  $B_k(x, 0, 0) \equiv 0$  for  $x \in \Omega$  or  $B_k$  satisfy (3.32) for  $x \in \Omega$ . All these conditions with exponent  $p_k$  and constants independent of  $k$  and with  $\|\tau_k\|_{L^\infty(\Omega)} \leq \tau_0$ , for some  $\tau_0 > 0$ .

Without loss of generality we will assume that  $\bar{x}_k = 0$ .

Let us define in  $B_1$ , for  $k$  large,  $\bar{u}_k(x) = \frac{1}{r_k} u_k(r_k x)$ ,  $\bar{p}_k(x) = p_k(r_k x)$  and  $\bar{\lambda}_k(x) = \lambda_k(r_k x)$ . Then  $p_{\min} \leq \bar{p}_k(x) \leq p_{\max}$ ,  $\|\nabla \bar{p}_k\|_{L^\infty(B_1)} \leq L r_k$  and  $0 \leq \bar{\lambda}_k(x) \leq \lambda_{\max}$ . Moreover,  $\bar{u}_k$  is a nonnegative minimizer in  $\bar{u}_k + W_0^{1, \bar{p}_k(\cdot)}(B_1)$  of the functional

$$\bar{J}_k(v) = \int_{B_1} \left( \bar{F}_k(x, v, \nabla v) + \bar{\lambda}_k(x) \chi_{\{v>0\}} \right) dx, \quad (4.18)$$

where

$$\bar{F}_k(x, s, \eta) = F_k(r_k x, r_k s, \eta),$$

with

$$\bar{u}_k(0) = 0 \quad \text{and} \quad \max_{\bar{B}_{1/4}} \bar{u}_k(x) > k.$$

Let  $d_k(x) = \text{dist}(x, \{\bar{u}_k = 0\})$  and  $O_k = \{x \in B_1 : d_k(x) \leq \frac{1-|x|}{3}\}$ . Since  $\bar{u}_k(0) = 0$  then  $\bar{B}_{1/4} \subset O_k$ , therefore

$$m_k := \sup_{O_k} (1-|x|)\bar{u}_k(x) \geq \max_{\bar{B}_{1/4}} (1-|x|)\bar{u}_k(x) \geq \frac{3}{4} \max_{\bar{B}_{1/4}} \bar{u}_k(x) > \frac{3}{4}k.$$

For each fixed  $k$ ,  $\bar{u}_k$  is bounded, then  $(1-|x|)\bar{u}_k(x) \rightarrow 0$  when  $|x| \rightarrow 1$  which means that there exists  $x_k \in O_k$  such that  $(1-|x_k|)\bar{u}_k(x_k) = \sup_{O_k} (1-|x|)\bar{u}_k(x)$ , and then

$$\bar{u}_k(x_k) = \frac{m_k}{1-|x_k|} \geq m_k > \frac{3}{4}k \quad (4.19)$$

as  $x_k \in O_k$ . Observe that  $\delta_k := d_k(x_k) \leq \frac{1-|x_k|}{3}$ . Let  $y_k \in \partial\{\bar{u}_k > 0\} \cap B_1$  such that  $|y_k - x_k| = \delta_k$ . Then,

$$(1) B_{2\delta_k}(y_k) \subset B_1,$$

$$\text{since if } y \in B_{2\delta_k}(y_k) \Rightarrow |y| < 3\delta_k + |x_k| \leq 1,$$

$$(2) B_{\frac{\delta_k}{2}}(y_k) \subset O_k,$$

$$\text{since if } y \in B_{\frac{\delta_k}{2}}(y_k) \Rightarrow |y| \leq \frac{3}{2}\delta_k + |x_k| \leq 1 - \frac{3}{2}\delta_k \Rightarrow d_k(y) \leq \frac{\delta_k}{2} \leq \frac{1-|y|}{3} \quad \text{and}$$

$$(3) \text{ if } z \in B_{\frac{\delta_k}{2}}(y_k) \Rightarrow 1-|z| \geq 1-|x_k| - |x_k - z| \geq 1-|x_k| - \frac{3}{2}\delta_k \geq \frac{1-|x_k|}{2}.$$

By (2) we have

$$\max_{O_k} (1-|x|)\bar{u}_k(x) \geq \max_{B_{\frac{\delta_k}{2}}(y_k)} (1-|x|)\bar{u}_k(x) \geq \max_{B_{\frac{\delta_k}{2}}(y_k)} \frac{(1-|x_k|)}{2} \bar{u}_k(x),$$



where in the last inequality we are using (3). Then,

$$2\bar{u}_k(x_k) \geq \max_{\bar{B}_{\frac{\delta_k}{2}}(y_k)} \bar{u}_k(x). \quad (4.20)$$

As  $B_{\delta_k}(x_k) \subset \{\bar{u}_k > 0\}$ , then  $B_{r_k\delta_k}(r_kx_k) \subset \{u_k > 0\}$ . Hence,  $\operatorname{div} A_k(x, u_k, \nabla u_k) = B_k(x, u_k, \nabla u_k)$  in  $B_{r_k\delta_k}(r_kx_k)$ . Recalling that  $\|u_k\|_{L^\infty(B_{r_k\delta_k}(r_kx_k))} \leq M$ , we can replace  $|s|^{\tau_k(x)}$  in (2.6) for  $B_k$  by  $M^{\tau_0}$ . Then we can apply Harnack's inequality (Theorem 3.2 in [23]) and we thus have

$$\max_{\bar{B}_{\frac{3}{4}r_k\delta_k}(r_kx_k)} u_k(x) \leq C \left[ \min_{\bar{B}_{\frac{3}{4}r_k\delta_k}(r_kx_k)} u_k(x) + r_k\delta_k \right], \quad (4.21)$$

with  $C$  a positive constant depending only on  $N, p_{\min}, p_{\max}, L, M, \lambda_0, \Lambda_0$  and  $\tau_0$ .

It follows that

$$\max_{\bar{B}_{\frac{3}{4}\delta_k}(x_k)} \bar{u}_k(x) \leq C \left[ \min_{\bar{B}_{\frac{3}{4}\delta_k}(x_k)} \bar{u}_k(x) + \delta_k \right]. \quad (4.22)$$

Recalling (4.19), we get from (4.22), for  $k$  large,

$$\min_{\bar{B}_{\frac{3}{4}\delta_k}(x_k)} \bar{u}_k(x) \geq c\bar{u}_k(x_k), \quad (4.23)$$

with  $c$  a positive constant depending only on  $N, p_{\min}, p_{\max}, L, M, \lambda_0, \Lambda_0$  and  $\tau_0$ . As  $\overline{B_{\frac{3}{4}\delta_k}(x_k)} \cap \overline{B_{\frac{\delta_k}{4}}(y_k)} \neq \emptyset$  we have by (4.23)

$$\max_{\bar{B}_{\frac{\delta_k}{4}}(y_k)} \bar{u}_k(x) \geq c\bar{u}_k(x_k). \quad (4.24)$$

Let  $w_k(x) = \frac{\bar{u}_k(y_k + \frac{\delta_k}{2}x)}{\bar{u}_k(x_k)}$ . Then,  $w_k(0) = 1$  and, by (4.20) and (4.24), we have

$$\max_{\bar{B}_1} w_k \leq 2 \quad \max_{\bar{B}_{1/2}} w_k \geq c > 0. \quad (4.25)$$

Now, recalling that  $\bar{u}_k$  is a nonnegative minimizer in  $\bar{u}_k + W_0^{1, \bar{p}_k(\cdot)}(B_1)$  of the functional  $\bar{J}_k$  in (4.18) and that  $B_{\frac{\delta_k}{2}}(y_k) \subset B_1$ , we see that  $w_k$  is a nonnegative minimizer of  $\hat{J}_k$  in  $w_k + W_0^{1, \bar{p}_k(y_k + \frac{\delta_k}{2}x)}(B_1)$ , where

$$\hat{J}_k(v) = \int_{B_1} \left( \hat{F}_k(x, v, \nabla v) + \hat{\lambda}_k(x) \chi_{\{v>0\}} \right) dx,$$

$$\hat{F}_k(x, s, \eta) = \bar{F}_k(y_k + \frac{\delta_k}{2}x, \bar{u}_k(x_k)s, \frac{2\bar{u}_k(x_k)}{\delta_k}\eta) \quad \text{and} \quad \hat{\lambda}_k(x) = \bar{\lambda}_k(y_k + \frac{\delta_k}{2}x).$$

We let  $c_k = \frac{2\bar{u}_k(x_k)}{\delta_k}$  and we notice that  $c_k \rightarrow \infty$ . So we define  $\tilde{p}_k(x) = \bar{p}_k(y_k + \frac{\delta_k}{2}x)$  and divide the functional  $\hat{J}_k$  by  $c_k^{\tilde{p}_k^-}$ , with  $\tilde{p}_k^- = \inf_{B_1} \tilde{p}_k$ . Then, it follows that  $w_k$  is a nonnegative minimizer of  $\tilde{J}_k$  in  $w_k + W_0^{1, \tilde{p}_k(\cdot)}(B_1)$ , where

$$\tilde{J}_k(v) = \int_{B_1} \left( \tilde{F}_k(x, v, \nabla v) + \tilde{\lambda}_k(x) \chi_{\{v>0\}} \right) dx,$$

$$\tilde{F}_k(x, s, \eta) = c_k^{-\tilde{p}_k} \hat{F}_k(x, s, \eta) \quad \text{and} \quad \tilde{\lambda}_k(x) = c_k^{-\tilde{p}_k} \hat{\lambda}_k(x).$$

We claim that

$$\tilde{\lambda}_k \rightarrow 0 \quad \text{uniformly in } B_1, \quad (4.26)$$

$$c_k^{\tilde{p}_k(x) - \tilde{p}_k} \rightarrow 1 \quad \text{uniformly,} \quad 1 \leq c_k^{\tilde{p}_k(x) - \tilde{p}_k} \leq M_1 \quad \text{in } B_1, \quad (4.27)$$

$$\tilde{p}_k \rightarrow p_0 \quad \text{uniformly} \quad \text{and} \quad p_{\min} \leq p_0 \leq p_{\max} \quad \text{in } B_1, \quad (4.28)$$

up to a subsequence, for some constants  $M_1$  and  $p_0$ , where  $M_1 = M_1(M, L)$ .

On the one hand,  $0 < \tilde{\lambda}_k(x) \leq \lambda_{\max} c_k^{-1} \rightarrow 0$  gives (4.26).

In addition, in  $B_1$  there holds, for  $k$  large, that  $1 \leq c_k^{\tilde{p}_k(x) - \tilde{p}_k} \leq e^{2\|\nabla \tilde{p}_k\|_{L^\infty} \log c_k}$ . But we have  $\|\nabla \tilde{p}_k\|_{L^\infty} \log c_k \leq L r_k \frac{\delta_k}{2} \log \left( \frac{2M}{r_k \delta_k} \right) \rightarrow 0$ , which implies (4.27).

To see (4.28) we observe that  $p_{\min} \leq p_k(x) \leq p_{\max}$  and  $\|\nabla p_k\|_{L^\infty(\Omega)} \leq L$  and then, for a subsequence,  $p_k \rightarrow p$  uniformly on compacts of  $\Omega$ , so  $\tilde{p}_k(x) = p_k(r_k(y_k + \frac{\delta_k}{2}x)) \rightarrow p_0 = p(0)$  uniformly in  $B_1$ .

We define  $\tilde{A}_k = \nabla_\eta \tilde{F}_k$  and  $\tilde{B}_k = (\tilde{F}_k)_s$  and we observe that

$$\begin{aligned} \tilde{p}_k(x) &= p_k(r_k(y_k + \frac{\delta_k}{2}x)), \quad \tilde{\tau}_k(x) = \tau_k(r_k(y_k + \frac{\delta_k}{2}x)), \\ \tilde{F}_k(x, s, \eta) &= c_k^{-\tilde{p}_k} \hat{F}_k(x, s, \eta) = c_k^{-\tilde{p}_k} \bar{F}_k(y_k + \frac{\delta_k}{2}x, \bar{u}_k(x_k)s, \frac{2\bar{u}_k(x_k)}{\delta_k}\eta) \\ &= c_k^{-\tilde{p}_k} F_k(r_k(y_k + \frac{\delta_k}{2}x), r_k\bar{u}_k(x_k)s, c_k\eta), \\ \tilde{A}_k(x, s, \eta) &= c_k^{-\tilde{p}_k} c_k A_k(r_k(y_k + \frac{\delta_k}{2}x), r_k\bar{u}_k(x_k)s, c_k\eta), \\ \tilde{B}_k(x, s, \eta) &= c_k^{-\tilde{p}_k} r_k\bar{u}_k(x_k) B_k(r_k(y_k + \frac{\delta_k}{2}x), r_k\bar{u}_k(x_k)s, c_k\eta). \end{aligned}$$

There holds that  $\tilde{p}_k$ ,  $\tilde{F}_k$ ,  $\tilde{A}_k$ ,  $\tilde{B}_k$  and  $\tilde{\tau}_k$  are under the assumptions of Section 2, with constants independent of  $k$ . In fact, recalling (4.27), we get for  $k$  large

$$\begin{aligned} p_{\min} &\leq \tilde{p}_k(x) \leq p_{\max}, \quad \|\nabla \tilde{p}_k\|_{L^\infty(\Omega)} \leq L, \quad \tilde{p}_k(x) \leq \tilde{\tau}_k(x) \leq \tau_0, \\ \tilde{A}_k(x, s, 0) &= 0, \\ \sum_{i,j} \frac{\partial(\tilde{A}_k)_i}{\partial \eta_j}(x, s, \eta) \xi_i \xi_j &= c_k^{-\tilde{p}_k} c_k^2 \sum_{i,j} \frac{\partial(A_k)_i}{\partial \eta_j}(r_k(y_k + \frac{\delta_k}{2}x), r_k\bar{u}_k(x_k)s, c_k\eta) \xi_i \xi_j \\ &\geq \lambda_0 c_k^{\tilde{p}_k(x) - \tilde{p}_k} |\eta|^{\tilde{p}_k(x) - 2} |\xi|^2 \geq \lambda_0 |\eta|^{\tilde{p}_k(x) - 2} |\xi|^2, \\ \sum_{i,j} \left| \frac{\partial(\tilde{A}_k)_i}{\partial \eta_j}(x, s, \eta) \right| &= c_k^{-\tilde{p}_k} c_k^2 \sum_{i,j} \left| \frac{\partial(A_k)_i}{\partial \eta_j}(r_k(y_k + \frac{\delta_k}{2}x), r_k\bar{u}_k(x_k)s, c_k\eta) \right| \\ &\leq \Lambda_0 c_k^{\tilde{p}_k(x) - \tilde{p}_k} |\eta|^{\tilde{p}_k(x) - 2} \leq \Lambda_0 M_1 |\eta|^{\tilde{p}_k(x) - 2}. \end{aligned} \quad (4.29)$$

Assuming, without loss of generality, that  $\tilde{p}_k(x_1) \geq \tilde{p}_k(x_2)$  and using that  $(r_k \frac{\delta_k}{2})^{\beta_1} \log c_k \leq (r_k \frac{\delta_k}{2})^{\beta_1} \log \left( \frac{2M}{r_k \delta_k} \right) \rightarrow 0$ , we get

$$\begin{aligned} |\tilde{A}_k(x_1, s, \eta) - \tilde{A}_k(x_2, s, \eta)| &\leq c_k^{-\tilde{p}_k} c_k \Lambda_0 (r_k \frac{\delta_k}{2})^{\beta_1} |x_1 - x_2|^\beta (|c_k \eta|^{\tilde{p}_k(x_1) - 1} + |c_k \eta|^{\tilde{p}_k(x_2) - 1}) \\ (1 + |\log |c_k \eta||) &\leq 2M_1 \Lambda_0 |x_1 - x_2|^\beta (|\eta|^{\tilde{p}_k(x_1) - 1} + |\eta|^{\tilde{p}_k(x_2) - 1}) (1 + |\log |\eta||). \end{aligned} \quad (4.31)$$

Finally, recalling that  $r_k \bar{u}_k(x_k) \leq M$ , we obtain

$$\begin{aligned} |\tilde{A}_k(x, s_1, \eta) - \tilde{A}_k(x, s_2, \eta)| &\leq c_k^{-\tilde{p}_k} c_k \Lambda_0 r_k \bar{u}_k(x_k) |s_1 - s_2| c_k \eta^{|\tilde{p}_k(x)-1|} \\ &\leq \Lambda_0 M_1 M |s_1 - s_2| |\eta|^{\tilde{p}_k(x)-1}, \end{aligned} \quad (4.32)$$

$$\begin{aligned} |\tilde{B}_k(x, s, \eta)| &\leq c_k^{-\tilde{p}_k} r_k \bar{u}_k(x_k) \Lambda_0 (1 + |c_k \eta|^{\tilde{p}_k(x)} + |r_k \bar{u}_k(x_k) s|^{\tilde{\tau}_k(x)}) \\ &\leq M \Lambda_0 (c_k^{-\tilde{p}_k} + M_1 |\eta|^{\tilde{p}_k(x)} + c_k^{-\tilde{p}_k} |M s|^{\tilde{\tau}_k(x)}) \leq M_1 M \Lambda_0 (1 + |\eta|^{\tilde{p}_k(x)} + M^{\tau_0} |s|^{\tilde{\tau}_k(x)}). \end{aligned} \quad (4.33)$$

On the other hand,  $\tilde{A}_k$  and  $\tilde{B}_k$  satisfy condition (3.14). In fact, since  $A_k$  and  $B_k$  satisfy condition (3.14),

$$\begin{aligned} &\frac{1}{2} \sum_{i,j} \frac{\partial(\tilde{A}_k)_i}{\partial \eta_j}(x, s, \eta) \xi_i \xi_j + (\tilde{B}_k)_s(x, s, \eta) w^2 \\ &= \frac{1}{2} c_k^{-\tilde{p}_k} c_k^2 \sum_{i,j} \frac{\partial(A_k)_i}{\partial \eta_j}(r_k(y_k + \frac{\delta_k}{2}x), r_k \bar{u}_k(x_k) s, c_k \eta) \xi_i \xi_j \\ &\quad + c_k^{-\tilde{p}_k} (r_k \bar{u}_k(x_k))^2 (B_k)_s(r_k(y_k + \frac{\delta_k}{2}x), r_k \bar{u}_k(x_k) s, c_k \eta) w^2 \\ &\geq c_k^{-\tilde{p}_k} 2 |(A_k)_s(r_k(y_k + \frac{\delta_k}{2}x), r_k \bar{u}_k(x_k) s, c_k \eta) \cdot (c_k \xi)(r_k \bar{u}_k(x_k) w)| \\ &= 2 |(\tilde{A}_k)_s(x, s, \eta) \cdot \xi w|. \end{aligned}$$

Also, since  $F_k$  satisfy (3.8) and (3.9) with  $\tau_k$  satisfying (2.7), with exponent  $p_k$  and constants independent of  $k$ , then  $\tilde{F}_k$  satisfy (3.8) and (3.9) with  $\tilde{\tau}_k$  satisfying (2.7), with exponent  $\tilde{p}_k$  and constants independent of  $k$ . In fact,

$$\begin{aligned} \tilde{F}_k(x, s, \eta) &\geq c_k^{-\tilde{p}_k} \mu |c_k \eta|^{\tilde{p}_k(x)} - c_k^{-\tilde{p}_k} c_1 (|r_k \bar{u}_k(x_k) s|^{\tilde{p}_k(x)-\delta} + 1) \\ &\geq \mu |\eta|^{\tilde{p}_k(x)} - c_1 M^{p_{\max}} (|s|^{\tilde{p}_k(x)-\delta} + 1). \end{aligned} \quad (4.34)$$

Analogously,

$$\begin{aligned} \tilde{F}_k(x, s, \eta) &\leq c_k^{-\tilde{p}_k} \mu^{-1} |c_k \eta|^{\tilde{p}_k(x)} + c_k^{-\tilde{p}_k} c_1 (|r_k \bar{u}_k(x_k) s|^{\tilde{\tau}_k(x)} + 1) \\ &\leq M_1 \mu^{-1} |\eta|^{\tilde{p}_k(x)} + c_1 M^{\tau_0} (|s|^{\tilde{\tau}_k(x)} + 1). \end{aligned} \quad (4.35)$$

If  $B_k(x, 0, 0) \equiv 0$  for  $x \in \Omega$ , then  $\tilde{B}_k(x, 0, 0) \equiv 0$  for  $x \in B_1$ .

On the other hand, if  $B_k$  satisfy (3.32) for  $x \in \Omega$  with exponent  $p_k$  and constant independent of  $k$ , then  $\tilde{B}_k$  satisfy (3.32) for  $x \in B_1$  with exponent  $\tilde{p}_k$  and constant independent of  $k$ . In fact,

$$\begin{aligned} |\tilde{B}_k(x, s, \eta)| &= c_k^{-\tilde{p}_k} r_k \bar{u}_k(x_k) |B_k(r_k(y_k + \frac{\delta_k}{2}x), r_k \bar{u}_k(x_k) s, c_k \eta)| \\ &\leq c_k^{-\tilde{p}_k} r_k \bar{u}_k(x_k) \Lambda_0 (1 + |r_k \bar{u}_k(x_k) s|^{\tilde{p}_k(x)-1} + |c_k \eta|^{\tilde{p}_k(x)-1}) \\ &\leq c_k^{-1} M_1 M^{p_{\max}} \Lambda_0 (1 + |s|^{\tilde{p}_k(x)-1} + |\eta|^{\tilde{p}_k(x)-1}) \\ &\leq M_1 M^{p_{\max}} \Lambda_0 (1 + |s|^{\tilde{p}_k(x)-1} + |\eta|^{\tilde{p}_k(x)-1}). \end{aligned} \quad (4.36)$$

We now take  $v_k$  the solution of

$$\operatorname{div} \tilde{A}_k(x, v_k, \nabla v_k) = \tilde{B}_k(x, v_k, \nabla v_k) \quad \text{in } B_{3/4}, \quad v_k - w_k \in W_0^{1, \tilde{p}_k(\cdot)}(B_{3/4}). \quad (4.37)$$

In fact, from Corollaries 3.3, 3.4 and 3.2 and the upper bound in (4.25), it follows that if  $k$  is large enough

$$\|v_k\|_{L^\infty(B_{3/4})} \leq \bar{C}, \quad (4.38)$$

where  $\bar{C}$  depends only on  $N, p_{\min}, p_{\max}, L, \lambda_0, \Lambda_0, \mu, c_1, \delta, M$  and  $\tau_0$ . Here we have used that  $\sup_{B_{3/4}} \tilde{p}_k - \inf_{B_{3/4}} \tilde{p}_k \leq \|\nabla \tilde{p}_k\|_{L^\infty} \frac{3}{2} \leq 3Lr_k \frac{\delta_k}{4} < \delta$  in (3.8), for  $k$  large.

Then, by (4.38), we can replace  $|s|^{\tilde{\tau}_k(x)}$  in (4.33) by  $1 + \bar{C}^{\tau_0}$  and applying Theorem 1.1 in [10] we obtain that, for  $k$  large,

$$\|v_k\|_{C^{1,\alpha}(\overline{B_{1/2}})} \leq \hat{C} \quad \text{with} \quad 0 < \alpha < 1 \quad (4.39)$$

where  $\hat{C}$  depends only on  $\beta, N, p_{\min}, p_{\max}, L, \lambda_0, \Lambda_0, \mu, c_1, \delta, M$  and  $\tau_0$ . Therefore, there is a function  $v_0 \in C^{1,\alpha}(\overline{B_{1/2}})$  such that, for a subsequence,

$$v_k \rightarrow v_0 \quad \text{and} \quad \nabla v_k \rightarrow \nabla v_0 \quad \text{uniformly in } \overline{B_{1/2}}. \quad (4.40)$$

Let us now show that

$$w_k - v_k \rightarrow 0 \quad \text{in } L^{p_{\min}}(B_{3/4}). \quad (4.41)$$

From the minimality of  $w_k$  we have

$$\int_{B_{3/4}} \tilde{F}_k(x, w_k \nabla w_k) - \tilde{F}_k(x, v_k \nabla v_k) \leq C(N) \|\tilde{\lambda}_k\|_{L^\infty(B_{3/4})}, \quad (4.42)$$

which together with Proposition 3.3 gives

$$\int_{A_2^k} |\nabla w_k - \nabla v_k|^{\tilde{p}_k(x)} dx \leq C \|\tilde{\lambda}_k\|_{L^\infty(B_{3/4})}, \quad (4.43)$$

$$\int_{A_1^k} |\nabla w_k - \nabla v_k|^2 (|\nabla w_k| + |\nabla v_k|)^{\tilde{p}_k(x)-2} dx \leq C \|\tilde{\lambda}_k\|_{L^\infty(B_{3/4})}, \quad (4.44)$$

where  $A_1^k = B_{3/4} \cap \{\tilde{p}_k(x) < 2\}$ ,  $A_2^k = B_{3/4} \cap \{\tilde{p}_k(x) \geq 2\}$  and  $C = C(p_{\min}, p_{\max}, N, \lambda_0)$ .

Applying Hölder's inequality (Theorem A.3) with exponents  $\frac{2}{\tilde{p}_k(x)}$  and  $\frac{2}{2-\tilde{p}_k(x)}$ , we get

$$\int_{A_1^k} |\nabla w_k - \nabla v_k|^{\tilde{p}_k(x)} dx \leq 2 \|G_k^a\|_{L^{2/\tilde{p}_k(\cdot)}(A_1^k)} \|G_k^b\|_{L^{2/(2-\tilde{p}_k(\cdot))}(A_1^k)}, \quad (4.45)$$

where

$$\begin{aligned} G_k^a &= |\nabla w_k - \nabla v_k|^{\tilde{p}_k} (|\nabla w_k| + |\nabla v_k|)^{(\tilde{p}_k-2)\tilde{p}_k/2} \\ G_k^b &= (|\nabla w_k| + |\nabla v_k|)^{(2-\tilde{p}_k)\tilde{p}_k/2}. \end{aligned}$$

Since

$$\int_{A_1^k} |G_k^a|^{2/\tilde{p}_k(x)} dx = \int_{A_1^k} |\nabla w_k - \nabla v_k|^2 (|\nabla w_k| + |\nabla v_k|)^{\tilde{p}_k(x)-2} dx,$$

then, from (4.44), (4.26) and Proposition A.1, we get, for  $k$  large,

$$\|G_k^a\|_{L^{2/\tilde{p}_k(\cdot)}(A_1^k)} \leq C \|\tilde{\lambda}_k\|_{L^\infty(B_{3/4})}^{p_{\min}/2}, \quad (4.46)$$

$C = C(p_{\min}, p_{\max}, N, \lambda_0)$ . On the other hand, (4.37) and the bounds (4.34), (4.35) and (4.38) give

$$\begin{aligned} C_1 \int_{B_{3/4}} |\nabla v_k|^{\tilde{p}_k(x)} &\leq \int_{B_{3/4}} \tilde{F}_k(x, v_k \nabla v_k) + C_2 \\ &\leq \int_{B_{3/4}} \tilde{F}_k(x, w_k \nabla w_k) + C_2 \\ &\leq C(1 + \int_{B_{3/4}} |\nabla w_k|^{\tilde{p}_k(x)}). \end{aligned}$$

This implies

$$\int_{A_1^k} |G_k^b|^{2/(2-\tilde{p}_k(x))} dx \leq C \int_{B_{3/4}} (|\nabla w_k|^{\tilde{p}_k(x)} + |\nabla v_k|^{\tilde{p}_k(x)}) dx \leq \tilde{C}(1 + \int_{B_{3/4}} |\nabla w_k|^{\tilde{p}_k(x)}), \quad (4.47)$$

for some  $\tilde{C} \geq 1$ , depending only on  $p_{\min}$ ,  $p_{\max}$  and the uniform constants and functions in (4.34), (4.35) and (4.38). Now (4.47) and Proposition A.1 give

$$\|G_k^b\|_{L^{2/(2-\tilde{p}_k(\cdot))}(A_1^k)} \leq \tilde{C}(1 + \int_{B_{3/4}} |\nabla w_k|^{\tilde{p}_k(x)}). \quad (4.48)$$

Let us show that the right hand side in (4.48) can be bounded independently of  $k$ .

In fact, let  $\tilde{v}_k$  be the solution of

$$\operatorname{div} \tilde{A}_k(x, \tilde{v}_k, \nabla \tilde{v}_k) = \tilde{B}_k(x, \tilde{v}_k, \nabla \tilde{v}_k) \quad \text{in } B_{7/8}, \quad \tilde{v}_k - w_k \in W_0^{1, \tilde{p}_k(\cdot)}(B_{7/8}). \quad (4.49)$$

Then, similar arguments to those leading to (4.38) and (4.39), give, for  $k$  large enough,

$$\|\tilde{v}_k\|_{L^\infty(B_{7/8})} \leq \bar{C}, \quad (4.50)$$

and

$$\|\tilde{v}_k\|_{C^{1, \alpha}(\overline{B_{3/4}})} \leq \hat{C} \quad \text{with} \quad 0 < \alpha < 1, \quad (4.51)$$

where  $\bar{C}$  and  $\hat{C}$  depend only  $\beta$ ,  $N$ ,  $p_{\min}$ ,  $p_{\max}$ ,  $L$ ,  $\lambda_0$ ,  $\Lambda_0$ ,  $\mu$ ,  $c_1$ ,  $\delta$ ,  $M$  and  $\tau_0$ .

Since  $w_k$  is a nonnegative minimizer of  $\tilde{J}_k$  in  $B_1$ , then we can argue as in the proof of Theorem 4.3 and get estimate (4.10) for  $u = w_k$ ,  $v = \tilde{v}_k$ ,  $p(x) = \tilde{p}_k(x)$ ,  $\lambda(x) = \tilde{\lambda}_k(x)$ ,  $r = 7/8$  and  $\rho = 3/4$ . That is,

$$\int_{B_{3/4}} |\nabla w_k|^{\tilde{p}_k(x)} dx \leq C + C \int_{B_{3/4}} |\nabla \tilde{v}_k|^{\tilde{p}_k(x)} dx, \quad (4.52)$$

where  $C = C(p_{\min}, p_{\max}, N, \lambda_{\max}, \lambda_0)$ . Therefore (4.52) and (4.51) give, for  $k$  large, a uniform bound for the right hand side in (4.48). That is,

$$\|G_k^b\|_{L^{2/(2-\tilde{p}_k(\cdot))}(A_1^k)} \leq \bar{C}, \quad (4.53)$$

with  $\bar{C}$  a constant depending only on  $\beta$ ,  $N$ ,  $p_{\min}$ ,  $p_{\max}$ ,  $L$ ,  $\lambda_0$ ,  $\Lambda_0$ ,  $\mu$ ,  $c_1$ ,  $\delta$ ,  $M$  and  $\tau_0$ .

Now, putting together (4.43), (4.45), (4.46), (4.53) and (4.26), we obtain

$$\int_{B_{3/4}} |\nabla w_k - \nabla v_k|^{\tilde{p}_k(x)} \rightarrow 0. \quad (4.54)$$

Thus, using Poincaré's inequality (Theorem A.4) and Theorem A.2, we get (4.41).

In order to conclude the proof, we now observe that, since  $\tilde{p}_k, \tilde{F}_k, \tilde{A}_k, \tilde{B}_k, \tilde{\tau}_k, \tilde{\lambda}_k$  and  $w_k$  fall (uniformly) under the assumption of Corollary 4.1 in  $B_1$ , there exists  $0 < \gamma < 1$ ,  $\gamma = \gamma(N, p_{\min})$ , such that

$$\|w_k\|_{C^\gamma(\overline{B_{1/2}})} \leq C$$

with  $C$  depending only on  $\beta, N, p_{\min}, p_{\max}, L, \lambda_{\max}, \lambda_0, \Lambda_0, \mu, c_1, \tau_0$  and  $\delta$  (recall that  $\|w_k\|_{L^\infty(B_1)} \leq 2$ ).

Therefore, there is a function  $w_0 \in C^\gamma(\overline{B_{1/2}})$  such that, for a subsequence,

$$w_k \rightarrow w_0 \quad \text{uniformly in } \overline{B_{1/2}}. \quad (4.55)$$

In addition, recalling (4.40) and (4.41), we get  $v_0 = w_0$  in  $\overline{B_{1/2}}$ .

We then observe that, since there holds that  $w_k \geq 0$ ,  $w_k(0) = 0$  and (4.25), then (4.55) implies

$$w_0 \geq 0, \quad w_0(0) = 0, \quad \max_{\overline{B_{1/2}}} w_0 \geq c > 0.$$

That is,

$$v_0 \geq 0, \quad v_0(0) = 0, \quad \max_{\overline{B_{1/2}}} v_0 \geq c > 0. \quad (4.56)$$

Let us show that (4.56) gives a contradiction. We will divide the proof in two cases.

*Case I.* Assume that  $\tilde{B}_k(x, 0, 0) \equiv 0$  for  $x \in B_1$ .

We first observe that, since  $w_k \geq 0$ , from Proposition 3.6 we deduce that  $v_k \geq 0$ .

Recalling (4.39), we choose  $M_0 > 0$  such that, for every  $k$ ,

$$\|v_k\|_{L^\infty(B_{1/2})} \leq M_0, \quad \|\nabla v_k\|_{L^\infty(B_{1/2})} \leq M_0,$$

and define

$$\begin{aligned} \tilde{\tilde{A}}_k(x, s, \eta) &= a(s, \eta)\tilde{A}_k(x, s, \eta) + (1 - a(s, \eta))|\eta|^{p_0-2}\eta, \\ \tilde{\tilde{B}}_k(x, s, \eta) &= a(s, \eta)\tilde{B}_k(x, s, \eta), \end{aligned}$$

where

$$a(s, \eta) = \chi_{\{|s| \leq M_0, |\eta| \leq M_0\}}.$$

Then,

$$\operatorname{div} \tilde{\tilde{A}}_k(x, v_k, \nabla v_k) = \tilde{\tilde{B}}_k(x, v_k, \nabla v_k) \quad \text{in } B_{1/2}.$$

From (4.29) and (4.30) (recall Remark 2.1) we deduce

$$\begin{aligned} |\tilde{A}_k(x, s, \eta)| &\leq \tilde{\Lambda}_0 |\eta|^{\tilde{p}_k(x)-1}, \\ \tilde{A}_k(x, s, \eta) \cdot \eta &\geq \tilde{\Lambda}_0^{-1} |\eta|^{\tilde{p}_k(x)}, \end{aligned} \quad (4.57)$$

for some constant  $\tilde{\Lambda}_0 > 0$  independent of  $k$ .

Let us now fix  $\varepsilon > 0$ . Then, if  $k \geq k_0(\varepsilon)$ , (4.57), (4.33) and (4.28) give, for large  $k$ ,

$$\begin{aligned} |\tilde{\tilde{A}}_k(x, s, \eta)| &\leq \tilde{\Lambda}_0 |\eta|^{p_0-1} + c\varepsilon, \\ \tilde{\tilde{A}}_k(x, s, \eta) \cdot \eta &\geq \tilde{\Lambda}_0^{-1} |\eta|^{p_0} - c\varepsilon, \\ |\tilde{\tilde{B}}_k(x, s, \eta)| &\leq \tilde{\Lambda}_0 |\eta|^{p_0-1} + c\varepsilon, \end{aligned}$$

for some positive constants  $\tilde{\Lambda}_0$  and  $c$  (independent of  $\varepsilon$  and  $k$ ).

Applying Harnack's inequality (see [22], Theorems 5 and 6 and Section 5), we get for any  $0 < r < 1$

$$\max_{\frac{B_{r/2}}{B_{r/2}}} v_k \leq C_r \left( \min_{\frac{B_{r/2}}{B_{r/2}}} v_k + \varepsilon^{\frac{1}{p_0}} \right),$$

with  $C_r$  a positive constant.

Now, letting  $k \rightarrow \infty$  first, and then  $\varepsilon \rightarrow 0$ , we get

$$\max_{\frac{B_{r/2}}{B_{r/2}}} v_0 \leq C_r \min_{\frac{B_{r/2}}{B_{r/2}}} v_0, \quad (4.58)$$

with

$$v_0 \geq 0, \quad v_0(0) = 0. \quad (4.59)$$

Since  $0 < r < 1$  is arbitrary, we get  $v_0 \equiv 0$  in  $B_{1/2}$ . This is in contradiction with (4.56) and concludes the proof of *Case I*.

*Case II.* Assume that  $\tilde{B}_k$  satisfy (3.32) for  $x \in B_1$  with exponent  $\tilde{p}_k$  and constant independent of  $k$ . Then, (4.30), (4.31), (4.32) and (4.36) imply that, for a subsequence,

$$\begin{aligned} \tilde{A}_k &\rightarrow \tilde{A} \quad \text{uniformly on compacts of } B_{1/2} \times \mathbb{R} \times \mathbb{R}^N \setminus \{0\} \text{ and pointwise on } B_{1/2} \times \mathbb{R} \times \mathbb{R}^N, \\ \tilde{B}_k &\rightarrow 0 \quad \text{uniformly on compacts of } B_{1/2} \times \mathbb{R} \times \mathbb{R}^N, \end{aligned}$$

and from (4.29) and (4.30) (recall Remark 2.1) we deduce

$$\begin{aligned} |\tilde{A}(x, s, \eta)| &\leq \tilde{\Lambda}_0 |\eta|^{p_0-1}, \\ \tilde{A}(x, s, \eta) \cdot \eta &\geq \tilde{\Lambda}_0^{-1} |\eta|^{p_0}, \end{aligned}$$

for some constant  $\tilde{\Lambda}_0 > 0$ . Then, (4.37) and (4.40) imply that

$$\operatorname{div} \tilde{A}(x, v_0, \nabla v_0) = 0 \quad \text{in } B_{1/2}.$$

Applying Harnack's inequality (see [22], Theorems 5 and 6 and Section 5), we get again, that (4.58) and (4.59) holds for any  $0 < r < 1$ . This contradicts once more (4.56) and concludes the proof.  $\square$

We can now prove the Lipschitz continuity of nonnegative local minimizers of (1.1)

**Theorem 4.5.** *Let  $p, F, A, B, \lambda$  and  $u$  be as in Corollary 4.1. Then  $u$  is locally Lipschitz continuous in  $\Omega$ . Moreover, for any  $\Omega' \subset \subset \Omega$  the Lipschitz constant of  $u$  in  $\Omega'$  can be estimated by a constant  $C$  depending only on  $\operatorname{dist}(\Omega', \partial\Omega)$ ,  $\beta$ ,  $N$ ,  $p_{\min}$ ,  $p_{\max}$ ,  $L$ ,  $\lambda_{\max}$ ,  $\lambda_0$ ,  $\Lambda_0$ ,  $\mu$ ,  $c_1$ ,  $\|u\|_{L^\infty(\Omega)}$ ,  $\|\tau\|_{L^\infty(\Omega)}$  and  $\delta$ .*

*Proof.* The result is a consequence of Corollary 4.1, Lemma 4.2 and Theorem 4.4 above, and Proposition 2.1 in [16]. We point out that, although the proof of Proposition 2.1 in [16] is written for the particular case in which  $A(x, s, \eta) = |\eta|^{p(x)-2}\eta$  and  $B(x, s, \eta) = f(x)$ , this same proof is valid for general  $A$  and  $B$  under the present assumptions, without changes.  $\square$

## 5. Examples

In this section we present some examples of application of our results.

**Theorem 5.1.** *Let  $f(x, s)$  be a measurable function such that  $f(x, \cdot) \in C^2(\mathbb{R})$  for every  $x \in \Omega$ . Let  $a(x, s)$  be a Hölder continuous function with exponent  $\alpha$ ,  $a(x, \cdot) \in C^2(\mathbb{R})$  for every  $x \in \Omega$ . Let  $p$ ,  $\tau$  and  $\lambda$  as in Section 2 and  $0 < \delta < p_{\min}$ . Assume that there exist positive constants  $a_0, a_1, a_2, c_1$  and  $\Lambda_0$  such that*

$$f1 \quad -c_1(1 + |s|^{p(x)-\delta}) \leq f(x, s) \leq c_1(1 + |s|^{\tau(x)}) \text{ in } \Omega \times \mathbb{R}.$$

$$f2 \quad f_s(x, 0) \equiv 0 \text{ in } \Omega.$$

$$f3 \quad f_{ss}(x, s) \geq 0 \text{ in } \Omega \times \mathbb{R}.$$

$$f4 \quad |f_s(x, s)| \leq \Lambda_0(1 + |s|^{\tau(x)}) \text{ in } \Omega \times \mathbb{R}.$$

And

$$a1 \quad 0 < a_0 \leq a(x, s) \leq a_1 < \infty \text{ in } \Omega \times \mathbb{R}.$$

$$a2 \quad |a_s(x, s)| \leq a_2 \text{ in } \Omega \times \mathbb{R}.$$

$$a3 \quad (a(x, s)^{1-\gamma(x)})_{ss} \leq 0 \text{ in } \Omega \times \mathbb{R} \text{ with } \gamma(x) = \frac{2p(x)}{\min\{1, p(x)-1\}} > 1.$$

Let

$$F(x, s, \eta) = a(x, s) \frac{|\eta|^{p(x)}}{p(x)} + f(x, s)$$

and let  $u \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$  a nonnegative, local minimizer of (1.1). Then,  $u$  is locally Lipschitz continuous in  $\Omega$ .

*Proof.* We only have to see that  $F, A, B$  satisfy the hypotheses of Theorem 4.5.

There holds that

$$A(x, s, \eta) = a(x, s)|\eta|^{p(x)-2}\eta, \quad B(x, s, \eta) = a_s(x, s) \frac{|\eta|^{p(x)}}{p(x)} + f_s(x, s).$$

And

$$\frac{a_0}{p_{\max}} |\eta|^{p(x)} - c_1(1 + |s|^{p(x)-\delta}) \leq F(x, s, \eta) \leq \frac{a_1}{p_{\min}} |\eta|^{p(x)} + c_1(1 + |s|^{\tau(x)}).$$

Moreover,

$$(1). \quad A(x, s, 0) = 0.$$

$$(2). \quad \sum_{i,j} \frac{\partial A_i}{\partial \eta_j}(x, s, \eta) \xi_i \xi_j \geq \lambda_0 |\eta|^{p(x)-2} |\xi|^2. \text{ In fact,}$$

$$\begin{aligned} \sum_{i,j} \frac{\partial A_i}{\partial \eta_j}(x, s, \eta) \xi_i \xi_j &= a(x, s) \left[ (p(x) - 2) |\eta|^{p(x)-4} \langle \eta, \xi \rangle^2 + |\eta|^{p(x)-2} |\xi|^2 \right] \\ &\geq a(x, s) \min\{1, p(x) - 1\} |\eta|^{p(x)-2} |\xi|^2 \geq \lambda_0 |\eta|^{p(x)-2} |\xi|^2 \end{aligned} \quad (5.1)$$

with  $\lambda_0 = a_0 \min\{1, p_{\min} - 1\}$ .

$$(3). \quad \sum_{i,j} \left| \frac{\partial A_i}{\partial \eta_j}(x, s, \eta) \right| \leq \Lambda_0 |\eta|^{p(x)-2} \text{ if } \Lambda_0 \geq a_1 N(p_{\max} + 3).$$



- (4).  $|A(x_1, s, \eta) - A(x_2, s, \eta)| \leq \Lambda_0 |x_1 - x_2|^\alpha (|\eta|^{p(x_1)-1} + |\eta|^{p(x_2)-1}) (1 + |\log |\eta||)$  for a big enough constant  $\Lambda_0$ . In fact, without loss of generality we may assume that  $p(x_1) \geq p(x_2)$ . There holds,

$$|A(x_1, s, \eta) - A(x_2, s, \eta)| \leq a(x_1, s) \left| |\eta|^{p(x_1)-1} - |\eta|^{p(x_2)-1} \right| + |a(x_1, s) - a(x_2, s)| |\eta|^{p(x_2)-1}.$$

Now, if  $|\eta| \geq 1$ ,

$$\left| |\eta|^{p(x_1)-1} - |\eta|^{p(x_2)-1} \right| \leq L |x_1 - x_2| |\eta|^{p(x_1)-1} |\log |\eta|| \leq L |x_1 - x_2| (|\eta|^{p(x_1)-1} + |\eta|^{p(x_2)-1}) |\log |\eta||.$$

A similar inequality holds if  $|\eta| \leq 1$ . So that,

$$|A(x_1, s, \eta) - A(x_2, s, \eta)| \leq a_1 L |x_1 - x_2| (|\eta|^{p(x_1)-1} + |\eta|^{p(x_2)-1}) |\log |\eta|| + C_a |x_1 - x_2|^\alpha |\eta|^{p(x_2)-1},$$

where  $C_a$  is the Holder constant of the function  $a$ . And the result follows if  $\Lambda_0 \geq a_1 L d(\Omega)^{1-\alpha} + C_a$  with  $d(\Omega)$  the diameter of  $\Omega$ .

- (5).  $|A(x, s_1, \eta) - A(x, s_2, \eta)| \leq a_2 |\eta|^{p(x)-1} |s_1 - s_2|$ .

We clearly have,

- (1).  $|B(x, s, \eta)| \leq \Lambda_0 (1 + |\eta|^{p(x)} + |s|^{\tau(x)})$  (as we may assume, without loss of generality that  $\Lambda_0 \geq \frac{a_2}{p_{\min}}$ ).  
 (2).  $B(x, 0, 0) = 0$ .

Finally, let us see that

$$2|A_s(x, s, \eta) \cdot \xi w| \leq \frac{1}{2} \sum_{i,j} \frac{\partial A_i}{\partial \eta_j}(x, s, \eta) \xi_i \xi_j + B_s(x, s, \eta) w^2.$$

In fact, let

$$\ell(x) = \frac{p(x) - 2}{2(p(x) - 1)} \quad \varepsilon(x, s) = a(x, s) \min\{1, p(x) - 1\}.$$

Then,

$$\begin{aligned} 2|A_s(x, s, \eta) \cdot \xi w| &\leq \left( \sqrt{\varepsilon(x, s)} |\eta|^{\ell(x)(p(x)-1)} |\xi| \right) \left( \frac{2}{\sqrt{\varepsilon(x, s)}} |a_s(x, s)| |\eta|^{(1-\ell(x))(p(x)-1)} |w| \right) \\ &\leq \frac{\varepsilon(x, s)}{2} |\eta|^{p(x)-2} |\xi|^2 + \frac{2}{\varepsilon(x, s)} a_s(x, s)^2 |\eta|^{p(x)} w^2 \\ &= \frac{1}{2} a(x, s) \min\{1, p(x) - 1\} |\eta|^{p(x)-2} |\xi|^2 + \frac{2a_s(x, s)^2}{a(x, s) \min\{1, p(x) - 1\}} |\eta|^{p(x)} w^2. \end{aligned}$$

By (5.1), we only have to check that

$$B_s(x, s, \eta) \geq \frac{2a_s(x, s)^2}{a(x, s) \min\{1, p(x) - 1\}} |\eta|^{p(x)}.$$

Since  $f_{ss}(x, s) \geq 0$  it is enough to check that

$$a_{ss}(x, s) \geq \gamma(x) \frac{a_s(x, s)^2}{a(x, s)} \quad \text{with} \quad \gamma(x) = \frac{2p(x)}{\min\{1, p(x) - 1\}} > 1. \quad (5.2)$$

And, (5.2) holds by hypothesis a3. □

If  $a(x, s)$  is smooth in  $-M_1 < s < M_2$  with  $M_1, M_2 > 0$ , condition a3 only holds in  $0 \leq s \leq M < M_2$  and the local minimizer  $u$  satisfies that  $0 \leq u \leq M$ , we can still apply the results in this paper and get that  $u$  is locally Lipschitz continuous.

**Theorem 5.2.** Let  $f(x, s)$  be a measurable function such that  $f(x, \cdot) \in C^2(\mathbb{R})$  for every  $x \in \Omega$ . Let  $a(x, s)$  be a Hölder continuous function with exponent  $\alpha$ ,  $a(x, \cdot) \in C^2(-M_1, M_2) \cap Lip(\mathbb{R})$  for almost every  $x \in \Omega$  with  $M_1, M_2 > 0$ . Let  $p, \tau$  and  $\lambda$  as in Section 2 and  $0 < \delta < p_{\min}$ . Assume that there exist positive constants  $a_0, a_1, a_2, c_1, \Lambda_0$  and  $0 < M < M_2$  such that

$$f1 \quad -c_1(1 + |s|^{p(x)-\delta}) \leq f(x, s) \leq c_1(1 + |s|^{\tau(x)}) \text{ in } \Omega \times \mathbb{R}.$$

$$f2 \quad f_s(x, 0) \equiv 0 \text{ in } \Omega.$$

$$f3 \quad f_{ss}(x, s) \geq 0 \text{ in } \Omega \times \mathbb{R}.$$

$$f4 \quad |f_s(x, s)| \leq \Lambda_0(1 + |s|^{\tau(x)}) \text{ in } \Omega \times \mathbb{R}.$$

And

$$a1 \quad 0 < a_0 \leq a(x, s) \leq a_1 < \infty \text{ in } \Omega \times \mathbb{R}.$$

$$a2 \quad |a_s(x, s)| \leq a_2 \quad \text{a.e. in } \Omega \times \mathbb{R}.$$

$$a3' \quad (a(x, s)^{1-\gamma(x)})_{ss} \leq 0 \text{ in } \Omega \times [0, M] \text{ with } \gamma(x) = \frac{2p(x)}{\min\{1, p(x)-1\}} > 1.$$

Let

$$F(x, s, \eta) = a(x, s) \frac{|\eta|^{p(x)}}{p(x)} + f(x, s)$$

and let  $u \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$  be a local minimizer of (1.1) such that  $0 \leq u \leq M$ . Then,  $u$  is locally Lipschitz continuous in  $\Omega$ .

*Proof.* By Proposition 3.7 for such a function  $f$  and with  $a$  satisfying a1 and a2, for every ball  $B_r(x_0) \subset \Omega$  with  $r$  small enough there exists a solution  $v \in u + W_0^{1,p(\cdot)}(B_r(x_0))$  of (1.2) such that  $0 \leq v \leq \|u\|_{L^\infty(B_r(x_0))}$ . And this result also holds for all the rescaled equations and functions that appear in the proofs of Section 4. Hence, condition (3.14) is only needed for  $s \in (0, M)$  and this is a consequence of a3'.  $\square$

**Example 5.1.** A possible example of functions  $a$  and  $f$  satisfying the assumptions of Theorem 5.2 is

$$a(x, s) = \begin{cases} (1 + s)^{-q(x)} & \text{if } -1/2 \leq s \leq M_2, \\ 2^{q(x)} & \text{if } s \leq -1/2, \\ (1 + M_2)^{-q(x)} & \text{if } s \geq M_2, \end{cases}$$

with  $M_2 > 0$  and  $q \in L^\infty(\Omega)$  a Hölder continuous function such that  $0 < q(x) < \frac{1}{\gamma(x)-1}$  and

$$f(x, s) = b(x)|s|^{\tau(x)}$$

with  $0 \leq b \in L^\infty(\Omega)$  and  $\tau(x) \geq 2$  in  $\Omega$  satisfying (2.7).

Another possible choice of  $f$  is

$$f(x, s) = b(x)\tilde{f}(x, s) \tag{5.3}$$

with  $0 \leq b \in L^\infty(\Omega)$  and

$$\tilde{f}(x, s) = \begin{cases} s^2 & \text{if } |s| \leq 1, \\ \tilde{a}(x)|s|^{\tau(x)} + \tilde{b}(x)|s| + \tilde{c}(x) & \text{if } |s| \geq 1, \end{cases}$$

where  $\tau(x)$  satisfies (2.7) and the functions  $\tilde{a}, \tilde{b}, \tilde{c} \in L^\infty(\Omega)$  are chosen in such a way that  $\tilde{f}(x, \cdot) \in C^2(\mathbb{R})$  for every  $x \in \Omega$ .

With this choice of  $a$  and  $f$ , for every  $0 < M < M_2$  there holds that any local minimizer  $u$  such that  $0 \leq u \leq M$  is locally Lipschitz continuous in  $\Omega$ .

Observe that, by Theorem 4.2, if  $\phi \in W^{1,p(\cdot)}(\Omega)$  is such that  $0 \leq \phi \leq M < M_2$ , such a minimizers always exists.

We have another example.

**Theorem 5.3.** *Let  $f(x, s)$  be a measurable function such that  $f(x, \cdot) \in C^2(\mathbb{R})$  for every  $x \in \Omega$ . Let  $G(x, \eta)$  be a measurable function such that  $G(x, \cdot) \in C^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N \setminus \{0\})$  for every  $x \in \Omega$ . Let  $p$  and  $\lambda$  as in Section 2 and assume that either  $f$  satisfies conditions f1,  $\dots$ , f4 in Theorem 5.1 or  $f$  satisfies f1, f3 in Theorem 5.1 and*

$$f4' \quad |f_s(x, s)| \leq \Lambda_0(1 + |s|^{p(x)-1}).$$

*On the other hand,  $G$  satisfies*

$$G1 \quad \mu(|\eta|^{p(x)} - 1) \leq G(x, \eta) \leq \mu^{-1}(|\eta|^{p(x)} + 1) \text{ with } \mu > 0.$$

$$G2 \quad \nabla_\eta G(x, 0) \equiv 0 \text{ in } \Omega.$$

$$G3 \quad \sum_{i,j} \frac{\partial^2 G}{\partial \eta_i \partial \eta_j} \xi_i \xi_j \geq \lambda_0 |\eta|^{p(x)-2} |\xi|^2.$$

$$G4 \quad \sum_{i,j} \left| \frac{\partial^2 G}{\partial \eta_i \partial \eta_j} \right| \leq \Lambda_0 |\eta|^{p(x)-2}.$$

$$G5 \quad |\nabla_\eta G(x_1, \eta) - \nabla_\eta G(x_2, \eta)| \leq \Lambda_0 |x_1 - x_2|^\beta (|\eta|^{p(x_1)-1} + |\eta|^{p(x_2)-1}) (1 + |\log |\eta||) \text{ for some } 0 < \beta \leq 1.$$

*Let*

$$F(x, s, \eta) = G(x, \eta) + f(x, s)$$

*and let  $u \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$  be a nonnegative, local minimizer of (1.1). Then,  $u$  is locally Lipschitz continuous in  $\Omega$ .*

*Proof.* There holds that

$$A(x, s, \eta) = \nabla_\eta G(x, \eta), \quad B(x, s, \eta) = f_s(x, s).$$

And it is clear that  $F, A$  and  $B$  satisfy the assumptions in Theorem 4.5. □

**Example 5.2.** A possible example of function  $G$  satisfying the assumptions of Theorem 5.3 is

$$G(x, \eta) = a(x) \tilde{G}(|\eta|^{p(x)}),$$

with  $p(x)$  as in Section 2,  $a(x)$  a Hölder continuous function such that  $a_0 \leq a(x) \leq a_1$ , with  $a_0, a_1$  positive constants and  $\tilde{G} \in C^2([0, \infty))$  a function satisfying:

$$\begin{aligned} c_0 &\leq \tilde{G}'(t) \leq C_0, \\ 0 &\leq \tilde{G}''(t) \leq \frac{C_0}{1+t} \quad c_0, C_0 \text{ positive constants.} \end{aligned}$$

In fact, since  $c_0 \leq \widetilde{G}'(t) \leq C_0$ , condition G1 in Theorem 5.3 holds. We have  $\nabla_\eta G(x, \eta) = a(x)\widetilde{G}'(|\eta|^{p(x)})p(x)|\eta|^{p(x)-2}\eta$ , so we get condition G2. We obtain condition G3 by reasoning as in (5.1), using that in the present case we have  $\widetilde{G}''(t) \geq 0$  and  $\widetilde{G}'(t) \geq c_0$ .

We get condition G4 by using in our computations that  $\widetilde{G}'(t) \leq C_0$  and  $\widetilde{G}''(t)t \leq C_0$ .

Finally, applying again that  $\widetilde{G}''(t)t \leq C_0$ , we can obtain the estimate

$$|\widetilde{G}'(|\eta|^{p(x_1)}) - \widetilde{G}'(|\eta|^{p(x_2)})| \leq C_0|p(x_1) - p(x_2)|\log|\eta|,$$

which combined with computations similar as those in (4) in Theorem 5.1 leads to condition G5.

A possible example of function  $f$  satisfying the assumptions of Theorem 5.3 is

$$f(x, s) = g(x)s, \quad \text{with } g \in L^\infty(\Omega).$$

In fact, it is immediate that  $f$  satisfies conditions f1, f3 and f4'.

On the other hand,  $f(x, s) = b(x)|s|^{\tau(x)}$  with  $b$  and  $\tau$  as in Example 5.1 and  $f(x, s)$  as in (5.3) are other possible choices.

Let us present another example

**Example 5.3.** Another possible example of function  $G$  satisfying the assumptions of Theorem 5.3 is

$$G(x, \eta) = \widetilde{A}(x)\eta \cdot \eta|\eta|^{p(x)-2},$$

with  $p(x)$  as in Section 2 and  $\widetilde{A}(x) \in \mathbb{R}^{N \times N}$ , symmetric, Hölder continuous in  $\Omega$  and such that

$$\lambda(x)I \leq \widetilde{A}(x) \leq \Lambda(x)I.$$

Here  $\lambda_0 \leq \lambda(x) \leq \Lambda(x) \leq \Lambda_0$  with  $\lambda_0, \Lambda_0$  positive constants and  $\Lambda(x) - \lambda(x) \leq c_0$ , with  $c_0$  a suitable positive constant depending only on  $N, p_{\min}, p_{\max}$  and  $\lambda_0$ .

In fact, conditions G1 and G2 in Theorem 5.3 are easy to verify. The computations leading to G4 and G5 are similar to the computations in Theorem 5.1.

In order to verify G3, we observe that, denoting  $a(x)$  the smaller eigenvalue of  $\widetilde{A}(x)$ , there holds that

$$\widetilde{A}(x) = a(x)I + \widetilde{B}(x), \quad \text{with } \|\widetilde{B}(x)\|_{L^\infty(\Omega)} \leq \|\Lambda(x) - \lambda(x)\|_{L^\infty(\Omega)}.$$

Then we can write

$$\begin{aligned} G(x, \eta) &= a(x)|\eta|^{p(x)} + \widetilde{B}(x)\eta \cdot \eta|\eta|^{p(x)-2} \\ &= G_1(x, \eta) + G_2(x, \eta). \end{aligned}$$

Now, proceeding as in Theorem 5.1, we get

$$\sum_{i,j} \frac{\partial^2 G_1}{\partial \eta_i \partial \eta_j} \xi_i \xi_j \geq c_{p_{\min}} \lambda_0 |\eta|^{p(x)-2} |\xi|^2. \quad (5.4)$$

It is not hard to see that

$$\sum_{i,j} \left| \frac{\partial^2 G_2}{\partial \eta_i \partial \eta_j} \right| \leq C \|\Lambda(x) - \lambda(x)\|_{L^\infty(\Omega)} |\eta|^{p(x)-2}, \quad (5.5)$$

with  $C$  depending only on  $N, p_{\min}$  and  $p_{\max}$ . Then, combining (5.4) and (5.5) we deduce that  $G(x, \eta)$  satisfies condition G3, if we take  $\|\Lambda(x) - \lambda(x)\|_{L^\infty(\Omega)} \leq c_0$ , with  $c_0$  depending only on  $\lambda_0, N, p_{\min}$  and  $p_{\max}$ .

For choices of suitable functions  $f(x, s)$  for this  $G(x, \eta)$  we refer to Example 5.2.

**Remark 5.1.** We can present further examples of functions satisfying our assumptions. Let  $p$  and  $\lambda$  be as in Section 2. Let  $F_1$  and  $F_2$  satisfy the assumptions on Theorem 4.5, with  $B_i = \partial_s F_i$  satisfying  $B_i(x, 0, 0) \equiv 0$  for  $x \in \Omega$ ,  $i = 1, 2$ . Then Theorem 4.5 also applies to the function

$$F(x, s, \eta) = a_1(x)F_1(x, s, \eta) + a_2(x)F_2(x, s, \eta),$$

for any choice of Hölder continuous functions  $a_1(x), a_2(x)$ , which are bounded from above and below by positive constants.

The same result holds if  $F_1$  and  $F_2$  satisfy the assumptions on Theorem 4.5, with  $B_i = \partial_s F_i$  satisfying (3.32) for  $x \in \Omega$ ,  $i = 1, 2$ .

Similar consideration applies to functions  $F_1$  and  $F_2$  under the assumptions of Theorem 5.2.

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## Conflict of interest

The authors declare no conflict of interest.

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## A. Appendix

In Section 1 we included some preliminaries on Lebesgue and Sobolev spaces with variable exponent. For the sake of completeness we collect here some additional results on these spaces.

**Proposition A.1.** *There holds*

$$\begin{aligned} \min \left\{ \left( \int_{\Omega} |u|^{p(x)} dx \right)^{1/p_{\min}}, \left( \int_{\Omega} |u|^{p(x)} dx \right)^{1/p_{\max}} \right\} &\leq \|u\|_{L^{p(\cdot)}(\Omega)} \\ &\leq \max \left\{ \left( \int_{\Omega} |u|^{p(x)} dx \right)^{1/p_{\min}}, \left( \int_{\Omega} |u|^{p(x)} dx \right)^{1/p_{\max}} \right\}. \end{aligned}$$

Some important results for these spaces are

**Theorem A.1.** Let  $p'(x)$  such that

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1.$$

Then  $L^{p'(\cdot)}(\Omega)$  is the dual of  $L^{p(\cdot)}(\Omega)$ . Moreover, if  $p_{\min} > 1$ ,  $L^{p(\cdot)}(\Omega)$  and  $W^{1,p(\cdot)}(\Omega)$  are reflexive.

**Theorem A.2.** Let  $q(x) \leq p(x)$ . If  $\Omega$  has finite measure, then  $L^{p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$  continuously.

We also have the following Hölder's inequality

**Theorem A.3.** Let  $p'(x)$  be as in Theorem A.1. Then there holds

$$\int_{\Omega} |f| |g| \, dx \leq 2 \|f\|_{p(\cdot)} \|g\|_{p'(\cdot)},$$

for all  $f \in L^{p(\cdot)}(\Omega)$  and  $g \in L^{p'(\cdot)}(\Omega)$ .

The following version of Poincaré's inequality holds

**Theorem A.4.** Let  $\Omega$  be bounded. Assume that  $p(x)$  is log-Hölder continuous in  $\Omega$  (that is,  $p$  has a modulus of continuity  $\omega(r) = C(\log \frac{1}{r})^{-1}$ ). For every  $u \in W_0^{1,p(\cdot)}(\Omega)$ , the inequality

$$\|u\|_{L^{p(\cdot)}(\Omega)} \leq C \|\nabla u\|_{L^{p(\cdot)}(\Omega)}$$

holds with a constant  $C$  depending only on  $N$ ,  $\text{diam}(\Omega)$  and the log-Hölder modulus of continuity of  $p(x)$ .

The following Sobolev embedding holds. We assume for simplicity that the domain is  $C^1$ , but the result holds with weaker assumptions on the smoothness of the boundary.

**Theorem A.5.** Let  $\Omega$  be a  $C^1$  bounded domain. Assume that  $p(x)$  is log-Hölder continuous in  $\Omega$  and  $1 < p_{\min} \leq p(x) \leq p_{\max} < \infty$ . Let  $\tau$  be such that  $\tau(x) \geq p(x)$  and  $\tau \in C(\overline{\Omega})$ . Assume moreover that  $\tau(x) \leq p^*(x) = \frac{Np(x)}{N-p(x)}$  if  $p_{\max} < N$ ,  $\tau(x)$  is arbitrary if  $p_{\min} > N$ ,  $\tau(x) = p(x)$  if  $p_{\min} \leq N \leq p_{\max}$ .

Then,  $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{\tau(\cdot)}(\Omega)$  continuously. The embedding constant depends only on  $N$ ,  $|\Omega|$ , the log-Hölder modulus of continuity of  $p(x)$ ,  $p_{\min}$ ,  $p_{\max}$ ,  $\|\tau\|_{L^\infty}$  and the  $C^1$  norm of  $\partial\Omega$ .

For the proof of these results and more about these spaces, see [9, 13, 14, 20] and the references therein.



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