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Research article

On the complete (permutation) symmetry of Cartesian tensors according to Young in a space of given finite arbitrary dimensions

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Abstract: Symmetry plays an important role in nature as well as in mathematics. Many properties of matter can be expressed in terms of Cartesian tensors. The analysis of tensor symmetry can be simplified by decomposing tensors into irreducible parts that possess complete (permutation) symmetry in the sense of Young. This paper compiles well-known results and formulas from group theory to construct an algorithm for such analytical decomposition and provides its explicit realization for Cartesian tensors of small rank (from 2 to 4). The complete results are presented in tabular form.

Keywords: irreducible representation; cartesian tensors; Euler and Gauss representation

1. Introduction

Symmetry plays an important role in nature and in mathematics. Many properties of matter can be expressed in terms of Cartesian tensors, which can be decomposed into irreducible parts possessing complete (permutation) symmetry according to Young (Euler form). From another point of view, these properties can be represented through the decomposition of tensor spherical harmonics in three-dimensional space (Gauss form). In spaces of other dimensions, the situation becomes more complex. Proponents and opponents of these representation forms continue to debate which approach is more appropriate. Each point of view is supported by serious and convincing arguments. In this author's opinion, both representations are valid and have their own field of application. The Euler representation is often used for analytical problems in which symmetry and irreducibility can be used to derive solutions. Another advantage of the Euler form is its relatively straightforward extension to problems in spaces of dimension higher than 3.

However, formulas of the Euler representation, especially for arbitrary fourth-rank tensors, are scarcely presented in the literature (except for second-rank tensors, which have been extensively studied and will also be discussed in the present paper). Although general formulas exist for certain stages of the decomposition, they are highly complicated, and users may easily make errors when using them. Therefore, the goal of this paper is to employ well-known results and formulas from group theory to build a complete algorithm for such analytical decomposition and to implement it for Cartesian tensors of small rank (from 2 to 4), presenting the results in tabular form. These tables may be used for the investigation of tensor symmetry properties.

The author has attempted to compile general formulas and algorithms that are useful for the practical study of symmetry properties and that enable readers to perform any decomposition, as well as to implement it computationally.

2. Methods and data sources

2.1. Notations and definitions

In this work, the following notations are used: *dim* is the dimension of the space; *GL(dim)* is the general group of linear transformations; *O(dim)* is the group of orthogonal transformations; and *SO(dim)* is the group of rotations (the special group of orthogonal transformations).

Most of the tensor-algebra notation employed in this paper can be found in [1,2], including definitions of multi-indices and multi-components according to Damour and Thorne; symmetric, antisymmetric, and trace-free tensors (both in terms of the group or groups of indices and in their entirety); definitions of dual components; and operations of symmetrization, antisymmetrization, deviatorization, and dualization (both in terms of the group or groups of indices and in their entirety). As such, these concepts are introduced here only to the extent necessary for the comprehension of the material presented.

In addition to widely accepted conventions, several further specific definitions, concepts, and notations from group theory, as well as some introduced by the author, will be needed.

2.1.1. Young tableaux and Young symmetry index

The central concept of group theory for describing Young-type symmetry is the Young tableaux, which is detailed in [3]. It is noteworthy to mention that here, only the line form of Young tableaux is used, as the Young symmetry index. It includes, in curly brackets, the content of every string of Young Tableaux, listed from biggest to smallest, in the form of string index numbers. Indices of each string are separated by commas. For example, for a third-rank tensor, the Young tableaux is as follows:

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\qquad \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\qquad \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\qquad \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array} \tag{1}$$

This can be written in line form as follows: {{1,2,3}}, {{1,2}{3}}, {{1,3}{2}}, and {{1}{2}{3}}, which can also be used as the Young symmetry index here.

Trace-free tensors are denoted by adding the suffix *TF* to the Young symmetry index (*YSI*). For example, the *YSI* of the trace-free part of a tensor with symmetry index {{1,2}{3}} is written as {{1,2}{3}}*TF*. Similarly, the traceless part of a tensor with Young symmetry index {{1,2,3}} is denoted as {{1,2,3}}*TF*.

2.1.2. Operation-history indices and other notations

Because the representation of high-rank tensors becomes increasingly complex, it is convenient to use not only the Young symmetry index but also a special coupled operation index (*OI*) and Young symmetry index (*YSI*) for all transformation steps, as explained below.

$$\underset{\left(\mathit{if\ operation(s)\ were\ executed\ or\ omitted}\right)}{OI\text{ for current component}}\left|\begin{matrix}YSI\text{ for start component}\\ \left(\mathit{if\ YSI\ exists\ or\ omitted}\right)\\ \hline OI\text{ for start component}\\ \left(\mathit{if\ operation(s)\ were\ executed\ or\ omitted}\right)\end{matrix}\right|\begin{matrix}YSI\text{ for result of first operation series}\\ \text{with start component}\\ \left(\mathit{if\ YSI\ exists\ or\ omitted}\right)\\ \hline OI\text{ for results of first operation series}\\ \text{with start component}\\ \left(\mathit{if\ operation(s)\ were\ executed\ or\ omitted}\right)\end{matrix}\left|\ \ldots T^{\overset{YSI\text{ of current component}}{\left(\mathit{if\ YSI\ exists\ or\ omitted}\right)}}_{\underbrace{i_1 i_2 \ldots i_n}_{\mathit{tensor\ indices}}}\right. \quad (2)$$

In most cases, the *OI* associated with the current component can (and, if possible, should) be shown using traditional bracket notation applied directly to tensor indices. However, additional notation is required to describe operations for other steps. Another reason is the need to perform operations not on all indices but only on individual groups. The author of this paper proposes the following notations by analogy with traditional notations.

Particular dualization. For every space dimension, a tensor with a certain Young symmetry is equivalent to a combination of components that are (dual) tensors or pseudotensors of smaller rank. These can be obtained by dualization using either the fully antisymmetric unit tensor of the space (named here as tensor dualization) or the fully antisymmetric unit symbol Levi-Civita (named as pseudotensor dualization). In the *OI,* tensor (pseudotensor) dualization is denoted by the symbol e (ε), followed by the enumeration of index groups in curly brackets, in parentheses and separated by commas. If dualization is executed for a group of indices in columns of a Young tableaux, the corresponding column numbers are written after symbol e (ε) in curly brackets and separated by commas.

Particular symmetrization, antisymmetrization, and deviatorization. In *OI,* the designations *S*, *A*, and *STF* are used for these operations. The groups of indices on which the operation if performed are listed in curly brackets; indices within each group are enclosed in parentheses and separated by commas.

Particular trace freedomization. In *OI,* the designation *TF* is used. Two groups of indices are specified in curly brackets. The first group includes the first indices of convolution in parentheses, separated by commas. The second group includes the second indices of convolution in parentheses, separated by commas.

Particular trace-part extraction. In *OI*, the designation *TP* is used. After this symbol, the groups of index numbers determining the trace part are listed in curly brackets. It should be noted that a completely antisymmetric tensor has no trace part, so the *ATP* notation is not meaningful.

Particular spoor. In *OI*, the designation *Sp* is used. Two parentheses are specified in curly brackets. The first parentheses list the first indices of the convolution, separated by commas, and the second parentheses list the corresponding second indices of the convolution. If the tensor is symmetric, it does not matter which pairs of indices are used for the trace. In this case, the Sp symbol in curly brackets only includes the number of pairs of indices used for the spoor, and the parentheses are omitted.

Extraction of a part multiplied by a given multiplier. In *OI*, the designation *PMB* (from *Part Multiplied By*) is used. The multiplier is written in curly brackets. This multiplier is usually an isotropic tensor. If the multiplier is a number (not tensor), it is written as a number before other operations in *OI*.

2.2. Method

First, an arbitrary tensor of a given rank is decomposed into a sum of certain Young symmetry index members (for the algorithm, see [4]). The Young operator is determined according to Appendix A of [4], yielding a decomposition into irreducible representations of *GL(dim)*. Subsequently, the rank of some members is reduced by dualization whenever possible. If, after dualization, reducible parts do not have a given *YSI*, the operation is repeated for each such component. The formulas for the number of independent components are derived from general expressions given in [5]. The first selection rule [5] is used to exclude zero members.

For traceless representations, after decomposition into components with certain *YSI*, the trace parts with certain *YSI* are extracted. All operations are then repeated for tensor parts appearing in these traces. Whenever possible, the rank of some components is reduced by dualization. As a result, the decomposition into traceless components is conducted. This yields irreducible representations of *O(dim)* (for tensor dualization) or *SO (dim)* (for pseudotensor dualization).

Because pseudotensor dualization is more frequently used and differs visually from tensor dualization only by the symbols e or ε (for Cartesian spaces), formulas are presented for pseudotensor dualization. The formulas for the number of independent components of traceless members are derived from general formulas [6,7]. The first and second selection rules [5] are used to exclude zero members.

In addition, numerical values for the number of independent components are provided for several commonly used space dimensions (2, 3, 4, 5, 10, 11, 16). The cases of 2-, 3-, and 4-dimensional spaces do not require comments. The Kaluza-Klein gravity [8] deals with a 5-dimensional space. The 10-, 11-, and 16-dimensional spaces are often used for theories of the Great Unification (string theory, M-theories, and others). Some theories require even higher dimensions (for example, in bosonic string theory, the space-time is 26-dimensional) [9].

2.3. Example

Formulas for second-rank tensors are trivial and do not require any special discussion. By contrast, formulas for fourth-rank tensors are very complicated, and a complete description of all operations requires extensive space. So let us consider an arbitrary third-rank tensor decomposition stage; for second- and fourth-rank tensors, only the final results will be presented.

2.3.1. *GL(dim)* decomposition

Using algorithm [4] for the decomposition of an arbitrary third-rank tensor into components with a certain *YSI*, one can obtain:

$$T_{i_1i_2i_3} = T_{i_1i_2i_3}^{\{\{1,2,3\}\}} + T_{i_1i_2i_3}^{\{\{1,2\}\{3\}\}} + T_{i_1i_2i_3}^{\{\{1,3\}\{2\}\}} + (1-\delta_{2\dim})T_{i_1i_2i_3}^{\{\{1\}\{2\}\{3\}\}}, \tag{3}$$

where

$$T^{\{\{1,2,3\}\}}_{k_1k_2k_3}=T_{(k_1k_2k_3)}=\frac{1}{6}\left(T_{k_1k_2k_3}+T_{k_2k_3k_1}+T_{k_3k_1k_2}+T_{k_2k_1k_3}+T_{k_1k_3k_2}+T_{k_3k_2k_1}\right)$$

$$T^{\{\{1\}\{2\}\{3\}\}}_{k_1k_2k_3}=T_{[k_1k_2k_3]}=\frac{1}{6}\left(T_{k_1k_2k_3}+T_{k_2k_3k_1}+T_{k_3k_1k_2}-T_{k_2k_1k_3}-T_{k_1k_3k_2}-T_{k_3k_2k_1}\right)$$

$$T^{\{\{1,2\}\{3\}\}}_{k_1k_2k_3}=\frac{1}{3}\left(T_{k_1k_2k_3}+T_{k_2k_1k_3}-T_{k_3k_2k_1}-T_{k_2k_3k_1}\right)=\frac{2}{3}\left(T_{[k_1|k_2|k_3]}+T_{k_2[k_1k_3]}\right) \tag{4}$$

$$T^{\{\{1,3\}\{2\}\}}_{k_1k_2k_3}=\frac{1}{3}\left(T_{k_1k_2k_3}+T_{k_3k_2k_1}-T_{k_2k_1k_3}-T_{k_3k_1k_2}\right)=\frac{2}{3}\left(T_{[k_1k_2]k_3}+T_{k_3[k_2k_1]}\right)$$

Taking into account that $T^{\{\{1,2\}\{3\}\}}_{k_1k_2k_3}+T^{\{\{1,3\}\{2\}\}}_{k_1k_2k_3}=\frac{1}{3}\left(2T_{k_1k_2k_3}-T_{k_2k_3k_1}-T_{k_3k_1k_2}\right)$, one can easily check that (3) is valid. The fully antisymmetric part is always equal to zero for a two-dimensional space, according to the first selection rule (for a two-dimensional space, a complex Fourier transform is more convenient, but, to preserve community, this case is also included).

Using a particular pseudotensor dualization, one can reduce the rank of some components for a space with *dim=2*:

$$T^{\{\{1,2\}\{3\}\}}_{k_1k_2k_3}=\varepsilon_{k_1k_3}\,{}^{\{\{1,2\}\{3\}\}}_{\varepsilon\{1\}}T_{k_2},\quad {}^{\{\{1,2\}\{3\}\}}_{\varepsilon\{1\}}T_{k_2}={}^{\{\{1,2\}\{3\}\}}_{\varepsilon\{(1,3)\}}T_{k_2}=\frac{1}{2}\varepsilon_{i_1i_3}T^{\{\{1,2\}\{3\}\}}_{i_1k_2i_3}=\frac{1}{3}\varepsilon_{i_1i_3}\left(T_{[i_1|k_2|i_3]}+T_{k_2[i_1i_3]}\right),$$

$$T^{\{\{1,3\}\{2\}\}}_{k_1k_2k_3}=\varepsilon_{k_1k_2}\,{}^{\{\{1,3\}\{2\}\}}_{\varepsilon\{1\}}T_{k_3},\quad {}^{\{\{1,3\}\{2\}\}}_{\varepsilon\{1\}}T_{k_3}={}^{\{\{1,3\}\{2\}\}}_{\varepsilon\{(1,2)\}}T_{k_3}=\frac{1}{2}\varepsilon_{i_1i_2}T^{\{\{1,3\}\{2\}\}}_{i_1i_2k_3}=\frac{1}{3}\varepsilon_{i_1i_2}\left(T_{[i_1i_2]k_3}+T_{k_3[i_2i_1]}\right).$$

For a space with *dim=3,* which is increasingly challenging, let us extract the trace part from the components with mixed symmetry:

$$\left.{}_{TP\{(k_1k_2k_3)\}}\right|T^{\{\{1,2\}\{3\}\}}_{k_1k_2k_3}\equiv T^{\{\{1,2\}\{3\}\}}_{k_1k_2k_3}-T^{\{\{1,2\}\{3\}\}TF}_{k_1k_2k_3}=\delta_{k_1k_2}\,{}^{\{\{1,2\}\{3\}\}}_{\frac{1}{2}Sp\{(1)\{2\}\}}T_{k_3}+\delta_{k_2k_3}\,{}^{\{\{1,2\}\{3\}\}}_{\frac{1}{2}Sp\{(2)\{3\}\}}T_{k_1}+$$

$$+\underbrace{\delta_{k_1k_3}\,{}^{\{\{1,2\}\{3\}\}}_{\frac{1}{2}Sp\{(1)\{3\}\}}T_{k}}_{\equiv 0}=\frac{1}{6}\delta_{k_1k_2}\left(2T_{ssk_3}-T_{k_3ss}-T_{sk_3s}\right)-\delta_{k_2k_3}\frac{1}{6}\left(2T_{ssk_1}-T_{k_1ss}-T_{sk_1s}\right).$$

If ${}^{\{\{1,2\}\{3\}\}}T^{(3)}_{k}\equiv\frac{1}{6}\left(2T_{ssk_3}-T_{k_3ss}-T_{sk_3s}\right)$, then $\left.{}_{TP\{(k_1k_2k_3)\}}\right|T^{\{\{1,2\}\{3\}\}}_{k_1k_2k_3}=\delta_{k_1k_2}\,{}^{\{\{1,2\}\{3\}\}}T^{(3)}_{k_3}-\delta_{k_2k_3}\,{}^{\{\{1,2\}\{3\}\}}T^{(3)}_{k_1}$.

The second trace-free part, as will be shown below (see next chapter), is

$$T^{\{\{1,2\}\{3\}\}TF}_{i_1i_2i_3}=\varepsilon_{i_1i_3k}\,{}^{\{\{1,2\}\{3\}\}TF}_{\varepsilon\{1\}}T^{\{\{1,2\}\}TF}_{ki_2},$$

where ${}^{\{\{1,2\}\{3\}\}TF}_{\varepsilon\{1\}}T^{\{\{1,2\}\}TF}_{ki_2}=\frac{1}{6}\left(\varepsilon_{ksj}\left(T_{si_2j}+T_{i_2sj}\right)+\varepsilon_{i_2sj}\left(T_{skj}+T_{ksj}\right)\right)$.

So, $T^{\{\{1,2\}\{3\}\}}_{k_1k_2k_3}=\varepsilon_{i_1i_3k}\,{}^{\{\{1,2\}\{3\}\}TF}_{\varepsilon\{1\}}T^{\{\{1,2\}\}TF}_{ki_2}+\delta_{k_1k_2}\,{}^{\{\{1,2\}\{3\}\}}T^{(3)}_{k_3}-\delta_{k_2k_3}\,{}^{\{\{1,2\}\{3\}\}}T^{(3)}_{k_1}$, which expresses the $T_{k_1k_2k_3}^{\{\{1,2\}\{3\}\}}$ component through one vector $T_{k_3}^{(3)}$ and the *STF*-tensor with the second-rank ${}^{\{\{1,2\}\{3\}\}TF}_{\varepsilon\{1\}}T_{ki_2}^{\{\{1,2\}\}TF}$. The corresponding number of independent components is 8 for decomposed parts and 3 and 5 for decomposition on equivalent components. So, a balance exists.

Analogically, $T_{k_1k_2k_3}^{\{\{1,3\}\{2\}\}} = \varepsilon_{k_1k_2i}\,{}^{\{\{1,3\}\{2\}\}TF}_{\varepsilon\{1\}}T_{ik_3}^{\{\{1,2\}\}TF} + \delta_{k_1k_3}\,{}^{\{\{1,3\}\{2\}\}}T_{k_2}^{(3)} - \delta_{k_2k_3}\,{}^{\{\{1,3\}\{2\}\}}T_{k_1}^{(3)}$,

where ${}^{\{\{1,3\}\{2\}\}TF}_{\varepsilon\{1\}}T_{ik_3}^{\{\{1,2\}\}TF} = \frac{1}{6}\left(\varepsilon_{isj}\left(T_{sjk_3} - T_{k_3sj}\right) + \varepsilon_{k_3sj}\left(T_{sji} - T_{isj}\right)\right)$ and ${}^{\{\{1,3\}\{2\}\}}T_{k}^{(3)} = \frac{1}{3(\dim-1)}\left(2T_{sks} - T_{kss} - T_{ssk}\right)$.

The rank of the fully antisymmetric part can be reduced by representation:

$T_{k_1k_2k_3}^{\{\{1\}\{2\}\{3\}\}} = \varepsilon_{k_1k_2k_3}\,{}^{\{\{1\}\{2\}\{3\}\}}T^{(3)}$, where the pseudo-scalar ${}^{\{\{1\}\{2\}\{3\}\}}T^{(3)} = \frac{1}{6}\varepsilon_{kjm}T_{kjm}$.

For a space with *dim* = 4 and higher, the dual to mixed components have more indices than the original component; as such, particular dualization for the mixed component does not make sense. A similar situation holds for fully antisymmetric members starting from *dim* = 6. The following formulas are valid for *dim* = 4:

$T_{k_1k_2k_3}^{\{\{1\}\{2\}\{3\}\}} = \varepsilon_{k_1k_2k_3m}\,{}^{\{\{1\}\{2\}\{3\}\}}T_m^{(4)}$, where ${}^{\{\{1\}\{2\}\{3\}\}}T_m^{(4)} = -\frac{1}{6}\varepsilon_{mijk}T_{ijk}$;

and, for dim = 5:,

$T_{k_1k_2k_3}^{\{\{1\}\{2\}\{3\}\}} = \frac{1}{2}\varepsilon_{k_1k_2k_3ml}\,{}^{\{\{1\}\{2\}\{3\}\}}T_{ml}^{(5)}$, where ${}^{\{\{1\}\{2\}\{3\}\}}T_{ml}^{(5)} \equiv \frac{1}{6}\varepsilon_{mlijk}T_{ijk}$.

The numbers of independent components in (3) are estimated, according to common formulas from [5], which, for an arbitrary dim, yield:
$\frac{(\dim+2)(\dim+1)\dim}{6}$, $\frac{\dim(\dim^2-1)}{3}$, $\frac{\dim(\dim^2-1)}{3}$, and $\frac{\dim(\dim-1)(\dim-2)}{6}$ for members in (3). The sum of these is equal to $\dim^3$ as expected.

2.3.2. *SO(dim)* decomposition

In this case, the formula (3) is used to start the following consideration. Tensor $T_{k_1k_2k_3}^{\{\{1,2,3\}\}} = T_{(k_1k_2k_3)}$ is fully symmetric. Therefore, it can be decomposed into a fully symmetric trace-free (*STF*) part and a trace part (*TP*) according to formula [10, 11]:

$$T_{k_1k_2k_3}^{\{\{1,2,3\}\}} = T_{k_1k_2k_3}^{\{\{1,2,3\}\}TF} + {}_{TP\{(1,2,3)\}}\Big|T_{k_1k_2k_3}^{\{\{1,2,3\}\}} =$$

$$= T_{k_1k_2k_3}^{\{\{1,2,3\}\}TF} + \delta_{k_1k_2}\,{}^{\{\{1,2,3\}\}}_{\frac{1}{(\dim+2)}Sp\{1\}}T_{k_3} + \delta_{k_1k_3}\,{}^{\{\{1,2,3\}\}}_{\frac{1}{(\dim+2)}Sp\{1\}}T_{k_2} + \delta_{k_2k_3}\,{}^{\{\{1,2,3\}\}}_{\frac{1}{(\dim+2)}Sp\{1\}}T_{k_1}$$

$$= T_{<k_1k_2k_3>} + \frac{1}{3(\dim+2)}\left(\delta_{k_1k_2}\left(T_{ssk_3} + T_{sk_3s} + T_{k_3ss}\right) + \delta_{k_1k_3}\left(T_{sk_2s} + T_{k_2ss} + T_{ssk_2}\right) + \delta_{k_2k_3}\left(T_{k_1ss} + T_{ssk_1} + T_{sk_1s}\right)\right), \quad (5)$$

where
$$T_{<k_1k_2k_3>} = \frac{1}{6}\left(T_{k_1k_2k_3} + T_{k_2k_3k_1} + T_{k_3k_1k_2} + T_{k_2k_1k_3} + T_{k_1k_3k_2} + T_{k_3k_2k_1}\right) - \\ -\frac{1}{3(\dim+2)}\left(\delta_{k_1k_2}\left(T_{ssk_3} + T_{sk_3s} + T_{k_3ss}\right) + \delta_{k_1k_3}\left(T_{sk_2s} + T_{k_2ss} + T_{ssk_2}\right) + \delta_{k_2k_3}\left(T_{k_1ss} + T_{ssk_1} + T_{sk_1s}\right)\right).$$

Let us note:

$$^{\{\{1,2,3\}\}}T_k = \frac{1}{3(\dim+2)}\left(T_{ssk} + T_{sks} + T_{kss}\right) \tag{6}$$

Then,
$$T_{k_1k_2k_3}^{\{\{1,2,3\}\}} = T_{k_1k_2k_3}^{\{\{1,2,3\}\}TF} + \delta_{k_1k_2}\,{}^{\{\{1,2,3\}\}}T_{k_3} + \delta_{k_1k_3}\,{}^{\{\{1,2,3\}\}}T_{k_2} + \delta_{k_2k_3}\,{}^{\{\{1,2,3\}\}}T_{k_1} \tag{7}$$

Let us consider the term with *YSI* $\{\{1,2\}\{3\}\}$:

$$T_{ssk}^{\{\{1,2\}\{3\}\}} = {}_{\frac{1}{(\dim-1)}Sp\{(1)\{2\}\}}^{\{\{1,2\}\{3\}\}}T_{ssk} = \frac{1}{3}\left(2T_{ssk} - T_{kss} - T_{sks}\right);\ T_{sks}^{\{\{1,2\}\{3\}\}} = {}_{\frac{1}{(\dim-1)}Sp\{(1)\{3\}\}}^{\{\{1,2\}\{3\}\}}T_k = 0;$$

$${}_{\frac{1}{(\dim-1)}Sp\{(2)\{3\}\}}^{\{\{1,2\}\{3\}\}}T_k = \frac{1}{(\dim-1)}T_{kss}^{\{\{1,2\}\{3\}\}} = -\frac{1}{3(\dim-1)}\left(2T_{ssk} - T_{kss} - T_{sks}\right) = -\frac{1}{(\dim-1)}T_{ssk}^{\{\{1,2\}\{3\}\}} = -{}_{\frac{1}{(\dim-1)}Sp\{(1)\{2\}\}}^{\{\{1,2\}\{3\}\}}T_k.$$

Then,

$$T_{k_1k_2k_3}^{\{\{1,2\}\{3\}\}} = T_{k_1k_2k_3}^{\{\{1,2\}\{3\}\}TF} + \delta_{k_1k_2}\,{}_{\frac{1}{(\dim-1)}Sp\{(1)\{2\}\}}^{\{\{1,2\}\{3\}\}}T_k + \delta_{k_2k_3}\,{}_{\frac{1}{(\dim-1)}Sp\{(2)\{3\}\}}^{\{\{1,2\}\{3\}\}}T_k + \delta_{k_1k_3}\underbrace{{}_{\frac{1}{(\dim-1)}Sp\{(1)\{3\}\}}^{\{\{1,2\}\{3\}\}}T_k}_{\equiv 0} = \\ = T_{k_1k_2k_3}^{\{\{1,2\}\{3\}\}TF} + \frac{1}{3(\dim-1)}\delta_{k_1k_2}\left(2T_{ssk_3} - T_{k_3ss} - T_{sk_3s}\right) - \delta_{k_2k_3}\frac{1}{3(\dim-1)}\left(2T_{ssk_1} - T_{k_1ss} - T_{sk_1s}\right) \tag{8}$$

where
$$T_{k_1k_2k_3}^{\{\{1,2\}\{3\}\}TF} = \frac{1}{3}\left(T_{k_1k_2k_3} + T_{k_2k_1k_3} - T_{k_3k_2k_1} - T_{k_2k_3k_1}\right) - \\ -\frac{1}{3(\dim-1)}\delta_{k_1k_2}\left(2T_{ssk_3} - T_{k_3ss} - T_{sk_3s}\right) + \delta_{k_2k_3}\frac{1}{3(\dim-1)}\left(2T_{ssk_1} - T_{k_1ss} - T_{sk_1s}\right)$$

Analogically,

$$T_{k_1k_2k_3}^{\{\{1,3\}\{2\}\}} = T_{k_1k_2k_3}^{\{\{1,3\}\{2\}\}TF} + \delta_{k_1k_3}\,{}_{\frac{1}{(\dim-1)}Sp\{(1)\{3\}\}}^{\{\{1,3\}\{2\}\}}T_k + {}_{\frac{1}{(\dim-1)}Sp\{(2)\{3\}\}}^{\{\{1,3\}\{2\}\}}T_k + \underbrace{{}_{\frac{1}{(\dim-1)}Sp\{(1)\{2\}\}}^{\{\{1,3\}\{2\}\}}T_k}_{\equiv 0} = \\ = T_{k_1k_2k_3}^{\{\{1,3\}\{2\}\}TF} + \frac{1}{3(\dim-1)}\delta_{k_1k_3}\left(2T_{sk_2s} - T_{k_2ss} - T_{ssk_2}\right) - \delta_{k_2k_3}\frac{1}{3(\dim-1)}\left(2T_{sk_1s} - T_{k_1ss} - T_{ssk_1}\right) \tag{9}$$

$$T_{k_1k_2k_3}^{\{\{1,3\}\{2\}\}TF} = \frac{1}{3}\left(T_{k_1k_2k_3} + T_{k_3k_2k_1} - T_{k_2k_1k_3} - T_{k_3k_1k_2}\right) - \\ -\frac{1}{3(\dim-1)}\delta_{k_1k_3}\left(2T_{sk_2s} - T_{k_2ss} - T_{ssk_2}\right) + \delta_{k_2k_3}\frac{1}{3(\dim-1)}\left(2T_{sk_1s} - T_{k_1ss} - T_{ssk_1}\right).$$

Now, one can insert expressions (6 – 9) in (3) and bring in similar members. As a result, for *dim* > 2:

$$T_{i_1i_2i_3} = T_{i_1i_2i_3}^{\{\{1,2,3\}\}TF} + T_{i_1i_2i_3}^{\{\{1,2\}\{3\}\}TF} + T_{i_1i_2i_3}^{\{\{1,3\}\{2\}\}TF} + T_{i_1i_2i_3}^{\{\{1\}\{2\}\{3\}\}TF} + \frac{1}{(\dim-1)(\dim+2)}\times$$
$$\times\left\{\delta_{i_1i_2}\left((\dim+1)T_{ssi_3} - T_{i_3ss} - T_{si_3s}\right) + \delta_{i_1i_3}\left((\dim+1)T_{si_2s} - T_{i_2ss} - T_{ssi_2}\right) + \delta_{i_2i_3}\left((\dim+1)T_{i_1ss} - T_{si_1s} - T_{ssi_1}\right)\right\}, \quad (10)$$

where

$$T_{i_1i_2i_3}^{\{\{1,2,3\}\}TF} = \frac{1}{6}\left(T_{k_1k_2k_3} + T_{k_2k_3k_1} + T_{k_3k_1k_2} + T_{k_2k_1k_3} + T_{k_1k_3k_2} + T_{k_3k_2k_1}\right) -$$
$$-\left(\delta_{k_1k_2}\, {}^{\{\{1,2,3\}\}}_{\frac{1}{(\dim+2)}Sp\{1\}}T_{k_3}^{\{1\}} + \delta_{k_1k_3}\, {}^{\{\{1,2,3\}\}}_{\frac{1}{(\dim+2)}Sp\{1\}}T_{k_2}^{\{1\}} + \delta_{k_2k_3}\, {}^{\{\{1,2,3\}\}}_{\frac{1}{(\dim+2)}Sp\{1\}}T_{k_1}^{\{1\}}\right),$$

$$\text{where}\quad {}^{\{\{1,2,3\}\}}_{\frac{1}{(\dim+2)}Sp\{1\}}T_{k} = \frac{1}{3(\dim+2)}\left(T_{ssk} + T_{sks} + T_{kss}\right);$$

$$T_{k_1k_2k_3}^{\{\{1,3\}\{2\}\}TF} = \frac{1}{3}\left(T_{k_1k_2k_3} + T_{k_3k_2k_1} - T_{k_2k_1k_3} - T_{k_3k_1k_2}\right) - \delta_{k_1k_3}\, {}^{\{\{1,3\}\{2\}\}}_{Sp\{(1)(3)\}}T_{k_2}^{\{\{1\}\}} + \delta_{k_2k_3}\, {}^{\{\{1,3\}\{2\}\}}_{Sp\{(1)(3)\}}T_{k_1}^{\{\{1\}\}},$$

$$\text{where}\quad {}^{\{\{1,3\}\{2\}\}}_{Sp\{(1)(3)\}}T_{k}^{\{\{1\}\}} = \frac{1}{3(\dim-1)}\left(2T_{sks} - T_{kss} - T_{ssk}\right);$$

$$T_{k_1k_2k_3}^{\{\{1,2\}\{3\}\}TF} = \frac{1}{3}\left(T_{k_1k_2k_3} + T_{k_2k_1k_3} - T_{k_3k_2k_1} - T_{k_2k_3k_1}\right) - \delta_{k_1k_2}\, {}^{\{\{1,2\}\{3\}\}}_{Sp\{(1)(2)\}}T_{k_3}^{\{\{1\}\}} + \delta_{k_2k_3}\, {}^{\{\{1,2\}\{3\}\}}_{Sp\{(1)(2)\}}T_{k_1}^{\{\{1\}\}},$$

$$\text{where}\quad {}^{\{\{1,2\}\{3\}\}}_{Sp\{(1)(2)\}}T_{k}^{\{\{1\}\}} = \frac{1}{3(\dim-1)}\left(2T_{ssk} - T_{kss} - T_{sks}\right)$$

Now, let us prove that ${}^{\{\{1,2\}\{3\}\}TF}_{\varepsilon\{1\}}T_{ki_2}^{\{\{1,2\}\}TF} = \frac{1}{6}\left(\varepsilon_{ksj}\left(T_{si_2j} + T_{i_2sj}\right) + \varepsilon_{i_2sj}\left(T_{skj} + T_{ksj}\right)\right)$ as suggested above.

The ${}^{\{\{1,2\}\{3\}\}TF}_{\varepsilon\{1\}}T_{ki_2}$ has mixed symmetry. Therefore, it has to be decomposed repeatedly:

$${}^{\{\{1,2\}\{3\}\}TF}_{\varepsilon\{1\}}T_{ki_2} = \frac{1}{2}\left({}^{\{\{1,2\}\{3\}\}TF}_{\varepsilon\{1\}}T_{ki_2} + {}^{\{\{1,2\}\{3\}\}TF}_{\varepsilon\{1\}}T_{i_2k}\right) + \frac{1}{2}\left({}^{\{\{1,2\}\{3\}\}TF}_{\varepsilon\{1\}}T_{ki_2} - {}^{\{\{1,2\}\{3\}\}TF}_{\varepsilon\{1\}}T_{i_2k}\right) =$$
$$= \frac{1}{6}\left(\varepsilon_{ksj}\left(T_{si_2j} + T_{i_2sj}\right) + \varepsilon_{i_2sj}\left(T_{skj} + T_{ksj}\right)\right) + \frac{1}{6}\left(\varepsilon_{ksj}\left(T_{si_2j} + T_{i_2sj}\right) - \varepsilon_{i_2sj}\left(T_{skj} + T_{ksj}\right)\right) +$$
$$+\frac{1}{6}\varepsilon_{kji_2}\left(2T_{ppj} - T_{jpp} - T_{pjp}\right)$$

$${}^{\{\{1,2\}\{3\}\}TF}_{\varepsilon\{1\}}T_{ki_2}^{\{\{1,2\}\}TF} = \frac{1}{6}\left(\varepsilon_{ksj}\left(T_{si_2j} + T_{i_2sj}\right) + \varepsilon_{i_2sj}\left(T_{skj} + T_{ksj}\right)\right) - \frac{1}{18}\delta_{ki_2}\underbrace{\left(-\varepsilon_{spj}T_{spj} + \varepsilon_{psj}T_{psj} - \varepsilon_{psj}T_{psj} + \varepsilon_{psj}T_{psj}\right)}_{=0}$$

$${}^{\{\{1,2\}\{3\}\}TF}_{\varepsilon\{1\}}T_{ki_2}^{\{\{1\}\{2\}\}TF} = \frac{1}{6}\left(\varepsilon_{ksj}\left(T_{si_2j} + T_{i_2sj}\right) - \varepsilon_{i_2sj}\left(T_{skj} + T_{ksj}\right)\right) + \frac{1}{6}\varepsilon_{kji_2}\left(2T_{ppj} - T_{jpp} - T_{pjp}\right)$$

With antisymmetric members, one can execute repeat dualization:

$$ {}_{\varepsilon\{1\}}\Big|^{\{\{1,2\}\{3\}\}TF}_{\varepsilon\{1\}} T^{\{\{1\}\{2\}\}}_{ki_2} = \varepsilon_{ki_2m}\ {}^{\{\{1,2\}\{3\}\}TF}_{\varepsilon\{1\}}\Big|^{\{\{1\}\{2\}\}}_{\varepsilon\{1\}} T_m $$

$$ {}^{\{\{1,2\}\{3\}\}TF}_{\varepsilon\{1\}}\Big|^{\{\{1\}\{2\}\}}_{\varepsilon\{1\}} T_m = -\varepsilon_{mwl}\frac{1}{12}\left(\varepsilon_{lsj}\left(T_{swj}+T_{wsj}\right)-\varepsilon_{wsj}\left(T_{slj}+T_{lsj}\right)\right)+\frac{1}{12}\varepsilon_{wml}\varepsilon_{ljw}\left(2T_{ppj}-T_{jpp}-T_{pjp}\right)= $$

$$ =-\frac{1}{12}\left(\delta_{ms}\delta_{wj}-\delta_{mj}\delta_{ws}\right)\left(T_{swj}+T_{wsj}\right)-\frac{1}{12}\left(\delta_{ms}\delta_{lj}-\delta_{mj}\delta_{ls}\right)\left(T_{slj}+T_{lsj}\right)-\frac{1}{12}2\delta_{mj}\left(2T_{ppj}-T_{jpp}-T_{pjp}\right)= $$

$$ =-\frac{1}{12}\left(T_{mjj}+T_{jmj}-2T_{ssm}\right)-\frac{1}{12}\left(T_{mss}+T_{sms}-2T_{ssm}\right)-\frac{1}{6}\left(2T_{ppm}-T_{mpp}-T_{pmp}\right)=0 $$

Then,

$$ \begin{aligned} T^{\{\{1,2\}\{3\}\}TF}_{i_1i_2i_3} &= \varepsilon_{i_1i_3k}\ {}^{\{\{1,2\}\{3\}\}TF}_{\varepsilon\{1\}}T_{ki_2} = \varepsilon_{i_1i_3k}\ {}^{\{\{1,2\}\{3\}\}TF}_{\varepsilon\{1\}}T^{\{\{1,2\}\}TF}_{ki_2} + \underbrace{\varepsilon_{i_1i_3k}\ {}^{\{\{1,2\}\{3\}\}TF}_{\varepsilon\{1\}}T^{\{\{1\}\{2\}\}}_{ki_2}}_{=0} = \\ &= \varepsilon_{i_1i_3k}\ {}^{\{\{1,2\}\{3\}\}TF}_{\varepsilon\{1\}}T^{\{\{1,2\}\}TF}_{ki_2} = \varepsilon_{i_1i_3k}\frac{1}{6}\left(\varepsilon_{ksj}\left(T_{si_2j}+T_{i_2sj}\right)+\varepsilon_{i_2sj}\left(T_{skj}+T_{ksj}\right)\right) \end{aligned} \qquad (11) $$

where ${}^{\{\{1,2\}\{3\}\}TF}_{\varepsilon\{1\}}T^{\{\{1,2\}\}TF}_{ki_2} = \frac{1}{6}\left(\varepsilon_{ksj}\left(T_{si_2j}+T_{i_2sj}\right)+\varepsilon_{i_2sj}\left(T_{skj}+T_{ksj}\right)\right)$ and the proof is concluded.

3. Results

The results of all evaluations (for all cases mentioned above) are shown below in tabular form for conveniency (see Tables 1–10).

3.1. Result for a second-rank tensor

These results (Tables 1 and 2) are trivial and often presented by several authors; however, for a better understanding of the other results and notations in this paper, they remain useful.

Table 1. Decomposition of a second-rank tensor.

GROUP DESIGNATION *GL(2)* *Direct decomposition [7]*	*O(2), SO(2)*
$T_{i_1i_2} = T^{\{\{1,2\}\}}_{i_1i_2} + T^{\{\{1\}\{2\}\}}_{i_1i_2}$	$T_{i_1i_2} = T^{\{\{1,2\}\}TF}_{i_1i_2} + T^{\{\{1\}\{2\}\}}_{i_1i_2} + \delta_{i_1i_2}\ {}^{\{\{1,2\}\}}_{\frac{1}{\dim}Sp\{(1)(2)\}}T$

where

$T^{\{\{1,2\}\}}_{i_1i_2} = \frac{1}{2}\left(T_{i_1i_2}+T_{i_2i_1}\right)$; $T^{\{\{1\}\{2\}\}}_{i_1i_2} = \frac{1}{2}\left(T_{i_1i_2}-T_{i_2i_1}\right)$; $T^{\{\{1,2\}\}TF}_{i_1i_2} = \frac{1}{2}\left(T_{i_1i_2}+T_{i_2i_1}\right)-\delta_{i_1i_2}\ {}^{\{\{1,2\}\}}_{\frac{1}{\dim}Sp\{(1)(2)\}}T$

$$ {}^{\{\{1,2\}\}}_{\frac{1}{\dim}Sp\{(1)(2)\}}T = \frac{1}{\dim}T^{\{\{1,2\}\}}_{kk} = {}_{\frac{1}{\dim}Sp\{(1)(2)\}}T = \frac{1}{\dim}T_{kk} $$

$$ {}_{TP\{(1,2)\}}\Big|T_{i_1i_2} = \delta_{i_1i_2}\ {}^{\{\{1,2\}\}}_{\frac{1}{\dim}Sp\{(1)(2)\}}T = \delta_{i_1i_2}\frac{1}{\dim}T^{\{\{1,2\}\}}_{kk} = \delta_{i_1i_2}\frac{1}{\dim}T_{kk} $$

with transformation by antisymmetric index group dualization if appropriate

Continued on next page

dim=2

$$T_{i_1i_2} = T^{\{\{1,2\}\}}_{i_1i_2} + e_{i_1i_2}\,{}^{\{\{1\}\{2\}\}}_{e\{1\}}T = $$
$$= T^{\{\{1,2\}\}}_{i_1i_2} + \varepsilon_{i_1i_2}\,{}^{\{\{1\}\{2\}\}}_{\varepsilon\{1\}}T$$

$$T_{i_1i_2} = T^{\{\{1,2\}\}TF}_{i_1i_2} + e_{i_1i_2}\,{}^{\{\{1\}\{2\}\}}_{e\{1\}}T + \delta_{i_1i_2}\,{}^{\{\{1,2\}\}}_{\frac{1}{\dim}Sp\{(1)(2)\}}T =$$
$$= T^{\{\{1,2\}\}TF}_{i_1i_2} + \varepsilon_{i_1i_2}\,{}^{\{\{1\}\{2\}\}}_{\varepsilon\{1\}}T + \delta_{i_1i_2}\,{}^{\{\{1,2\}\}}_{\frac{1}{\dim}Sp\{(1)(2)\}}T$$

where ${}^{\{\{1\}\{2\}\}}_{e\{1\}}T = \frac{1}{2} e_{kj} T_{kj}$, ${}^{\{\{1\}\{2\}\}}_{\varepsilon\{1\}}T = \frac{1}{2} \varepsilon_{kj} T_{kj}$

dim=3

$$T_{i_1i_2} = T^{\{\{1,2\}\}}_{i_1i_2} + e_{i_1i_2k}\,{}^{\{\{1\}\{2\}\}}_{e\{1\}}T_k = $$
$$= T^{\{\{1,2\}\}}_{i_1i_2} + \varepsilon_{i_1i_2k}\,{}^{\{\{1\}\{2\}\}}_{\varepsilon\{1\}}T_k$$

$$T_{i_1i_2} = T^{\{\{1,2\}\}TF}_{i_1i_2} + e_{i_1i_2k}\,{}^{\{\{1\}\{2\}\}}_{e\{1\}}T_k + \delta_{i_1i_2}\,{}^{\{\{1,2\}\}}_{\frac{1}{\dim}Sp\{(1)(2)\}}T =$$
$$= T^{\{\{1,2\}\}TF}_{i_1i_2} + \varepsilon_{i_1i_2k}\,{}^{\{\{1\}\{2\}\}}_{\varepsilon\{1\}}T_k + \delta_{i_1i_2}\,{}^{\{\{1,2\}\}}_{\frac{1}{\dim}Sp\{(1)(2)\}}T$$

where ${}^{\{\{1\}\{2\}\}}_{e\{1\}}T_k = \frac{1}{2} e_{kps} T_{ps}$, ${}^{\{\{1\}\{2\}\}}_{\varepsilon\{1\}}T_k = \frac{1}{2} \varepsilon_{kps} T_{ps}$

The dualization does not provide a simplification for a second-rank fully antisymmetric tensor for dim > 3

Finally, expressions in which components with certain *YSI* are obtained from the starting tensor by operations of symmetrization, antisymmetrization, and deviatorization, are denoted by brackets:

$$T^{\{\{1,2\}\}}_{i_1i_2} = T_{(i_1i_2)};\ T^{\{\{1\}\{2\}\}}_{i_1i_2} = T_{[i_1i_2]};\quad T^{\{\{1,2\}\}TF}_{i_1i_2} = T_{<i_1i_2>}.$$

Table 2. Numbers of independent components in spaces of a certain dimension.

Space dimension	Symmetry index {{1,2}}	{{1,2}}*TF*	{{1}{2}}	{}
dim	$\frac{\dim(\dim+1)}{2}$	$\frac{\dim(\dim+1)}{2}-1$	$\frac{\dim(\dim-1)}{2}$	1
2	3	2	1	1
3	6	5	3	1
4	10	9	6	1
5	15	14	10	1
10	55	54	45	1
11	66	65	55	1
16	136	135	120	1

In fact, the scalar and vector do not have *YSI*, as they do not have enough indices for any permutation. Still, it is possible to define *YSI* {1} (Young tableaux with one cell only) and {} (empty set of Young Tableaux) for complete representation.

3.2. Result for a third-rank tensor

As pseudotensor dualization visually differs from tensor dualization only by changing the fully unit antisymmetric tensor of space by the fully unit antisymmetric symbol Levi-Civita, in the following formulas, only the pseudotensor dualization form will be presented.

Results for of a third-rank tensor are presented in Tables 3 and 4 for GL(dim) and in Tables 5 and 6 for decomposition on trace-free components.

Table 3. Decomposition of a third-rank tensor.

Direct decomposition [7], GL(dim)

$$T_{i_1i_2i_3}=T_{i_1i_2i_3}^{\{\{1,2,3\}\}}+T_{i_1i_2i_3}^{\{\{1,2\}\{3\}\}}+T_{i_1i_2i_3}^{\{\{1,3\}\{2\}\}}+(1-\delta_{2\,\mathrm{dim}})T_{i_1i_2i_3}^{\{\{1\}\{2\}\{3\}\}}$$

where:

$$T_{k_1k_2k_3}^{\{\{1,2,3\}\}}=T_{(k_1k_2k_3)}=\frac{1}{6}\left(T_{k_1k_2k_3}+T_{k_2k_3k_1}+T_{k_3k_1k_2}+T_{k_2k_1k_3}+T_{k_1k_3k_2}+T_{k_3k_2k_1}\right)$$

$$T_{k_1k_2k_3}^{\{\{1\}\{2\}\{3\}\}}=T_{[k_1k_2k_3]}=\frac{1}{6}\left(T_{k_1k_2k_3}+T_{k_2k_3k_1}+T_{k_3k_1k_2}-T_{k_2k_1k_3}-T_{k_1k_3k_2}-T_{k_3k_2k_1}\right)$$

$$T_{k_1k_2k_3}^{\{\{1,2\}\{3\}\}}=\frac{1}{3}\left(T_{k_1k_2k_3}+T_{k_2k_1k_3}-T_{k_3k_2k_1}-T_{k_2k_3k_1}\right)=\frac{2}{3}\left(T_{[k_1|k_2|k_3]}+T_{k_2[k_1k_3]}\right)$$

$$T_{k_1k_2k_3}^{\{\{1,3\}\{2\}\}}=\frac{1}{3}\left(T_{k_1k_2k_3}+T_{k_3k_2k_1}-T_{k_2k_1k_3}-T_{k_3k_1k_2}\right)=\frac{2}{3}\left(T_{[k_1k_2]k_3}+T_{k_3[k_2k_1]}\right)$$

with transformation by antisymmetric index group dualization if appropriate

dim=2

$$T_{k_1k_2k_3}^{\{\{1,2\}\{3\}\}}=\varepsilon_{k_1k_3}\,{}^{\{\{1,2\}\{3\}\}}_{\varepsilon\{1\}}T_{k_2}\,,\qquad T_{k_1k_2k_3}^{\{\{1,3\}\{2\}\}}=\varepsilon_{k_1k_2}\,{}^{\{\{1,3\}\{2\}\}}_{\varepsilon\{1\}}T_{k_3}\,;$$

where:

$${}^{\{\{1,2\}\{3\}\}}_{\varepsilon\{1\}}T_{k_2}=\frac{1}{2}\varepsilon_{i_1i_3}T_{i_1k_2i_3}^{\{\{1,2\}\{3\}\}}=\frac{1}{3}\varepsilon_{i_1i_3}\left(T_{[i_1|k_2|i_3]}+T_{k_2[i_1i_3]}\right),\quad {}^{\{\{1,3\}\{2\}\}}_{\varepsilon\{1\}}T_{k_3}=\frac{1}{2}\varepsilon_{i_1i_2}T_{i_1i_2k_3}^{\{\{1,3\}\{2\}\}}=\frac{1}{3}\varepsilon_{i_1i_2}\left(T_{[i_1i_2]k_3}+T_{k_3[i_2i_1]}\right).$$

($T_{i_1i_2i_3}^{\{\{1\}\{2\}\{3\}\}}\equiv 0$ for *dim*=2, according to the first selection rule)

dim=3

$$T_{k_1k_2k_3}^{\{\{1,2\}\{3\}\}}=\varepsilon_{i_1i_3k}\,{}^{\{\{1,2\}\{3\}\}TF}_{\varepsilon\{1\}}T_{ki_2}^{\{\{1,2\}\}TF}+\delta_{k_1k_2}\,{}^{\{\{1,2\}\{3\}\}}T_{k_3}^{(3)}-\delta_{k_2k_3}\,{}^{\{\{1,2\}\{3\}\}}T_{k_1}^{(3)}\,;$$

$$T_{k_1k_2k_3}^{\{\{1,3\}\{2\}\}}=\varepsilon_{k_1k_2i}\,{}^{\{\{1,3\}\{2\}\}TF}_{\varepsilon\{1\}}T_{ik_3}^{\{\{1,2\}\}TF}+\delta_{k_1k_3}\,{}^{\{\{1,3\}\{2\}\}}T_{k_2}^{(3)}-\delta_{k_2k_3}\,{}^{\{\{1,3\}\{2\}\}}T_{k_1}^{(3)}\,;$$

$$T_{k_1k_2k_3}^{\{\{1\}\{2\}\{3\}\}}=\varepsilon_{k_1k_2k_3}\,{}^{\{\{1\}\{2\}\{3\}\}}T^{(3)}\,;$$

where:

$${}^{\{\{1,2\}\{3\}\}TF}_{\varepsilon\{1\}}T_{ki_2}^{\{\{1,2\}\}TF}=\frac{1}{6}\left(\varepsilon_{ksj}\left(T_{si_2j}+T_{i_2sj}\right)+\varepsilon_{i_2sj}\left(T_{skj}+T_{ksj}\right)\right),\quad {}^{\{\{1,2\}\{3\}\}}T_k^{(3)}\equiv\frac{1}{6}\left(2T_{ssk_3}-T_{k_3ss}-T_{sk_3s}\right);$$

$${}^{\{\{1,3\}\{2\}\}TF}_{\varepsilon\{1\}}T_{ik_3}^{\{\{1,2\}\}TF}=\frac{1}{6}\left(\varepsilon_{isj}\left(T_{sjk_3}-T_{k_3sj}\right)+\varepsilon_{k_3sj}\left(T_{sji}-T_{isj}\right)\right),\quad {}^{\{\{1,3\}\{2\}\}}T_k^{(3)}\equiv\frac{1}{6}\left(2T_{sks}-T_{kss}-T_{ssk}\right);$$

$${}^{\{\{1\}\{2\}\{3\}\}}T^{(3)}=\frac{1}{6}\varepsilon_{kjm}T_{kjm}\,.$$

The dualization does not provide a simplification for third-rank tensors with mixed symmetry for dim > 3

dim=4

$$T_{k_1k_2k_3}^{\{\{1\}\{2\}\{3\}\}}=\varepsilon_{k_1k_2k_3m}\,{}^{\{\{1\}\{2\}\{3\}\}}T_m^{(4)}\,,\ \textit{where: }\ {}^{\{\{1\}\{2\}\{3\}\}}T_m^{(4)}=-\frac{1}{6}\varepsilon_{mijk}T_{ijk}\,;$$

dim=5

$$T_{k_1k_2k_3}^{\{\{1\}\{2\}\{3\}\}}=\frac{1}{2}\varepsilon_{k_1k_2k_3ml}\,{}^{\{\{1\}\{2\}\{3\}\}}T_{ml}^{(5)}\,,\ \textit{where: }\ {}^{\{\{1\}\{2\}\{3\}\}}T_{ml}^{(5)}\equiv\frac{1}{6}\varepsilon_{mlijk}T_{ijk}\,.$$

The dualization does not provide a simplification for a third-rank fully antisymmetric tensor for dim > 5

Table 4. Numbers of independent components in spaces of certain dimension.

Space dimension	Young symmetry index {{1,2,3}}	{{1,2}{3}}	{{1,3}{2}}	{{1}{2}{3}}	SUM
Dim	$\frac{(\dim+2)(\dim+1)\dim}{6}$	$\frac{\dim(\dim^2-1)}{3}$	$\frac{\dim(\dim^2-1)}{3}$	$\frac{\dim(\dim-1)(\dim-2)}{6}$	$\dim^3$
2	4	2	2	–	8
3	10	8	8	1	27
4	20	20	20	4	64
5	35	40	40	10	125
10	220	330	330	120	1000
11	286	440	440	165	1331
16	816	1360	1360	560	4096

Table 5. Decomposition of a third-rank tensor.

Direct decomposition according to algorithm [7], SO(dim)

$$T_{i_1i_2i_3} = T^{\{\{1,2,3\}\}TF}_{i_1i_2i_3} + (1-\delta_{2\dim})\left(T^{\{\{1,2\}\{3\}\}TF}_{i_1i_2i_3} + T^{\{\{1,3\}\{2\}\}TF}_{i_1i_2i_3} + T^{\{\{1\}\{2\}\{3\}\}TF}_{i_1i_2i_3}\right)+$$

$$+\frac{1}{(\dim-1)(\dim+2)}\left\{\delta_{i_1i_2}\left((\dim+1)T_{ssi_3} - T_{i_3ss} - T_{si_3s}\right)+\right.$$

$$\left.+\delta_{i_1i_3}\left((\dim+1)T_{si_2s} - T_{i_2ss} - T_{ssi_2}\right) + \delta_{i_2i_3}\left((\dim+1)T_{i_1ss} - T_{si_1s} - T_{ssi_1}\right)\right\}$$

(according to the first and second selection rules, $T^{\{\{1,2\}\{3\}\}TF}_{i_1i_2i_3}$, $T^{\{\{1,3\}\{2\}\}TF}_{i_1i_2i_3}$, and $T^{\{\{1\}\{2\}\{3\}\}TF}_{i_1i_2i_3}$ are equal to zero and identical in a two-dimensional space),

where

$$T^{\{\{1,2,3\}\}TF}_{i_1i_2i_3} = \frac{1}{6}\left(T_{k_1k_2k_3} + T_{k_2k_3k_1} + T_{k_3k_1k_2} + T_{k_2k_1k_3} + T_{k_1k_3k_2} + T_{k_3k_2k_1}\right) -$$

$$-\left(\delta_{k_1k_2}\,{}^{\{\{1,2,3\}\}}_{\frac{1}{(\dim+2)}Sp\{1\}}T^{\{1\}}_{k_3} + \delta_{k_1k_3}\,{}^{\{\{1,2,3\}\}}_{\frac{1}{(\dim+2)}Sp\{1\}}T^{\{1\}}_{k_2} + \delta_{k_2k_3}\,{}^{\{\{1,2,3\}\}}_{\frac{1}{(\dim+2)}Sp\{1\}}T^{\{1\}}_{k_1}\right)$$

Where

$${}^{\{\{1,2,3\}\}}_{\frac{1}{(\dim+2)}Sp\{1\}}T_k = \frac{1}{3(\dim+2)}\left(T_{ssk} + T_{sks} + T_{kss}\right)$$

for dim > 2:

$$T^{\{\{1,3\}\{2\}\}TF}_{k_1k_2k_3} = \frac{1}{3}\left(T_{k_1k_2k_3} + T_{k_3k_2k_1} - T_{k_2k_1k_3} - T_{k_3k_1k_2}\right) -$$

$$-\delta_{k_1k_3}\,{}^{\{\{1,3\}\{2\}\}}_{Sp\{(1)(3)\}}T^{\{\{1\}\}}_{k_2} + \delta_{k_2k_3}\,{}^{\{\{1,3\}\{2\}\}}_{Sp\{(1)(3)\}}T^{\{\{1\}\}}_{k_1}$$

Where

$${}^{\{\{1,3\}\{2\}\}}_{Sp\{(1)(3)\}}T^{\{\{1\}\}}_{k} = \frac{1}{3(\dim-1)}\left(2T_{sks} - T_{kss} - T_{ssk}\right)$$

$$T^{\{\{1,2\}\{3\}\}TF}_{k_1k_2k_3} = \frac{1}{3}\left(T_{k_1k_2k_3} + T_{k_2k_1k_3} - T_{k_3k_2k_1} - T_{k_2k_3k_1}\right) -$$

$$-\delta_{k_1k_2}\,{}^{\{\{1,2\}\{3\}\}}_{Sp\{(1)(2)\}}T^{\{\{1\}\}}_{k_3} + \delta_{k_2k_3}\,{}^{\{\{1,2\}\{3\}\}}_{Sp\{(1)(2)\}}T^{\{\{1\}\}}_{k_1}$$

Where

Continued on next page

$$ {}^{\{\{1,2\}\{3\}\}}_{Sp\{(1)(2)\}}T^{\{\{1\}\}}_{k} = \frac{1}{3(\dim-1)}\left(2T_{ssk} - T_{kss} - T_{sks}\right) $$

$$ T^{\{\{1\}\{2\}\{3\}\}}_{k_1k_2k_3} = \frac{1}{6}\left(T_{k_1k_2k_3} + T_{k_2k_3k_1} + T_{k_3k_1k_2} - T_{k_2k_1k_3} - T_{k_1k_3k_2} - T_{k_3k_2k_1}\right) $$

with transformation by antisymmetric index group dualization if appropriate

dim=2

According to the first and second selection rules, $T^{\{\{1\}\{2\}\{3\}\}TF}_{i_1i_2i_3} \equiv 0$, $T^{\{\{1,3\}\{2\}\}TF}_{i_1i_2i_3} \equiv 0$, and $T^{\{\{1,2\}\{3\}\}TF}_{i_1i_2i_3} \equiv 0$ for *dim* = 2; therefore, components with antisymmetric index groups do not exist

dim=3

$T^{\{\{1,2\}\{3\}\}TF}_{k_1k_2k_3} = \varepsilon_{k_1k_3k}\,{}^{\{\{1,2\}\{3\}\}}_{\varepsilon\{1\}}T^{\{\{1,2\}\}TF}_{kk_2}$, $T^{\{\{1,3\}\{2\}\}TF}_{k_1k_2k_3} = \varepsilon_{k_1k_2k}\,{}^{\{\{1,3\}\{2\}\}}_{\varepsilon\{1\}}T^{\{\{1,2\}\}TF}_{kk_3}$,

$T^{\{\{1\}\{2\}\{3\}\}}_{k_1k_2k_3} = \varepsilon_{k_1k_2k_3}\,{}^{\{\{1\}\{2\}\{3\}\}}_{\varepsilon\{1\}}T$.

where

$$ {}^{\{\{1,3\}\{2\}\}}_{\varepsilon\{1\}}T^{\{\{1,2\}\}TF}_{kk_3} = \frac{1}{6}\left(\varepsilon_{kjp}\left(T_{jpk_3} - T_{k_3jp}\right) + \varepsilon_{k_3jp}\left(T_{jpk} - T_{kjp}\right)\right), $$

$$ {}^{\{\{1,2\}\{3\}\}}_{\varepsilon\{1\}}T^{\{\{1,2\}\}TF}_{kk_2} = \frac{1}{6}\left(\varepsilon_{kjp}\left(T_{jk_2p} + T_{k_2jp}\right) + \varepsilon_{k_2jp}\left(T_{jkp} + T_{kjp}\right)\right), \quad {}^{\{\{1\}\{2\}\{3\}\}}_{\varepsilon\{1\}}T^{\{\}} = \frac{1}{6}\varepsilon_{kjm}T_{kjm}. $$

In that case,

$$ T_{i_1i_2i_3} = T^{\{\{1,2,3\}\}TF}_{i_1i_2i_3} + \varepsilon_{i_1i_3k}\,{}^{\{\{1,2\}\{3\}\}}_{\varepsilon\{1\}}T^{\{\{1,2\}\}TF}_{ki_2} + \varepsilon_{i_1i_2k}\,{}^{\{\{1,3\}\{2\}\}}_{\varepsilon\{1\}}T^{\{\{1,2\}\}TF}_{ki_3} + \varepsilon_{i_1i_2i_3}\,{}^{\{\{1\}\{2\}\{3\}\}}_{\varepsilon\{1\}}T + $$

$$ + \frac{1}{10}\left\{\delta_{i_1i_2}\left(4T_{ssi_3} - T_{i_3ss} - T_{si_3s}\right) + \delta_{i_1i_3}\left(4T_{si_2s} - T_{i_2ss} - T_{ssi_2}\right) + \delta_{i_2i_3}\left(4T_{i_1ss} - T_{si_1s} - T_{ssi_1}\right)\right\} $$

The dualization does not provide a simplification for third-rank tensors with mixed symmetry for dim > 3

dim = 4

$T^{\{\{1\}\{2\}\{3\}\}TF}_{k_1k_2k_3} = \varepsilon_{k_1k_2k_3m}\,{}^{\{\{1\}\{2\}\{3\}\}}T^{(4)}_{m}$, *where* ${}^{\{\{1\}\{2\}\{3\}\}}T^{(4)}_{m} = -\frac{1}{6}\varepsilon_{mijk}T_{ijk}$;

dim = 5

$T^{\{\{1\}\{2\}\{3\}\}TF}_{k_1k_2k_3} = \frac{1}{2}\varepsilon_{k_1k_2k_3ml}\,{}^{\{\{1\}\{2\}\{3\}\}}T^{(5)}_{ml}$, *where* ${}^{\{\{1\}\{2\}\{3\}\}}T^{(5)}_{ml} \equiv \frac{1}{6}\varepsilon_{mlijk}T_{ijk}$.

The dualization does not provide a simplification for a third-rank fully antisymmetric tensor for dim > 5

Table 6. Numbers of independent components in spaces of certain dimensions.

Space dimension	Young symmetry index for trace-free tensors				SUM
	$\{\{1,2,3\}\}TF$	$\{\{1,2\}\{3\}\}TF$	$\{\{1,3\}\{2\}\}TF$	$\{\{1\}\{2\}\{3\}\}TF$	
dim	$\frac{\dim}{6}\left(\dim^2+3\dim-4\right)$	$\frac{(\dim+2)\dim(\dim-2)}{3}$	$\frac{(\dim+2)\dim(\dim-2)}{3}$	$\frac{\dim(\dim-1)(\dim-2)}{6}$	$\dim\left(\dim^2-3\right)$
2	2	–	–	–	2
3	7	5	5	1	18
4	16	16	16	4	52
5	30	35	35	10	110
10	210	320	320	120	970
11	275	429	429	165	1298
16	800	1344	1344	560	4048

3.3. Result for a fourth-rank tensor

Results for of a fourth-rank tensor are presented in Tables 7 and 8 for GL(dim) and in Tables 5 and 6 for decomposition on trace-free components.

Table 7. Decomposition of a fourth-rank tensor.

Direct decomposition according to algorithm [7], GL(dim)

$$T_{i_1i_2i_3i_4}=T^{\{\{1,2,3,4\}\}}_{i_1i_2i_3i_4}+T^{\{\{1,2,3\}\{4\}\}}_{i_1i_2i_3i_4}+T^{\{\{1,2,4\}\{3\}\}}_{i_1i_2i_3i_4}+T^{\{\{1,2\}\{3,4\}\}}_{i_1i_2i_3i_4}+T^{\{\{1,3,4\}\{2\}\}}_{i_1i_2i_3i_4}+T^{\{\{1,3\}\{2,4\}\}}_{i_1i_2i_3i_4}+$$
$$+(1-\delta_{2\dim})\left(T^{\{\{1,2\}\{3\}\{4\}\}}_{i_1i_2i_3i_4}+T^{\{\{1,3\}\{2\}\{4\}\}}_{i_1i_2i_3i_4}+T^{\{\{1,4\}\{2\}\{3\}\}}_{i_1i_2i_3i_4}+(1-\delta_{3\dim})T^{\{\{1\}\{2\}\{3\}\{4\}\}}_{i_1i_2i_3i_4}\right)$$

(according to the first selection rules, $T^{\{\{1,2\}\{3\}\{4\}\}}_{i_1i_2i_3i_4}$, $T^{\{\{1,3\}\{2\}\{4\}\}}_{i_1i_2i_3i_4}$, $T^{\{\{1,4\}\{2\}\{3\}\}}_{i_1i_2i_3i_4}$, and $T^{\{\{1\}\{2\}\{3\}\{4\}\}}_{i_1i_2i_3i_4}$ are equal to zero and identical in a two-dimensional space),
where

$$T^{\{\{1,2,3,4\}\}}_{i_1i_2i_3i_4}=\frac{1}{24}\Big\{T_{i_1i_2i_3i_4}+T_{i_1i_2i_4i_3}+T_{i_1i_3i_2i_4}+T_{i_1i_3i_4i_2}+T_{i_1i_4i_3i_2}+T_{i_1i_4i_2i_3}+$$
$$+T_{i_2i_1i_3i_4}+T_{i_2i_1i_4i_3}+T_{i_2i_3i_1i_4}+T_{i_2i_3i_4i_1}+T_{i_2i_4i_3i_1}+T_{i_2i_4i_1i_3}+$$
$$+T_{i_3i_2i_1i_4}+T_{i_3i_2i_4i_1}+T_{i_3i_1i_2i_4}+T_{i_3i_1i_4i_2}+T_{i_3i_4i_1i_2}+T_{i_3i_4i_2i_1}+$$
$$+T_{i_4i_2i_3i_1}+T_{i_4i_2i_1i_3}+T_{i_4i_3i_2i_1}+T_{i_4i_3i_1i_2}+T_{i_4i_1i_3i_2}+T_{i_4i_1i_2i_3}\Big\}$$

$$T^{\{\{1,2,3\}\{4\}\}}_{i_1i_2i_3i_4}=\frac{1}{8}\Big\{T_{i_1i_2i_3i_4}-T_{i_4i_2i_3i_1}+T_{i_1i_3i_2i_4}-T_{i_4i_3i_2i_1}+T_{i_2i_1i_3i_4}-T_{i_2i_4i_3i_1}+$$
$$+T_{i_2i_3i_1i_4}-T_{i_2i_3i_4i_1}+T_{i_3i_2i_1i_4}-T_{i_3i_2i_4i_1}+T_{i_3i_1i_2i_4}-T_{i_3i_4i_2i_1}\Big\}$$

$$T^{\{\{1,2,4\}\{3\}\}}_{i_1i_2i_3i_4}=\frac{1}{8}\Big\{T_{i_1i_2i_3i_4}-T_{i_3i_2i_1i_4}+T_{i_1i_4i_3i_2}-T_{i_3i_4i_1i_2}+T_{i_2i_1i_3i_4}-T_{i_2i_3i_1i_4}+$$
$$+T_{i_2i_4i_3i_1}-T_{i_2i_4i_1i_3}+T_{i_4i_2i_3i_1}-T_{i_4i_2i_1i_3}+T_{i_4i_1i_3i_2}-T_{i_4i_3i_1i_2}\Big\}$$

$$T^{\{\{1,2\}\{3,4\}\}}_{i_1i_2i_3i_4}=\frac{1}{12}\Big\{T_{i_1i_2i_3i_4}-T_{i_1i_4i_3i_2}-T_{i_3i_2i_1i_4}+T_{i_3i_4i_1i_2}+T_{i_1i_2i_4i_3}-T_{i_1i_4i_2i_3}-$$
$$-T_{i_3i_2i_4i_1}+T_{i_3i_4i_2i_1}+T_{i_2i_1i_3i_4}-T_{i_4i_1i_3i_2}-T_{i_2i_3i_1i_4}+T_{i_4i_3i_1i_2}+$$
$$+T_{i_2i_1i_4i_3}-T_{i_4i_1i_2i_3}-T_{i_2i_3i_4i_1}+T_{i_4i_3i_2i_1}\Big\}$$

$$T^{\{\{1,3,4\}\{2\}\}}_{i_1i_2i_3i_4}=\frac{1}{8}\Big\{T_{i_1i_2i_3i_4}-T_{i_2i_1i_3i_4}+T_{i_1i_2i_4i_3}-T_{i_2i_1i_4i_3}+T_{i_3i_2i_1i_4}-T_{i_3i_1i_2i_4}+$$
$$+T_{i_3i_2i_4i_1}-T_{i_3i_1i_4i_2}+T_{i_4i_2i_3i_1}-T_{i_4i_1i_3i_2}+T_{i_4i_2i_1i_3}-T_{i_4i_1i_2i_3}\Big\}$$

Continued on next page

$$T^{\{\{1,3\}\{2,4\}\}}_{i_1i_2i_3i_4} = \frac{1}{12}\Big\{T_{i_1i_2i_3i_4} - T_{i_1i_2i_4i_3} - T_{i_2i_1i_3i_4} + T_{i_2i_1i_4i_3} + T_{i_1i_4i_3i_2} - T_{i_1i_3i_4i_2} -$$
$$- T_{i_2i_4i_3i_1} + T_{i_2i_3i_4i_1} + T_{i_3i_2i_1i_4} - T_{i_4i_2i_1i_3} - T_{i_3i_1i_2i_4} + T_{i_4i_1i_2i_3} +$$
$$+ T_{i_3i_4i_1i_2} - T_{i_4i_3i_1i_2} - T_{i_3i_4i_2i_1} + T_{i_4i_3i_2i_1}\Big\}$$

for *dim* > 2:

$$T^{\{\{1,2\}\{3\}\{4\}\}}_{i_1i_2i_3i_4} = \frac{1}{8}\Big\{T_{i_1i_2i_3i_4} - T_{i_1i_2i_4i_3} - T_{i_3i_2i_1i_4} + T_{i_3i_2i_4i_1} - T_{i_4i_2i_3i_1} + T_{i_4i_2i_1i_3} +$$
$$+ T_{i_2i_1i_3i_4} - T_{i_2i_1i_4i_3} - T_{i_2i_3i_1i_4} + T_{i_2i_3i_4i_1} - T_{i_2i_4i_3i_1} + T_{i_2i_4i_1i_3}\Big\}$$

$$T^{\{\{1,3\}\{2\}\{4\}\}}_{i_1i_2i_3i_4} = \frac{1}{8}\Big\{T_{i_1i_2i_3i_4} - T_{i_1i_4i_3i_2} - T_{i_2i_1i_3i_4} + T_{i_2i_4i_3i_1} - T_{i_4i_2i_3i_1} + T_{i_4i_1i_3i_2} +$$
$$+ T_{i_3i_2i_1i_4} - T_{i_3i_4i_1i_2} - T_{i_3i_1i_2i_4} + T_{i_3i_4i_2i_1} - T_{i_3i_2i_4i_1} + T_{i_3i_1i_4i_2}\Big\}$$

$$T^{\{\{1,4\}\{2\}\{3\}\}}_{i_1i_2i_3i_4} = \frac{1}{8}\Big\{T_{i_1i_2i_3i_4} - T_{i_1i_3i_2i_4} - T_{i_2i_1i_3i_4} + T_{i_2i_3i_1i_4} - T_{i_3i_2i_1i_4} + T_{i_3i_1i_2i_4} +$$
$$+ T_{i_4i_2i_3i_1} - T_{i_4i_3i_2i_1} - T_{i_4i_1i_3i_2} + T_{i_4i_3i_1i_2} - T_{i_4i_2i_1i_3} + T_{i_4i_1i_2i_3}\Big\}$$

for *dim* > 3:

$$T^{\{\{1\}\{2\}\{3\}\{4\}\}}_{i_1i_2i_3i_4} = \frac{1}{24}\Big\{T_{i_1i_2i_3i_4} - T_{i_1i_2i_4i_3} - T_{i_1i_3i_2i_4} + T_{i_1i_3i_4i_2} - T_{i_1i_4i_3i_2} + T_{i_1i_4i_2i_3} -$$
$$- T_{i_2i_1i_3i_4} + T_{i_2i_1i_4i_3} + T_{i_2i_3i_1i_4} - T_{i_2i_3i_4i_1} + T_{i_2i_4i_3i_1} - T_{i_2i_4i_1i_3} -$$
$$- T_{i_3i_2i_1i_4} + T_{i_3i_2i_4i_1} + T_{i_3i_1i_2i_4} - T_{i_3i_1i_4i_2} + T_{i_3i_4i_1i_2} - T_{i_3i_4i_2i_1} -$$
$$- T_{i_4i_2i_3i_1} + T_{i_4i_2i_1i_3} + T_{i_4i_3i_2i_1} - T_{i_4i_3i_1i_2} + T_{i_4i_1i_3i_2} - T_{i_4i_1i_2i_3}\Big\}$$

with transformation by antisymmetric index group dualization if appropriate

dim=2

According to the first selection rules, $T^{\{\{1,2\}\{3\}\{4\}\}}_{i_1i_2i_3i_4} \equiv 0$, $T^{\{\{1,3\}\{2\}\{4\}\}}_{i_1i_2i_3i_4} \equiv 0$, $T^{\{\{1,4\}\{2\}\{3\}\}}_{i_1i_2i_3i_4} \equiv 0$, and $T^{\{\{1\}\{2\}\{3\}\{4\}\}}_{i_1i_2i_3i_4} \equiv 0$.

for dim=2:

$$T^{\{\{1,2,3\}\{4\}\}}_{i_1i_2i_3i_4} = \varepsilon_{i_1i_4}\,{}^{\{\{1,2,3\}\{4\}\}}_{\varepsilon\{1\}}T^{\{\{1,2\}\}}_{i_2i_3};\quad T^{\{\{1,2,4\}\{3\}\}}_{i_1i_2i_3i_4} = \varepsilon_{i_1i_3}\,{}^{\{\{1,2,4\}\{3\}\}}_{\varepsilon\{1\}}T^{\{\{1,2\}\}}_{i_2i_4};\; T^{\{\{1,3,4\}\{2\}\}}_{i_1i_2i_3i_4} = \varepsilon_{i_1i_2}\,{}^{\{\{1,3,4\}\{2\}\}}_{\varepsilon\{1\}}T^{\{\{1,2\}\}}_{i_3i_4};$$

$$T^{\{\{1,3\}\{2,4\}\}}_{i_1i_2i_3i_4} = \varepsilon_{i_1i_2}\varepsilon_{i_3i_4}\,{}^{\{\{1,3\}\{2,4\}\}}_{\varepsilon\{1,2\}}T^{(2)};\quad T^{\{\{1,3\}\{2,4\}\}}_{i_1i_2i_3i_4} = \varepsilon_{i_1i_2}\varepsilon_{i_3i_4}\,{}^{\{\{1,3\}\{2,4\}\}}_{\varepsilon\{1,2\}}T^{(2)}$$

where

$${}^{\{\{1,2,3\}\{4\}\}}_{\varepsilon\{1\}}T^{\{\{1,2\}\}}_{i_2i_3} = \frac{1}{2}\varepsilon_{i_1i_4}T^{\{\{1,2,3\}\{4\}\}}_{i_1i_2i_3i_4} = {}^{\{\{1,2,3\}\{4\}\}}_{\varepsilon\{1\}}T^{\{\{1,2\}\}TF}_{i_2i_3} + \delta_{i_2i_3}\,{}^{\{\{1,2,3\}\{4\}\}}_{\varepsilon\{1\}}\Big|{}^{\{\{1,2\}\}}_{\frac{1}{2}Sp\{1\}}T^{(2)};$$

with ${}^{\{\{1,2,3\}\{4\}\}}_{\varepsilon\{1\}}T^{\{\{1,2\}\}TF}_{i_2i_3} = \frac{1}{2}\varepsilon_{i_1i_4}T^{\{\{1,2,3\}\{4\}\}}_{i_1i_2i_3i_4} - \frac{1}{4}\delta_{i_2i_3}\varepsilon_{i_1i_4}T^{\{\{1,2,3\}\{4\}\}}_{i_1kki_4}$; ${}^{\{\{1,2,3\}\{4\}\}}_{\varepsilon\{1\}}\Big|{}^{\{\{1,2\}\}}_{\frac{1}{2}Sp\{1\}}T^{(2)} = \frac{1}{4}\varepsilon_{i_1i_4}T^{\{\{1,2,3\}\{4\}\}}_{i_1kki_4}$;

$${}^{\{\{1,2,4\}\{3\}\}}_{\varepsilon\{1\}}T^{\{\{1,2\}\}}_{i_2i_4} = \frac{1}{2}\varepsilon_{i_1i_3}T^{\{\{1,2,4\}\{3\}\}}_{i_1i_2i_3i_4} = {}^{\{\{1,2,4\}\{3\}\}}_{\varepsilon\{1\}}T^{\{\{1,2\}\}TF}_{i_2i_4} + \delta_{i_2i_4}\,{}^{\{\{1,2,4\}\{3\}\}}_{\varepsilon\{1\}}\Big|{}^{\{\{1,2\}\}}_{\frac{1}{2}Sp\{1\}}T^{(2)};$$

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with ${}^{\{\{1,2,4\}\{3\}\}}_{\varepsilon\{1\}}T^{\{\{1,2\}\}TF}_{i_2i_4}=\frac{1}{2}\varepsilon_{i_1i_3}T^{\{\{1,2,4\}\{3\}\}}_{i_1i_2i_3i_4}-\frac{1}{4}\delta_{i_2i_4}\varepsilon_{i_1i_3}T^{\{\{1,2,4\}\{3\}\}}_{i_1ki_3k}$; ${}^{\{\{1,2,4\}\{3\}\}}_{\varepsilon\{1\}}\Big|^{\{\{1,2\}\}}_{\frac{1}{2}Sp\{1\}}T^{(2)}=\frac{1}{4}\varepsilon_{i_1i_3}T^{\{\{1,2,4\}\{3\}\}}_{i_1ki_3k}$;

${}^{\{\{1,3,4\}\{2\}\}}_{\varepsilon\{1\}}T^{\{\{1,2\}\}}_{i_3i_4}=\frac{1}{2}\varepsilon_{i_1i_2}T^{\{\{1,3,4\}\{2\}\}}_{i_1i_2i_3i_4}={}^{\{\{1,3,4\}\{2\}\}}_{\varepsilon\{1\}}T^{\{\{1,2\}\}TF}_{i_3i_4}+\delta_{i_3i_4}\,{}^{\{\{1,3,4\}\{2\}\}}_{\varepsilon\{1\}}\Big|^{\{\{1,2\}\}}_{\frac{1}{2}Sp\{1\}}T^{(2)}$;

with ${}^{\{\{1,3,4\}\{2\}\}}_{\varepsilon\{1\}}T^{\{\{1,2\}\}TF}_{i_3i_4}=\frac{1}{2}\varepsilon_{i_1i_2}T^{\{\{1,3,4\}\{2\}\}}_{i_1i_2i_3i_4}-\frac{1}{4}\delta_{i_3i_4}\varepsilon_{i_1i_2}T^{\{\{1,3,4\}\{2\}\}}_{i_1i_2kk}$; ${}^{\{\{1,3,4\}\{2\}\}}_{\varepsilon\{1\}}\Big|^{\{\{1,2\}\}}_{\frac{1}{2}Sp\{1\}}T^{(2)}=\frac{1}{4}\varepsilon_{i_1i_2}T^{\{\{1,3,4\}\{2\}\}}_{i_1i_2kk}$.

${}^{\{\{1,2\}\{3,4\}\}}_{\varepsilon\{1,2\}}T^{(2)}=\frac{1}{4}\varepsilon_{i_1i_3}\varepsilon_{i_2i_4}T^{\{\{1,2\}\{3,4\}\}}_{i_1i_2i_3i_4}$; ${}^{\{\{1,3\}\{2,4\}\}}_{\varepsilon\{1,2\}}T^{(2)}=\frac{1}{4}\varepsilon_{i_1i_2}\varepsilon_{i_3i_4}T^{\{\{1,3\}\{2,4\}\}}_{i_1i_2i_3i_4}$.

---

dim = 3

According to the first selection rules, $T^{\{\{1\}\{2\}\{3\}\{4\}\}}_{i_1i_2i_3i_4}\equiv 0$.

In that case,

$T^{\{\{1,2\}\{3\}\{4\}\}}_{i_1i_2i_3i_4}=\varepsilon_{i_1i_3i_4}\,{}^{\{\{1,2\}\{3\}\{4\}\}}_{\varepsilon\{1\}}T_{i_2}$; $T^{\{\{1,3\}\{2\}\{4\}\}}_{i_1i_2i_3i_4}=\varepsilon_{i_1i_2i_4}\,{}^{\{\{1,3\}\{2\}\{4\}\}}_{\varepsilon\{1\}}T_{i_3}$; $T^{\{\{1,4\}\{2\}\{3\}\}}_{i_1i_2i_3i_4}=\varepsilon_{i_1i_2i_3}\,{}^{\{\{1,3\}\{2\}\{4\}\}}_{\varepsilon\{1\}}T_{i_4}$;

$T^{\{\{1,3\}\{2,4\}\}}_{i_1i_2i_3i_4}=\varepsilon_{i_1i_2i}\varepsilon_{i_3i_4j}\,{}^{\{\{1,3\}\{2,4\}\}}_{\varepsilon\{1,2\}}T_{ij}$; $T^{\{\{1,3\}\{2,4\}\}}_{i_1i_2i_3i_4}=\varepsilon_{i_1i_3k}\varepsilon_{i_2i_4j}\,{}^{\{\{1,2\}\{3,4\}\}}_{\varepsilon\{1,2\}}T_{kj}$;

$T^{\{\{1,2,3\}\{4\}\}}_{i_1i_2i_3i_4}=\varepsilon_{i_1i_4k}\,{}^{\{\{1,2,3\}\{4\}\}}_{\varepsilon\{1\}}T_{ki_2i_3}$; $T^{\{\{1,2,4\}\{3\}\}}_{i_1i_2i_3i_4}=\varepsilon_{i_1i_3k}\,{}^{\{\{1,2,4\}\{3\}\}}_{\varepsilon\{1\}}T_{ki_2i_4}$; $T^{\{\{1,3,4\}\{2\}\}}_{i_1i_2i_3i_4}=\varepsilon_{i_1i_2k}\,{}^{\{\{1,3,4\}\{2\}\}}_{\varepsilon\{1\}}T_{ki_3i_4}$.

where

${}^{\{\{1,2\}\{3\}\{4\}\}}_{\varepsilon\{1\}}T_{i_2}=\frac{1}{6}\varepsilon_{ijk}T^{\{\{1,2\}\{3\}\{4\}\}}_{ii_2jk}$; ${}^{\{\{1,3\}\{2\}\{4\}\}}_{\varepsilon\{1\}}T_{i_3}=\frac{1}{6}\varepsilon_{ijk}T^{\{\{1,3\}\{2\}\{4\}\}}_{iji_3k}$; ${}^{\{\{1,4\}\{2\}\{3\}\}}_{\varepsilon\{1\}}T_{i_4}=\frac{1}{6}\varepsilon_{ijk}T^{\{\{1,4\}\{2\}\{3\}\}}_{ijki_4}$;

${}^{\{\{1,3\}\{2,4\}\}}_{\varepsilon\{1,2\}}T_{ij}=\frac{1}{4}\varepsilon_{ii_1i_2}\varepsilon_{ji_3i_4}T^{\{\{1,3\}\{2,4\}\}}_{i_1i_2i_3i_4}$; ${}^{\{\{1,2\}\{3,4\}\}}_{\varepsilon\{1,2\}}T_{ij}=\frac{1}{4}\varepsilon_{ii_1i_3}\varepsilon_{ji_2i_4}T^{\{\{1,3\}\{2,4\}\}}_{i_1i_2i_3i_4}$;

${}^{\{\{1,2,3\}\{4\}\}}_{\varepsilon\{1\}}T_{ki_2i_3}=\frac{1}{2}\varepsilon_{ki_1i_4}T^{\{\{1,2,3\}\{4\}\}}_{i_1i_2i_3i_4}$; ${}^{\{\{1,2,4\}\{3\}\}}_{\varepsilon\{1\}}T_{ki_2i_4}=\frac{1}{2}\varepsilon_{ki_1i_3}T^{\{\{1,2,4\}\{3\}\}}_{i_1i_2i_3i_4}$; ${}^{\{\{1,2,3\}\{4\}\}}_{\varepsilon\{1\}}T_{ki_2i_3}=\frac{1}{2}\varepsilon_{ki_1i_4}T^{\{\{1,2,3\}\{4\}\}}_{i_1i_2i_3i_4}$,

The dualization does not provide a simplification for fourth-rank tensors with symmetry other than {{1,2}{3}{4}}, {{1,3}{2}{4}}, {{1,4}{2}{3}} и {{1}{2}{3}{4}} for dim > 3.

---

dim = 4

$T^{\{\{1\}\{2\}\{3\}\{4\}\}}_{i_1i_2i_3i_4}=\varepsilon_{i_1i_2i_3i_4}\,{}^{\{\{1\}\{2\}\{3\}\{4\}\}}_{\varepsilon\{1\}}T^{(4)}$; *where* ${}^{\{\{1\}\{2\}\{3\}\{4\}\}}_{\varepsilon\{1\}}T^{(4)}=\frac{1}{24}\varepsilon_{i_1i_2i_3i_4}T^{\{\{1\}\{2\}\{3\}\{4\}\}}_{i_1i_2i_3i_4}$;

$T^{\{\{1,2\}\{3\}\{4\}\}}_{i_1i_2i_3i_4}=\varepsilon_{i_1i_3i_4k}\,{}^{\{\{1,2\}\{3\}\{4\}\}}_{\varepsilon\{1\}}T_{ki_2}$; $T^{\{\{1,3\}\{2\}\{4\}\}}_{i_1i_2i_3i_4}=\varepsilon_{i_1i_2i_4k}\,{}^{\{\{1,3\}\{2\}\{4\}\}}_{\varepsilon\{1\}}T_{ki_3}$; $T^{\{\{1,4\}\{2\}\{3\}\}}_{i_1i_2i_3i_4}=\varepsilon_{i_1i_2i_3k}\,{}^{\{\{1,3\}\{2\}\{4\}\}}_{\varepsilon\{1\}}T_{ki_4}$;

where

${}^{\{\{1,2\}\{3\}\{4\}\}}_{\varepsilon\{1\}}T_{ki_2}=-\frac{1}{6}\varepsilon_{ijsk}T^{\{\{1,2\}\{3\}\{4\}\}}_{ii_2js}$; ${}^{\{\{1,3\}\{2\}\{4\}\}}_{\varepsilon\{1\}}T_{ki_3}=-\frac{1}{6}\varepsilon_{ijsk}T^{\{\{1,3\}\{2\}\{4\}\}}_{iji_3s}$;

${}^{\{\{1,4\}\{2\}\{3\}\}}_{\varepsilon\{1\}}T_{ki_4}=-\frac{1}{6}\varepsilon_{ijsk}T^{\{\{1,4\}\{2\}\{3\}\}}_{ijsi_4}$;

---

dim = 5

$T^{\{\{1\}\{2\}\{3\}\{4\}\}}_{i_1i_2i_3i_4}=\varepsilon_{i_1i_2i_3i_4k}\,{}^{\{\{1\}\{2\}\{3\}\{4\}\}}_{\varepsilon\{1\}}T^{(5)}_{k}$; *where* ${}^{\{\{1\}\{2\}\{3\}\{4\}\}}_{\varepsilon\{1\}}T^{(5)}_{k}=\frac{1}{24}\varepsilon_{ki_1i_2i_3i_4}T^{\{\{1\}\{2\}\{3\}\{4\}\}}_{i_1i_2i_3i_4}$;

$T^{\{\{1,2\}\{3\}\{4\}\}}_{i_1i_2i_3i_4}=\frac{1}{2}\varepsilon_{i_1i_3i_4kp}\,{}^{\{\{1,2\}\{3\}\{4\}\}}_{\varepsilon\{1\}}T_{kpi_2}$; $T^{\{\{1,3\}\{2\}\{4\}\}}_{i_1i_2i_3i_4}=\frac{1}{2}\varepsilon_{i_1i_2i_4kp}\,{}^{\{\{1,3\}\{2\}\{4\}\}}_{\varepsilon\{1\}}T_{kpi_3}$;

$T^{\{\{1,4\}\{2\}\{3\}\}}_{i_1i_2i_3i_4}=\frac{1}{2}\varepsilon_{i_1i_2i_3kp}\,{}^{\{\{1,3\}\{2\}\{4\}\}}_{\varepsilon\{1\}}T_{kpi_4}$;

where

${}^{\{\{1,2\}\{3\}\{4\}\}}_{\varepsilon\{1\}}T_{kpi_2}=\frac{1}{6}\varepsilon_{ijskp}T^{\{\{1,2\}\{3\}\{4\}\}}_{ii_2js}$; ${}^{\{\{1,3\}\{2\}\{4\}\}}_{\varepsilon\{1\}}T_{kpi_3}=\frac{1}{6}\varepsilon_{ijskp}T^{\{\{1,3\}\{2\}\{4\}\}}_{iji_3s}$; ;

The dualization does not provide a simplification for fourth-rank tensors with symmetry other than {{1}{2}{3}{4}} for dim > 5.

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dim = 6

$$T_{i_1i_2i_3i_4}^{\{\{1\}\{2\}\{3\}\{4\}\}} = \frac{1}{2}\varepsilon_{i_1i_2i_3i_4kp}\;{}^{\{\{1\}\{2\}\{3\}\{4\}\}}_{\varepsilon\{1\}}T_{kp}^{\{\{1\}\{2\}\}(6)}\text{; where }{}^{\{\{1\}\{2\}\{3\}\{4\}\}}_{\varepsilon\{1\}}T_{kp}^{\{\{1\}\{2\}\}(6)} = \frac{1}{24}\varepsilon_{kpi_1i_2i_3i_4}T_{i_1i_2i_3i_4}^{\{\{1\}\{2\}\{3\}\{4\}\}};$$

dim = 7

$$T_{i_1i_2i_3i_4}^{\{\{1\}\{2\}\{3\}\{4\}\}} = \frac{1}{6}\varepsilon_{i_1i_2i_3i_4ksp}\;{}^{\{\{1\}\{2\}\{3\}\{4\}\}}_{\varepsilon\{1\}}T_{ksp}^{(7)\{\{1\}\{2\}\{3\}\}}\text{; where }{}^{\{\{1\}\{2\}\{3\}\{4\}\}}_{\varepsilon\{1\}}T_{ksp}^{(7)\{\{1\}\{2\}\{3\}\}} = \frac{1}{24}\varepsilon_{kspi_1i_2i_3i_4}T_{i_1i_2i_3i_4}^{\{\{1\}\{2\}\{3\}\{4\}\}};$$

The dualization does not provide a simplification for fourth-rank fully antisymmetric tensor for dim > 7.

Table 8. Numbers of independent components in spaces of certain dimensions.

YSI	*Space dimension* 2	3	4	5	10	11	16	*dim*
{{1,2,3,4}}	5	15	35	70	715	1001	3876	$\frac{(\dim+3)(\dim+2)(\dim+1)\dim}{24}$
{{1,2,3}{4}}	3	15	45	105	1485	2145	9180	$\frac{(\dim+2)(\dim+1)\dim(\dim-1)}{8}$
{{1,2,4}{3}}	3	15	45	105	1485	2145	9180	
{{1,3,4}{2}}	3	15	45	105	1485	2145	9180	
{{1,2}{3,4}}	1	6	20	50	825	1210	5440	$\frac{(\dim+1)\dim^2(\dim-1)}{12}$
{{1,3}{2,4}}	1	6	20	50	825	1210	5440	
{{1,2}{3}{4}}	–	3	15	45	990	1485	7140	$\frac{(\dim+1)\dim(\dim-1)(\dim-2)}{8}$
{{1,3}{2}{4}}	–	3	15	45	990	1485	7140	
{{1,4}{2}{3}}	–	3	15	45	990	1485	7140	
{{1}{2}{3}{4}}	–	–	1	5	210	330	1820	$\frac{\dim(\dim-1)(\dim-2)(\dim-3)}{24}$
SUM	16	81	256	625	10000	14641	65536	$\dim^4$

Table 9. Decomposition of a fourth-rank tensor.

Direct decomposition according to algorithm [7], SO(dim)

$$T_{i_1i_2i_3i_4} = T_{i_1i_2i_3i_4}^{\{\{1,2,3,4\}\}TF} + \left(1-\delta_{2\dim}\right)\Big\{T_{i_1i_2i_3i_4}^{\{\{1,2,3\}\{4\}\}TF} + T_{i_1i_2i_3i_4}^{\{\{1,2,4\}\{3\}\}TF} +$$
$$+T_{i_1i_2i_3i_4}^{\{\{1,2\}\{3\}\{4\}\}TF} + \left(1-\delta_{3\dim}\right)\Big[T_{i_1i_2i_3i_4}^{\{\{1,2\}\{3,4\}\}TF} + T_{i_1i_2i_3i_4}^{\{\{1,3,4\}\{2\}\}TF} +$$
$$+T_{i_1i_2i_3i_4}^{\{\{1,3\}\{2,4\}\}TF} + T_{i_1i_2i_3i_4}^{\{\{1,3\}\{2\}\{4\}\}TF} + T_{i_1i_2i_3i_4}^{\{\{1,4\}\{2\}\{3\}\}TF} +$$
$$+T_{i_1i_2i_3i_4}^{\{\{1\}\{2\}\{3\}\{4\}\}TF}\Big]\Big\} + {}_{TP\{(i_1,i_2,i_3,i_4)\}}T_{k_1k_2k_3k_4}$$

(according to the second selection rule, all components in curly brackets are identical to zero in a two-dimensional space. According to the first and second selection rules, all components in square brackets are identical to zero in a three-dimensional space)

where

$$T_{i_1i_2i_3i_4}^{\{\{1,2,3,4\}\}TF} = T_{i_1i_2i_3i_4}^{\{\{1,2,3,4\}\}} - \delta_{i_1i_2}\;{}^{\{\{1,2,3,4\}\}}_{\frac{1}{(\dim+4)}Sp\{1\}}T_{i_3i_4}^{\{\{1,2\}\}TF} - \delta_{i_1i_3}\;{}^{\{\{1,2,3,4\}\}}_{\frac{1}{(\dim+4)}Sp\{1\}}T_{i_2i_4}^{\{\{1,2\}\}TF} -$$
$$-\delta_{i_1i_4}\;{}^{\{\{1,2,3,4\}\}}_{\frac{1}{(\dim+4)}Sp\{1\}}T_{i_2i_3}^{\{\{1,2\}\}TF} - \delta_{i_2i_3}\;{}^{\{\{1,2,3,4\}\}}_{\frac{1}{(\dim+4)}Sp\{1\}}T_{i_1i_4}^{\{\{1,2\}\}TF} - \delta_{i_2i_4}\;{}^{\{\{1,2,3,4\}\}}_{\frac{1}{(\dim+4)}Sp\{1\}}T_{i_1i_3}^{\{\{1,2\}\}TF} -$$
$$-\delta_{i_3i_4}\;{}^{\{\{1,2,3,4\}\}}_{\frac{1}{(\dim+4)}Sp\{1\}}T_{i_1i_2}^{\{\{1,2\}\}TF} - \left(\delta_{i_1i_2}\delta_{i_3i_4} + \delta_{i_1i_3}\delta_{i_2i_4} + \delta_{i_1i_4}\delta_{i_2i_3}\right)\;{}^{\{\{1,2,3,4\}\}}_{\frac{1}{(\dim+2)\dim}Sp\{2\}}T,$$

with

Continued on next page

$$ {}^{\{\{1,2,3,4\}\}}_{\frac{1}{(\dim+4)}Sp\{1\}}T^{\{\{1,2\}\}TF}_{i_3i_4} = \frac{1}{6(\dim+4)}\left\{T_{ss(i_3i_4)} + T_{s(i_3|s|i_4)} + T_{s(i_3i_4)s} + T_{(i_3|ss|i_4)} + T_{(i_3|s|i_4)s} + T_{(i_3i_4)ss}\right\} - $$

$$ -\frac{1}{3\dim(\dim+4)}\delta_{i_3i_4}\left(T_{ssmm} + T_{smsm} + T_{smms}\right) $$

$$ {}^{\{\{1,2,3,4\}\}}_{\frac{1}{(\dim+2)\dim}Sp\{2\}}T = \frac{1}{3\left(\dim+2\right)\dim}\left(T_{ssmm} + T_{smsm} + T_{smms}\right) $$

According to the second selection rule, $T^{\{\{1,2,3\}\{4\}\}TF}_{i_1i_2i_3i_4} \equiv 0$ in a two-dimensional space.

for *dim* > 2:

$$ T^{\{\{1,2,3\}\{4\}\}TF}_{i_1i_2i_3i_4} = \frac{1}{8}\left\{T_{i_1i_2i_3i_4} - T_{i_4i_2i_3i_1} + T_{i_1i_3i_2i_4} - T_{i_4i_3i_2i_1} + T_{i_2i_1i_3i_4} - T_{i_2i_4i_3i_1} + \right. $$

$$ \left. + T_{i_2i_3i_1i_4} - T_{i_2i_3i_4i_1} + T_{i_3i_2i_1i_4} - T_{i_3i_2i_4i_1} + T_{i_3i_1i_2i_4} - T_{i_3i_4i_2i_1}\right\} - $$

$$ -\left(\delta_{i_1i_2}\,{}^{\{\{1,2,3\}\{4\}\}}_{\frac{1}{\dim}Sp\{(1)(2)\}}T^{\{\{1,2\}\}TF}_{i_3i_4} + \delta_{i_1i_3}\,{}^{\{\{1,2,3\}\{4\}\}}_{\frac{1}{\dim}Sp\{(1)(2)\}}T^{\{\{1,2\}\}TF}_{i_2i_4} - \delta_{i_2i_4}\,{}^{\{\{1,2,3\}\{4\}\}}_{\frac{1}{\dim}Sp\{(1)(2)\}}T^{\{\{1,2\}\}TF}_{i_1i_3} - \delta_{i_3i_4}\,{}^{\{\{1,2,3\}\{4\}\}}_{\frac{1}{\dim}Sp\{(1)(2)\}}T^{\{\{1,2\}\}TF}_{i_1i_2}\right) - $$

$$ -\left(\delta_{i_2i_3}\,{}^{\{\{1,2,3\}\{4\}\}}_{\frac{1}{(\dim+2)}Sp\{(2)(3)\}}T^{\{\{1\}\{2\}\}}_{i_1i_2} + \delta_{i_1i_2}\,{}^{\{\{1,2,3\}\{4\}\}}_{\frac{1}{2(\dim+2)}Sp\{(2)(3)\}}T^{\{\{1\}\{2\}\}}_{i_3i_4} + \delta_{i_1i_3}\,{}^{\{\{1,2,3\}\{4\}\}}_{\frac{1}{2(\dim+2)}Sp\{(2)(3)\}}T^{\{\{1\}\{2\}\}}_{i_2i_4} + \right. $$

$$ \left. + \delta_{i_3i_4}\,{}^{\{\{1,2,3\}\{4\}\}}_{\frac{1}{2(\dim+2)}Sp\{(2)(3)\}}T^{\{\{1\}\{2\}\}}_{i_1i_2} + \delta_{i_2i_4}\,{}^{\{\{1,2,3\}\{4\}\}}_{\frac{1}{2(\dim+2)}Sp\{(2)(3)\}}T^{\{\{1\}\{2\}\}}_{i_1i_3}\right), $$

with

$$ {}^{\{\{1,2,3\}\{4\}\}}_{\frac{1}{\dim}Sp\{(1)(2)\}}T^{\{\{1,2\}\}TF}_{ij} = \frac{1}{4\dim}\left\{T_{ss(ij)} + T_{(i|ss|j)} + T_{s(i|s|j)} - T_{(i|s|j)s} - T_{(ij)ss} - T_{s(ij)s}\right\}, $$

$$ {}^{\{\{1,2,3\}\{4\}\}}_{\frac{1}{(\dim+2)}Sp\{(2)(3)\}}T^{\{\{1\}\{2\}\}}_{ij} = \frac{1}{2(\dim+2)}\left(T_{s[i|s|j]} + T_{ss[ij]} + T_{[i|ss|j]}\right), $$

$$ {}^{\{\{1,2,3\}\{4\}\}}_{\frac{1}{2(\dim+2)}Sp\{(2)(3)\}}T^{\{\{1\}\{2\}\}}_{ij} = \frac{1}{4(\dim+2)}\left(T_{s[i|s|j]} + T_{ss[ij]} + T_{[i|ss|j]}\right). $$

According to the second selection rule, $T^{\{\{1,2,4\}\{3\}\}TF}_{i_1i_2i_3i_4} \equiv 0$ in a two-dimensional space.

for *dim* > 2:

$$ T^{\{\{1,2,4\}\{3\}\}TF}_{i_1i_2i_3i_4} = T^{\{\{1,2,4\}\{3\}\}}_{i_1i_2i_3i_4} + \delta_{i_2i_3}\,{}^{\{\{1,2,4\}\{3\}\}}_{\frac{1}{\dim}Sp\{(1)(2)\}}T^{\{\{1,2\}\}}_{i_1i_4} + \delta_{i_3i_4}\,{}^{\{\{1,2,4\}\{3\}\}}_{\frac{1}{\dim}Sp\{(1)(2)\}}T^{\{\{1,2\}\}}_{i_1i_2} - $$

$$ -\delta_{i_1i_2}\,{}^{\{\{1,2,4\}\{3\}\}}_{\frac{1}{\dim}Sp\{(1)(2)\}}T^{\{\{1,2\}\}}_{i_3i_4} - \delta_{i_1i_4}\,{}^{\{\{1,2,4\}\{3\}\}}_{\frac{1}{\dim}Sp\{(1)(2)\}}T^{\{\{1,2\}\}}_{i_2i_3} + $$

$$ +\delta_{i_1i_2}\,{}^{\{\{1,2,4\}\{3\}\}}_{\frac{1}{2(\dim+2)}Sp\{(2)(4)\}}T^{\{\{1\}\{2\}\}}_{i_3i_4} - \delta_{i_1i_4}\,{}^{\{\{1,2,4\}\{3\}\}}_{\frac{1}{2(\dim+2)}Sp\{(2)(4)\}}T^{\{\{1\}\{2\}\}}_{i_2i_3} - \delta_{i_2i_3}\,{}^{\{\{1,2,4\}\{3\}\}}_{\frac{1}{2(\dim+2)}Sp\{(2)(4)\}}T^{\{\{1\}\{2\}\}}_{i_1i_4} - $$

$$ -\delta_{i_3i_4}\,{}^{\{\{1,2,4\}\{3\}\}}_{\frac{1}{2(\dim+2)}Sp\{(2)(4)\}}T^{\{\{1\}\{2\}\}}_{i_1i_2} - \delta_{i_2i_4}\,{}^{\{\{1,2,4\}\{3\}\}}_{\frac{1}{(\dim+2)}Sp\{(2)(4)\}}T^{\{\{1\}\{2\}\}}_{i_1i_3} $$

with

$$ {}^{\{\{1,2,4\}\{3\}\}}_{\frac{1}{(\dim+2)}Sp\{(2)(4)\}}T^{\{\{1\}\{2\}\}}_{i_1i_2} = \frac{1}{2(\dim+2)}\left\{T_{[i_1|s|i_2]s} + T_{s[i_1i_2]s} - T_{ss[i_1i_2]}\right\} $$

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$$ {}^{\{\{1,2,4\}\{3\}\}}_{\frac{1}{2(\dim+2)}Sp\{(2)(4)\}}T^{\{\{1\}\{2\}\}}_{i_1i_2} = \frac{1}{4(\dim+2)}\left\{T_{[i_1|s|i_2]s} + T_{s[i_1i_2]s} - T_{ss[i_1i_2]}\right\} $$

According to the second selection rule, $T^{\{\{1,3,4\}\{2\}\}TF}_{i_1i_2i_3i_4} \equiv 0$ in a two-dimensional space.

for dim > 2:

$$ T^{\{\{1,3,4\}\{2\}\}TF}_{i_1i_2i_3i_4} = T^{\{\{1,3,4\}\{2\}\}}_{i_1i_2i_3i_4} + \delta_{i_1i_3}\, {}^{\{\{1,3,4\}\{2\}\}}_{\frac{1}{\dim}Sp\{(2)(3)\}}T^{\{\{1,2\}\}}_{i_2i_4} + \delta_{i_1i_4}\, {}^{\{\{1,3,4\}\{2\}\}}_{\frac{1}{\dim}Sp\{(2)(3)\}}T^{\{\{1,2\}\}}_{i_2i_3} - $$

$$ -\delta_{i_2i_3}\, {}^{\{\{1,3,4\}\{2\}\}}_{\frac{1}{\dim}Sp\{(2)(3)\}}T^{\{\{1,2\}\}}_{i_1i_4} - \delta_{i_2i_4}\, {}^{\{\{1,3,4\}\{2\}\}}_{\frac{1}{\dim}Sp\{(2)(3)\}}T^{\{\{1,2\}\}}_{i_1i_3} + $$

$$ +\delta_{i_1i_3}\, {}^{\{\{1,3,4\}\{2\}\}}_{\frac{1}{2(\dim+2)}Sp\{(3)(4)\}}T^{\{\{1\}\{2\}\}}_{i_2i_4} + \delta_{i_1i_4}\, {}^{\{\{1,3,4\}\{2\}\}}_{\frac{1}{2(\dim+2)}Sp\{(3)(4)\}}T^{\{\{1\}\{2\}\}}_{i_2i_3} - \delta_{i_2i_3}\, {}^{\{\{1,3,4\}\{2\}\}}_{\frac{1}{2(\dim+2)}Sp\{(3)(4)\}}T^{\{\{1\}\{2\}\}}_{i_1i_4} - $$

$$ -\delta_{i_2i_4}\, {}^{\{\{1,3,4\}\{2\}\}}_{\frac{1}{2(\dim+2)}Sp\{(3)(4)\}}T^{\{\{1\}\{2\}\}}_{i_1i_3} - \delta_{i_3i_4}\, {}^{\{\{1,3,4\}\{2\}\}}_{\frac{1}{(\dim+2)}Sp\{(3)(4)\}}T^{\{\{1\}\{2\}\}}_{i_1i_2} $$

with

$$ {}^{\{\{1,3,4\}\{2\}\}}_{\frac{1}{\dim}Sp\{(2)(3)\}}T^{\{\{1,2\}\}TF}_{i_1i_2} = \frac{1}{4\dim}\left\{T_{(i_1|ss|i_2)} + T_{(i_1|s|i_2)s} + T_{ss(i_1i_2)} - T_{s(i_1|s|i_2)} - T_{s(i_1i_2)s} - T_{(i_1i_2)ss}\right\} $$

$$ {}^{\{\{1,3,4\}\{2\}\}}_{\frac{1}{(\dim+2)}Sp\{(3)(4)\}}T^{\{\{1\}\{2\}\}}_{i_1i_2} = \frac{1}{2(\dim+2)}\left(T_{[i_1i_2]ss} - T_{s[i_1i_2]s} - T_{s[i_1|s|i_2]}\right) $$

$$ {}^{\{\{1,3,4\}\{2\}\}}_{\frac{1}{2(\dim+2)}Sp\{(3)(4)\}}T^{\{\{1\}\{2\}\}}_{i_1i_2} = \frac{1}{4(\dim+2)}\left(T_{[i_1i_2]ss} - T_{s[i_1i_2]s} - T_{s[i_1|s|i_2]}\right) $$

According to the second selection rule, $T^{\{\{1,2\}\{3,4\}\}TF}_{i_1i_2i_3i_4} \equiv 0$ in two- and three-dimensional spaces.

for dim > 3:

$$ T^{\{\{1,2\}\{3,4\}\}TF}_{i_1i_2i_3i_4} = T^{\{\{1,2\}\{3,4\}\}}_{i_1i_2i_3i_4} - \delta_{i_1i_2}\, {}^{\{\{1,2\}\{3,4\}\}}_{\frac{1}{(\dim-2)}Sp\{(1)(2)\}}T^{\{\{1,2\}\}TF}_{i_3i_4} - \delta_{i_3i_4}\, {}^{\{\{1,2\}\{3,4\}\}}_{\frac{1}{(\dim-2)}Sp\{(1)(2)\}}T^{\{\{1,2\}\}TF}_{i_1i_2} + $$

$$ +\delta_{i_1i_4}\, {}^{\{\{1,2\}\{3,4\}\}}_{\frac{1}{(\dim-2)}Sp\{(1)(2)\}}T^{\{\{1,2\}\}TF}_{i_2i_3} + \delta_{i_2i_3}\, {}^{\{\{1,2\}\{3,4\}\}}_{\frac{1}{(\dim-2)}Sp\{(1)(2)\}}T^{\{\{1,2\}\}TF}_{i_1i_4} - $$

$$ -(\delta_{i_1i_2}\delta_{i_3i_4} - \delta_{i_1i_4}\delta_{i_2i_3})\, {}^{\{\{1,2\}\{3,4\}\}}_{\frac{1}{\dim(\dim-1)}Sp\{(1,3)(2,4)\}}T, $$

with

$$ {}^{\{\{1,2\}\{3,4\}\}}_{Sp\{(1)(2)\}}T^{\{\{1,2\}\}TF}_{i_3i_4} = \frac{1}{6}\left\{2T_{ss(i_3i_4)} + 2T_{(i_3i_4)ss} - T_{s(i_3i_4)s} - T_{(i_3|ss|i_4)} - T_{(i_3|s|i_4)s} - T_{s(i_3|s|i_4)}\right\} - \delta_{i_3i_4}\, {}^{\{\{1,2\}\{3,4\}\}}_{\frac{1}{\dim}Sp\{(1,3)(2,4)\}}T^{\{\{1,2\}\{3,4\}\}}_{ssmm} , $$

$$ {}^{\{\{1,2\}\{3,4\}\}}_{\frac{1}{\dim}Sp\{(1,3)(2,4)\}}T = \frac{1}{\dim}T^{\{\{1,2\}\{3,4\}\}}_{ssmm} = \frac{1}{3\dim}\left\{2T_{ssmm} - T_{smms} - T_{msms}\right\}. $$

According to the second selection rule, $T^{\{\{1,3\}\{2,4\}\}TF}_{i_1i_2i_3i_4} \equiv 0$ in two- and three-dimensional spaces.

for dim > 3:

$$ T^{\{\{1,3\}\{2,4\}\}TF}_{i_1i_2i_3i_4} = T^{\{\{1,3\}\{2,4\}\}}_{i_1i_2i_3i_4} - \delta_{i_1i_3}\, {}^{\{\{1,3\}\{2,4\}\}}_{\frac{1}{(\dim-2)}Sp\{(1)(3)\}}T^{\{\{1,2\}\}TF}_{i_2i_4} - \delta_{i_2i_4}\, {}^{\{\{1,3\}\{2,4\}\}}_{\frac{1}{(\dim-2)}Sp\{(1)(3)\}}T^{\{\{1,2\}\}TF}_{i_1i_3} + $$

$$ +\delta_{i_1i_4}\, {}^{\{\{1,3\}\{2,4\}\}}_{\frac{1}{(\dim-2)}Sp\{(1)(3)\}}T^{\{\{1,2\}\}TF}_{i_2i_3} + \delta_{i_2i_3}\, {}^{\{\{1,3\}\{2,4\}\}}_{\frac{1}{(\dim-2)}Sp\{(1)(3)\}}T^{\{\{1,2\}\}TF}_{i_1i_4} - $$

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$$-(\delta_{i_1i_3}\delta_{i_2i_4}-\delta_{i_1i_4}\delta_{i_2i_3})\;{}^{\{\{1,3\}\{2,4\}\}}_{\frac{1}{(\dim-1)\dim}Sp\{(1,2)(3,4)\}}T$$

with

$${}^{\{\{1,3\}\{2,4\}\}}_{\frac{1}{(\dim-2)}Sp\{(1)(3)\}}T^{\{\{1,2\}\}TF}_{i_1i_2}=\frac{1}{6(\dim-2)}\left(2T_{s(i_1|s|i_2)}+2T_{(i_1|s|i_2)s}-T_{s(i_1i_2)s}-T_{(i_1|ss|i_2)}-T_{ss(i_1i_2)}-T_{(i_1i_2)ss}\right)-\frac{1}{3\dim(\dim-2)}\delta_{i_1i_2}\left(2T_{smsm}-T_{mssm}-T_{ssmm}\right)$$

$${}^{\{\{1,3\}\{2,4\}\}}_{\frac{1}{\dim(\dim-1)}Sp\{(1,2)(3,4)\}}T=\frac{1}{\dim(\dim-1)}T^{\{\{1,3\}\{2,4\}\}}_{smsm}=\frac{1}{3\dim(\dim-1)}\left(2T_{smsm}-T_{mssm}-T_{ssmm}\right)$$

According to the second selection rule, $T^{\{\{1,2\}\{3\}\{4\}\}TF}_{i_1i_2i_3i_4}\equiv 0$ in two- and three-dimensional spaces.

for dim > 3:

$$T^{\{\{1,2\}\{3\}\{4\}\}TF}_{i_1i_2i_3i_4}=T^{\{\{1,2\}\{3\}\{4\}\}}_{i_1i_2i_3i_4}-\delta_{i_1i_2}\;{}^{\{\{1,2\}\{3\}\{4\}\}}_{\frac{1}{(\dim-2)}Sp\{(1)(2)\}}T^{\{\{1\}\{2\}\}TF}_{i_3i_4}+$$

$$+\delta_{i_2i_3}\;{}^{\{\{1,2\}\{3\}\{4\}\}}_{\frac{1}{(\dim-2)}Sp\{(1)(2)\}}T^{\{\{1\}\{2\}\}TF}_{i_1i_4}-\delta_{i_2i_4}\;{}^{\{\{1,2\}\{3\}\{4\}\}}_{\frac{1}{(\dim-2)}Sp\{(1)(2)\}}T^{\{\{1\}\{2\}\}TF}_{i_1i_3}$$

with

$${}^{\{\{1,2\}\{3\}\{4\}\}}_{\frac{1}{(\dim-2)}Sp\{(1)(2)\}}T^{\{\{1\}\{2\}\}TF}_{i_3i_4}=\frac{1}{(\dim-2)}T^{\{\{1,2\}\{3\}\{4\}\}}_{ssi_3i_4}=\frac{1}{4(\dim-2)}\left\{T_{ss[i_3i_4]}+T_{[i_3|s|i_4]s}+T_{ss[i_3i_4]}+T_{s[i_3i_4]s}-T_{[i_3|ss|i_4]}-T_{s[i_3|s|i_4]}\right\}$$

According to the second selection rule, $T^{\{\{1,3\}\{2\}\{4\}\}TF}_{i_1i_2i_3i_4}\equiv 0$ in two- and three-dimensional spaces.

for dim > 3:

$$T^{\{\{1,3\}\{2\}\{4\}\}TF}_{i_1i_2i_3i_4}=T^{\{\{1,3\}\{2\}\{4\}\}}_{i_1i_2i_3i_4}-\delta_{i_1i_3}\;{}^{\{\{1,3\}\{2\}\{4\}\}}_{\frac{1}{(\dim-2)}Sp\{(3)(4)\}}T^{\{\{1\}\{2\}\}}_{i_2i_4}+$$

$$+\delta_{i_2i_3}\;{}^{\{\{1,3\}\{2\}\{4\}\}}_{\frac{1}{(\dim-2)}Sp\{(3)(4)\}}T^{\{\{1\}\{2\}\}}_{i_1i_4}-\delta_{i_3i_4}\;{}^{\{\{1,3\}\{2\}\{4\}\}}_{\frac{1}{(\dim-2)}Sp\{(3)(4)\}}T^{\{\{1\}\{2\}\}}_{i_1i_2}$$

with

$${}^{\{\{1,3\}\{2\}\{4\}\}}_{\frac{1}{(\dim-2)}Sp\{(3)(4)\}}T^{\{\{1\}\{2\}\}}_{i_1i_2}=\frac{1}{(\dim-2)}T^{\{\{1,3\}\{2\}\{4\}\}}_{i_1i_2ss}=\frac{1}{4(\dim-2)}\left\{2T_{s[i_1|s|i_2]}+T_{[i_1i_2]ss}-T_{[i_1|ss|i_2]}-T_{ss[i_1i_2]}-T_{s[i_1i_2]s}\right\}$$

According to the second selection rule, $T^{\{\{1,4\}\{2\}\{3\}\}TF}_{i_1i_2i_3i_4}\equiv 0$ in two- and three-dimensional spaces.

for dim > 3:

$$T^{\{\{1,4\}\{2\}\{3\}\}TF}_{i_1i_2i_3i_4}=T^{\{\{1,4\}\{2\}\{3\}\}}_{i_1i_2i_3i_4}-\delta_{i_1i_4}\;{}^{\{\{1,4\}\{2\}\{3\}\}}_{\frac{1}{(\dim-2)}Sp\{(3)(4)\}}T^{\{\{1\}\{2\}\}}_{i_2i_3}+$$

$$+\delta_{i_2i_4}\;{}^{\{\{1,4\}\{2\}\{3\}\}}_{\frac{1}{(\dim-2)}Sp\{(3)(4)\}}T^{\{\{1\}\{2\}\}}_{i_1i_3}-\delta_{i_3i_4}\;{}^{\{\{1,4\}\{2\}\{3\}\}}_{\frac{1}{(\dim-2)}Sp\{(3)(4)\}}T^{\{\{1\}\{2\}\}}_{i_1i_2}$$

with

$${}^{\{\{1,4\}\{2\}\{3\}\}}_{\frac{1}{(\dim-2)}Sp\{(3)(4)\}}T^{\{\{1\}\{2\}\}}_{i_1i_2}=\frac{1}{(\dim-2)}T^{\{\{1,4\}\{2\}\{3\}\}}_{i_1i_2ss}=\frac{1}{4(\dim-2)}\left\{2T_{s[i_1i_2]s}+T_{ss[i_1i_2]}+T_{[i_1i_2]ss}-T_{[i_1|s|i_2]s}-T_{s[i_1|s|i_2]}\right\}$$

According to the first selection rule, $T^{\{\{1\}\{2\}\{3\}\{4\}\}TF}_{i_1i_2i_3i_4}\equiv 0$ in two- and three-dimensional spaces.

for dim > 3:

Continued on next page

$$T_{i_1i_2i_3i_4}^{\{\{1\}\{2\}\{3\}\{4\}\}} = T_{i_1i_2i_3i_4}^{\{\{1\}\{2\}\{3\}\{4\}\}TF} = \frac{1}{24}\Big\{T_{i_1i_2i_3i_4} - T_{i_1i_2i_4i_3} - T_{i_1i_3i_2i_4} + T_{i_1i_3i_4i_2} - T_{i_1i_4i_3i_2} + T_{i_1i_4i_2i_3} -$$

$$- T_{i_2i_1i_3i_4} + T_{i_2i_1i_4i_3} + T_{i_2i_3i_1i_4} - T_{i_2i_3i_4i_1} + T_{i_2i_4i_3i_1} - T_{i_2i_4i_1i_3} -$$

$$- T_{i_3i_2i_1i_4} + T_{i_3i_2i_4i_1} + T_{i_3i_1i_2i_4} - T_{i_3i_1i_4i_2} + T_{i_3i_4i_1i_2} - T_{i_3i_4i_2i_1} -$$

$$- T_{i_4i_2i_3i_1} + T_{i_4i_2i_1i_3} + T_{i_4i_3i_2i_1} - T_{i_4i_3i_1i_2} + T_{i_4i_1i_3i_2} - T_{i_4i_1i_2i_3}\Big\}$$

The trace part can be expressed by the following general formulas:

$$ {}_{TP\{(i_1,i_2,i_3,i_4)\}}T_{k_1k_2k_3k_4} = \sum_{\substack{(i,j,s,p)=\{(1,2,3,4),(1,3,2,4),(1,4,2,3),\\ (2,3,1,4),(2,4,1,3),(3,4,1,2)\}}} \delta_{k_ik_j}\left({}_{PMB\{\delta_{k_ik_j}\}TP\{(i_1,i_2,i_3,i_4)\}}T_{k_sk_p}^{\{\{1,2\}\}TF} + {}_{PMB\{\delta_{k_ik_j}\}TP\{(i_1,i_2,i_3,i_4)\}}T_{k_sk_p}^{\{\{1\}\{2\}\}} \right) +$$

$$+\delta_{k_1k_2}\delta_{k_3k_4}\ {}_{PMB\{\delta_{k_1k_2}\delta_{k_3k_4}\}TP\{(i_1,i_2,i_3,i_4)\}}T + \delta_{k_1k_3}\delta_{k_2k_4}\ {}_{PMB\{\delta_{k_1k_3}\delta_{k_2k_4}\}TP\{(i_1,i_2,i_3,i_4)\}}T + \delta_{k_1k_4}\delta_{k_2k_3}\ {}_{PMB\{\delta_{k_1k_3}\delta_{k_2k_4}\}TP\{(i_1,i_2,i_3,i_4)\}}T$$

with:
for dim = 2:

$$ {}_{PMB\{\delta_{i_1i_2}\delta_{i_3i_4}\}TP\{(i_1,i_2,i_3,i_4)\}}T = \frac{(3T_{ssmm} - T_{smsm} - T_{smms})}{8}; \quad {}_{PMB\{\delta_{i_1i_3}\delta_{i_2i_4}\}TP\{(i_1,i_2,i_3,i_4)\}}T = \frac{(3T_{smsm} - T_{ssmm} - T_{smms})}{8};$$

$$ {}_{PMB\{\delta_{i_1i_4}\delta_{i_2i_3}\}TP\{(i_1,i_2,i_3,i_4)\}}T = \frac{(3T_{smms} - T_{smsm} - T_{ssmm})}{8}.$$

$$ {}_{PMB\{\delta_{i_1i_2}\}TP\{(i_1,i_2,i_3,i_4)\}}T_{i_3i_4}^{\{\{1\}\{2\}\}} = \frac{1}{16}\left(2T_{ss[i_3i_4]} + T_{[i_3|ss|i_4]} - T_{s[i_3i_4]s} + T_{s[i_3|s|i_4]} - T_{[i_3|s|i_4]s}\right)$$

$$ {}_{PMB\{\delta_{i_1i_3}\}TP\{(i_1,i_2,i_3,i_4)\}}T_{i_2i_4}^{\{\{1\}\{2\}\}} = \frac{1}{16}\left(2T_{s[i_2|s|i_4]} + T_{[i_2|ss|i_4]} + T_{s[i_2i_4]s} + T_{ss[i_2i_4]} - T_{[i_2i_4]ss}\right)$$

$$ {}_{PMB\{\delta_{i_1i_4}\}TP\{(i_1,i_2,i_3,i_4)\}}T_{i_2i_3}^{\{\{1\}\{2\}\}} = \frac{1}{16}\left(2T_{s[i_2i_3]s} + T_{[i_2|s|i_3]s} + T_{s[i_2|s|i_3]} - T_{ss[i_2i_3]} - T_{[i_2i_3]ss}\right)$$

$$ {}_{PMB\{\delta_{i_2i_3}\}TP\{(i_1,i_2,i_3,i_4)\}}T_{i_1i_4}^{\{\{1\}\{2\}\}} = \frac{1}{16}\left(2T_{[i_1|ss|i_4]} + T_{s[i_1|s|i_4]} + T_{ss[i_1i_4]} + T_{[i_1|s|i_4]s} + T_{[i_1i_4]ss}\right)$$

$$ {}_{PMB\{\delta_{i_2i_4}\}TP\{(i_1,i_2,i_3,i_4)\}}T_{i_1i_3}^{\{\{1\}\{2\}\}} = \frac{1}{16}\left(2T_{[i_1|s|i_3]s} + T_{[i_1|ss|i_3]} + T_{[i_1i_3]ss} - T_{ss[i_1i_3]} + T_{s[i_1i_3]s}\right)$$

$$ {}_{PMB\{\delta_{i_3i_4}\}TP\{(i_1,i_2,i_3,i_4)\}}T_{i_1i_2}^{\{\{1\}\{2\}\}} = \frac{1}{16}\left(2T_{[i_1i_2]ss} + T_{[i_1|ss|i_2]} - T_{s[i_1i_2]s} + T_{[i_1|s|i_2]s} - T_{s[i_1|s|i_2]}\right)$$

$$ {}_{PMB\{\delta_{i_1i_2}\}TP\{(i_1,i_2,i_3,i_4)\}}T_{i_3i_4}^{\{\{1,2\}\}TF} =$$

$$= \frac{1}{36}\Big\{10T_{ss(i_3i_4)} - 8T_{(i_3i_4)ss} + \left(T_{s(i_3|s|i_4)} + T_{s(i_3i_4)s} + T_{(i_3|ss|i_4)} + T_{(i_3|s|i_4)s}\right) - \delta_{i_3i_4}\left(T_{ssmm} + T_{smsm} + T_{smms}\right)\Big\}$$

$$ {}_{PMB\{\delta_{i_1i_3}\}TP\{(i_1,i_2,i_3,i_4)\}}T_{i_2i_4}^{\{\{1,2\}\}TF} =$$

$$== \frac{1}{36}\Big(10T_{s(i_2|s|i_4)} - 8T_{(i_2|s|i_4)s} + \left(T_{s(i_2i_4)s} + T_{(i_2i_4)ss} + T_{ss(i_2i_4)} + T_{(i_2|ss|i_4)}\right) - \delta_{i_3i_4}\left(T_{ssmm} + T_{smsm} + T_{smms}\right)\Big)$$

$$ {}_{PMB\{\delta_{i_1i_4}\}TP\{(i_1,i_2,i_3,i_4)\}}T_{i_2i_3}^{\{\{1,2\}\}TF} =$$

$$= \frac{1}{36}\Big\{10T_{s(i_2i_3)s} - 8T_{(i_2|ss|i_3)} + \left(T_{ss(i_2i_3)} + T_{(i_2i_3)ss} + T_{s(i_2|s|i_3)} + T_{(i_2|s|i_3)s}\right) - \delta_{i_3i_4}\left(T_{ssmm} + T_{smsm} + T_{smms}\right)\Big\}$$

$$ {}_{PMB\{\delta_{i_2i_3}\}TP\{(i_1,i_2,i_3,i_4)\}}T_{i_1i_4}^{\{\{1,2\}\}TF} =$$

$$= \frac{1}{36}\Big\{10T_{(i_1|ss|i_4)} - 8T_{s(i_1i_4)s} + \left(T_{ss(i_1i_4)} + T_{(i_1i_4)ss} + T_{s(i_1|s|i_4)} + T_{(i_1|s|i_4)s}\right) - \delta_{i_3i_4}\left(T_{ssmm} + T_{smsm} + T_{smms}\right)\Big\}$$

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$$ {}_{PMB\left\{\delta_{i_2 i_4}\right\}TP\{(i_1,i_2,i_3,i_4)\}}T^{\{\{1,2\}\}TF}_{i_1 i_3} = $$

$$ =\frac{1}{36}\left\{10T_{(i_1|s|i_3)s}-8T_{s(i_1|s|i_3)}+\left(T_{ss(i_1i_3)}+T_{(i_1i_3)ss}+T_{s(i_1i_3)s}+T_{(i_1|ss|i_3)}\right)-\delta_{i_3i_4}\left(T_{ssmm}+T_{smsm}+T_{smms}\right)\right\} $$

$$ {}_{PMB\left\{\delta_{i_3 i_4}\right\}TP\{(i_1,i_2,i_3,i_4)\}}T^{\{\{1,2\}\}TF}_{i_1 i_2} = $$

$$ =\frac{1}{36}\left\{10T_{(i_1i_2)ss}-8T_{ss(i_1i_2)}+\left(T_{s(i_1i_2)s}+T_{(i_1|ss|i_2)}+T_{(i_1|s|i_2)s}+T_{s(i_1|s|i_2)}\right)-\delta_{i_1i_2}\left(T_{ssmm}+T_{smsm}+T_{smms}\right)\right\} $$

for *dim* > *2*:

$$ {}_{PMB\left\{\delta_{i_1i_2}\delta_{i_3i_4}\right\}TP\{(i_1,i_2,i_3,i_4)\}}T=\frac{\left(T_{ssmm}\left(\dim+1\right)-T_{smsm}-T_{smms}\right)}{\left(\dim+2\right)\dim(\dim-1)};\quad {}_{PMB\left\{\delta_{i_1i_3}\delta_{i_2i_4}\right\}TP\{(i_1,i_2,i_3,i_4)\}}T=\frac{\left(T_{smsm}\left(\dim+1\right)-T_{ssmm}-T_{smms}\right)}{\left(\dim+2\right)\dim(\dim-1)}; $$

$$ {}_{PMB\left\{\delta_{i_1i_4}\delta_{i_2i_3}\right\}TP\{(i_1,i_2,i_3,i_4)\}}T=\frac{\left(T_{smms}\left(\dim+1\right)-T_{smsm}-T_{ssmm}\right)}{\left(\dim+2\right)\dim(\dim-1)}. $$

$$ {}_{PMB\left\{\delta_{i_1i_2}\right\}TP\{(i_1,i_2,i_3,i_4)\}}T^{\{\{1\}\{2\}\}}_{i_3i_4}=\frac{1}{(\dim+2)(\dim-2)}\left\{\dim T_{ss[i_3i_4]}-T_{[i_2|ss|i_4]}-T_{s[i_2|s|i_4]}+T_{[i_1|s|i_3]s}+T_{s[i_1i_3]s}\right\} $$

$$ {}_{PMB\left\{\delta_{i_1i_3}\right\}TP\{(i_1,i_2,i_3,i_4)\}}T^{\{\{1\}\{2\}\}}_{i_2i_4}=\frac{1}{(\dim+2)(\dim-2)}\left\{\dim T_{s[i_2|s|i_4]}+T_{[i_2i_4]ss}-T_{[i_2|ss|i_4]}-T_{s[i_2i_4]s}-T_{ss[i_2i_4]}\right\} $$

$$ {}_{PMB\left\{\delta_{i_1i_4}\right\}TP\{(i_1,i_2,i_3,i_4)\}}T^{\{\{1\}\{2\}\}}_{i_2i_3}=\frac{1}{(\dim+2)(\dim-2)}\left\{\dim T_{s[i_2i_3]s}+T_{ss[i_2i_3]}+T_{[i_2i_3]ss}-T_{[i_2|s|i_3]s}-T_{s[i_2|s|i_3]}\right\} $$

$$ {}_{PMB\left\{\delta_{i_2i_3}\right\}TP\{(i_1,i_2,i_3,i_4)\}}T^{\{\{1\}\{2\}\}}_{i_1i_4}=\frac{1}{(\dim+2)(\dim-2)}\left\{\dim T_{[i_1|ss|i_4]}-T_{s[i_1|s|i_4]}-T_{ss[i_1i_4]}-T_{[i_1|s|i_4]s}-T_{[i_1i_4]ss}\right\} $$

$$ {}_{PMB\left\{\delta_{i_2i_4}\right\}TP\{(i_1,i_2,i_3,i_4)\}}T^{\{\{1\}\{2\}\}}_{i_1i_3}=\frac{1}{(\dim+2)(\dim-2)}\left\{\dim T_{[i_1|s|i_3]s}+T_{ss[i_1i_3]}-T_{[i_1|ss|i_3]}-T_{s[i_1i_3]s}-T_{[i_1i_2]ss}\right\} $$

$$ {}_{PMB\left\{\delta_{i_3i_4}\right\}TP\{(i_1,i_2,i_3,i_4)\}}T^{\{\{1\}\{2\}\}}_{i_1i_2}=\frac{1}{(\dim+2)(\dim-2)}\left\{\dim T_{[i_1i_2]ss}+T_{s[i_1|s|i_2]}+T_{s[i_1i_2]s}-T_{[i_1|ss|i_2]}-T_{[i_1|s|i_2]s}\right\} $$

$$ {}_{PMB\left\{\delta_{i_1i_2}\right\}TP\{(i_1,i_2,i_3,i_4)\}}T^{\{\{1,2\}\}TF}_{i_3i_4}=\frac{1}{(\dim+4)(\dim-2)\dim}\left\{(\dim^2+2\dim-4)T_{ss(i_3i_4)}+4T_{(i_3i_4)ss}-\right. $$

$$ \left.-\dim(T_{s(i_3|s|i_4)}+T_{s(i_3i_4)s}+T_{(i_3|ss|i_4)}+T_{(i_3|s|i_4)s})-\delta_{i_3i_4}\left(\left(\dim+2\right)T_{ssmm}-2\left(T_{smsm}+T_{smms}\right)\right)\right\} $$

$$ {}_{PMB\left\{\delta_{i_1i_3}\right\}TP\{(i_1,i_2,i_3,i_4)\}}T^{\{\{1,2\}\}TF}_{i_2i_4}=\frac{1}{(\dim+4)(\dim-2)\dim}\left\{(\dim^2+2\dim-4)T_{s(i_2|s|i_4)}+4T_{(i_2|s|i_4)s}-\right. $$

$$ \left.-\dim(T_{ss(i_2i_4)}+T_{(i_2i_4)ss}+T_{s(i_3i_4)s}+T_{(i_3|ss|i_4)})-\delta_{i_2i_4}\left(\left(\dim+2\right)T_{smsm}-2\left(T_{ssmm}+T_{smms}\right)\right)\right\} $$

$$ {}_{PMB\left\{\delta_{i_1i_4}\right\}TP\{(i_1,i_2,i_3,i_4)\}}T^{\{\{1,2\}\}TF}_{i_2i_3}=\frac{1}{(\dim+4)(\dim-2)\dim}\left\{(\dim^2+2\dim-4)T_{s(i_2i_3)s}+4T_{(i_2|ss|i_3)}-\right. $$

$$ \left.-\dim(T_{ss(i_2i_3)}+T_{(i_2i_3)ss}+T_{s(i_2|s|i_3)}+T_{(i_2|s|i_3)s})-\delta_{i_2i_3}\left(\left(\dim+2\right)T_{smms}-2\left(T_{ssmm}+T_{smsm}\right)\right)\right\} $$

$$ {}_{PMB\left\{\delta_{i_2i_3}\right\}TP\{(i_1,i_2,i_3,i_4)\}}T^{\{\{1,2\}\}TF}_{i_1i_4}=\frac{1}{(\dim+4)(\dim-2)\dim}\left\{(\dim^2+2\dim-4)T_{(i_1|ss|i_4)}+4T_{s(i_1i_4)s}-\right. $$

$$ \left.-\dim(T_{ss(i_1i_4)}+T_{(i_1i_4)ss}+T_{s(i_1|s|i_4)}+T_{(i_1|s|i_4)s})-\delta_{i_1i_4}\left(\left(\dim+2\right)T_{smms}-2\left(T_{ssmm}+T_{smsm}\right)\right)\right\} $$

$$ {}_{PMB\left\{\delta_{i_2i_4}\right\}TP\{(i_1,i_2,i_3,i_4)\}}T^{\{\{1,2\}\}TF}_{i_1i_3}=\frac{1}{(\dim+4)(\dim-2)\dim}\left\{(\dim^2+2\dim-4)T_{(i_1|s|i_3)s}+4T_{s(i_1|s|i_3)}-\right. $$

$$ \left.-\dim(T_{ss(i_1i_3)}+T_{(i_1i_3)ss}+T_{s(i_1i_3)s}+T_{(i_1|ss|i_3)})-\delta_{i_1i_3}\left(\left(\dim+2\right)T_{smsm}-2\left(T_{ssmm}+T_{smms}\right)\right)\right\} $$

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$$_{PMB\{\delta_{i_3i_4}\}TP\{(i_1,i_2,i_3,i_4)\}}T^{\{\{1,2\}\}TF}_{i_1i_2} = \frac{1}{(\dim+4)(\dim-2)\dim}\Big\{(\dim^2+2\dim-4)T_{(i_1i_2)ss}+4T_{ss(i_1i_2)}-$$

$$-\dim(T_{s(i_1i_2)s}+T_{(i_1|ss|i_2)}+T_{(i_1|s|i_2)s}+T_{s(i_1|s|i_2)})-\delta_{i_1i_2}\left((\dim+2)T_{smms}-2\left(T_{ssmm}+T_{smsm}\right)\right)\Big\}$$

with transformation by antisymmetric index group dualization if appropriate

dim = 2

According to the first and second selection rules, $T^{\{\{1,3,4\}\{2\}\}TF}_{i_1i_2i_3i_4}\equiv 0$; $T^{\{\{1,2,4\}\{3\}\}TF}_{i_1i_2i_3i_4}\equiv 0$; $T^{\{\{1,2,3\}\{4\}\}TF}_{i_1i_2i_3i_4}\equiv 0$; $T^{\{\{1,2\}\{3,4\}\}TF}_{i_1i_2i_3i_4}\equiv 0$; $T^{\{\{1,3\}\{2,4\}\}TF}_{i_1i_2i_3i_4}\equiv 0$; $T^{\{\{1,2\}\{3\}\{4\}\}TF}_{i_1i_2i_3i_4}\equiv 0$, $T^{\{\{1,3\}\{2\}\{4\}\}TF}_{i_1i_2i_3i_4}\equiv 0$, $T^{\{\{1,4\}\{2\}\{3\}\}TF}_{i_1i_2i_3i_4}\equiv 0$, and $T^{\{\{1\}\{2\}\{3\}\{4\}\}TF}_{i_1i_2i_3i_4}\equiv 0$. Therefore,

for dim = 2:

$$T_{i_1i_2i_3i_4}=T^{\{\{1,2,3,4\}\}TF}_{i_1i_2i_3i_4}+_{TP\{(i_1,i_2,i_3,i_4)\}}T_{k_1k_2k_3k_4}$$

dim = 3

According to the first and second selection rules, $T^{\{\{1,2\}\{3,4\}\}TF}_{i_1i_2i_3i_4}\equiv 0$; $T^{\{\{1,3\}\{2,4\}\}TF}_{i_1i_2i_3i_4}\equiv 0$; $T^{\{\{1,2\}\{3\}\{4\}\}TF}_{i_1i_2i_3i_4}\equiv 0$, $T^{\{\{1,3\}\{2\}\{4\}\}TF}_{i_1i_2i_3i_4}\equiv 0$, $T^{\{\{1,4\}\{2\}\{3\}\}TF}_{i_1i_2i_3i_4}\equiv 0$, and $T^{\{\{1\}\{2\}\{3\}\{4\}\}TF}_{i_1i_2i_3i_4}\equiv 0$.

Therefore, *for dim = 3:*

$$T_{i_1i_2i_3i_4}=T^{\{\{1,2,3,4\}\}TF}_{i_1i_2i_3i_4}+T^{\{\{1,2,3\}\{4\}\}TF}_{i_1i_2i_3i_4}+T^{\{\{1,2,4\}\{3\}\}TF}_{i_1i_2i_3i_4}+T^{\{\{1,2\}\{3\}\{4\}\}TF}_{i_1i_2i_3i_4}+_{TP\{(i_1,i_2,i_3,i_4)\}}T_{k_1k_2k_3k_4}$$

In that case,

$T^{\{\{1,2,3\}\{4\}\}TF}_{i_1i_2i_3i_4}=\varepsilon_{i_1i_4k}\,{}^{\{\{1,2,3\}\{4\}\}TF}_{\varepsilon\{1\}}T_{ki_2i_3}$; $T^{\{\{1,2,4\}\{3\}\}TF}_{i_1i_2i_3i_4}=\varepsilon_{i_1i_3k}\,{}^{\{\{1,2,4\}\{3\}\}TF}_{\varepsilon\{1\}}T_{ki_2i_4}$;

$T^{\{\{1,3,4\}\{2\}\}TF}_{i_1i_2i_3i_4}=\varepsilon_{i_1i_2k}\,{}^{\{\{1,3,4\}\{2\}\}TF}_{\varepsilon\{1\}}T^{\{\{1,2\}\}}_{ki_3i_4}$.

where

${}^{\{\{1,2,3\}\{4\}\}TF}_{\varepsilon\{1\}}T_{ki_2i_3}=\frac{1}{2}\varepsilon_{ki_1i_4}T^{\{\{1,2,3\}\{4\}\}TF}_{i_1i_2i_3i_4}$; ${}^{\{\{1,2,4\}\{3\}\}TF}_{\varepsilon\{1\}}T_{ki_2i_4}=\frac{1}{2}\varepsilon_{ki_1i_3}T^{\{\{1,2,4\}\{3\}\}TF}_{i_1i_2i_3i_4}$;

${}^{\{\{1,2,3\}\{4\}\}TF}_{\varepsilon\{1\}}T_{ki_2i_3}=\frac{1}{2}\varepsilon_{ki_1i_4}T^{\{\{1,2,3\}\{4\}\}TF}_{i_1i_2i_3i_4}$,

The dualization does not provide a simplification for fourth-rank tensors with symmetry other than {{1,2}{3}{4}}, {{1,3}{2}{4}}, {{1,4}{2}{3}} и {{1}{2}{3}{4}} for dim > 3.

dim = 4

$T^{\{\{1\}\{2\}\{3\}\{4\}\}}_{i_1i_2i_3i_4}=\varepsilon_{i_1i_2i_3i_4}\,{}^{\{\{1\}\{2\}\{3\}\{4\}\}}_{\varepsilon\{1\}}T^{(4)}$; *where* ${}^{\{\{1\}\{2\}\{3\}\{4\}\}}_{\varepsilon\{1\}}T^{(4)}=\frac{1}{24}\varepsilon_{i_1i_2i_3i_4}T^{\{\{1\}\{2\}\{3\}\{4\}\}}_{i_1i_2i_3i_4}$;

$T^{\{\{1,2\}\{3\}\{4\}\}TF}_{i_1i_2i_3i_4}=\varepsilon_{i_1i_3i_4k}\,{}^{\{\{1,2\}\{3\}\{4\}\}TF}_{\varepsilon\{1\}}T_{ki_2}$; $T^{\{\{1,3\}\{2\}\{4\}\}TF}_{i_1i_2i_3i_4}=\varepsilon_{i_1i_2i_4k}\,{}^{\{\{1,3\}\{2\}\{4\}\}TF}_{\varepsilon\{1\}}T_{ki_3}$;

$T^{\{\{1,4\}\{2\}\{3\}\}TF}_{i_1i_2i_3i_4}=\varepsilon_{i_1i_2i_3k}\,{}^{\{\{1,3\}\{2\}\{4\}\}TF}_{\varepsilon\{1\}}T_{ki_4}$;

where

${}^{\{\{1,2\}\{3\}\{4\}\}TF}_{\varepsilon\{1\}}T_{ki_2}=-\frac{1}{6}\varepsilon_{ijsk}T^{\{\{1,2\}\{3\}\{4\}\}TF}_{ii_2js}$; ${}^{\{\{1,3\}\{2\}\{4\}\}TF}_{\varepsilon\{1\}}T_{ki_3}=-\frac{1}{6}\varepsilon_{ijsk}T^{\{\{1,3\}\{2\}\{4\}\}TF}_{iji_3s}$;

${}^{\{\{1,4\}\{2\}\{3\}\}TF}_{\varepsilon\{1\}}T_{ki_4}=-\frac{1}{6}\varepsilon_{ijsk}T^{\{\{1,4\}\{2\}\{3\}\}TF}_{ijsi_4}$;

dim = 5

$T^{\{\{1\}\{2\}\{3\}\{4\}\}}_{i_1i_2i_3i_4}=\varepsilon_{i_1i_2i_3i_4k}\,{}^{\{\{1\}\{2\}\{3\}\{4\}\}}_{\varepsilon\{1\}}T^{(5)}_k$; *where* ${}^{\{\{1\}\{2\}\{3\}\{4\}\}}_{\varepsilon\{1\}}T^{(5)}_k=\frac{1}{24}\varepsilon_{ki_1i_2i_3i_4}T^{\{\{1\}\{2\}\{3\}\{4\}\}}_{i_1i_2i_3i_4}$;

$T^{\{\{1,2\}\{3\}\{4\}\}TF}_{i_1i_2i_3i_4}=\frac{1}{2}\varepsilon_{i_1i_3i_4kp}\,{}^{\{\{1,2\}\{3\}\{4\}\}TF}_{\varepsilon\{1\}}T_{kpi_2}$; $T^{\{\{1,3\}\{2\}\{4\}\}TF}_{i_1i_2i_3i_4}=\frac{1}{2}\varepsilon_{i_1i_2i_4kp}\,{}^{\{\{1,3\}\{2\}\{4\}\}TF}_{\varepsilon\{1\}}T_{kpi_3}$;

Continued on next page

$$T_{i_1i_2i_3i_4}^{\{\{1,4\}\{2\}\{3\}\}TF} = \frac{1}{2}\varepsilon_{i_1i_2i_3kp}\;{}^{\{\{1,3\}\{2\}\{4\}\}TF}_{\varepsilon\{1\}}T_{kpi_4};$$

where

$${}^{\{\{1,2\}\{3\}\{4\}\}TF}_{\varepsilon\{1\}}T_{kpi_2} = \frac{1}{6}\varepsilon_{ijskp}T_{ii_2js}^{\{\{1,2\}\{3\}\{4\}\}TF}; \quad {}^{\{\{1,3\}\{2\}\{4\}\}TF}_{\varepsilon\{1\}}T_{kpi_3} = \frac{1}{6}\varepsilon_{ijskp}T_{iji_3s}^{\{\{1,3\}\{2\}\{4\}\}TF};$$

$${}^{\{\{1,4\}\{2\}\{3\}\}TF}_{\varepsilon\{1\}}T_{kpi_4} = \frac{1}{6}\varepsilon_{ijskp}T_{ijsi_4}^{\{\{1,4\}\{2\}\{3\}\}TF};$$

The dualization does not provide a simplification for fourth-rank tensors with symmetry that differs from $\{\{1\}\{2\}\{3\}\{4\}\}$ for dim > 5.

dim = 6

$$T_{i_1i_2i_3i_4}^{\{\{1\}\{2\}\{3\}\{4\}\}} = \frac{1}{2}\varepsilon_{i_1i_2i_3i_4kp}\;{}^{\{\{1\}\{2\}\{3\}\{4\}\}}_{\varepsilon\{1\}}T_{kp}^{\{\{1\}\{2\}\}(6)};\ where\ {}^{\{\{1\}\{2\}\{3\}\{4\}\}}_{\varepsilon\{1\}}T_{kp}^{\{\{1\}\{2\}\}(6)} = \frac{1}{24}\varepsilon_{kpi_1i_2i_3i_4}T_{i_1i_2i_3i_4}^{\{\{1\}\{2\}\{3\}\{4\}\}};$$

dim = 7

$$T_{i_1i_2i_3i_4}^{\{\{1\}\{2\}\{3\}\{4\}\}} = \frac{1}{6}\varepsilon_{i_1i_2i_3i_4ksp}\;{}^{\{\{1\}\{2\}\{3\}\{4\}\}}_{\varepsilon\{1\}}T_{ksp}^{(7)\{\{1\}\{2\}\{3\}\}};\ where\ {}^{\{\{1\}\{2\}\{3\}\{4\}\}}_{\varepsilon\{1\}}T_{ksp}^{(7)\{\{1\}\{2\}\{3\}\}} = \frac{1}{24}\varepsilon_{kspi_1i_2i_3i_4}T_{i_1i_2i_3i_4}^{\{\{1\}\{2\}\{3\}\{4\}\}};$$

The dualization does not provide a simplification for a fourth-rank fully antisymmetric tensor for dim > 7.

Table 10. Numbers of independent components in spaces of certain dimensions.

YSI	*Space dimension*							
	2	3	4	5	10	11	16	*dim*
{{1,2,3,4}}*TF*	2	9	25	55	660	935	3740	$\frac{(\dim+1)\dim}{24}\left(\dim^2+5\dim-6\right)$
{{1,2,3}{4}}*TF*	–	7	30	81	1386	2025	8925	$\frac{1}{8}(\dim+4)\left(\dim^2-1\right)(\dim-2)$, $\dim\geq 3$
{{1,2,4}{3}}*TF*	–	7	30	81	1386	2025	8925	
{{1,3,4}{2}}*TF*	–	7	30	81	1386	2025	8925	
{{1,2}{3,4}}*TF*	–	–	10	35	770	1144	5304	$\frac{1}{12}(\dim+2)(\dim+1)\dim(\dim-3)$, $\dim\geq 4$
{{1,3}{2,4}}*TF*	–	–	10	35	770	1144	5304	
{{1,2}{3}{4}}*TF*	–	–	9	30	945	1430	7020	$\frac{1}{8}(\dim+2)\dim(\dim-1)(\dim-3)$, $\dim\geq 4$
{{1,3}{2}{4}}*TF*	–	–	9	30	945	1430	7020	
{{1,4}{2}{3}}*TF*	–	–	9	30	945	1430	7020	
{{1}{2}{3}{4}}*TF*	–	–	1	5	210	330	1820	$\frac{\dim(\dim-1)(\dim-2)(\dim-3)}{24}$, $\dim\geq 4$
SUM	2	30	163	463	9403	13918	64003	$\dim^4-6\dim^2+3$, $\dim\geq 4$

4. Discussion

The dual simplification parts of Tables 7 and 9 are not fully completed because, for some *YSI* components, dual tensors do not possess a *YSI* and need to be decomposed again. Therefore, these have no current *YSI* in the present notation.

Fully antisymmetric components with *YSI* {{1}{2}}, {{1}{2}{3}}, and {{1}{2}{3}{4}} have no trace and thus are identical to tensors with *YSI* {{1}{2}}*TF*, {{1}{2}{3}}*TF,* and {{1}{2}{3}{4}}*TF*. For this reason, the sign *TF* was omitted in most cases.

5. Conclusion

A complete algorithm for analytical decomposition, based on well-known results and formulas from group theory, was proposed. The algorithm was applied for Cartesian tensors of small rank (from 2 to 4), and results are presented in tabular form for use in the investigation of tensor symmetry properties. The results for second-rank tensors are well-known and have been reported in many previous studies; they were repeated here for clarification of the proposed notation. Therefore, the main goal of this investigation was achieved.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (*AI*) tools in the creation of this article. The content of Table 7 before dual simplification was verified by program *decompgl* developed by the author [12]. This program can generated *GL*-decomposition of tensor of given rank and space dimension (but not greater than 99) given by user. But, for decomposition on trace-free component author still does not develop the program. So, this decomposition as such as content for second and third rank tensors had been evaluated analytically and was not verified by the computer program.

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Conflict of Interest

The authors declare no conflict of interest.

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