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Research article

Alexandroff-Bakelman-Pucci estimate and first exit time in cylindrical domains

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Abstract: This paper is concerned with elliptic equations satisfied by the mean first exit time of a stochastic process and higher moments. We prove an estimate for all moments of the first exit time in a cylindrical domain, using viscosity solutions. As a further result, we show the existence and the uniqueness of a more general fully nonlinear Dirichlet problem for partial trace equations, which can be associated to nonlinear diffusion.

Keywords: fully nonlinear elliptic equations; stochastic processes; first exit time; ABP estimates; unbounded domains

1. Introduction

Let us consider a stochastic process $\{X_t^x,\ t \geq 0\}$ in domain Ω of $\mathbb{R}^n$ starting from $x \in \Omega$, namely $X_0^x = x$. The *first exit time* or *first passage time* (FPT) is the random variable

$$\tau_{x,\Omega} := \inf\{t \geq 0 : X_t^x \notin \Omega\}.$$

Letting $\mathscr{L}$ be the (infinitesimal) generator of the process, it is well known that the mean first exit time $\tau(x) = \mathbb{E}(\tau_{x,\Omega})$ is solution of the Dirichlet problem

$$\begin{cases} \mathscr{L}\tau = -1 & \text{in } \Omega \\ \tau(x) = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

If the process is governed by the Stochastic Differential Equation

$$dX_t^x = b(X_t^x)dt + \boldsymbol{\sigma}(X_t^x)dW_t$$

with drift b and diffusion matrix $\boldsymbol{\sigma}$, by setting $A(x) = \boldsymbol{\sigma}(x)\boldsymbol{\sigma}^T(x)$ we have:

$$\begin{aligned} \mathscr{L}u &= \tfrac{1}{2}\,\mathrm{Tr}\,(A(x)\nabla^2 u) + b(x)\cdot\nabla u \\ &= \tfrac{1}{2}\,a_{ij}(x)u_{x_i x_j} + b_i(x)u_{x_i}. \end{aligned} \tag{1.2}$$

where, for a C^2 function u, we denote by ∇u and $\nabla^2 u$ the gradient and the Hessian matrix of u, respectively.

It is also well known that in general, provided that the first passage occurs with probability 1, for any $k \geq 1$ the moments $\tau_k = E(\tau^k_{x,\Omega})$ of FPT satisfy recursive equations, more precisely

$$\begin{cases} \frac{1}{2}\,a_{ij}(x)\frac{\partial^2\tau_k}{\partial x_i\partial x_j} + b_i(x)\frac{\partial\tau_k}{\partial x_i} = -k\tau_{k-1} & \text{in } \Omega \\ \tau_k = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\tau_0 = 1$ and $\tau_1(x) = \tau(x)$, the mean FPT defined above [1, 2]. Therefore higher moments τ_k, $k \geq 2$, depend on the existence and uniqueness of solutions of the Dirichlet problem (1.1) for the mean first exit time $\tau(x)$, as well as on suitable estimates of $\tau(x)$.

Our purpose is to provide estimates for the moments τ_k in domains of cylindrical type, which will be defined later, including cylinders and slabs, possibly infinite, but also more general domains. Applications of interest for biosciences encompass the escape of diffusive particles from membrane-enclosed structures, as in the case of calcium ions from the endoplasmic reticulum of cells [3]. A related issue is the narrow escape problem [4], where the narrow escape time is estimated for a Brownian particle to reach a small target located on the basis of a cylinder having a large radius with respect to the height. For applications in engineering we refer to chemical reactions, fluid dynamics and heat transfer in a pipe.

By cylindrical domain we mean an open connected set Ω that satisfies the measure-geometric condition $(\boldsymbol{G})$ introduced by Cabré [5], following insights of [6]: it is satisfied by bounded domains, infinite cylinders or slabs, but also by different domains, apparently not so close to cylinders and slabs as for instance, in dimension 2, the complement of periodic lattices of balls. We will give the precise definition of condition $(\boldsymbol{G})$ and further examples in Section 2.

The estimation of $\tau(x)$ in cylindrical domains is done by means of improved Alexandroff-Bakelman-Pucci estimates (ABP) in domains Ω satisfying condition $(\boldsymbol{G})$. Such estimates can be obtained in the more general case of fully nonlinear operators $\mathscr{F}u = F(x, \nabla u, \nabla^2 u)$, which are defined through a mapping $F : \Omega \times \mathbb{R}^n \times \mathcal{S}^n \to \mathbb{R}$.

Here $\mathcal{S}^n$ is the space on real $n\times n$ symmetric matrices, endowed with the partial ordering induced by the semidefinite positivity, and throughout we assume that $\mathscr{F}$ is elliptic, according to the next section.

As a consequence of ABP estimates, see Theorem 2.1 of the next section, we can deduce for solutions of Dirichlet problem (1.1) the following estimation of the mean first exit time.

Proposition 1.1. *Let Ω be a domain of $\mathbb{R}^n$ satisfying condition $(\boldsymbol{G})$ with parameters $R > 0$, $\sigma, \zeta \in (0,1)$. Suppose that the coefficients a_{ij}, b_i of operator $\mathscr{L}$ are continuous in $\overline{\Omega}$, with $(a_{ij}(x))$ satisfying the uniform ellipticity condition*

$$\lambda|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \quad \textit{for all } \xi \in \mathbb{R}^n,\ x \in \Omega, \tag{1.4}$$

where λ and $\Lambda \geq \lambda$ are positive constants, and $b_i(x)$ uniformly bounded, namely

$$|b(x)| := (b_i(x)b_i(x))^{1/2} \leq \overline{b} \quad \textit{for all } \xi \in \mathbb{R}^n,\ x \in \Omega, \tag{1.5}$$

for some $\overline{b} \geq 0$. *If* $\tau \in C^0(\overline{\Omega})$ *is a viscosity solution of Dirichlet problem* (1.1)*, bounded in* Ω*, then we have* $\tau \geq 0$ *in* Ω *and*

$$\sup_{\Omega} \tau \leq CR^2, \tag{1.6}$$

where $C = C(n, \lambda, \Lambda, \overline{b}R, \sigma, \zeta)$ *is a positive constant.*

Note that in Proposition 1.1 the existence of solutions τ continuous and bounded in Ω is given as an assumption. However, to estimate higher moments we need that τ is well defined by the Dirichlet problem (1.1), requiring existence and uniqueness of a continuous and bounded solution.

Sufficient conditions in order that this holds true in the more general setting of fully nonlinear uniformly elliptic operators are provided for cylindrical domains in a previous work [7] for viscosity solutions of the more general Dirichlet problem

$$\begin{cases} \mathscr{F}u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \tag{1.7}$$

with $f \in C^0(\Omega) \cap L^\infty(\Omega)$ and $g \in C^0(\partial\Omega)$.

This demands additional conditions with respect to the ABP estimate of Theorem 2.1, behind Proposition 1.1, both on the domain Ω and on the operator $\mathscr{F}$.

More precisely, we will require some regularity to the cylindrical domain Ω, assuming a geometric condition, which implies condition (***G***), see [7], and will be called regular geometric condition (***RG***): Ω has finite inner radius and satisfies a uniform exterior cone property. We refer to Section 2 for more detailed definitions.

On the other hand, we complement condition (***SC***) assuming the uniform ellipticity, which in the case of fully nonlinear operators means, for $X, Y \in \mathcal{S}^n$:

$$X \leq Y \quad \Rightarrow \quad \lambda \mathrm{Tr}(Y - X) \leq F(x, \xi, Y) - F(x, \xi, X) \leq \Lambda \mathrm{Tr}(Y - X) \tag{\textbf{\textit{UE}}}$$

for all $(x, \xi) \in \Omega \times \mathbb{R}^n$.

Adding the stronger condition (***RG***) on Ω and the uniform ellipticity (***UE***) of $\mathscr{F}$ to the assumptions of Theorem 2.1, we obtained in [7] the existence of viscosity solutions $u \in C^0(\overline{\Omega}) \cap L^\infty(\Omega)$ for the Dirichlet problem (1.7), and consequent uniform estimates for solutions [7, Theorem 1.1].

Here we will specialize this result in the case of Dirichlet problem (1.1) for the mean FPT. As we will see, the linear structure of operator $\mathscr{L}$ defined in (1.2) also allows to get uniqueness.

Theorem 1.2. *Let* Ω *be a domain of* $\mathbb{R}^n$ *endowed with property* (***RG***) *with inner radius R, and a finite cone of slant height h and aperture* θ. *Under the assumptions of Proposition 1.1 on the coefficient* $a_{ij}(x)$ *and* $b_i(x)$ *of* $\mathscr{L}$, *there exists a unique viscosity solution* $\tau \in C^0(\overline{\Omega}) \cap L^\infty(\Omega)$ *of Dirichlet problem* (1.1)*, which is non-negative and satisfies the ABP estimate* (1.6) *with a positive constant* $C = C(n, \theta, \lambda, \Lambda, \overline{b}R)$.

It is worth noting that indeed, by continuity assumptions, the solutions are $C^{1,\alpha}_{loc}(\Omega)$ with $\alpha(0, 1)$. In fact, looking at the proof, $\tau \in W^{2,n}_{loc}(\Omega)$ with $p > n$. To obtain classical solutions (twice differentiable) we should require $a_{ij}, b_i, f \in C^\alpha_{loc}(\Omega)$, see [8, 9].

We are now ready to establish estimates for higher moments τ_k.

Theorem 1.3. *Suppose that the assumptions of Theorem 1.2 hold. Then the moments $\tau_k = E(\tau^k_{x,\Omega})$, $k \in \mathbb{N}$, of the first exit time are well defined, non-negative, and satisfy the following estimates:*

$$\sup_{\Omega} \tau_k \leq Ck!R^{2k}, \tag{1.8}$$

where C is the positive constant of Theorem 1.2.

In the ending part of this work, we will show that the general result concerning the Dirichlet problem (1.7) can be also extended to a class of elliptic operators that are not uniformly elliptic and associated to nonlinear diffusion.

In this respect, let $\boldsymbol{i}_k = (i_1, \dots, i_k)$ be a k-ple of positive integers in $\{1, \dots, n\}$ such that $i_1 < \cdots < i_k$. We consider the partial trace operators

$$\mathrm{Tr}_{\boldsymbol{i}_k}(\nabla^2 u) = \lambda_{i_1}(\nabla^2 u) + \cdots + \lambda_{i_k}(\nabla^2 u),$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of the Hessian matrix, repeated according to the multiplicity and arranged in non-decreasing order.

Such operators appear in geometric problems of mean partial curvature [10, 11] and have an interpretation in two-players zero-sum differential games [12, 13, 14, 15].

If $k = n$, then $\mathrm{Tr}_{\boldsymbol{i}_k}(\nabla^2 u) = \mathrm{Tr}(\nabla^2 u) = \Delta u$ is linear and uniformly elliptic, so that it falls within the framework of the aforementioned [7, Theorem 1.1].

On the contrary, if $k < n$, then $\mathrm{Tr}_{\boldsymbol{i}_k}(X)$ is neither linear nor uniformly elliptic, see [16] for counterexamples, and it is not covered by [7, Theorem 1.1].

Nonetheless, we will show that ABP estimates still hold when

$$\mathscr{F} u = \mathrm{Tr}_{\boldsymbol{i}_k}(\nabla^2 u) + b(x) \cdot \nabla u,$$

provided that $i_1 = 1$ and $i_k = n$, namely the partial trace operator contains the lowest and the highest eigenvalue of X, as for instance $\mathrm{Tr}_{\boldsymbol{i}_2}(X) = \lambda_1(X) + \lambda_n(X)$, largely studied in [16].

Using this, we establish existence and uniqueness of a bounded solution in cylindrical domain for the Dirichlet problem (1.10) of Theorem 1.4 below with bounded continuous boundary conditions.

Theorem 1.4. *Let Ω be as in Theorem 1.2. Let $\mathrm{Tr}_{\boldsymbol{i}_k}(X)$ be the partial trace operator defined above with $i_1 = 1$ and $i_k = n$, $b_i(x)$ continuous functions bounded as in* (1.5)*, such that*

$$(b(x) - b(y)) \cdot (x - y) \leq \overline{b}|x - y|^2, \tag{1.9}$$

with $\overline{b} \geq 0$, $f \in C^0(\overline{\Omega})$ such that the norm $\|f\|_{\Omega,R}$, defined in (2.7)*, is finite, and $g \in C^0(\partial\Omega) \cap L^\infty(\Omega)$.*

There exists a unique viscosity solution $u \in C^0(\overline{\Omega}) \cap L^\infty(\Omega)$ of the Dirichlet problem

$$\begin{cases} \mathrm{Tr}_{\boldsymbol{i}_k}(\nabla^2 u) + b(x) \cdot \nabla u = f(x) & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \tag{1.10}$$

which satisfies the ABP estimate

$$\sup_{\Omega} |u| \leq \sup_{\partial\Omega} |g| + CR\|f\|_{\Omega,R} \tag{1.11}$$

with a positive constant $C = C(n, \theta, \lambda, \Lambda, \overline{b}R)$.

The proof of Theorem 1.4 will be done by approximation with solutions in an increasing sequence of bounded domains invading Ω. So we primarily need to prove an existence and uniqueness result in bounded domains, since we cannot appeal, as in [7], to [17, Theorem 1.1], due to missing uniform ellipticity.

This is possible since the mapping $F(X) = \mathrm{Tr}_{i_k}(X)$ is not merely elliptic, but something more, namely non-totally degenerate elliptic, according to [18],

$$F(X + tI) - F(X) \geq kt, \quad t \geq 0, \quad \text{for } \; k > 0,$$

with respect to the basic ellipticity:

$$F(Y) - F(X) \geq 0, \quad Y \geq X.$$

The paper is organized as follows: in Section 2 we discuss geometric properties of cylindrical domains and structure conditions on operators that guarantee the validity of ABP estimates for viscosity solutions, in Section 3 we show estimates for the mean first exit time and higher moments, and in Section 4 we give existence, uniqueness and estimates for solutions of partial trace equations.

2. Notations and preparatory results

Firstly, we specify the definition of cylindrical domain considered in the previous section. What we call cylindrical domain is an open connected set Ω endowed with the geometric condition ($\boldsymbol{G}$), introduced by Cabré [5], depending on parameters $R > 0$, $\sigma, \zeta \in (0,1)$: for all $x \in \Omega$ there exist a ball $B^x_{R_x}$, containing x, of radius $R_x \leq R$, and a concentric ball $B^x_{R_x/\zeta}$, such that

$$\frac{|B^x_{R_x} \backslash \Omega_{x,\zeta}|}{|B^x_{R_x}|} \geq \sigma, \tag{$\boldsymbol{G}$}$$

where $|\,|$ is the Lebesgue measure, and $\Omega_{x,\zeta}$ the (connected) component of $\Omega \cap B^x_{R_x/\zeta}$ that contains x.

A right cylinder, finite or infinite, satisfies this condition with $R > r$ for some $\sigma, \zeta \in (0,1)$, where r is the radius of the circular section. The same holds for slabs, finite or infinite, of thickness r. As a matter of fact bounded sets and more generally domains of finite Lebesgue measure satisfy condition ($\boldsymbol{G}$). However, we can also find domains with complements having zero Lebesgue measure among domains satisfying condition ($\boldsymbol{G}$) as for instance, in dimension 2, the complement of Archimedean spirals. It is also worth noting that different variants of condition ($\boldsymbol{G}$) have been considered in [19, 20, 21, 22, 23].

A remarkable fact is that an improved ABP estimate holds in cylindrical domains, see [5]. Indeed, the ABP estimate holds not only in the linear case that we are considering for the mean first exit time, but in the more general case of fully nonlinear elliptic operators

$$\mathscr{F}u = F(x, \nabla u, \nabla^2 u).$$

Here $F : \Omega \times \mathbb{R}^n \times \mathcal{S}^n \to \mathbb{R}$, where $\mathcal{S}^n$ is the space of $n \times n$ real symmetric matrices endowed with the partial ordering $X \leq Y \Leftrightarrow Y - X$ semidefinite positive, and we assume that $\mathscr{F}$ is elliptic, namely

$$X \leq Y \quad \Rightarrow \quad F(x, \xi, X) \leq F(x, \xi, Y)$$

for all $(x, \xi) \in \Omega \times \mathcal{S}^n$, and the structure conditions

$$F(x, \xi, X) \leq \Lambda \mathrm{Tr}(X) + \overline{b}|\xi|, \tag{$\boldsymbol{SC^+}$}$$

$$F(x, \xi, X) \geq \lambda \mathrm{Tr}(X) - \overline{b}|\xi|, \tag{$\boldsymbol{SC^-}$}$$

for all $x \in \Omega$, $\xi \in \mathbb{R}^n$, $X \in \mathcal{S}^n$ such that $X \geq 0$, where λ and Λ are positive constants such that $\lambda \leq \Lambda$, and $\overline{b} \in \mathbb{R}_+$. We also set

$$(\boldsymbol{SC}) = (\boldsymbol{SC^+}) \cap (\boldsymbol{SC^-}).$$

Note that linear operators $\mathscr{L}$ as in (1.2) fit into our framework, corresponding to $\mathscr{F}$ when choosing

$$F(x, \xi, X) = \tfrac{1}{2}\,\mathrm{Tr}(AX) + b(x) \cdot \xi, \tag{2.1}$$

where $\mathrm{Tr}(X)$ denotes the trace of matrices $X \in \mathcal{S}^n$. Moreover, $\mathscr{L}$ satisfies condition ($\boldsymbol{SC}$) if the $a_{ij}(x)$'s are uniformly elliptic with ellipticity constants λ, Λ and the coefficients $b_i(x)$ are bounded in Ω.

Nonetheless, condition ($\boldsymbol{SC}$) is also satisfied by fully nonlinear operators like

$$F(x, \xi, X) = \mathcal{M}^{\pm}_{\lambda,\Lambda}(X) \pm \overline{b}|\xi|,$$

for any sign combination, where $\mathcal{M}^{\pm}_{\lambda,\Lambda}$ are the Pucci extremal operators and $\overline{b}$ is a positive constant. Indeed, recalling the definition of the Pucci maximal and minimal operators,

$$\mathcal{M}^{+}_{\lambda,\Lambda}(X) := \sup_{\lambda I \leq A \leq \Lambda I} \mathrm{Tr}(AX), \quad \mathcal{M}^{-}_{\lambda,\Lambda}(X) := \inf_{\lambda I \leq A \leq \Lambda I} \mathrm{Tr}(AX),$$

we see that condition ($\boldsymbol{SC}$) is equivalent to

$$\mathcal{M}^{-}_{\lambda,\Lambda}(X) - \overline{b}|\xi| \leq F(x, \xi, X) \leq \mathcal{M}^{+}_{\lambda,\Lambda}(X) + \overline{b}|\xi| \tag{2.2}$$

for all $x \in \Omega$, $\xi \in \mathbb{R}^n$, $X \in \mathcal{S}^n$.

ABP estimates of interest for our purpose were obtained in [5] for strong solutions $u \in W^{2,n}_{loc}(\Omega) \cap C^0(\overline{\Omega})$ of linear uniformly elliptic equations, and in [21] for viscosity solutions of fully nonlinear equations with F satisfying (2.2), which is the approach used here.

Let u be an upper (resp. lower) semicontinuous function in $\overline{\Omega}$. We say that u is a viscosity subsolution (resp. supersolution) of equation $\mathscr{F}u = f$ in Ω, equivalently a viscosity solution of the differential inequality $\mathscr{F}u \geq f$ (resp. $\mathscr{F}u \leq f$), if for all $x \in \Omega$ and smooth functions φ, touching from above (resp. below) the graph of u at x, we have

$$\mathscr{F}\varphi(x) \geq f(x) \quad (\text{resp. } \mathscr{F}\varphi(x) \leq f(x)). \tag{2.3}$$

A function $u \in C^0(\overline{\Omega})$ is a viscosity solution if it is both a viscosity subsolution and supersolution. For a detailed account on viscosity solutions we refer to [24] and [25].

Theorem 2.1. *Let Ω be a domain of $\mathbb{R}^n$ satisfying condition ($\boldsymbol{G}$) with parameters $R > 0$, $\sigma, \zeta \in (0, 1)$, and $\mathscr{F}$ satisfy ($\boldsymbol{SC^+}$), resp. ($\boldsymbol{SC^-}$), with constants $\lambda > 0$ and $\Lambda \geq \lambda$. Suppose also $f \in C^0(\Omega)$. If $u \in C^0(\overline{\Omega})$ is a viscosity subsolution, bounded above, resp. supersolution, bounded below, of equation*

$$\mathscr{F}u = f(x) \quad \text{in } \Omega, \tag{2.4}$$

then the following ABP estimate holds:

$$\sup_{\Omega} u^+ \leq \sup_{\partial\Omega} u^+ + CR\|f^-\|_{\Omega,R}, \tag{2.5}$$

resp.

$$\sup_{\Omega} u^- \leq \sup_{\partial\Omega} u^- + CR\|f^+\|_{\Omega,R}, \tag{2.6}$$

where

$$\|f\|_{\Omega,R} := \sup_{x\in\Omega} \|f\|_{L^n(\Omega\cap B^x_{R/\zeta})} \tag{2.7}$$

and C is a positive constant depending only on $n, \lambda, \Lambda, \overline{b}R, \sigma, \zeta$.

In particular, (2.5) and (2.6) also yield the inequalities

$$\sup_{\Omega} u^+ \leq \sup_{\partial\Omega} u^+ + CR^2\|f^-\|_{L^\infty(\Omega)} \tag{2.8}$$

and

$$\sup_{\Omega} u^- \leq \sup_{\partial\Omega} u^- + CR^2\|f^+\|_{L^\infty(\Omega)}. \tag{2.9}$$

As a further particular case, we get the sign propagation property (SPP) for subsolutions of equation $\mathscr{F}u = 0$,

$$u \leq 0 \quad \text{on } \partial\Omega \quad \Rightarrow \quad u \leq 0 \quad \text{in } \Omega,$$

and for supersolutions,

$$u \geq 0 \quad \text{on } \partial\Omega \quad \Rightarrow \quad u \geq 0 \quad \text{in } \Omega.$$

Moreover, for viscosity solutions $u \in C^0(\Omega)$, bounded in Ω, of equation (2.4), Theorem 2.1 yields

$$\sup_{\Omega} |u| \leq \sup_{\partial\Omega} |u| + CR\|f\|_{\Omega,R}, \tag{2.10}$$

which implies

$$\sup_{\Omega} |u| \leq \sup_{\partial\Omega} |u| + CR^2\|f\|_{L^\infty(\Omega)}. \tag{2.11}$$

In particular, when $f = 0$, the above inequalities provide the maximum modulus principle

$$\sup_{\Omega} |u| \leq \sup_{\partial\Omega} |u|. \tag{2.12}$$

Proposition 1.1 is an application of Theorem 2.1 and its consequences.

Proof. (Proposition 1.1) Firstly observe that the linear operator $\mathscr{L}$ of (1.1) corresponds to $\mathscr{F}$ when choosing

$$F(x, \xi, X) = \tfrac{1}{2}\,\mathrm{Tr}(A(x)X) + b(x) \cdot \xi,$$

where by uniform ellipticity we have $\lambda I \leq A(x) \leq \Lambda I$ and I is the $n \times n$ identity matrix. From this

$$\frac{\lambda}{2}\,\mathrm{Tr}(X) - \overline{b}|\xi| \leq F(x, t, \xi, X) \leq \frac{\Lambda}{2}\,\mathrm{Tr}(X) + \overline{b}|\xi|,$$

and therefore conditions ($\boldsymbol{SC^+}$) and ($\boldsymbol{SC^-}$) of Theorem 2.1 are satisfied with ellipticity constants $\lambda/2$ and $\Lambda/2$. Taking into account that $f = -1$ in Ω, and $u = 0$ on $\partial\Omega$, we have that τ is a C-viscosity supersolution of equation $\frac{1}{2}\,\mathrm{Tr}(A\nabla^2 u) + b(x)\cdot\nabla u = 0$ in Ω such that $u \geq 0$ on $\partial\Omega$ so that, by the sign propagation property for supersolutions deriving from (2.6), we get $\tau \geq 0$ in Ω, as we expect. Being u a viscosity (sub)solution of equation $\frac{1}{2}\,\mathrm{Tr}(A\nabla^2 u) + b(x)\cdot\nabla u = -1$ in Ω such that $u = 0$ on $\partial\Omega$, then (1.6) follows from (2.5). □

To obtain the existence and uniqueness result of Theorem 1.2 from [7, Theorem 1.1], we need condition (***RG***) and uniform ellipticity (***UE***).

Condition (***RG***) consists of finite inner radius and uniform exterior cone condition, where we recall that the inner radius of a domain Ω is defined as

$$\sup\{r \in \mathbb{R}_+ : \exists \text{ a ball } B \text{ of radius } r \text{ such that } B \subset \Omega\},$$

and Ω satisfies a uniform exterior cone condition with aperture θ and slant height h if there exist an open circular cone $\Sigma \subset \mathbb{R}^n$ of aperture θ and an open ball B of radius h such that the following condition holds: for each point $x_0 \in \partial\Omega$, we have a finite cone Σ_{x_0} with vertex x_0, obtained from $\Sigma \cap B$ by a roto-translation, such that

$$\Sigma_{x_0} \subset \mathscr{C}\overline{\Omega}.$$

Let $f \in L^p_{loc}(\Omega)$, for $p > \frac{n}{2}$. We say that $u \in C^0(\Omega)$ is an L^p-viscosity subsolution, resp. supersolution, of equation $\mathscr{F}u = f$ if for all $x_0 \in \Omega$ and test functions $\varphi \in W^{2,p}_{loc}(\Omega)$ such that $u - \varphi$ has a local maximum, resp. minimum, at x_0, we have

$$\operatorname{ess}\limsup_{x\to x_0}\left\{F(x, D^2\varphi(x)) + b(x)\cdot\nabla\varphi(x) - f(x)\right\} \geq 0, \tag{2.13}$$

resp.

$$\operatorname{ess}\liminf_{x\to x_0}\left\{F(x, D^2\varphi(x)) + b(x)\cdot\nabla\varphi(x) + -f(x)\right\} \leq 0. \tag{2.14}$$

An L^p-viscosity solution $u \in C^0(\Omega)$ is both an L^p-viscosity subsolution and supersolution. For a detailed treatment of L^p-viscosity solutions see [26].

Here we only notice that, when F is continuous, as in our case, L^p-viscosity agrees with C-viscosity [26, Proposition 2.9].

3. Estimation of higher moments

We firstly prove Theorem 1.2, providing that τ is well defined by the Dirichlet problem (1.1), which is essential to obtain estimates for moments τ_k with $k > 1$.

Proof. (Theorem 1.2) As remarked above with (2.1), the linear operator $\mathscr{L}$ fits in our framework and satisfies both conditions (***SC***) and (***UE***), while $c = 0$ and $f = -1$. Moreover, in [7] it is shown that the geometric property (***RG***), there (***GEC***), implies condition (***G***) with $R + \varepsilon$ in the place of R for any $\varepsilon \in (0, h)$ and some $\sigma, \zeta \in (0, 1)$ depending on n, R, h and θ.

Therefore [7, Theorem 1.1] provides the existence of a C-viscosity solution $\tau \in C^0(\overline{\Omega}) \cap L^\infty(\Omega)$ satisfying the ABP estimate (1.6) with $C = C(n, R, h, \theta, \lambda, \Lambda, \overline{b})$.

In order to deal with uniqueness, we start considering the following Dirichlet problem for the linear operator $\mathscr{L}$ in a ball B_r of radius r such that $\overline{B}_r \subset \Omega$:

$$\begin{cases} \frac{1}{2}\, a_{ij}(x)\, \dfrac{\partial^2 u}{\partial x_i \partial x_j} + b_i(x)\, \dfrac{\partial u}{\partial x_i} = -1 & \text{in } B_r \\ u(x) = \tau(x) & \text{on } \partial B_r \end{cases}. \tag{3.1}$$

Consider now any $p \geq n$. Since $a_{ij}, b_i \in C^0(\overline{\Omega})$, by [27, Theorem 3.1], there exists a unique L^p-viscosity solution $u \in W^{2,p}_{loc}(B_r) \cap C^0(\overline{B}_r)$ of (3.4), which is by [26, Proposition 3.5] a strong solution with Sobolev derivatives satisfying the equation almost everywhere:

$$\frac{1}{2}\, a_{ij}(x)\, \frac{\partial^2 u}{\partial x_i \partial x_j} + b_i(x)\, \frac{\partial u}{\partial x_i} = -1 \quad \text{a.e. in } B_r.$$

On the other hand, τ is also an L^p-viscosity solution of (3.4), as noticed above, so that by uniqueness $\tau = u \in W^{2,p}_{loc}(B_r) \cap C^0(\overline{B}_r)$, and so

$$\frac{1}{2}\, a_{ij}(x)\, \frac{\partial^2 \tau}{\partial x_i \partial x_j} + b_i(x)\, \frac{\partial \tau}{\partial x_i} = -1 \quad \text{a.e. in } B_r.$$

By the arbitrariness of B_r, we have in particular $\tau \in W^{2,p}_{loc}(\Omega)$, so that

$$\frac{1}{2}\, a_{ij}(x)\, \frac{\partial^2 \tau}{\partial x_i \partial x_j} + b_i(x)\, \frac{\partial \tau}{\partial x_i} = -1 \quad \text{a.e. in } \Omega. \tag{3.2}$$

If $\tau' \in C^0(\overline{\Omega})$ were another C-viscosity solution of (3.4), we should conclude in similar manner that $\tau' \in W^{2,p}_{loc}(\Omega)$, and

$$\frac{1}{2}\, a_{ij}(x)\, \frac{\partial^2 \tau'}{\partial x_i \partial x_j} + b_i(x)\, \frac{\partial \tau'}{\partial x_i} = -1 \quad \text{a.e. in } \Omega, \tag{3.3}$$

so that from (3.2) and (3.3), setting $v = \tau - \tau'$, we have $v \in W^{2,p}_{loc}(\Omega) \cap C^0(\overline{\Omega})$ and

$$\begin{cases} \frac{1}{2}\, a_{ij}(x)\, \dfrac{\partial^2 v}{\partial x_i \partial x_j} + b_i(x)\, \dfrac{\partial v}{\partial x_i} = 0 & \text{a.e. in } \Omega \\ v(x) = 0 & \text{on } \partial\Omega \end{cases}. \tag{3.4}$$

Since condition (***RG***) implies condition (***G***), then the improved ABP estimate for strong solutions, see for instance [5, Theorem 1.1], yields $v = 0$ in Ω, namely $\tau' = \tau$, which proves uniqueness.

Finally, observing that condition (***RG***) implies condition (***G***) with a slightly larger R and some $\sigma, \tau \in (0, 1)$, the ABP estimate (1.6) follows from Proposition 1.1. $\square$

We are now ready to prove the estimates (1.8) for higher moments τ_k.

Proof. (Theorem 1.3) Firstly, we show by induction that τ_k is well defined for all $k \in \mathbb{N}$.

In fact, for $k = 1$, $\tau_1 = \tau \geq 0$ is the unique bounded viscosity solution of Dirichlet problem (1.1), by Theorem 1.2, and satisfies the ABP estimate (1.6).

Next, suppose that τ_k is well defined as the unique viscosity solution of Dirichlet problem (1.3), such that $\tau_k \geq 0$ and inequality (1.8) is satisfied. Then τ_{k+1} is well defined as the unique bounded viscosity solution of Dirichlet problem

$$\begin{cases} \frac{1}{2}\, a_{ij}(x)\, \dfrac{\partial^2 \tau_{k+1}}{\partial x_i \partial x_j} + b_i(x)\, \dfrac{\partial \tau_{k+1}}{\partial x_i} = -(k+1)\tau_k & \text{in } \Omega \\ v(x) = 0 & \text{on } \partial\Omega \end{cases}, \tag{3.5}$$

by [7, Theorem 1.1]. In particular

$$\frac{1}{2}\, a_{ij}(x)\, \frac{\partial^2 \tau_{k+1}}{\partial x_i \partial x_j} + b_i(x)\, \frac{\partial \tau_{k+1}}{\partial x_i} \leq 0 \quad \text{in } \Omega, \quad \tau_{k+1} = 0 \quad \text{on } \partial\Omega,$$

and so, by the sign propagating property (SPP) for supersolutions, deriving from (2.6), we infer that $\tau_{k+1} \geq 0$.

Then from (2.5) we obtain:

$$\sup_{\Omega} \tau_{k+1} \leq C(k+1)R^2 \sup_{\Omega} \tau_k \leq C(k+1)!R^{2(k+1)},$$

which shows the validity of estimate (1.8) for τ_{k+1}, completing the induction procedure, and finishing the proof. □

4. Partial trace operators

We begin with ABP estimates, in cylindrical domains, which follow from Theorem 2.1.

Corollary 4.1. *Suppose that the assumptions of Theorem 2.1 are verified, but*

$$\mathscr{F} u = \mathrm{Tr}_{\boldsymbol{i}_k}(\nabla^2 u) + b(x) \cdot \nabla u \tag{4.1}$$

with $i_1 = 1$ and $i_k = n$. Then estimates (2.5), (2.6) *and* (2.10) *holds, respectively, for viscosity bounded above subsolutions, bounded below supersolutions and bounded solutions $u \in C^0(\overline{\Omega})$ of equation*

$$\mathrm{Tr}_{\boldsymbol{i}_k}(\nabla^2 u) + b(x) \cdot \nabla u = f(x), \tag{4.2}$$

where $\mathrm{Tr}_{\boldsymbol{i}_k}(\nabla^2 u) := \lambda_{i_1}(\nabla^2 u) + \cdots + \lambda_{i_k}(\nabla^2 u)$.

Proof. The proof relies on the following inequalities. Since $i_1 = 1$, then $\lambda_{i_1} + \cdots + \lambda_{i_k} \leq \kappa_j \lambda_j$, where, observing that $\lambda_1 \leq \frac{1}{n}(\lambda_1 + \cdots + \lambda_n)$, we have $k_j = 1 + \frac{1}{n}$ if $j \in \{i_2, \ldots, i_k\}$, $k_j = \frac{1}{n}$ otherwise. From this we deduce, see [16], that

$$\mathrm{Tr}_{\boldsymbol{i}_k}(X) = \lambda_{i_1}(X) + \cdots + \lambda_{i_k}(X) \leq \mathcal{M}^+_{\frac{1}{n}, 1+\frac{1}{n}}(X). \tag{4.3}$$

On the other hand, since $i_1 = 1$, then $\lambda_{i_1} + \cdots + \lambda_{i_k} \leq \kappa_j \lambda_j$, where, observing that $\lambda_1 \leq \frac{1}{n}(\lambda_1 + \cdots + \lambda_n)$, we have $k_j = 1 + \frac{1}{n}$ if $j \in \{i_2, \ldots, i_k\}$, $k_j = \frac{1}{n}$ otherwise. From this we deduce, see [16], that

$$\mathrm{Tr}_{\boldsymbol{i}_k}(X) = \lambda_{i_1}(X) + \cdots + \lambda_{i_k}(X) \geq \mathcal{M}^-_{\frac{1}{n}, 1+\frac{1}{n}}(X). \tag{4.4}$$

Now, let us consider a viscosity subsolution u of equation (4.2). By inequality (4.3), being $|b(x)| \le \overline{b}$, it is in turn a subsolution of equation

$$\mathcal{M}^+_{\frac{1}{n},1+\frac{1}{n}}(\nabla^2 u) + \overline{b}\,|\nabla u| = -f^-(x),$$

and Theorem 2.1 yields inequality (2.5).

In similar manner, using inequality (4.4), a viscosity supersolution u is also a supersolution of equation

$$\mathcal{M}^-_{\frac{1}{n},1+\frac{1}{n}}(\nabla^2 u) - \overline{b}\,|\nabla u| = f^+(x)$$

and Theorem 2.1 yields inequality (2.6).

Gathering together inequalities (2.5) and (2.6), inequality (2.10) follows for viscosity solutions u. □

Next, we state an existence and uniqueness result in bounded domains.

Theorem 4.2. *Let Ω be a bounded domain of $\mathbb{R}^n$ endowed with a uniform exterior cone property. Let Tr_{i_k} be the partial trace operator defined above with $i_1 = 1$ and $i_k = n$, $b_i \in C^0(\overline{\Omega})$, such that* (1.9) *holds, i.e.*

$$(b(x) - b(y)) \cdot (x - y) \le \overline{b}|x - y|^2,$$

and $f \in C^0(\overline{\Omega})$. There exists a unique viscosity solution $u \in C^0(\overline{\Omega})$ of Dirichlet problem (1.10)*, namely*

$$\begin{cases} \mathrm{Tr}_{\boldsymbol{i}_k}(\nabla^2 u) + b(x) \cdot \nabla u = f(x) & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases},$$

where $g \in C^0(\partial\Omega)$.

Proof. Existence and uniqueness of viscosity solutions are provided by Theorem 4.1 of [24] (see also [28]). So our proof will be done if the assumptions of this theorem are verified: comparison principle between viscosity subsolutions and supersolutions; existence of viscosity subsolutions and supersolutions of the equation satisfying the boundary condition.

Comparison principle. We infer that comparison principle follows from [18, Theorem 3.1], observing that the assumptions are satisfied. In fact, it is plain that the mapping

$$G(x, \xi, X) = \mathrm{Tr}_{\boldsymbol{i}_k}(X) + b(x) \cdot \xi - f(x)$$

is basically continuous and elliptic by assumptions, and in addition non-totally degenerate elliptic, just like the mapping $\mathrm{Tr}_{\boldsymbol{i}_k}(X)$. Moreover it is Lipschitz-continuous in ξ with Lipschitz constant $\overline{b}$, given by (1.5), by the assumption $b \in C^0(\overline{\Omega}; \mathbb{R}^n)$, and satisfies a typical viscosity condition for comparison principles:

$$G(x, \alpha(x - y), X) - G(y, \alpha(x - y), Y) \le \omega(\alpha|x - y|^2 + |x - y|),$$

with $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\omega(r) \to 0$ as $r \to 0^+$, for all $\alpha \in \mathbb{R}_+$, $x, y \in \overline{\Omega}$ and $X, Y \in \mathcal{S}^n$ such that

$$-3\alpha \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix},$$

by the ellipticity of Tr_{i_k}, the monotonicity assumptions on b, see [24], and the continuity of $f \in C^0(\overline{\Omega})$.

Concerning the existence of subsolutions and supersolutions, firstly observe that the mappings

$$\tilde{M}^{\pm}_{\lambda,\Lambda}(x, \xi, X) = \mathcal{M}^{\pm}_{\lambda,\Lambda}(X) \pm \overline{b}|\xi| \pm f^{\mp}(x),$$

where $\mathcal{M}^{\pm}_{\lambda,\Lambda}$ are the Pucci extremal operators with $0 < \lambda \leq \Lambda$, satisfy condition (SC) of [17], so that by [17, Theorem 1.1], see also [28], there exist viscosity solutions in $C^0(\overline{\Omega})$ of the Dirichlet problems

$$\begin{cases} \mathscr{M}^{\pm}u = \mp f^{\mp}(x) & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}, \tag{4.5}$$

where $\mathscr{M}^{\pm}u = M^{\pm}_{\lambda,\Lambda}(\nabla^2 u) \pm \overline{b}\,|\nabla u|$.

Subsolutions. If $u \in C^0(\overline{\Omega})$ is a viscosity solution of the Dirichlet problem (4.5) for $\mathscr{M}^-$ with $\lambda = \frac{1}{n}$ and $\Lambda = 1 + \frac{1}{n}$, then from inequality (4.4) we get

$$\mathrm{Tr}_{i_k}(\nabla^2 u) + b(x) \cdot \nabla u \geq \mathcal{M}^{-}_{\frac{1}{n},1+\frac{1}{n}}(\nabla^2 u) - \overline{b}|\nabla u| = f^+(x)$$

in the viscosity sense, i.e. u is a continuous viscosity subsolution of equation $\mathrm{Tr}_{i_k}(\nabla^2 u)+b(x)\cdot\nabla u = f(x)$ satisfying the boundary condition $u = g$ on $\partial\Omega$.

Supersolutions. Analogously, if $u \in C^0(\overline{\Omega})$ is a viscosity solution of the Dirichlet problem (4.5) for $\mathscr{M}^+$ with $\lambda = \frac{1}{n}$ and $\Lambda = 1 + \frac{1}{n}$, then from inequality (4.3) we get

$$\mathrm{Tr}_{i_k}(\nabla^2 u) + b(x) \cdot \nabla u \leq \mathcal{M}^{+}_{\frac{1}{n},1+\frac{1}{n}}(\nabla^2 u) + \overline{b}|\nabla u| = -f^-(x)$$

in the viscosity sense, i.e. u is a continuous viscosity supersolution of equation $\mathrm{Tr}_{i_k}(\nabla^2 u) + b(x) \cdot \nabla u = f(x)$ satisfying the boundary condition $u = g$ on $\partial\Omega$. □

We are now ready to prove existence and uniqueness in a possibly infinite, regular cylindrical domain.

Proof. (Theorem 1.4) We proceed by approximation, considering a viscosity solution u_j of the Dirichlet problem

$$\begin{cases} \mathrm{Tr}_{i_k}(\nabla^2 u) + b(x) \cdot \nabla u = f(x) & \text{in } \Omega \cap B_j \\ u = \overline{g}_j & \text{on } \partial(\Omega \cap B_j) \end{cases}, \tag{4.6}$$

where B_j is the ball of radius $j \in \mathbb{N}$ centered at the origin, and $\overline{g}_j \in C^0(\mathbb{R}^n)$ is the Tietze-Urysohn-Brower extension of $g_j := g|_{\overline{\Omega \cap B_j}}$, such that $\sup_{\mathbb{R}^n} |\overline{g}_j| \leq \sup_{\partial\Omega} |g_j| \leq \sup_{\partial\Omega} |g|$.

The existence of a viscosity solution $u_j \in C^0(\overline{\Omega \cap B_j})$ follows from Theorem 4.2 since $\Omega \cap B_j$ is endowed with the same uniform exterior cone property of Ω that holds by assumptions. Hence all domains $\Omega \cap B_j$ satisfy condition ($\boldsymbol{G}$) with the same parameters $R \in \mathbb{R}_+$, $\sigma, \tau \in (0, 1)$.

Then by Corollary 4.1 we also get the following estimates, see (2.10):

$$\sup_{\Omega\cap B_j} |u_j| \leq \sup_{\partial(\Omega\cap B_j)} |\overline{g}_j| + C\|f\|_{\Omega\cap B_j,R},$$

from which the uniform estimates, with respect to $j \in \mathbb{N}$,

$$\sup_{\Omega \cap B_j} |u_j| \leq \sup_{\partial\Omega} |g| + C\|f\|_{\Omega,R}. \tag{4.7}$$

Next, let us consider a compact set K such that $K \subset \Omega$. By interior C^α-estimates, see [29, 30], taking $\delta > 0$ such that $\delta < \mathrm{dist}(K, \partial\Omega)$, we have, for sufficiently large $j \in \mathbb{N}$,

$$\sup_{\substack{|x-y|<\delta/2 \\ x \neq y}} \frac{|u_j(x) - u_j(y)|}{|x-y|^\alpha} \leq C\,(\delta^{-\alpha} \sup_{B_\delta(y)} |u_j| + \|f\|_{L^n(B_\delta(y))})$$

for all $y \in K$, with $\alpha \in (0,1)$ and $C > 0$ independent of $j \in \mathbb{N}$. This, combined with (4.7), yields uniform interior Hölder estimates:

$$|u_j(x) - u_j(y)| \leq c_K |x-y|^\alpha \quad \text{for all} \quad x, y \in K. \tag{4.8}$$

Inequalities (4.7) and (4.8) show that the functions u_j are equi-bounded and equi-continuous over K. Therefore, by Ascoli-Arzelà theorem, we can extract a uniformly converging subsequence.

Using an increasing sequence of domains $\Omega_i \Subset \Omega$ covering Ω, we obtain by a Cantor diagonalization process a subsequence that uniformly converges, on any compact set of Ω, to a function $u \in C^0(\Omega)$.

By stability results for viscosity solutions, see [24, Section 6], the function u is in turn viscosity solution of equation in (1.10), namely

$$\mathrm{Tr}_{i_k}(\nabla^2 u) + b(x) \cdot u = f(x) \quad \text{in} \quad \Omega. \tag{4.9}$$

We remain with the task to verify that the boundary values are taken from the interior with continuity.

For this purpose, let $x_0 \in \partial\Omega$. By the exterior cone property, there exists $r_0 > 0$ such that for any ball B_r of radius $r \leq r_0$, centered at x_0, the concentric ball $B_{r/2}$ satisfies the measure inequality

$$|B_{r/2} \backslash \Omega| \geq \sigma |B_{r/2}|$$

with $\sigma \in (0,1)$. Since $u_j(x) - u_j(x_0)$ is a viscosity solution of equation (4.9), then $(u_j(x) - u_j(x_0))^+$ is in turn a subsolution, in the viscosity sense, of the same equation with $-f^-(x)$ instead of $f(x)$, see [25].

Setting

$$\omega_j(r) = \sup_{B_r \cap \Omega} (u_j(x) - u_j(x_0))^+, \quad v_j(x) = \omega_j(r) - (u_j(x) - u_j(x_0))^+,$$

we obtain a non-negative function v_j such that, from inequality (4.4):

$$\mathcal{M}^+_{\frac{1}{n}, 1+\frac{1}{n}}(-\nabla^2 v_j) - b(x) \cdot v_j \geq -f^-(x) \quad \text{in} \quad \Omega, \tag{4.10}$$

in the viscosity sense. From this, by duality, we get

$$\mathcal{M}^-_{\lambda,\Lambda}(\nabla^2 v_j) + b(x) \cdot v_j \leq f^-(x) \quad \text{in} \quad \Omega. \tag{4.11}$$

Next, we extend v_j to B_r setting

$$\ell_j = \liminf_{x \to B_r \cap \partial\Omega} v_j(x)$$

and

$$\overline{v}_j = \begin{cases} \min(v_j, \ell_j) & \text{in } \Omega \\ \ell_j & \text{outside } \Omega \end{cases},$$

and apply, in the viscosity setting, the boundary weak Harnack inequality, see [5, 21]:

$$\left(\frac{1}{|B_{r/2}|}\int_{B_{r/2}} \overline{v}_j^p\,dx\right)^{1/p} \leq C\left(\inf_{B_{r/2}\cap\Omega} v_j + \|f^-\|_{L^n(\Omega\cap B_r)}\right) \tag{4.12}$$

where C is a positive constant independent of $j \in \mathbb{N}$.

Now, being $\overline{v}_j = \ell_j$ outside Ω and $|B_{r/2}\backslash\Omega| \geq \sigma|B_{r/2}|$, we have

$$\sigma^{1/p}\ell_j \leq C\left(\inf_{B_{r/2}\cap\Omega} v_j + \|f^-\|_{L^n(\Omega\cap B_r)}\right). \tag{4.13}$$

Here, explicating the definition of ℓ_j and v_j,

$$\begin{aligned} \ell_j &= \liminf_{x\to B_r\cap\partial\Omega}(\omega_j(r) - (u_j(x) - u_j(x_0))^+) \\ &= \omega_j(r) - \limsup_{x\to B_r\cap\partial\Omega}(u_j(x) - u_j(x_0))^+ \end{aligned}$$

and

$$\inf_{B_{r/2}\cap\Omega} v_j = \omega_j(r) - \sup_{B_{r/2}\cap\Omega}(u_j(x) - u_j(x_0))^+$$

so that from (4.13) we get

$$\sigma^{1/p}\omega_j(r) - \sigma^{1/p}\limsup_{x\to B_r\cap\partial\Omega}(u_j(x) - u_j(x_0))^+ \leq C\omega_j(r) - C\omega_j(r/2) + C\|f^-\|_{L^n(\Omega\cap B_r)},$$

and therefore

$$\omega_j(r/2) \leq \gamma\omega_j(r) + \gamma' \sup_{x\in B_r\cap\partial\Omega}(g(x) - g(x_0))^+ + \|f^-\|_{L^n(\Omega\cap B_r)}, \tag{4.14}$$

where $\gamma = 1 - \frac{\sigma^{1/p}}{C} \in (0,1)$ and $\gamma' = 1 - \gamma$. Then, rewriting the above inequality as

$$\omega(\tau r) \leq \gamma\omega(r) + \sigma(r)$$

with $\tau = 1/2$, γ as above, and $\sigma(r) = \gamma' \sup_{x\in B_r\cap\partial\Omega}(g(x) - g(x_0))^+ + \|f^-\|_{L^n(\Omega\cap B_r)}$, from Lemma 8.23 of [8] we infer that

$$\omega_j(r) \leq C'\left(\frac{\omega_j(r_0)}{r_0^\alpha}r^\alpha + \gamma' \sup_{x\in B_{r'}\cap\partial\Omega}(g(x) - g(x_0))^+ + \|f^-\|_{L^n(\Omega\cap B_{r'})}\right), \tag{4.15}$$

where C' and α are positive constants independent of r and j, and $r' = r_0^{1/2}r^{1/2}$.

Next, we observe that by (4.7),

$$\omega_j(r_0) \leq 2(\sup_{\partial\Omega}|g| + C\|f\|_{\Omega,R}) =: Kr_0^\alpha,$$

so that for $x \in B_r \cap \Omega$ we have:

$$(u_j(x) - u_j(x_0))^+ \leq C' \left(K\, r^\alpha + \gamma' \sup_{x \in B_{r'} \cap \partial\Omega} (g(x) - g(x_0))^+ + \|f^-\|_{L^n(\Omega \cap B_{r'})} \right),$$

and sending $j \to \infty$, we obtain

$$(u(x) - g(x_0))^+ \leq C' \left(K\, r^\alpha + \gamma' \sup_{x \in B_{r'} \cap \partial\Omega} (g(x) - g(x_0))^+ + \|f^-\|_{L^n(\Omega \cap B_{r'})} \right) \tag{4.16}$$

with $r' = r_0^{1/2} r^{1/2}$, for every $x \in B_r \cap \Omega$. Therefore, being $g \in C^0(\partial\Omega)$ and $\|f\|_{\Omega,R} < \infty$, we get

$$\lim_{x \to x_0} (u(x) - g(x_0))^+ = 0 \tag{4.17}$$

On the other hand, we can use a similar argument observing that $u_j(x_0) - u_j(x)$ is a viscosity solution of equation (4.9) with $-f(x)$ instead of $f(x)$, and therefore $(u_j(x_0) - u_j(x))^+$ is in turn a subsolution, in the viscosity sense, of the same equation with $-f^+(x)$ instead of $f(x)$.

Reasoning as above for $u_j(x) - u_j(x_0)$, we get in similar manner

$$(u_j(x_0) - u_j(x))^+ \leq C' \left(K\, r^\alpha + \gamma' \sup_{x \in B_{r'} \cap \partial\Omega} (g(x_0) - g(x))^+ + \|f^+\|_{L^n(\Omega \cap B_{r'})} \right)$$

and sending $j \to \infty$, we obtain

$$(g(x_0) - u(x))^+ \leq C' \left(K\, r^\alpha + \gamma' \sup_{x \in B_{r'} \cap \partial\Omega} (g(x_0) - g(x))^+ + \|f^+\|_{L^n(\Omega \cap B_{r'})} \right) \tag{4.18}$$

with $r' = r_0^{1/2} r^{1/2}$, for every $x \in B_r \cap \Omega$. Therefore, being $g \in C^0(\partial\Omega)$ and $\|f\|_{\Omega,R} < \infty$, we also get

$$\lim_{x \to x_0} (g(x_0) - u(x))^+ = 0. \tag{4.19}$$

From (4.17) and (4.19) we conclude that

$$\lim_{x \to x_0} (u(x) - g(x_0)) = 0,$$

and so $u(x)$ approaches with continuity the boundary value $g(x_0)$ as $x \to x_0$, for any $x_0 \in \partial\Omega$, thereby proving that $u \in C^0(\overline{\Omega})$ with boundary values $u = g$ on $\partial\Omega$. □

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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