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Research article

Global stability for a Lotka-Volterra competition system with symmetric diffusion matrices

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Abstract: The global stability of a two-species Lotka-Volterra competition n-patch system with symmetric diffusion matrices is a conjecture proposed by Hofbauer, So, and Takeuchi. The cases n = 3, 4, 5 have been demonstrated to hold validity. In this article, it is shown that the Hofbauer-So-Takeuchi conjecture holds true for general n. A concrete example is given to display the global stability for one of the equilibria of the system depending on the magnitude of the parameters. Furthermore, an algebraic curve problem in the system's parameter space for determining the stability regions is proposed.

Keywords: Lotka-Volterra; competition system; symmetric diffusion; type K monotone; global stability

1. Introduction

Various mathematical models have been developed to investigate the population dynamics behaviors among distinct patches under heterogeneous resource distribution. For example, the local stability of boundary equilibria and the nonexistence of positive equilibria are discussed for a Lotka-Volterra patch model of two competing aquatic species with combined random and advective movements [1]. A Lotka–Volterra competitive patch model composed of two competing species living in a stream that is divided into n patches along a straight line is investigated for a complete understanding on the global dynamics [2]. The two-species Lotka–Volterra competition patch model with diffusion is given and analyzed to classify the global dynamics under some symmetric assumptions for diffusion coefficients matrices [3]. The richer dynamics of a two-species competition system in a two-patch environment are revealed with the diversity of average birth rates between two species [4]. The Lotka–Volterra competition model in discrete habitats is a typical example of patch models, which can be used to explore the relationship between local dynamics in each patch and the dispersal dynamics among patches.

Now we consider the following system which is composed of n patches connected by diffusion and occupied by two species:

$$\begin{aligned} \dot{u}_i &= u_i\,(r_i - a_i u_i - b_i v_i) + \sum_{j=1, j\neq i}^{n} D_{ij}(u_j - \alpha_{ij} u_i), \\ \dot{v}_i &= v_i\,(s_i - c_i u_i - d_i v_i) + \sum_{j=1, j\neq i}^{n} E_{ij}(v_j - \beta_{ij} v_i), \end{aligned} \tag{1.1}$$

where $i = 1, \ldots, n$, D_{ij} (or E_{ij}) is a nonnegative diffusion coefficient for species u (or v) from the j-th patch to the i-th patch ($i \neq j$). $r_i, s_i > 0$ measure the intrinsic growth rates of u_i, v_i in patch i, respectively. $a_i, d_i > 0$ are the intra-specific competition rates of two species; $b_i, c_i > 0$ are the inter-specific competition rates of two species. The parameters $\alpha_{ij} \geq 0$ and $\beta_{ij} \geq 0$ correspond to the boundary conditions of the continuous diffusion case: $\alpha_{ij} = 1$ and $\beta_{ij} = 1$ for Neumann conditions, and $\alpha_{ij} \neq 1$ and $\beta_{ij} \neq 1$ for Dirichlet or Robin conditions. Dispersal by linear diffusion means that the species is able to move to the interconnected patches with equal probability [5, 6].

Let $D = (d_{ij})$ and $E = (e_{ij})$ be the diffusion matrices of (1.1) with the elements as follows:

$$d_{ij} = \begin{cases} D_{ij} & \text{for } i \neq j, \\ -\sum\limits_{k=1, k\neq i}^{n} D_{ik}\alpha_{ik} & \text{for } i = j. \end{cases}$$

and

$$e_{ij} = \begin{cases} E_{ij} & \text{for } i \neq j, \\ -\sum\limits_{k=1, k\neq i}^{n} E_{ik}\beta_{ik} & \text{for } i = j. \end{cases}$$

Then, system (1.1) can be rewritten as

$$\begin{aligned} \dot{u}_i &= u_i\,(r_i - a_i u_i - b_i v_i) + \sum_{j=1}^{n} d_{ij} u_j, \\ \dot{v}_i &= v_i\,(s_i - c_i u_i - d_i v_i) + \sum_{j=1}^{n} e_{ij} v_j. \end{aligned} \tag{1.2}$$

Here, matrices D and E are supposed to be irreducible, which implies that the species can reach any i-th patch from any j-th patch for $i, j = 1, \ldots, n$. A symmetric diffusion matrix D (or E), where the rate from patch i to j equals that from j to i, models scenarios like island populations with unbiased movement. This symmetry arises from spatially regular configurations or purely random diffusion.

System (1.2) is said to describe a tolerant competition in each patch, if the interaction matrix is Volterra-Liapunov stable. Otherwise, it is said to describe a severe competition [7]. For a system composed of two or three competitive patches, [4] demonstrates that global stability is preserved under biologically reasonable assumptions in heterogeneous environments and proposed a conjecture which we refer to as the HST conjecture (details in Section 3). This conjecture states that if the system is tolerant and under symmetric diffusion and competition conditions independent of patches, then the system must be globally stable. Specifically, the existence of a positive equilibrium guarantees its global stability. Thus, spatial heterogeneity alone is capable of maintaining stable coexistence and does not necessitate complex dynamical outcomes. In this paper, we aim to resolve this conjecture.

Goh [7] proposed two examples of two-patch Lotka-Volterra competitive systems (1.1): one representing tolerant competition (labeled as system (3.3) in [8]) and the other severe competition (labeled as system (3.2) in [8]). The authors [4, 8] proved that system (3.3) in [8] is globally stable, which satisfies the conditions of the HST conjecture. However, when the conjecture's conditions are not met, the results are more complicated. System (3.2) in [8] , although its diffusion matrix is symmetric but it belongs to the severe type, does not meet the conditions of the HST conjecture. Nevertheless, due to the uniqueness of the positive equilibrium point and the K-monotonicity of the system itself, the system remains globally stable [8]. In contrast, system (3.4) in [8], under asymmetric diffusion conditions, is permanent but not globally stable because it has two positive equilibrium points instead.

The remainder of this paper is organized as follows. In Section 2, we present some preliminary results. In Section 3, we give the main result of the global dynamics of system (1.2). In Section 4, we give an example to display the global stability for one of the equilibria of the system depending on the magnitude of the parameters.

2. Preliminaries

In this section, we list some basic results about type K monotone systems below for later use.

In the sense of Smith [9], system (1.2) is a $2n$-dimensional type K monotone system with respect to the following ordering on $\mathbb{R}_+^n \times \mathbb{R}_+^n$:

$$\begin{aligned} &\text{I).}(u, v) \leq_K (\bar{u}, \bar{v}), \quad \text{iff} \quad u \leq \bar{u} \quad and \quad v \geq \bar{v}, \\ &\text{II).}\ (u, v) <_K (\bar{u}, \bar{v}), \quad \text{iff} \quad (u, v) \leq_K (\bar{u}, \bar{v})\ and\ (u, v) \neq (\bar{u}, \bar{v}). \end{aligned}$$

It is easy to check that solutions starting in $\mathbb{R}_+^n \times \mathbb{R}_+^n$ remain there, that is, the state space $\mathbb{R}_+^n \times \mathbb{R}_+^n$ is positively invariant. Clearly, the origin $E_0 = (0, \ldots, 0; 0, \ldots, 0)$ of system (1.2) is an equilibrium.

The Jacobian matrix of system (1.2) is given by

$$J = \begin{pmatrix} d_{ij} + \delta_{ij}(f_i - a_i u_i) & -\delta_{ij} b_i u_i \\ -\delta_{ij} c_i v_i & e_{ij} + \delta_{ij}(g_i - d_i v_i) \end{pmatrix},$$

where $i, j = 1, ..., n$ in each of the four blocks. Here, $f_i = r_i - a_i u_i - b_i v_i$, $g_i = s_i - c_i u_i - d_i v_i$, and δ_{ij} is the Kronecker delta.

The spectrum of matrix A, written as $\delta(A)$, is the set of eigenvalues of A. Define the stability modulus of A, $s(A)$, as

$$s(A) = \max\{\mathrm{Re}\, \lambda : \lambda \in \delta(A)\}.$$

Then, we have the following lemma [10].

Lemma 1. *Consider the u-subsystem (with v=0) of system* (1.2) *as follows:*

$$\dot{u}_i = u_i(r_i - a_i u_i) + \sum_{j=1}^{n} d_{ij} u_j, \quad i = 1, \ldots, n,$$

and denote $A_D = D + \mathrm{diag}(r_1, \ldots, r_n)$. *Then, we have the following:*

(1) *If* $s(A_D) \le 0$, *then the origin* $(0, \ldots, 0)$ *is globally stable in the positive u subspace.*

(2) *If* $s(A_D) > 0$, *then there exists a positive equilibrium* $E^n_{\bar{u}} = (\bar{u}_1, \ldots, \bar{u}_n)$ *which is globally stable in the positive u-subspace, i.e., system* (1.2) *has a boundary equilibrium with the form* $E^{2n}_{\bar{u}} = (\bar{u}_1, \ldots, \bar{u}_n; 0, \ldots, 0)$.

For the positive v-subspace, a similar result holds true for the following v-subsytem of system (1.2) depending on the sign of $s(A_E)$:

$$\dot{v}_i = v_i(s_i - d_i v_i) + \sum_{j=1}^{n} e_{ij} v_j, \quad i = 1, \ldots, n.$$

Here $A_E = E + \mathrm{diag}(s_1, \ldots, s_n)$. The other boundary equilibrium of system (1.2) has the form $E^{2n}_{\bar{v}} = (0, \ldots, 0; \bar{v}_1, \ldots, \bar{v}_n)$.

Setting the right side of system (1.2) to be zero, we obtain the following equilibrium equations:

$$\begin{aligned} u_i(r_i - a_i u_i - b_i v_i) + \sum_{j=1}^{n} d_{ij} u_j = 0, \\ v_i(s_i - c_i u_i - d_i v_i) + \sum_{j=1}^{n} e_{ij} v_j = 0. \end{aligned} \tag{2.1}$$

Lemma 1 tells us that system (1.2) may have three boundary equilibria (i.e., the origin E_0, $E^{2n}_{\bar{u}}$, and $E^{2n}_{\bar{v}}$).

Lemma 2. [4,9,11]. *If there is no positive equilibrium in* $\mathbb{R}^n_+ \times \mathbb{R}^n_+$, *then one of the boundary equilibria must be globally stable. More precisely:*

(1) *If* $s(A_D) \le 0$, *and* $s(A_E) \le 0$ *(i.e., neither* $E^{2n}_{\bar{u}}$ *nor* $E^{2n}_{\bar{v}}$ *exists), then the origin* E_0 *is globally stable.*

(2) *If* $s(A_D) > 0$, *and* $s(A_E) \le 0$ *(i.e.,* $E^{2n}_{\bar{u}}$ *exists but not* $E^{2n}_{\bar{v}}$*), then* $E^{2n}_{\bar{u}}$ *is globally stable. Similarly, if* $s(A_E) > 0$ *and* $s(A_D) \le 0$, *then* $E^{2n}_{\bar{v}}$ *is globally stable.*

(3) *If both* $E^{2n}_{\bar{u}}$ *and* $E^{2n}_{\bar{v}}$ *exist and there is no positive equilibrium, then one of them is globally stable.*

Lemma 2 is a theoretically elegant result, providing a comprehensive classification of the system's dynamical behavior in the absence of a positive equilibrium, although it may be challenging to verify in practice. For specific systems, even in two-patch case, the stability analysis of the boundary equilibrium is quite difficult, see works [12–14]. The existence and global stability of a positive equilibrium are more important and meaningful, and more difficult. According to the basic theory of monotone systems [9, 11], the following permanence result of system (1.2) is obtained.

Lemma 3. [8]. *If both* $s(J(E^{2n}_{\bar{u}})) > 0$ *and* $s(J(E^{2n}_{\bar{v}})) > 0$ *(i.e., both* $E^{2n}_{\bar{u}}$ *and* $E^{2n}_{\bar{v}}$ *are unstable), then system* (1.2) *is permanent and there exist positive equilibria* E_* *and* E^* *of system* (1.2) *with* $E^{2n}_{\bar{u}} >_K E^* \ge_K E_* >_K E^{2n}_{\bar{v}}$ *such that all the solutions of system* (1.2) *in* $\mathrm{int}\,\mathbb{R}^n_+ \times \mathbb{R}^n_+$ *ultimately enter and remain in the ordered interval* $[E_*, E^*]$. *Furthermore, if the positive equilibrium is unique, then it is globally stable.*

Based on the above lemma, analyzing the global dynamics of a type K competitive monotone system can be reduced to determining the stability of its finitely many equilibria. The two examples proposed by Goh [7] are proved to be permanent and have a unique positive equilibrium which imply the global staility of the systems [8]. The uniqueness of the positive quilibrium of the systems is based on the homotopy algorithm theory developed in [15] of numerically determining solutions of polynomial systems, it is found by Li et al. that 16 roots to each system of polynomial equilibrium equations of the right-hand side of the systems, respectively [8]. Since the Bézoat number of each system of polynomial equations is 16, the 16 solutions are all solutions of each system, and only one is positive in both cases, respectively. Later, by using the multivariable realroots isolation algorithm proposed by the authors [16, 17], a unique positive real root in the interval form for both polynomials in Goh's examples are obtained again. Goh's examples show that it is possible to check the stability condition and uniqueness of a positive equilibrium for a given system with constant coefficients. In general, it is not easy to verify the uniqueness without specifying the values of parameters of the systems [12–14].

The authors [13, 18] consider an n-competitor Lotka-Volterra diffusion system with two identical patches.

$$
\begin{aligned}
\dot{u}_i &= u_i\left(r_i - \sum_{j=1}^{n} a_{ij}u_j\right) + D_i(v_i - u_i),\\
\dot{v}_i &= v_i\left(r_i - \sum_{j=1}^{n} a_{ij}v_j\right) + D_i(u_i - v_i),
\end{aligned}
\tag{2.2}
$$

where $i = 1, \ldots, n$. If $n = 2$ (a speciel case of system (1.2)), and if the isolated patches are in severe competition, which means that $(a_{ij})_{n\times n}$ is not Volterra-Liapunv stable, Levin [19] has shown by some examples that system (2.2) can have locally stable positive equilibrium points. Since a severe system (2.2) either possesses at least one locally stable boundary equilibrium point or has (if there are some positive equilibrium points) a continuum of positive equilibrium points, in these cases the system cannot be globally stable. In another aspect, Hastings [18] has shown that for any n, if system (2.2) is in tolerant competition which means that $(a_{ij})_{n\times n}$ is Volterra-Liapunv stable and has a positive equilibrium point, then it is globally stable by using a Liapunov method. Therefore, for $n = 2$, system (2.2) is globally stable if and only if it is in tolerant competition.

If the type K monotone system (1.2) has two or more positive equilibria, then there must be a stable one and an unstable one [9, 11]. Therefore, if we can show that each equilibrium is locally asymptotically stable, then the uniqueness of the positive equilibrium is guaranteed.

From [4, 9, 11]and Lemma 1, we have the following.

Lemma 4. *The following statements are equivalent:*

(1) *System* (1.2) *has a globally stable positive equilibrium.*

(2) *Each positive equilibrium is locally asymptotically stable.*

(3) *At each positive equilibrium* $(\bar{u}, \bar{v})$, $s(J(\bar{u}, \bar{v})) < 0$.

(4) *At each positive equilibrium* $(\bar{u}, \bar{v})$, *the Jacobian matrix* $J(\bar{u}, \bar{v})$ *is stable, that is, the leading principal minors of* $J(\bar{u}, \bar{v})$ *alternate in sign (starting from negative).*

The equivalence conditons (3) and (4) are used in [3, 20, 21] to prove the global stability of system (1.2).

3. The main result

In the present paper, we consider a special case of system (1.2) as follows:

$$\begin{aligned}\dot{u}_i &= u_i\,(r_i - au_i - bv_i) + \sum_{j=1}^{n} d_{ij}u_j,\\ \dot{v}_i &= v_i\,(s_i - cu_i - dv_i) + \sum_{j=1}^{n} e_{ij}v_j.\end{aligned} \tag{3.1}$$

Here the basic growth rates may depend on the patch but the interaction matrix is patch independent, and in each patch the system is tolerant (i.e., $ad - bc > 0$).

In [4], Hofbauer et al., under the conditions

$$d_{ii} + \sum_{j\neq i} d_{ij} \leq 0, \qquad e_{ii} + \sum_{j\neq i} e_{ij} \leq 0, \tag{3.2}$$

posed the following conjecture.

HST Conjecture: Under condition (3.2) with $ad - bc > 0$, the n-patch system (3.1) is globally stable, provided that the dispersal matrices $D = (d_{ij})_{n\times n}$ and $E = (e_{ij})_{n\times n}$ are symmetric.

This conjecture is shown to be true in the cases when $n = 2$ and 3 in [18] and when $n = 4$ in [20] and $n = 5$ in [9], by proving a series of polynomials to be positive definiteness.

Under the conditions that the corresponding elements of two diffusion matrices are proportional, the matrices are Laplacian ones, that is, $d_{ij}/e_{ij} =$ Const. and $\alpha_{ij} = \beta_{ij} = 1$, that is, $\sum d_{ij} = 0$ and $\sum e_{ij} = 0$, for $i, j = 1, \ldots, n$. The result in [3] implies that at each positive equilibrium $\bar{u}, \bar{v}$, $s(J(\bar{u}, \bar{v})) < 0$. The main result of the present paper is to show that the HST conjecture holds true for system (3.1) by using the technique as in [3].

Theorem 1. *HST Conjecture holds true for general n .*

Proof. By Lemma 4, it is clear that to show the conjecture, one just needs to show that a positive equilibrium $\mathbf{E}^*$ of (3.1), if it exists, is locally asymptotically stable.

Let $\mathbf{E}^* = (u, v)$ be a positive equilibrium of (3.1). Then, it satisfies the following equilibrum equations:

$$\begin{aligned}&\sum_{j=1}^{n} d_{ij}u_j + u_i(r_i - au_i - bv_i) = 0,\\ &\sum_{j=1}^{n} e_{ij}v_j + v_i(s_i - cu_i - dv_i) = 0,\end{aligned} \tag{3.3}$$

where $i = 1, \ldots, n$.

Linearising (3.1) at $\mathbf{E}^*$, we have the following eigenvalue problem:

$$\begin{aligned}&\sum_{j=1}^{n} d_{ij}\varphi_j + (r_i - au_i - bv_i)\varphi_i - u_i(a\varphi_i + b\psi_i) - \lambda\varphi_i = 0,\\ &\sum_{j=1}^{n} e_{ij}\psi_j + (s_i - cu_i - dv_i)\psi_i - v_i(c\varphi_i + d\psi_i) - \lambda\psi_i = 0,\end{aligned} \tag{3.4}$$

such that $\varphi_i > 0$ and $\psi_i < 0$ for all $i = 1, \ldots, n$. Here $(\varphi, \psi) = (\varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_n)$ is a principal eigenvector associated with the principal eigenvalue λ of (3.4). We will show that $\lambda < 0$.

Multiplying the first equation of (3.4) by u_i, the first equation of (3.3) by φ_i, and taking the difference, we have

$$\sum_{j=1, j\neq i}^{n} d_{ij}(\varphi_j u_i - \varphi_i u_j) = u_i^2(a\varphi_i + b\psi_i) + \lambda u_i \varphi_i. \tag{3.5}$$

Multiplying both sides of (3.5) by φ_i^2/u_i^2 and summing up all the equations, we obtain

$$\sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} d_{ij}\left(\frac{\varphi_i^2\varphi_j}{u_i} - \frac{\varphi_i^3 u_j}{u_i^2}\right) = \sum_{i=1}^{n} \varphi_i^2(a\varphi_i + b\psi_i) + \lambda \sum_{i=1}^{n} \frac{\varphi_i^3}{u_i}. \tag{3.6}$$

Since matrice $D = [d_{ij}]$ is symmetric, we have

$$\sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} d_{ij}\left(\frac{\varphi_i^2\varphi_j}{u_i} - \frac{\varphi_i^3 u_j}{u_i^2}\right) = \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} d_{ij}\left(\frac{\varphi_j^2\varphi_i}{u_j} - \frac{\varphi_j^3 u_i}{u_j^2}\right). \tag{3.7}$$

It follows that

$$\begin{aligned}
&\sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} d_{ij}\left(\frac{\varphi_i^2\varphi_j}{u_i} - \frac{\varphi_i^3 u_j}{u_i^2}\right) + \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} d_{ij}\left(\frac{\varphi_j^2\varphi_i}{u_j} - \frac{\varphi_j^3 u_i}{u_j^2}\right) \\
&= \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} d_{ij}\left\{\left(\frac{\varphi_i^2\varphi_j}{u_i} - \frac{\varphi_i^3 u_j}{u_i^2}\right) + \left(\frac{\varphi_i\varphi_j^2}{u_j} - \frac{\varphi_j^3 u_i}{u_j^2}\right)\right\} \\
&= \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} d_{ij}\left\{-u_i u_j\left(\frac{\varphi_i}{u_i} - \frac{\varphi_j}{u_j}\right)^2\left(\frac{\varphi_i}{u_i} + \frac{\varphi_j}{u_j}\right)\right\} \leq 0.
\end{aligned} \tag{3.8}$$

So by combining (3.6) with (3.8), we have

$$\sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} d_{ij}\left(\frac{\varphi_i^2\varphi_j}{u_i} - \frac{\varphi_i^3 u_j}{u_i^2}\right) = \sum_{i=1}^{n} \varphi_i^2(a\varphi_i + b\psi_i) + \lambda \sum_{i=1}^{n} \frac{\varphi_i^3}{u_i} \leq 0. \tag{3.9}$$

Similarly, we can obtain

$$\sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} e_{ij}\left(\frac{\psi_i^2\psi_j}{v_i} - \frac{\psi_i^3 v_j}{v_i^2}\right) = \sum_{i=1}^{n} \psi_i^2(c\varphi_i + d\psi_i) + \lambda \sum_{i=1}^{n} \frac{\psi_i^3}{v_i} \geq 0. \tag{3.10}$$

Hence, we have

$$\begin{aligned}
&\lambda \sum_{i=1}^{n}\left(\frac{\varphi_i^3}{au_i} - \frac{\bar{b}^3}{d}\frac{\psi_i^3}{v_i}\right) \leq -\sum_{i=1}^{n} \varphi_i^2(\varphi_i + \bar{b}\psi_i) + \sum_{i=1}^{n} \bar{b}^2\psi_i^2(\bar{c}\bar{b}\varphi_i + \bar{b}\psi_i) \\
&< -\sum_{i=1}^{n} \varphi_i^2(\varphi_i + \bar{b}\psi_i) + \sum_{i=1}^{n} \bar{b}^2\psi_i^2(\varphi_i + \bar{b}\psi_i) = -\sum_{i=1}^{n}(\varphi_i - \bar{b}\psi_i)(\varphi_i + \bar{b}\psi_i)^2 \leq 0,
\end{aligned} \tag{3.11}$$

where we have used $\bar{b} = b/a$, $\bar{c} = c/d$ This implies that $\lambda < 0$ and it is concluded that the positive equilibrium $\mathbf{E}^*$ is locally asymptotically stable.

This completes the proof of the theorem.

Remark. Based on the proof, it is known that the HST conjecture also holds without assumptions of the condition (3.2).

4. An example

We consider the following system which is composed of two patches connected by diffusion and occupied by two species:

$$\begin{aligned}\dot{u}_1 &= u_1\,(3 - 2u_1 - v_1) + (u_2 - 4u_1),\\ \dot{u}_2 &= u_2\,(3 - 2u_2 - v_2) + (u_1 - \alpha u_2),\\ \dot{v}_1 &= v_1\,(3 - u_1 - v_1) + (v_2 - \beta v_1),\\ \dot{v}_2 &= v_2\,(1 - u_2 - v_2) + (v_1 - 2v_2).\end{aligned} \tag{4.1}$$

Theorem 2. *System* (4.1) *is globally stable, or more precisely,*

(i) *If both* $\alpha \geq 4$ *and* $\beta \geq 4$*, then the origin* $(0, 0, 0, 0)$ *is globally stable.*

(ii) *If* $\alpha \geq 4$ *and* $\beta < 4$*, then the boundary equilibrium* $E_{\bar{v}}$ *is globally stable, and if* $\alpha < 4$ *and* $\beta \geq 4$*, then the boundary equilibrium* $E_{\bar{u}}$ *is globally stable.*

(iii) *If both* $\alpha < 4$ *and* $\beta < 4$*, then system* (4.1) *must have a boundary equilibrium or a positive equilibrium which is globally stable.*

Proof. (i) When $\alpha \geq 4$ and $\beta \geq 4$, there is no other equilibrium except the origin for system (4.1). By Lemma 2(1) , the origin $(0, 0, 0, 0)$ is globally stable.

(ii) When $\alpha \geq 4$ and $\beta < 4$, the boundary equilibrium $E_{\bar{v}}$ exists but not $E_{\bar{u}}$. Lemma 2(2) implies that $E_{\bar{v}}$ is globally stable. The case of that $\alpha < 4$ and $\beta \geq 4$ is similar.

(iii) When $\alpha < 4$ and $\beta < 4$, both boundary equilibria $E_{\bar{u}}$ and $E_{\bar{v}}$ exist. If one of them is an attractor, it must be globally stable. If both $E_{\bar{u}}$ and $E_{\bar{v}}$ are repellers, there must be a unique positive equilibrium which is globally stable by Theorem 1. This completes the proof ot Theorem 2.

Although the positive equilibrium is globally stable when $0 < \alpha = \beta < 4$, it seems challenging to obtain a complete result for case (iii) in Theorem 2.

We now perform a numerical investigation of the system's equilibrium points and their stability properties for various parameter values α and β.

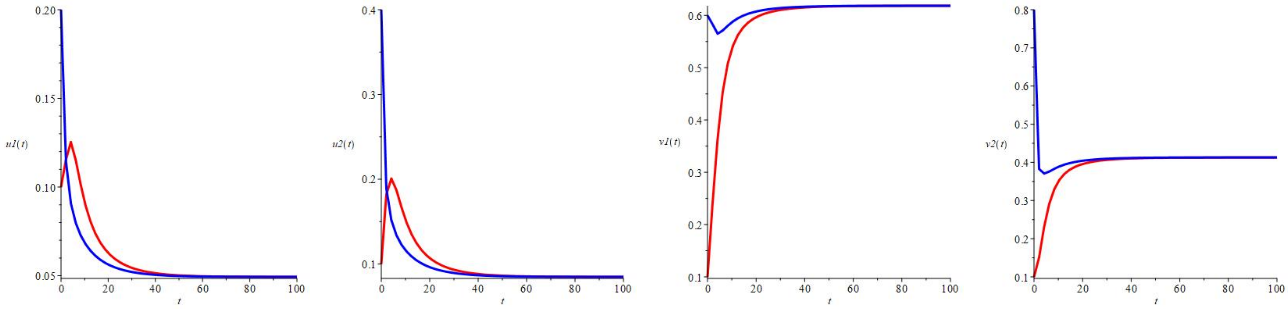
(I). For the parameter set $(\alpha,\beta) = (3, 3)$, numerical computation reveals the existence of four distinct equilibrium points. These equilibria are: the trivial equilibrium $\mathbf{E}_0 = (0, 0, 0, 0)$, two boundary equilibria $\mathbf{E}_{\bar{v}} = (0,\ 0,\ 0.6823278038,\ 0.4655712319)$, $\mathbf{E}_{\bar{u}} = (0.2327856159,\ 0.3411639019,\ 0,\ 0)$, and a positive interior equilibrium $\mathbf{E}^* = (0.0493558,\ 0.0847475,\ 0.618362,\ 0.412891)$.

By applying Theorem 2, we conclude that the positive equilibrium $\mathbf{E}^*$ is globally asymptotically stable for this parameter configuration.

A local stability analysis was conducted by evaluating the eigenvalues of the Jacobian matrix linearized at each equilibrium point. The eigenvalue spectra show that $\mathbf{E}_0$, $E_{\bar{v}}$ and $E_{\bar{u}}$ are unstable and $\mathbf{E}^*$ is stable, consistent with the global stability result from Theorem 2.

To visualize the global attracting behavior of $\mathbf{E}^*$, we compute numerical solutions for two distinct initial conditions: $\mathrm{IC}_1(u_1(0), u_2(0), v_1(0), v_2(0)) = (0.1, 0.1, 0.1, 0.1)$ and $\mathrm{IC}_2(u_1(0), u_2(0), v_1(0), v_2(0)) = (0.2, 0.4, 0.6, 0.8)$.

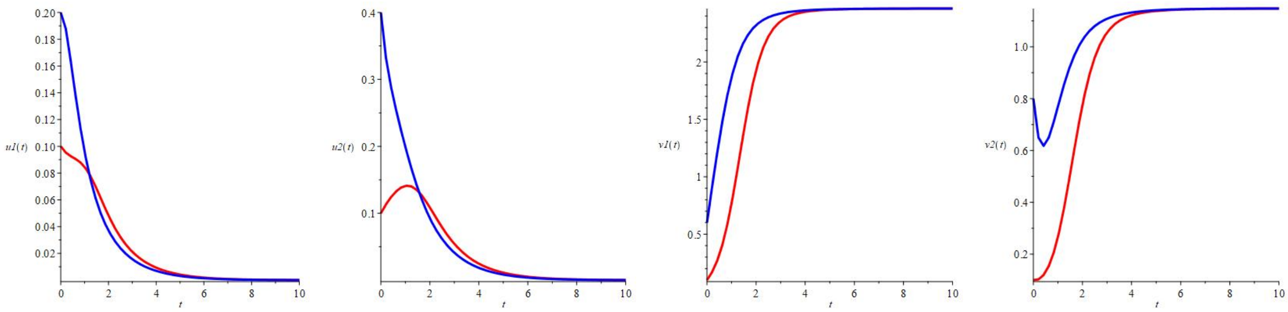
Figure 1 displays the solution curves for different initial conditions. Both trajectories asymptotically approach $\mathbf{E}^*$. This provides numerical evidence supporting the analytical conclusion that $\mathbf{E}^*$ is a global attractor for the system when $(\alpha, \beta) = (3, 3)$.



(a) $u_1(t)$ versus time t. **(b)** $u_2(t)$ versus time t. **(c)** $v_1(t)$ versus time t. **(d)** $v_2(t)$ versus time t.

Figure 1. Time series of the four state variables for parameter values $(\alpha, \beta) = (3, 3)$. Both solutions corresponding to initial conditions IC_1 (red line) and IC_2 (blue line) are shown. All four components converge to their respective equilibrium values at $\mathbf{E}^* = (0.0494, 0.0847, 0.6184, 0.4129)$.

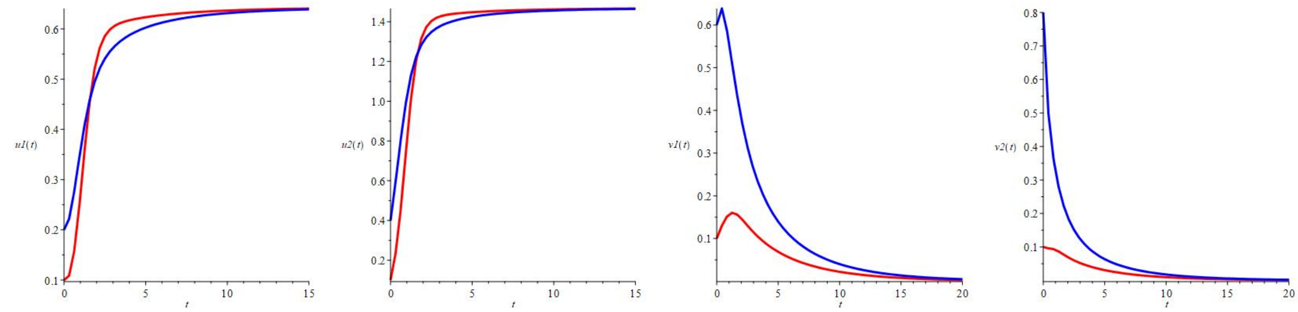
(II). For $(\alpha, \beta) = (3, 1)$, numerical analysis reveals three equilibria: $\mathbf{E}_0$, $\mathbf{E}_{\bar{v}} = (0, 0, 2.4656, 1.1479)$, and $\mathbf{E}_{\bar{u}} = (0.2328, 0.3412, 0, 0)$. No positive interior equilibrium exists in this region. Eigenvalue analysis shows that $\mathbf{E}_{\bar{v}}$ is locally asymptotically stable, while $\mathbf{E}_0$ and $\mathbf{E}_{\bar{u}}$ are unstable. Numerical integration from the same initial conditions as before (IC_1 and IC_2) confirms that all trajectories converge to $\mathbf{E}_{\bar{v}}$ (see Figure 2), indicating global stability of this boundary equilibrium and extinction of the u-species.



(a) $u_1(t)$ versus time t. **(b)** $u_2(t)$ versus time t. **(c)** $v_1(t)$ versus time t. **(d)** $v_2(t)$ versus time t.

Figure 2. Time series of the four state variables for parameter values $(\alpha, \beta) = (3, 1)$. All four components converge to their respective equilibrium values at $E_{\bar{v}} = (0, 0, 2.4656, 1.1479)$.

(III). For $(\alpha, \beta) = (0.5, 3)$, numerical computation yields three equilibria: $\mathbf{E}_0$, $\mathbf{E}_{\bar{v}} = (0, 0, 0.6823, 0.4656)$, and $\mathbf{E}_{\bar{u}} = (0.6427, 1.4688, 0, 0)$. No positive interior equilibrium exists. Eigenvalue analysis confirms that only $\mathbf{E}_{\bar{u}}$ is locally asymptotically stable. Figure 3 indicates extinction of the v-species.



(a) $u_1(t)$ versus time t. **(b)** $u_2(t)$ versus time t. **(c)** $v_1(t)$ versus time t. **(d)** $v_2(t)$ versus time t.

Figure 3. Time series of the four state variables for parameter values $(\alpha,\beta) = (0.5, 3)$. All four components converge to their respective equilibrium values at $E_{\bar{u}} = (0.6427, 1.4688, 0, 0)$.

Following the above procedure, we conduct a systematic numerical investigation of the equilibrium structure and stability properties of system (4.1) across a comprehensive parameter domain. The parameters α and β are varied independently over the interval $(0, 4)$ with a step size of $\Delta\alpha = \Delta\beta = 0.1$.

For each pair (α_i, β_j), the system's equilibria are computed numerically, and their stability is determined by eigenvalue analysis of the corresponding Jacobian matrices. The long-term dynamical outcome for generic initial conditions is then classified into three distinct regions, each associated with a globally stable equilibrium:

- **Coexistence (Yellow)**: A positive interior equilibrium $\mathbf{E}^* > 0$ exists and is globally asymptotically stable. Both species u and v persist.
- **Competitive Exclusion by v (Red)**: No positive interior equilibrium exists. The equilibrium $\mathbf{E}_{\bar{v}} = (0, 0, \bar{v}_1, \bar{v}_2)$, with $\bar{v}_1, \bar{v}_2 > 0$, is globally stable, leading to the extinction of the u-species.
- **Competitive Exclusion by u (Blue)**: No positive interior equilibrium exists. The equilibrium $\mathbf{E}_{\bar{u}} = (\bar{u}_1, \bar{u}_2, 0, 0)$, with $\bar{u}_1, \bar{u}_2 > 0$, is globally stable, leading to the extinction of the v-species.

The resulting stability classification for each parameter pair is recorded and visualized in the (α,β)-plane using the color scheme described above.

Figure 4 presents the resulting bifurcation diagram. The most salient feature is the clear tripartite partition of the parameter space $(0, 4) \times (0, 4)$ into three non-overlapping subregions, each corresponding to a unique asymptotic outcome. The yellow coexistence region is bounded by two smooth bifurcation curves.

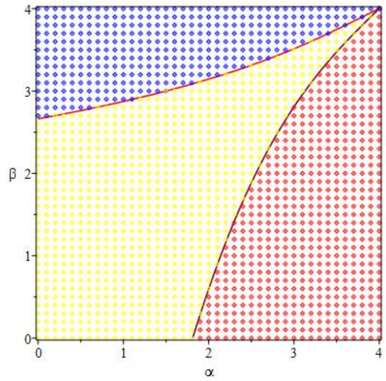


Figure 4. Bifurcation diagram in the (α, β) parameter space for system (4.1). Each point is colored according to the globally stable equilibrium type: yellow (coexistence), red (exclusion by v-species), and blue (exclusion by u-species). The parameter space $(0, 4) \times (0, 4)$ is clearly partitioned into three disjoint subregions separated by distinct bifurcation curves.

To derive the analytical boundaries separating the stability regions observed in Figure 4, we employ symbolic computation methods. Specifically, we consider the parameter-dependent polynomial system describing the equilibria of (4.1). For each equilibrium point, we compute the corresponding Jacobian matrix and its characteristic polynomial $P(\lambda; \alpha, \beta)$.

The Routh–Hurwitz criterion provides necessary and sufficient conditions for all roots of $P(\lambda; \alpha, \beta)$ to have negative real parts, thereby ensuring local asymptotic stability. Applying this criterion symbolically yields a set of polynomial inequalities in the parameters α and β. The boundaries of the stability regions correspond to the algebraic varieties where these inequalities become equalities, that is, where the stability condition undergoes a critical change.

Through this symbolic computation, we obtain two key polynomials: $P_1(\alpha, \beta) = -16\,\alpha^2\beta^3 - 4\,\alpha^3\beta + 176\,\alpha^2\beta^2 + 168\,\alpha\beta^3 + 18\,\alpha^3 - 582\,\alpha^2\beta - 1904\,\alpha\beta^2 - 448\,\beta^3 + 513\,\alpha^2 + 6876\,\alpha\beta + 5216\,\beta^2 - 7696\,\alpha - 19744\,\beta + 24048$, $P_2(\alpha, \beta) = -\alpha^3\beta^2 + 9\,\alpha^3\beta + 12\,\alpha^2\beta^2 - \alpha\beta^3 - 21\,\alpha^3 - 109\,\alpha^2\beta - 35\,\alpha\beta^2 + 5\,\beta^3 + 256\,\alpha^2 + 384\,\alpha\beta - 960\,\alpha - 320\,\beta + 1024$.

The two red lines in Figure 4 displays the curves of $P_1(\alpha, \beta) = 0$ and $P_2(\alpha, \beta) = 0$. These algebraic curves coincide with the observed boundaries between the three dynamical regions. This provides numerical evidence that the bifurcation curves are indeed given by $P_1(\alpha, \beta)$ and $P_2(\alpha, \beta)$.

While the visual coincidence is compelling, a rigorous mathematical proof that these polynomials fully characterize the bifurcation boundaries requires additional analytical work. We therefore formulate the following conjecture:

Conjecture 1. *The stability regions of system* (4.1) *in the* (α, β)*-parameter space are exactly determined by the algebraic curves defined by* $P_1(\alpha, \beta) = 0$ *and* $P_2(\alpha, \beta) = 0$.

This conjecture establishes a precise algebraic connection between the symbolic stability conditions and the observed dynamical transitions. Further analytical study of these polynomial conditions may yield deeper insights into the bifurcation structure of the system. It seems a challenge to prove this conjecture. We leave this as a future consideration.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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