Mathematical Biosciences and Engineering

http://www.aimspress.com/journal/mbe

MBE, 23(3): 567–593.
DOI: 10.3934/mbe.2026022
Received: 31 July 2025
Revised: 07 December 2025
Accepted: 26 December 2025
Published: 27 January 2026

Research article

On the problem of minimizing the epidemic final size for SIR model by social distancing

Pierre-Alexandre Bliman[1], **Anas Bouali**[2], **Patrice Loisel**[2], **Alain Rapaport**[2,*] **and Arnaud Virelizier**[2,3]

[1] Sorbonne Université, Inria, CNRS, Université Paris Cité, Laboratoire Jacques-Louis Lions UMR 7598, Equipe MUSCLEES, Paris, France
[2] MISTEA, Univ. Montpellier, INRAE, Institut Agro, Montpellier, France
[3] MIA, INRAE, AgroParisTech, Univ. Paris-Saclay, Palaiseau, France

* **Correspondence:** Email: alain.rapaport@inrae.fr.

Abstract: We revisit the problem of minimizing the epidemic final size in the SIR model through social distancing of a bounded intensity. In the existing literature, this problem was considered imposing a priori interval structure on the time period when interventions are enforced. We show that the support of the optimal control is still a single time interval when considering the more general class of controls with an L^1 constraint on the confinement effort that reduces the infection rate. There is thus no benefit in splitting interventions on several disjoint time periods. However, if the infection rate is known beforehand to change with time once from one value to another one, then we show that the optimal solution may consist in splitting the interventions in at most two disjoint time periods.

Keywords: epidemiological models; optimal control; L1 constraints; infinite horizon

1. Introduction

Several works have considered the specific issue of optimal control of the classical SIR model through non-pharmaceutical interventions, such as a lockdown, that reduce the transmission rate β by a proportion $u(t)$. This quantity plays the role of a control variable, usually during a limited period. The corresponding controlled system is as follows.

$$\begin{cases} \dot{S} = -(1-u(t))\beta S I, \\ \dot{I} = (1-u(t))\beta S I - \gamma I, \end{cases} \qquad u(t) \in U := [0, \bar{u}], \tag{1.1}$$

where S and I stand for the proportions of the susceptible and infected populations, respectively, and $\bar{u}$ is a fixed number in $(0, 1]$. As usual, the evolution of the third compartment R that represents the proportion of *recovered* individuals has been omitted, and is such that $S+I+R \equiv 1$. These contributions differ through the choice of the cost function, the state constraint, and the class of admissible input functions. The cost functions that are commonly considered are the *peak prevalence* (i.e., the maximal value of $I(\cdot)$, abbreviated PP below), the *epidemic final size* (i.e., the total number of individuals $S(0) - S(\infty)$ infected during the outbreak, abbreviated FS below), a measure of the *control effort* and the *time to reach the sub-epidemic 'safe' region*, or some combination of them. The constraints are usually on the peak prevalence or on the cumulative global control effort.

The class of control inputs among which optimization is completed may be constant or piecewise constant during the lockdown phases, possibly with values chosen in a prescribed finite set, or taken in full generality in the set $\mathcal{U}$ of measurable functions $u(\cdot)$ defined on $[0, +\infty)$ by taking values in U defined in (1.1) (in practice one may look for piecewise continuous controls). Their support, denoted $\mathtt{supp}(u)$, which represents the temporal set of effective interventions, may be imposed as a single interval or a union of intervals of prescribed length, or may be simply included in a fixed interval ($\mathtt{supp}(u) \subset [0, \delta]$, for a given $\delta > 0$ which then represents the instants where intervention is possible, but not necessarily effective) or initially unspecified. Tables 1 to 4 show the diversity of approaches considered and the results obtained.

Here, we are concerned by the issue of reducing the final size (see Table 1). This quantity is crucial, as it determines how close the epidemic can be brought to the herd-immunity threshold while still preserving the largest possible number of susceptible individuals. In the absence of vaccinations or treatments, adjusting social distancing is usually the only available means to reduce the overall number of infections. References [1] and [2] have shown that the problem of imposing $\mathtt{supp}(u)$ to belong to an interval $[0, T]$ has an optimal solution of bang-bang structure which consists in a single sub-interval of effective intervention. The solution consists of at most three phases. In the first phase (possibly of zero duration), no action is taken, until a certain optimal initiation time, which, in any case, is located before reaching the prevalence peak. Then, the control is activated at its largest value $\bar{u}$ until it reaches the time horizon δ. Lastly, the control is null after the end of the permitted confinement interval. The fact that the optimal intervention may have to start after the beginning of the latter, that is, at a positive time, suggests that a more efficient use of a prescribed confinement duration may exist. This is the subject of [3], in which intervals of the type $[t, t+\delta]$ have been considered; here, $t \geq 0$ is a decision variable. This corresponds to the following problem :

$$\mathcal{Q}_S : \quad \sup_{t \geq 0} \sup_{u(\cdot) \in \mathcal{U}} S(+\infty) \text{ subject to } \mathtt{supp}(u) \subset [t, t+\delta].$$

Not quite surprisingly, it is demonstrated therein that there exists an optimal initiation time $t^\star$, and that the optimal cost is obtained by taking $u \equiv \bar{u}$ on the whole interval $[t^\star, t^\star + \delta]$. Therefore, if confinement is only allowed for a given amount of time δ, in order to reduce the epidemic final size, then it may be more beneficial to wait for some time, and then completely exhaust the lockdown ability. However, this does not answer the more involved issue: would it not be even more efficient to split the confinement period, and to consider more general control phases, while keeping a total treatment time at most equal to δ? Compared to the previous works [1, 2, 3], the goal (and novelty) of the present paper is to precisely answer this question, by considering the optimal control problem.

$$\mathcal{R}_S : \quad \sup_{u(\cdot) \in \mathcal{U}} S(+\infty) \text{ subject to } \mathtt{meas}\ \mathtt{supp}(u) \leq \delta,$$

for a given non trivial initial condition (S_0, I_0) of system (1.1), that is $S_0 > 0$, $I_0 > 0$ with $S_0 + I_0 \leq 1$. Our main contribution is to establish that Problem $\mathcal{R}_S$ has the same optimal solution as Problem Q_S. In other words, there is no gain in attempting to split the control period: the solution of Problem $\mathcal{R}_S$ is also uniquely obtained by taking $u \equiv \bar{u}$ on the interval $[t^\star, t^\star + \delta]$, for some optimal time $t^\star$ that may be computed.

Our approach is related to the works of [4, 5]. In [5], the authors impose an L^1 constraint on the control over an infinite horizon, without restricting the intervention times and without using the Pontryagin Maximum Principle, contrary to what we do here. In [4], the authors only consider the case of a constant transmission rate, in which the optimal intervention time is obtained by computing the partial derivative of the objective function with respect to the intervention time. In the present paper, the two main contributions are as follows. First, we provide a different characterization of the optimal intervention period (in the case of a constant transmission rate) based on a sensitivity analysis. More precisely, we show that the intervention time is the maximizer of a functional that depends on the initial conditions, model parameters, budget, and control upper bound. This result was previously presented in the conference paper [6] without proof; a complete and rigorous proof is provided here. Furthermore, we consider a piecewise constant transmission rate (one jump), for which we prove that it is optimal to have at most two interventions. Such an extension follows (mostly) the same steps as the constant transmission rate case; however, to ensure convergence of the objective function, we rely on a Γ-convergence property. Moreover, this allows us to preserve the structure of optimal controls when considering an infinite horizon.

Table 1. Contributions with final size as cost function. Reference [2] uses a more general, FS + state/control term, cost.

	Constraint	Input set
[1]	—	$\mathtt{supp}(u) \subset [0, \delta]$
[3]	—	$\mathtt{supp}(u) \subset [t, t+\delta],\ t \geq 0$
[7]	—	$\mathtt{supp}(u) = [0, \delta]$
[8]	—	$\mathtt{supp}(u) = [0, \delta]$, $u\|_{\mathtt{supp}(u)}$ constant
[2]	—	$\mathtt{supp}(u) \subset [0, \delta]$
[9]	PP	$\mathtt{meas}\ \mathtt{supp}(u) = \delta$, $I \equiv \max I(t)$ on $\mathtt{supp}(u)$
[5]		$\|\|u\|\| \leq K$
[4]	$\|\|u\|\|_{L^1} \leq K$	—
[6]	$\|\|u\|\|_{L^1} \leq K$	—

Table 2. Contributions with peak prevalence as cost function.

	Constraint	Input set
[7]	—	$\mathtt{supp}(u) = [0, \delta]$
[10]	—	$\mathtt{meas}\ \mathtt{supp}(u) = \delta$, $u\|_{\mathtt{supp}(u)}$ constant
[11]	—	$\mathtt{supp}(u) \subset I$, $\mathtt{meas}\ I = \delta$
[12]	—	$\mathtt{supp}(u) = \cup_{k=1}^{K} I_k$, $\mathtt{meas}\ I_k = \delta_k$, $u\|_{\mathtt{supp}(u)}$ constant
[13]	$\|\|u\|\|_1$	—

Table 3. Contributions taking L^1-norm of u as cost function.

	Cost	Constraint	Input set
[14]	$\|u\|_{L^1}$	PP	$\mathrm{supp}(u) \subset [0,\delta]$
[15]	$\|u\|_{L^1}$	PP	$\mathrm{supp}(u) \subset [0,\delta]$

Table 4. Contribution taking as cost function the time to the 'safe zone', i.e. the zone in which absence of control cannot yield violating of the constraint.

	Constraint	Input set
[16]	PP	—

The paper is organized as follows: in Section 2, we introduce an auxiliary problem $\mathcal{P}_S$ in finite horizon, for which we are able to study the optimal solutions with the help of the Pontryagin Maximum Principle; then in Section 3, we study in the limiting optimal solution of this auxiliary problem when the time horizon tends to $+\infty$; this allows us to show the equivalence between Problems $\mathcal{R}_S$ and Q_S in Section 4, thereby taking advantage of the characterization of the optimal solution over an infinite horizon obtained in Section 3; then in Section 5, we propose a slight extension, to the case where the transmission rate parameter β only changes only, from one value to another one, at a time instant known in advance, where this latter case is motivated by a sudden modification in the contamination mode, typically due to an adaptation or to the apparition of a variant that could rapidly supplant the original one; and finally, in Section 6, we illustrate the theoretical results obtained in the preceding sections through a series of numerical examples.

2. Formulation and analysis of the auxiliary Problem $\mathcal{P}_S$

The rationale adopted in this paper in order to prove the main claim is as follows. First, we introduce an auxiliary problem on a finite horizon (problem $\mathcal{P}_S$ below), in which a restriction is introduced on the maximal value of $\|u\|_1$, instead of a constraint on $\mathrm{supp}(u)$. Then, on the one hand, the idea of the argument consists in showing, for large enough time horizons, that the optimal solution of this problem has a structure independent of the time horizon; on the other hand the argument consists in showing that the optimal control has a connected support. This is the main result of the present section, which is stated in Proposition 1 after the proof of several technical steps. First, let us introduce the auxiliary problem. Given a positive initial condition $(S(0), I(0)) = (S_0, I_0)$, a time horizon $T > 0$, constant positive rates β and γ, and a budget constraint $K > 0$ on the control $u(\cdot)$, we consider the following optimal control problem:

$$\mathcal{P}_S : \quad \sup_{u(\cdot)\in\mathcal{U}} S(T) \text{ subject to } \|u(\cdot)\|_1 \leq K.$$

The number K represents the maximal amount of efforts to reduce the incidence that is allowed without fixing any *a priori* time window of action on the interval $[0, T]$. What is expected is that, for large values of T (larger than $\frac{K}{\bar{u}}$), the constraint on $\|u\|_1$ will be active, with the optimal solution having the same structure as whatever the value of T.

The L^1 constraint on the control variable

$$\|u(\cdot)\|_1 \leq K \tag{2.1}$$

can be easily tackled by augmenting the state of the system

$$\begin{cases} \dot{S} = -(1-u)\beta S I, \\ \dot{I} = (1-u)\beta S I - \gamma I, \\ \dot{C} = -u, \end{cases} \tag{2.2}$$

with $C(0) = K$, so that the integral constraint (2.1) is transformed into the terminal constraint $C(T) \geq 0$. Then, the optimal control Problem $\mathcal{P}_S$ is formulated as a Mayer problem with a fixed initial condition (S_0, I_0, K) and a target $\{(S, I, C) \in \mathbb{R}^3_+ \mid S + I \leq 1\}$.

The existence of optimal controls is guaranteed by classical existence results, known as Filippov's existence Theorem (see e.g. [17, Theorem 5.1.1.]), as the following properties are fulfilled:

- dynamics and cost functions are of class C^1,
- any admissible solution remains in the compact set

$$\mathcal{A} := \{(S, I, C) \in \mathbb{R}^3_+ \mid S + I \leq S_0 + I_0,\ C \leq K\},$$

  (and thus the dynamics is with at most linear growth on $\mathcal{A}$),
- controls take values in the compact set $[0, \bar{u}]$,
- at any $(S, I, C) \in \mathcal{A}$, the velocity set

$$\mathcal{V}(S, I, C) := \bigcup_{u \in [0,\bar{u}]} \{(-(1-u)\beta S I, (1-u)\beta S I - \gamma I, -u)^\top\},$$

  is convex (since the dynamics is linear with respect to the control u).

This implies that the set of admissible solutions, as absolutely continuous functions from $[0, T]$ to $\mathcal{A}$, is compact for the strong-weak topology.

We begin by writing the necessary optimality conditions of Pontryagin's Maximum Principle for problem $\mathcal{P}_S$. The Hamiltonian writes

$$H = (p_I - p_S)(1-u)\beta S I - p_I \gamma I - p_C u, \tag{2.3}$$

and the adjoint equations

$$\begin{aligned} \dot{p}_S &= -\partial_S H = (p_S - p_I)(1-u)\beta I, \\ \dot{p}_I &= -\partial_I H = (p_S - p_I)(1-u)\beta S + p_I \gamma, \\ \dot{p}_C &= -\partial_C H = 0, \end{aligned} \tag{2.4}$$

along with the following transversality conditions:

$$\begin{aligned} & p_S(T) = p^0, \\ & p_I(T) = 0, \\ & p_C(T) \geq 0 \ (= 0 \text{ if the terminal constraint is not saturated}), \end{aligned} \tag{2.5}$$

with $p^0 \in \{0, 1\}$. An optimal control $u(\cdot)$ is a maximizer of the Hamiltonian for almost any t, which allows us to claim that one has the following for any optimum control u:

$$\begin{aligned} \phi(t) > 0 &\Rightarrow u(t) = \bar{u}, \\ \phi(t) < 0 &\Rightarrow u(t) = 0, \end{aligned} \tag{2.6}$$

for almost any t, where ϕ is the switching function

$$\phi = (p_S - p_I)\beta S I - p_C, \tag{2.7}$$

with the property that $H = u\phi + (p_I - p_S)\beta S I - p_I\gamma I$. Moreover, for any optimal solution, there exists a number $\bar{H}$ such that $H(t) = \bar{H}$ a.e. $t \in [0, T]$ (a property which is called the conservation of the Hamiltonian).

Before giving our main result (Proposition 1 below), which characterizes the structure of the optimal solutions, we begin with several lemmas that provide relevant information that will be also used in Section 5, which is devoted to the extension with the piecewise constant parameter β.

The following lemma rules out the existence of abnormal trajectories.

Lemma 1. *Every optimal extremal for problem $\mathcal{P}_S$ is normal (i.e., $p^0 = 1$).*

Proof. Assume by contradiction, that there exists an optimal extremal that is abnormal. Let (p_S, p_I, p_C) denote the corresponding adjoint vector. The transversality conditions read

$$p_S(T) = p^0 = 0 \quad \text{(abnormal)}, \qquad p_I(T) = 0, \qquad p_C(T) = \begin{cases} 0, & \text{if } C(T) > 0, \\ \nu, & \text{if } C(T) = 0, \end{cases}$$

for some $\nu \geq 0$. From the transversality conditions above and the linear Cauchy system (see (2.4)) satisfied by (p_S, p_I), we obtain that $p_S(t) = p_I(t) = 0$ for all $t \in [0, T]$. Now, we consider the remaining (possible) cases:

- If $C(T) > 0$, then, from the linear Cauchy system and the transversality condition for p_C, we obtain $p_C(t) = 0$ for all $t \in [0, T]$. This contradicts the nontriviality condition of the PMP, namely $(p_S, p_I, p_C, p^0) \neq (0, 0, 0, 0)$.
- If $C(T) = 0$, then $p_C(T) \geq 0$. If $p_C(T) = 0$, then the proof is the same as in the previous case. If $p_C(T) = \nu > 0$, then the switching function is $\phi = -\nu < 0$, hence $u(t) = 0$ for a.e. $t \in [0, T]$, and thus the constraint (2.1) is not saturated, so $C(T) > 0$, which is a contradiction.

This ends the proof of Lemma 1. □

Now, let us study the switching function ϕ, which is the key component to determine bang and singular arcs. A straightforward computation gives the following:

$$\begin{aligned} \dot{\phi} &= (\dot{p}_S - \dot{p}_I)\beta S I + (p_S - p_I)\beta(\dot{S} I + S \dot{I}) - \dot{p}_C \\ &= ((p_S - p_I)(1-u)\beta(I - S) - p_I\gamma)\beta S I \\ &\quad + (p_S - p_I)\beta\left(-(1-u)\beta S I^2 + (1-u)\beta S^2 I - \gamma S I\right) \\ &= (p_S - p_I)(1-u)\,(I - S - I + S)\,\beta^2 S I + \beta\gamma S I(-p_I - p_S + p_I), \end{aligned}$$

and thus ϕ is C^1 with

$$\dot{\phi} = -\gamma\beta S I p_S. \tag{2.8}$$

Additionally, at any point of continuity of u,

$$-\frac{1}{\beta\gamma}\ddot{\phi} = (\dot{S} I + S \dot{I}) p_S + S I \dot{p}_S$$

$$= \Big(-(1-u)\beta S I^2 + (1-u)\beta S^2 I - \gamma S I\Big) p_S + S I (p_S - p_I)(1-u)\beta I$$
$$= (1-u)\beta S I(-I+S)p_S + \frac{1}{\beta}\dot{\phi} + (1-u)I(\phi + p_C)$$
$$= -\frac{1}{\gamma}(1-u)(S-I)\dot{\phi} + \frac{1}{\beta}\dot{\phi} + (1-u)I(\phi + p_C);$$

thus, using (2.8),

$$\ddot{\phi} = -(\phi + p_C)(1-u)\beta\gamma I + \dot{\phi}(\beta(1-u)(S-I) - \gamma). \tag{2.9}$$

Lemma 2. *For $K < T\bar{u}$, one has $p_C > 0$.*

Proof. Assume by contradiction that $p_C = 0$. One has

$$\phi(T) = (p_S(T) - p_I(T))\beta S(T)I(T) = \beta S(T)I(T) > 0.$$

By continuity, $\phi(t)$ is positive for $t \in [T-\eta, T]$ and for some $\eta > 0$; then, any optimal control verifies $u(t) = \bar{u}$ for a.e. $t \in [T-\eta, T]$ for some $\eta > 0$.

If $\bar{u} = 1$, then one obtains $\dot{p}_S = 0$, $\dot{p}_I = \gamma p_I$ from the adjoint equations; additionally, from the transversality conditions (2.5), we deduce that $p_S = 1$ and $p_I = 0$ on this time interval, and this constant solution is propagated on the whole interval $[0, T]$. Then, any optimal control has to satisfy $u(t) = \bar{u}$ for a.e. $t \in [0, T]$.

If $\bar{u} < 1$, then as p_I is continuous with $p_I(T) = 0$, one obtains the following identity from the conservation of the Hamiltonian along an optimal solution :

$$H = -(1-u(t))\phi(t) - p_I(t)\gamma I(t) = H(T) \quad \text{a.e. } t \in [0, T].$$

On the other hand, using the assumption that $p_C = 0$,

$$H(T) = -p_S(T)(1-\bar{u})\beta S(T)I(T) - p_C\bar{u} = -(1-\bar{u})\beta S(T)I(T) < 0;$$

thus,

$$H = -(1-u(t))\phi(t) - p_I(t)\gamma I(t) < 0 \quad \text{a.e. } t \in [0, T]. \tag{2.10}$$

Now, considering the following set

$$E := \{t \in [0, T];\ \phi(t) \le 0\}.$$

If E is non empty, then let $t_c = \sup E < T$. By continuity of ϕ, one has $\phi(t_c) = 0$, and thus $p_I(t_c) = p_S(t_c) > 0$ using inequality (2.10). In view of (2.8), this implies $\dot{\phi}(t_c) < 0$, which contradicts the definition of t_c. We conclude that E has to be empty and that any optimal control also has to satisfy $u(t) = \bar{u}$ for a.e. $t \in [0, T]$.

For the cases $\bar{u} = 1$ or $\bar{u} < 1$, we have shown that $p_C = 0$ implies that $u(t) = \bar{u}$ for a.e. $t \in [0, T]$. The, one obtains the following :

$$\int_0^T u(t)dt = T\bar{u} > K,$$

which violates the constraint (2.1). Therefore, one has $p_C > 0$, which demonstrates Lemma 2. □

As the objective is to consider the criterion for large values of T, we assume in the following that the condition $T\bar{u} > K$ is verified, so that an optimal control saturates the constraint (2.1).

Lemma 3. *For $\bar{u} < 1$, an optimal solution has no singular arc. For $\bar{u} = 1$, a singular arc is given by $u = 1$.*

Proof. If ϕ is null on a non-empty open time interval J, then one should have $\dot{\phi}(t) = 0$ for $t \in J$ and $\ddot{\phi}(t) = 0$ for a.e. $t \in J$. From expression (2.9), one obtains the following :

$$\ddot{\phi}(t) = -p_C(1 - u(t))\beta\gamma I(t) = 0 \quad \text{a.e. } t \in J.$$

If $\bar{u} < 1$, then one obtains a contradiction due to the fact that $p_C > 0$ (see Lemma 2): there is no singular arc. If $\bar{u} = 1$, then $u = 1$ is optimal on J. □

The key point to establish that the optimal solution consists in a single intervention interval, is to show that the switching function remains positive for a single continuous period.

Lemma 4. *For any optimal solution, the set*

$$\{t \in [0, T]; \phi(t) \geq 0\}$$

is connected.

Proof. To prove this result, we proceed by contradiction. We assume that there exists $t_1 < t_2$ in $(0, T)$ such that $\phi(t_1) = \phi(t_2) = 0$ with $\phi(t) < 0$ for $t \in J := (t_1, t_2)$. From (2.8), we know that ϕ is a C^1 function, and then one necessarily has $\dot{\phi}(t_1) \leq 0 \leq \dot{\phi}(t_2)$. Again, from (2.8), one gets the inequalities $p_S(t_1) \geq 0 \geq p_S(t_2)$. On the other hand, we know that an optimal solution verifies $u(t) = 0$ a.e. on J. Therefore, from (2.4), p_S is right-differentiable at t_1 and left-differentiable at t_2, with

$$\dot{p}_S(t_1^+) = \frac{\phi(t_1) + p_C}{S(t_1)} = \frac{p_C}{S(t_1)} > 0, \quad \dot{p}_S(t_2^-) = \frac{\phi(t_2) + p_C}{S(t_2)} = \frac{p_C}{S(t_2)} > 0.$$

Consequently, the function p_S possesses a local maximum at a certain $\hat{t}$ in J with $p_S(\hat{t}) > 0$, and a local minimum at a certain $\check{t} > \hat{t}$ in J with $p_S(\check{t}) < 0$. This implies $\dot{p}_S(\hat{t}) = \dot{p}_S(\check{t}) = 0$, and from (2.4), one has $p_S(\hat{t}) - p_I(\hat{t}) = p_S(\check{t}) - p_I(\check{t}) = 0$.

Finally, we use the conservation of the Hamiltonian (2.3) to write

$$H(\hat{t}) = -p_S(\hat{t})\gamma I(\hat{t}) = H(\check{t}) = -p_S(\check{t})\gamma I(\check{t});$$

however, as $p_S(\hat{t})$ and $p_S(\check{t})$ have opposite signs, we thus obtain a contradiction. □

Then, we obtain the following '0-$\bar{u}$-0' optimal synthesis result.

Proposition 1. *For any positive initial condition and time horizon $T > K/\bar{u}$, there exists $t^\star \in [0, T - K/\bar{u}]$ such that the control*

$$u(t) = \begin{cases} \bar{u} & t \in [t^\star, t^\star + K/\bar{u}], \\ 0 & \text{otherwise}, \end{cases} \tag{2.11}$$

is optimal for Problem $\mathcal{P}_S$.

Proof. From Lemma 3, we know that an optimal control can only take the value 0 or $\bar{u}$, at almost any time. With Lemma 4, we know that the optimal solution has at most one arc with $u = \bar{u}$, and with Lemma 2, we know that constraint (2.1) is saturated. Therefore, an optimal control has necessarily the structure (2.11). □

3. Problem $\mathcal{P}_S$ over an infinite horizon

In this section, we aim to extend the results of Section 2, originally established for a finite time horizon (with constant β), to the infinite horizon setting. This extension is stated below in Proposition 2.

We begin by a Lemma which characterizes the optimal intervention time $t^\star$ given in Proposition 1 for large time horizons T. Define the following numbers :

$$S_h := \frac{\gamma}{\beta}, \quad \delta := \frac{K}{\bar{u}}$$

One may recognize the *herd immunity threshold* in S_h, that is, the proportion of susceptible individuals below which the number of infected starts decreasing (for the SIR model without control).

Lemma 5. *For any positive initial condition and finite time horizon $T > \delta$, define the function*

$$\Gamma_T(t_c) = \begin{cases} (1 - e^{-\gamma K})I(t_c) + I(T), & \bar{u} = 1 \\ \dfrac{\bar{u}}{1 - \bar{u}} \left(\log(S(t_c) - \log(S(t_c + \delta)\right) + I(T), & \bar{u} < 1 \end{cases} \tag{3.1}$$

along the solution for the bang-bang control "0-$\bar{u}$-0" with commutations at t_c and $t_c+\delta$. Then, for large enough values of T, an optimal intervention time $t^\star$ that yields the maximal value $S(T)$ maximizes the function Γ_T, and one has $S(T) < S_h$.

Proof. For a finite horizon $T > \delta = K/\bar{u}$, for which the structure of an optimal control is given by (2.11), one may characterize the optimal value of $S(T)$ as follows. Define, for convenience, the following function :

$$F(S) := S - S_h \log(S). \tag{3.2}$$

Fix an intervention time $t_c \in [0, T - \delta]$ and consider the control

$$u(t) = \begin{cases} \bar{u} & t \in [t_c, t_c + \delta], \\ 0 & \text{otherwise.} \end{cases}$$

On the time intervals $[0, t_c)$ and $(t_c + \delta, T]$, we can write the invariant properties of system (1.1) with constant control :

$$F(S(t_c)) + I(t_c) = F(S(0)) + I(0) =: W_0, \tag{3.3}$$

$$F(S(T)) + I(T) = F(S(t_c + \delta)) + I(t_c + \delta). \tag{3.4}$$

On the time interval $(t_c, t_c + \delta)$, the control u is constant equal to $\bar{u}$. We distinguish two cases:

1. $\bar{u} = 1$. Then, one has

$$S(t_c + \delta) = S(t_c), \quad I(t_c + \delta) = e^{-\gamma\delta} I(t_c), \tag{3.5}$$

2. $\bar{u} < 1$. We can write the invariant property

$$S(t_c + \delta) + I(t_c + \delta) - \frac{S_h}{1 - \bar{u}} \log(S(t_c + \delta)) = S(t_c) + I(t_c) - \frac{S_h}{1 - \bar{u}} \log(S(t_c)). \tag{3.6}$$

Let us combine expressions (3.3), (3.5) or (3.6), and (3.4). We obtain the following (with W_0 defined in (3.3)):

1. for $\bar{u} = 1$
$$F(S(T)) = W_0 - (1 - e^{-\gamma\delta})I(t_c) - I(T), \tag{3.7}$$

2. for $\bar{u} < 1$
$$F(S(T)) = W_0 - \frac{\bar{u}}{1-\bar{u}} S_h \left(\log(S(t_c)) - \log(S(t_c + \delta))\right) - I(T). \tag{3.8}$$

Then, from (3.7) and (3.8), $S(T)$ and t_c verify the following :

$$F(S(T)) = W_0 - \Gamma_T(t_c). \tag{3.9}$$

First, let us show that for a large enough T, one necessarily has $S(T) < S_h$ for any admissible control. If $S(t) \geq S_h$ for any $t > 0$, the one should have :

$$\frac{d}{dt} \log I = \beta S(1 - u(t)) - \gamma \geq -\gamma u(t) \Rightarrow I(t) \geq I(0)e^{-\gamma K} > 0, \quad t > 0;$$

however, from the equality

$$\dot{S} + \dot{I} = -\gamma I \Rightarrow S(t) + I(t) = S(0) + I(0) - \gamma \int_0^t I(\tau)d\tau,$$

$S + I$ would take negative values for large t, which is a contradiction.

Note that the function F is decreasing for $S < S_h$. From (3.9), for any large enough value of T, we dedice that the optimal commutation time $t^\star$, which gives the maximal value $S(T)$, has to maximize the function Γ_T. □

Now, we investigate the optimal solution of the limiting problem $\mathcal{P}_S$ when $T \to +\infty$. Let us denote $t_T^\star$ an optimal intervention time for the Problem $\mathcal{P}_S$ with finite horizon $T > 0$.

Proposition 2. *The optimal solution for the the limiting problem $\mathcal{P}_S$ when $T \to +\infty$ is given by the control* (2.11)*, where $t^\star = 0$ if $S_0 \leq S_h$ and $t^\star = t_\infty^\star$ for $S_0 > S_h$, such that*

1. *for $\bar{u} = 1$, $t_\infty^\star$ is uniquely defined as $S(t_\infty^\star) = S_h$;*
2. *for $\bar{u} < 1$, $t_\infty^\star$ is the unique minimizer of the function $t_c \mapsto \log S(t_c + \delta) - \log S(t_c)$. Moreover, one has $S(t_\infty^\star) > S_h > S(t_\infty^\star + \delta)$.*

Furthermore, $t_T^\star$ converges to $t_\infty^\star$ when $T \to +\infty$.

Proof. An optimal solution to Problem $\mathcal{P}_S$ in an infinite horizon with a control of the form (2.11) has to maximize the following function :

$$\Gamma_\infty(t_c) = \begin{cases} (1 - e^{-\gamma\delta})I(t_c), & \bar{u} = 1, \\ \dfrac{\bar{u}}{1-\bar{u}} \left(\log(S(t_c) - \log(S(t_c + \delta))\right), & \bar{u} < 1, \end{cases} \tag{3.10}$$

which is the pointwise limit of the functions Γ_T when $T \to \infty$.

For $\bar{u} = 1$, the function Γ_∞ is maximized for $t^\star_\infty = 0$ if $S_0 \leq S_h$, because $I(\cdot)$ is always decreasing, and for the unique $t^\star_\infty$ such that $S(t^\star_\infty) = S_h$, which gives the peak of $I(\cdot)$, if $S_0 > S_h$.

For $\bar{u} < 1$, a maximizer $t^\star_\infty$ of Γ_∞ maximizes $t_c \mapsto \log(S(t_c)) - \log(S(t_c + \delta))$. As this last quantity tends to 0 when t_c tends to $+\infty$, we conclude that its maximum is reached for $t^\star_\infty < +\infty$. Let us show that $t^\star_\infty$ is unique.

Posit $\Sigma := \log S$ and $Z := I + S - S_h\Sigma$. On the interval where $u = \bar{u}$, one obtains the following :

$$\begin{cases} \dot{\Sigma} = -\beta I(1-\bar{u}) = \beta(1-\bar{u})(e^\Sigma - S_h\Sigma - Z), \\ \dot{Z} = -\gamma I + S_h\beta(1-\bar{u})I = -\gamma\bar{u}I = \frac{\bar{u}}{1-\bar{u}} S_h \dot{\Sigma}. \end{cases}$$

In particular, the quantity $Z - \frac{\bar{u}}{1-\bar{u}} S_h\Sigma$ appears invariant along the trajectories. Then, one has $\log S(t_c + \delta) = \xi(\delta, \xi_c)$ for $\xi_c = \log S(t_c)$, and $\xi(t, \xi_c)$ is solution of the following :

$$\begin{aligned} \frac{\partial \xi}{\partial t} &= \dot{\Sigma}(t) = \beta(1-\bar{u})(e^\Sigma - S_h\Sigma - Z) \\ &= \beta(1-\bar{u})\left(e^\xi - S_h\xi - Z_0 - \tfrac{\bar{u}}{1-\bar{u}} S_h(\xi - \xi_c)\right) \\ &:= F(\xi, \xi_c), \end{aligned}$$

with $\xi(0, \xi_c) = \xi_c$, and $Z_0 = I_0 + S_0 - S_h \log S_0$. Minimizing $\log S(t_c + \delta) - \log S(t_c)$ amounts to looking for the minimum of $J : \xi_c \mapsto \xi(\delta, \xi_c) - \xi_c$ on $(-\infty, \log S_0]$. From the sensitivity equation, one obtains the following :

$$\begin{aligned} \frac{d}{dt}\frac{\partial \xi}{\partial \xi_c} &= \tfrac{\partial F}{\partial \xi}(\xi(t,\xi_c),\xi_c)\tfrac{\partial \xi}{\partial \xi_c} + \tfrac{\partial F}{\partial \xi_c}(\xi(t,\xi_c),\xi_c) \\ &= \beta(1-\bar{u})\left(e^{\xi(t,\xi_c)} - \tfrac{S_h}{1-\bar{u}}\right)\tfrac{\partial \xi}{\partial \xi_c} + \gamma\bar{u} \\ &:= G\left(\xi(t,\xi_c), \tfrac{\partial \xi}{\partial \xi_c}\right). \end{aligned} \tag{3.11}$$

Note that one has $G(\cdot, 0) > 0$, which, together with the positive initial condition $\frac{\partial \xi}{\partial \xi_c}(0, \xi_c) = 1$, implies that $\frac{\partial \xi}{\partial \xi_c}(t, \xi_c) > 0$, for any $t > 0$. This, in turn, implies that $\xi_c \mapsto \xi(t, \xi_c)$ is increasing, for any $t > 0$. In particular, the map $\xi_c \mapsto G(\xi(t, \xi_c), \lambda)$ is increasing for any $(t, \lambda) \in \mathbb{R}^2_+$. Therefore, the solution $\frac{\partial \xi}{\partial \xi_c}(\delta, \xi_c)$ of (3.11) at time δ is increasing w.r.t. ξ_c. We deduce that J' is increasing, so that J is strictly convex. This proves the existence of a unique minimum $\xi^\star_c$ on $(-\infty, \log S_0]$, which is reached at the unique value $t^\star_\infty$ such that $\log S(t^\star_\infty) = \xi^\star_c$ (any solution $S(\cdot)$ being monotonic).

Note that as $\xi^\star_c$ is a minimizer of the convex function J on the interval $(-\infty, \log S_0]$, one has the following inequality :

$$J'(\xi^\star_c) = \frac{\partial \xi}{\partial \xi_c}(\delta, \xi^\star_c) - 1 \leq 0, \tag{3.12}$$

with equality when $\xi^\star_c < \log S_0$.

If $S(t^\star_\infty + \delta) \geq S_h$, then one should have $S(t) > S_h$ for any $t \in [t^\star_\infty, t^\star_\infty + \delta)$, as the function $S(\cdot)$ is decreasing. Then, from (3.11), one obtains the following :

$$\frac{d}{dt}\frac{\partial \xi}{\partial \xi_c}(t, \xi^\star_c) > \beta(1-\bar{u})S_h\frac{-\bar{u}}{1-\bar{u}}\frac{\partial \xi}{\partial \xi_c}(t, \xi^\star_c) + \gamma\bar{u} = \gamma\bar{u}\left(1 - \frac{\partial \xi}{\partial \xi_c}(t)\right), \quad t \in [t^\star_\infty, t^\star_\infty + \delta).$$

As $\frac{\partial \xi}{\partial \xi_c}(0, \xi^\star_c) = 1$, we deduce that $\frac{\partial \xi}{\partial \xi_c}(\cdot, \xi^\star_c)$ is increasing on $(t^\star_\infty, t^\star_\infty + \delta)$, and thus $\frac{\partial \xi}{\partial \xi_c}(\delta, \xi^\star_c) > 1$, which contradicts (3.12). This shows that the inequality $S(t^\star_\infty + \delta) < S_h$ is necessarily fulfilled.

In the same manner, if $S(t^\star_\infty) \le S_h$, then one should have $S(t) < S_h$ for any $t \in (t^\star_\infty, t^\star_\infty + \delta]$, and

$$\frac{d}{dt}\frac{\partial \xi}{\partial \xi_c}(t, \xi^\star_c) < \gamma \bar{u}\left(1 - \frac{\partial \xi}{\partial \xi_c}(t)\right), \quad t \in (t^\star_\infty, t^\star_\infty + \delta],$$

which implies $\frac{\partial \xi}{\partial \xi_c}(\delta, \xi^\star_c) < 1$. According to (3.11), this is only possible when $\xi^\star_c = \log S_0$, that is, when $S_0 \le S_h$.

Therefore, when $S_0 > S_h$, we conclude that the maximizer $t^\star_\infty$ is such that $S(t^\star_\infty) > S_h > S(t^\star_\infty + \delta)$. When $S_0 \le S_h$, one necessarily has $S(t^\star_\infty) \le S_h$, and then $S(t^\star_\infty) = S_0$ (i.e., $t^\star_\infty = 0$).

Now, we prove the convergence of $t^\star_T$ to $t^\star_\infty$. For this purpose, we show that the family $\{t^\star_T\}_{T>0}$ is bounded and has a unique accumulation point, which happens to be $t^\star_\infty$.

By contradiction, assume $\{t^\star_T\}_{T>0}$ is unbounded. Then, one can extract a sequence $t^\star_{T_n} \to \infty$, each maximizing Γ_{T_n}. Since $t^\star_{T_n} \le T_n$, we also have $T_n \to \infty$. Therefore,

$$\lim_{n\to\infty} \|\Gamma_{T_n}\|_\infty = \lim_{n\to\infty} \Gamma_{T_n}(t^\star_{T_n}) = 0,$$

and thus the limit Γ_∞ is identically null, which is clearly false.

Observe that the functions Γ_T $(T > 0)$ are continuous, and for each given t_c, the map $T > t_c + \delta \mapsto I(T)$ is decreasing. Thus, $\{\Gamma_T(\cdot)\}_{T>0}$ is a decreasing family of continuous functions, which converges pointwise to the continuous function Γ_∞. Therefore, this family of functions Γ-converges to Γ_∞ (see [18, Prop 5.4]). Consequently, any converging subsequence of maximizers $t^\star_T$ of Γ_T has to converge to a maximizer of Γ_∞, that is, $t^\star_\infty$. □

Remark 1. *Hereafter, we provide an interpretation of Proposition 2 and a characterization of an intervention time $t^\star_T$ for Problem $\mathcal{P}_S$ with finite horizon T.*

- *Proposition 2 shows that when a complete stop of the transmission is possible (i.e., $\bar{u} = 1$), the optimal solution for $\mathcal{P}_S$ over an infinite horizon consists of waiting the proportion S to reach the herd immunity value S_h (whatever is the initial condition), and then to block the transmission as long as the L^1 budget is not completely used. Differently, when only a partial reduction of the transmission rate is possible (i.e., $\bar{u} < 1$), the optimal intervention time anticipates the time instant when S reaches S_h (depending on the initial condition). Figure 2b illustrates how to obtain the optimal switching time as realizing the maximum of function Γ_∞ defined in* (3.10) *and the corresponding optimal solution.*

- *From the convergence $t^\star_T \to t^\star_\infty < +\infty$ as $T \to +\infty$, there exists $\tilde{T} > t^\star_\infty + \delta + 1$ such that*

$$\forall\, T > \tilde{T}, \qquad |t^\star_T - t^\star_\infty| < \tfrac{1}{2}.$$

In particular, for such T, one has $T > t^\star_T + \delta$, so the interval $[t^\star_T + \delta,\, T]$ is of a positive measure. Since one cannot apply the control $\bar{u}$ beyond a duration $\delta = K/\bar{u}$ after the last activation time $t^\star_T$, the optimal control u that corresponds to $t^\star_T$ has to satisfy

$$u(t) = 0, \qquad \text{for a.e. } t \in [t^\star_T + \delta,\, T]. \tag{3.13}$$

This shows that, whatever is the value of T, an optimal solution of Problem $\mathcal{P}_S$ with horizon $T > 0$ is given by an optimal control with a support contained in $[0, \tilde{T}]$.

4. Equivalence of Problems $\mathcal{Q}_S$ and $\mathcal{R}_S$

Using the results about the auxiliary problem $\mathcal{P}_S$, we are now in position to prove that the optimal costs of the Problems $\mathcal{Q}_S$ and $\mathcal{R}_S$ are the same.

Let us fix a positive initial condition $(S(0), I(0)) = (S_0, I_0)$, a constant β, and a number $\delta > 0$. Let us recall that the set of admissible controls for Problems $\mathcal{Q}_S$, $\mathcal{R}_S$, and $\mathcal{P}_S$ (for $T = +\infty$) are as follows:

$$
\begin{aligned}
\mathcal{U}(\mathcal{Q}_S) &:= \{\, u : [0, +\infty) \to [0, \bar{u}] \mid \mathtt{supp}(u) \subset [t, t+\delta] \text{ for some } t \geq 0\},\\
\mathcal{U}(\mathcal{R}_S) &:= \{\, u : [0, +\infty) \to [0, \bar{u}] \mid \mathtt{meas}(\mathtt{supp}(u)) \leq \delta \,\},\\
\mathcal{U}(\mathcal{P}_S) &:= \{\, u : [0, +\infty) \to [0, \bar{u}] \mid \|u\|_{L^1} \leq \bar{u}\delta \,\}.
\end{aligned}
$$

One can observe the following inclusion :

$$
\mathcal{U}(\mathcal{Q}_S) \subset \mathcal{U}(\mathcal{R}_S) \subset \mathcal{U}(\mathcal{P}_S). \tag{4.1}
$$

Therefore, the optimal value of Problem $\mathcal{Q}_S$ is not larger than the optimal cost of Problem $\mathcal{R}_S$. In the following result, we provide equivalence between problems $\mathcal{Q}_S$ and $\mathcal{R}_S$ thanks to the structure of the optimal control of $\mathcal{P}_S$ (when T tends to $+\infty$) obtained in Proposition 2.

Theorem 1. *The Problems $\mathcal{Q}_S$ and $\mathcal{R}_S$ have the same optimal cost, achieved by a unique optimal trajectory generated by the control $u^\star$ of type* (2.11)*, such that* $\mathtt{meas}\ \mathtt{supp}(u^\star) = \delta$ *and $u^\star = \bar{u}$ on* $\mathtt{supp}(u^\star)$*. Moreover, $u^\star$ is optimal for Problem $\mathcal{P}_S$ with $T = +\infty$.*

Proof. From the inclusion (4.1), one immediately obtains the following :

$$
\sup_{u \in \mathcal{U}(\mathcal{Q}_S)} S(\infty, u) \leq \sup_{u \in \mathcal{U}(\mathcal{R}_S)} S(\infty, u) \leq \sup_{u \in \mathcal{U}(\mathcal{P}_S)} S(\infty, u);
$$

however, we have shown that an optimal solution of Problem $\mathcal{P}_S$ is generated by a control $u^\star$ of the form (2.11), and thus belongs to $\mathcal{U}(\mathcal{Q}_S)$, so

$$
\sup_{u \in \mathcal{U}(\mathcal{P}_S)} S(\infty, u) \leq \sup_{u \in \mathcal{U}(\mathcal{Q}_S)} S(\infty, u),
$$

which implies the equality of all optimal values.

The optimal trajectory for Problems $\mathcal{Q}_S$ and $\mathcal{R}_S$ is also unique because if there existed another optimal trajectory with a control in $\mathcal{U}(\mathcal{Q}_S)$ or $\mathcal{U}(\mathcal{R}_S)$, then it would contradict the uniqueness of the optimal trajectory for Problem $\mathcal{P}_S$ generated by the control $u^\star$. It follows that $u^\star$ is also optimal for Problems $\mathcal{Q}_S$ and $\mathcal{R}_S$. □

5. Extension to a case of piecewise constant β

As previously shown, the solution of Problem $\mathcal{R}_S$ is quite simple when the parameter β is constant: it corresponds to applying the maximal possible control effort during a time interval of length δ, beginning at an optimally chosen time instant. In an attempt to investigate whether this property may be generalized, now we investigate Problem $\mathcal{R}_S$ when the parameter β undergoes a unique jump, specified as follows:

$$\beta(t) = \begin{cases} \beta_1 & t \in [0, T_c), \\ \beta_2 & t \in [T_c, +\infty), \end{cases} \tag{5.1}$$

for some $T_c > 0$ and non-negative $\beta_1 \neq \beta_2$.

As before, we first consider the auxiliary Problem $\mathcal{P}_S$ over a finite (large) horizon T, and then obtain a convergence result that characterizes the optimal solutions of Problem $\mathcal{R}_S$ with changing β. First, we show that an optimal solution for Problem $\mathcal{P}_S$ with large time horizon consists in at most two distinct intervals of intervention. This result exploits former results (Lemmas 2, 3 and 4, Proposition 1).

Proposition 3. *Fix a large enough $T > T_c$. For any optimal solution of Problem $\mathcal{P}_S$ with β defined in* (5.1)*, there exist numbers $K_c \in [0, K]$, $t_1^\star \in [0, T_c - (K - K_c)/\bar{u}]$, and $t_2^\star \in [T_c, T - K_c/\bar{u}]$ such that the optimal control verifies the following :*

$$u(t) = \begin{cases} \bar{u} & t \in [t_1^\star, t_1^\star + (K - K_c)/\bar{u}] \cup [t_2^\star, t_2^\star + K_c/\bar{u}], \\ 0 & \textit{otherwise}, \end{cases} \tag{5.2}$$

for a.e. $t \in [0, T]$. Moreover, one has $t_2^\star + K/\bar{u} < T$.

Proof. For Problem $\mathcal{P}_S$ with (finite) time horizon $T > T_c$, one can apply the Hybrid Maximum Principle (see e.g., [19, Th. 22.20]). The corresponding necessary conditions for optimality are stated as follows. We introduce the Hamiltonian, defined by

$$H = \begin{cases} (p_S - p_I)(1-u)\beta_1 S I \; - \; p_I \gamma I \; - \; p_C u, & \text{over } [0, T_c), \\ (p_S - p_I)(1-u)\beta_2 S I \; - \; p_I \gamma I \; - \; p_C u, & \text{over } (T_c, T], \end{cases} \tag{5.3}$$

and the adjoint equations

$$\begin{cases} \dot{p}_S = (p_S - p_I)(1-u)\beta_1 I, \\ \dot{p}_I = (p_S - p_I)(1-u)\beta_1 S + p_I \gamma, \\ \dot{p}_C = 0, \end{cases} \text{over } [0, T_c), \quad \begin{cases} \dot{p}_S = (p_S - p_I)(1-u)\beta_2 I, \\ \dot{p}_I = (p_S - p_I)(1-u)\beta_2 S + p_I \gamma, \\ \dot{p}_C = 0, \end{cases} \text{over } (T_c, T]. \tag{5.4}$$

along with the transversality conditions:

$$\begin{aligned} &p_S(T) = p^0, \\ &p_I(T) = 0, \\ &p_C(T) \geq 0 \; (= 0 \text{ if the terminal constraint is not saturated}), \end{aligned} \tag{5.5}$$

with $p^0 \in \{0, 1\}$. Note that since T_c is fixed and no switching condition occurs at time T_c, the transversality conditions are the same as those obtained in Section 2. Moreover, the adjoint variables p_S, p_I, and p_C are absolutely continuous on $[0, T]$; in particular, p_C is constant on $[0, T]$. Therefore, abnormal extremals can be ruled out by the same argument used in the proof of Lemma 1, and we conclude that $p^0 = 1$.

Notice that, when considering $(S(T_c), I(T_c), C(T_c))$ as an initial condition of system (2.2) at time T_c, one faces exactly Problem $\mathcal{P}_S$ with constant $\beta = \beta_2$ over the time interval $[T_c, T]$, with K replaced by $K_c = C(T_c)$ (possibly null), and Proposition 1 shows the optimality of the control (2.11) for $t \geq T_c$. Additionally, notice that Lemma 2 shows the positivity of the constant p_C (provided that T is large enough) for the Problem $\mathcal{P}_S$ over the whole time interval $[0, T]$. Moreover, Remark 1 ensures that, for any large enough value of $T > 0$, one has $t_2^\star + K/\bar{u} < T$.

On the time interval $[0, T_c]$, Lemma 3 applies (since its proof does not use the transversality conditions). Moreover, Lemma 4 applies for Problem $\mathcal{P}_S$ over $[0, T_c]$ with $\beta = \beta_1$. Therefore, there exists an optimal solution on the interval $[0, T_c]$ of the form (2.11) with $K - K_c$ instead of K, where K_c is the optimal value of $C(T_c)$. □

Remark 2. *In the statement of Proposition 3, K_c can be interpreted as the remaining budget at time T_c. The times $t_1^\star$ or $t_2^\star$ are irrelevant when $K_c = K$ or $K_c = 0$. When $t_1^\star + (K - K_c)/\bar{u} = t_2^\star = T_c$, then the optimal solution consists simply in a single time interval of intervention $[t_1^\star, t_1^\star + K/\bar{u}]$ with $T_c \in (t_1^\star, t_1^\star + K/\bar{u})$.*

Proposition 3 reduces the problem of characterizing an optimal control to identify the three decision variables $(t_1^\star, t_2^\star, K_c)$ for which the control (5.2) is optimal. Furthermore, we can provide an additional characterization of the optimal intervention times $t_1^\star$ and $t_2^\star$. For this purpose, we define the following property of a control with at most two intervention intervals, which claims that when there are two distinct intervention intervals, a commutation has to occur exactly at time T_c.

Property A. *A control function $u^\star$ of the form* (5.2) *with $T > T_c$ (where T can be equal to $+\infty$) is said to fulfil Property A when the following is fulfilled :*

(i) If $K_c \in \{0, K\}$, then $u^\star$ reduces to (2.11) *and its support belongs to either* $[0, T_c]$ *or* $[T_c, T]$.

(ii) If $K_c \in (0, K)$, then $t_1^\star + (K - K_c)/\bar{u} = T_c$ or $t_2^\star = T_c$. Furthermore, one has $t_2^\star = T_c$ if $\beta_2 > \beta_1$, and $t_1^\star + (K - K_c)/\bar{u} = T_c$ if $\beta_2 < \beta_1$.

Now, we are in a position to give the following result.

Proposition 4. *For Problem $\mathcal{P}_S$ with a large enough T, an optimal control is of the form* (5.2) *for a.e. $t \in [0, T]$ and satisfies Property A.*

Proof. Let $u^\star$ be an optimal control. From Proposition 3, we know that it has the structure (5.2) for a.e. t. If $K_c \in \{0, K\}$, then item (i) of Property A follows immediately from Proposition 3. Now, consider the case for which one has $K_c \in (0, K)$. We proceed by contradiction to show that an optimal solution fulfills point (ii) of Property A for a.e. t. If not, then both conditions $t_2^\star > T_c$ and $t_1^\star + (K - K_c)/\bar{u} < T_c$ hold. This means that the switching function

$$\phi(t) = \begin{cases} (p_S(t) - p_I(t))\beta_1 S(t) I(t) - p_C, & t < T_c, \\ (p_S(t) - p_I(t))\beta_2 S(t) I(t) - p_C, & t \geq T_c, \end{cases}$$

has to be negative on some (nonempty) interval (t_1, t_2) that contains T_c with $\phi(t_1) = \phi(t_2) = 0$. Similarly to the proof of Lemma 4, and the fact that ϕ is C^1 over $[0, T_c)$ and $(T_c, T]$, one has $\dot{\phi}(t_1) \leq 0 \leq \dot{\phi}(t_2)$, which implies $p_S(t_1) \geq 0 \geq p_S(t_2)$, and as u is equal to 0 for a.e. t in (t_1, t_2), one obtains the following

$$\dot{p}_S(t_1^+) = \frac{\phi(t_1) + p_C}{S(t_1)} > 0, \quad \dot{p}_S(t_2^-) = \frac{\phi(t_2) + p_C}{S(t_2)} > 0.$$

Therefore, the function p_S, which is continuous, possesses a local maximum at a certain $\hat{t}$ in (t_1, t_2) with $p_S(\hat{t}) > 0$, and a local minimum at a certain $\check{t} > \hat{t}$ in (t_1, t_2) with $p_S(\check{t}) < 0$, such that p_S is decreasing on $(\hat{t}, \check{t})$. The adjoint variable p_S is absolutely continuous over (t_1, t_2), and $u(t)$ is equal to 0 for almost any $t \in (t_1, t_2)$. Therefore, p_s is left- and right-differentiable at $\hat{t}$ and $\check{t}$. Therefore, we have $\dot{p}_S(\hat{t}^-) \geq 0 \geq \dot{p}_S(\hat{t}^+)$ and $\dot{p}_S(\check{t}^-) \leq 0 \leq \dot{p}_S(\check{t}^+)$; from (5.4), one recovers $p_S(\hat{t}) - p_I(\hat{t}) = p_S(\check{t}) - p_I(\check{t}) = 0$. Finally, we obtain the following :

$$H(\check{t}) = -p_S(\check{t})\gamma I(\check{t}) > 0 > H(\hat{t}) = -p_S(\hat{t})\gamma I(\hat{t}).$$

Recall that the Hamiltonian along an optimal solution is piecewise constant, i.e., there exist numbers H_1, H_2 such that

$$H(t) = \begin{cases} H_1, & t < T_c, \\ H_2, & t > T_c. \end{cases}$$

If $\check{t} \leq T_c$ or $\hat{t} \geq T_c$, then one should have $H(\hat{t}) = H(\check{t})$, which is a contradiction. If $\hat{t} < T_c < \check{t}$, then one has

$$H_1 = H(T_c^-) = H(\hat{t}) < 0 < H(\check{t}) = H(T_c^+) = H_2;$$

however, one has $H(T) = -\beta_2 S(T)I(T) < 0$ from the transversality conditions (5.5) and Proposition 3, which ensures that u vanishes over $[t_2^\star + K/\bar{u}, T]$ and thus a contradiction with $H(T) = H_2 > 0$.

Therefore, we have shown that ϕ cannot be negative on a neighborhood of T_c. Hence, either ϕ remains positive on a neighborhood of T_c, or its sign changes at T_c. In the first case, the control has the form (2.11), so one has $t_1^\star + (K - K_c)/\bar{u} = t_2^\star = T_c$ and item (ii) holds. In the second case, we cannot have $p_S(T_c) - p_I(T_c) \leq 0$, because then ϕ would be negative on a neighborhood of T_c. Hence, we necessarily have $p_S(T_c) - p_I(T_c) > 0$. The function ϕ has thus a jump at T_c in the following way :

$$\phi(T_c^+) - \phi(T_c^-) = \underbrace{(p_S(T_c) - p_I(T_c))S(T_c)I(T_c)}_{>0}(\beta_2 - \beta_1);$$

consequently, we get that

1. if $\beta_2 > \beta_1$, then $\phi(T_c^-) < 0 < \phi(T_c^+)$, which implies that one has necessarily $t_2^\star = T_c$;
2. if $\beta_2 < \beta_1$, then $\phi(T_c^-) > 0 > \phi(T_c^-)$, which implies that one has necessarily $t_1^\star + (K - K_c)/\bar{u} = T_c$,

which proves item (ii). Both (i) and (ii) hold, hence the control $u^\star$ fulfills Property A. □

Remark 3. *Proposition 4 shows the optimality of a control that has one or two intervention intervals. In the first case, this interval may start at any time. In the second case, the first intervention must end exactly at time T_c, or the second intervention must start exactly at time T_c, or both (but this latter case may be viewed as a single intervention period). However, when there are two separated interventions, there must be a commutation at T_c.*

Now, based on our preceding analysis, we aim to characterize the optimal solutions for Problem $\mathcal{R}_S$ in an infinite horizon. Formally, we consider the limit of the optimal solutions of Problem $\mathcal{P}_S$ when $T \to +\infty$, as a candidate for the optimal solution of Problem $\mathcal{R}_S$. This is established by the following convergence result.

Proposition 5. *Consider a positive initial condition and an increasing sequence* $\{T_n\}_{n\in\mathbb{N}}$ *that tends to* $+\infty$ *when* $n \to +\infty$. *Let* $\{u_n^\star\}_{n\in\mathbb{N}}$ *be a family of optimal controls for Problem* $\mathcal{P}_S$ *with a finite horizon* T_n. *Then, up to a sub-sequence, the sequence* $\{u_n^\star\}_{n\in\mathbb{N}}$ *converges in* L^1 *when* $n \to +\infty$ *to some function* $u_\infty^\star \in \mathcal{U}$ *of the form* (5.2) *that satisfies Property A for a.e. t, and is an optimal solution of Problem* $\mathcal{R}_S$ *with variable* β *given in* (5.1).

Furthermore, the optimal value of Problem $\mathcal{R}_S$ *is the limit of the optimal value of Problem* $\mathcal{P}_S$ *in finite horizon* T_n *as* $n \to +\infty$.

Proof. Let us fix a positive initial condition (S_0, I_0, C_0) and consider the system (2.2). The proof is done in three steps.

Step 1. We show that there exists a number $\tilde{T} \geq T_c$, independent of n, such that for Problem $\mathcal{P}_S$ in horizon T_n, there exists an optimal control which is null for $t > \tilde{T}$. To do so, consider the system (2.2) with $\beta = \beta_1$. From the Theorem of compactness of the solutions of controlled dynamical systems (see e.g., [19, Th. 23.2]), the attainable set $\mathcal{A}(T_c)$ at time T_c, with controls $u(\cdot)$ in $\mathcal{U}|_{[0,T_c]}$ such that $||u(\cdot)||_1 \leq C_0$, is compact. From any state (S_c, I_c, C_c) in $\mathcal{A}(T_c)$, the optimal solution over the time interval $[T_c, T]$ (with $T > T_c$) is exactly given by the optimal solution of the auxiliary Problem $\mathcal{P}_S$ with constant $\beta = \beta_2$ over $[T_c, T]$. Consider the following exit time function :

$$t_h(S_c, I_c) := \inf\{t \geq T_c;\ S(t) \leq S_h\}, \quad (S_c, I_c) \in O := (\mathbb{R}_+ \setminus \{0\})^2,$$

where $S_h = \gamma/\beta_2$, and $(S(\cdot), I(\cdot))$ denotes the solution of (1.1) with $(S(T_c), I(T_c)) = (S_c, I_c), \beta = \beta_2$, and the null control. From the analysis of Section 2, we know that an optimal solution for Problem $\mathcal{P}_S$ with a large enough time horizon consists of taking $u(t) = 0$ for a t larger than $t_h(S_c, I_c) + C_c/\bar{u}$ (for some optimal state (S_c, I_c, C_c) at intermediate time T_c). From classical results on the Bellman equation for minimal time with a controllable target (see e.g., [20, Chap. 4, Sec. 1]), the function t_h is continuous on O, and one can then define the following number :

$$\tilde{T} := \max_{(S_c, I_c, C_c)\in\mathcal{A}(T_c)} t_h(S_c, I_c) + C_c/\bar{u} < +\infty.$$

Therefore, for $T_n > \tilde{T}$, an optimal solution $u(\cdot)$ for Problem $\mathcal{P}_S$ is null on $(\tilde{T}, T_n]$, and its restriction to $[0, \tilde{T}]$ realizes the maximum of the map

$$\Psi_n : \tilde{u}(\cdot) \in \tilde{\mathcal{U}} \mapsto \Phi_n(S(\tilde{T}), I(\tilde{T})),$$

where $\tilde{\mathcal{U}}$ is the set of controls in $L^1([0, \tilde{T}], U)$ such that $||u||_1 \leq C_0$, and $\Phi_n(\tilde{S}, \tilde{I})$ is defined as the solution $S(T_n)$ of (1.1) with $(S(\tilde{T}), I(\tilde{T})) = (\tilde{S}, \tilde{I}), \beta = \beta_2$ and null control.

Step 2. We establish a Γ-convergence result for the family of functions $\{\Psi_n\}_{n\in\mathbb{N}}$. For any $n \in \mathbb{N}$, Ψ_n is continuous on $\tilde{\mathcal{U}}$ because, on the one hand, from the continuous dependency of the solutions with respect to the control, the map $\tilde{u}(\cdot) \in \tilde{\mathcal{U}} \mapsto (S(\tilde{T}), I(\tilde{T}))$ is continuous (see e.g. [17, Th. 3.2.1]); on the other hand, the function Φ_n is continuous with respect to the initial condition $(\tilde{S}, \tilde{I})$. For any increasing sequence of (large enough) numbers T_n that tends to $+\infty$, $\{-\Psi_n\}_{n\in\mathbb{N}}$ is an increasing family of continuous functions on $\tilde{\mathcal{U}}$. Therefore, it Γ-converges to its pointwise limit $-\Psi_\infty$ (see [18, Prop 5.4]). From this, we know that any converging sub-sequence of minimizers of $\{-\Psi_n\}_{n\in\mathbb{N}}$ converges in L^1 to a minimizer of $-\Psi_\infty$.

Step 3. We show that a converging sub-sequence exists and has the form (5.2) that satisfies Property A. For simplicity, let us assume $\beta_1 < \beta_2$ (the converse case is exactly the same, up to the definition of $\tilde{\mathcal{U}}_1$ and $\tilde{\mathcal{U}}_2$ below). Denote $\delta = K/\bar{u}$ and define the following subsets :

$$\tilde{\mathcal{U}}_1 := \Big\{\tilde{u} \in \tilde{\mathcal{U}};\ \tilde{u}(\cdot) = \bar{u}\mathbb{1}_{[t_c, t_c+\delta]}(\cdot),\ t_c \in [T_c, \tilde{T} - \delta]\Big\},$$
$$\tilde{\mathcal{U}}_2 := \Big\{\tilde{u} \in \tilde{\mathcal{U}};\ \tilde{u}(\cdot) = \bar{u}\mathbb{1}_{[t_1, t_1+d]}(\cdot) + \bar{u}\mathbb{1}_{[T_c, T_c+\delta-d]}(\cdot),$$
$$(t_1, d) \in [0, T_c] \times [0, \delta],\ t_1 + d \le T_c\Big\};$$

which are relatively compact in $\tilde{\mathcal{U}}$. From Propositions 3 and 4, we know that for any n large enough, there exists $\tilde{u}_n^\star$ in $\tilde{\mathcal{U}}_1 \cup \tilde{\mathcal{U}}_2$ which maximizes Ψ_n over $\tilde{\mathcal{U}}$, and whose extension $u_n^\star$ on $\mathcal{U}$ by 0 for $t > \tilde{T}$ is optimal for the problem with horizon T_n. As $\tilde{\mathcal{U}}_1 \cup \tilde{\mathcal{U}}_2$ is relatively compact, we deduce that there exists a sub-sequence of $\{\tilde{u}_n^\star\}_{n\in\mathbb{N}}$ that converges to some $\tilde{u}_\infty^\star \in \tilde{\mathcal{U}}_1 \cup \tilde{\mathcal{U}}_2$. Moreover, the Γ-convergence ensures that one has $\Psi_\infty(\tilde{u}_\infty^\star) = \lim_{n\to+\infty} \Psi_n(\tilde{u}_n^\star)$. Recall that for each control u in $\mathcal{U}$, the corresponding solution $S(T_n)$ is non-increasing with respect to n. Therefore, the optimal value $\Psi_n(\tilde{u}_n^\star)$ of Problem $\mathcal{P}_S$ with time horizon T_n is an upper bound of Problem $\mathcal{R}_S$ in infinite horizon. As Ψ_∞ is the map that gives $S(+\infty)$ for controls in $\tilde{\mathcal{U}}$, we conclude that the extension $u_\infty^\star$ of $\tilde{u}_\infty^\star$ on $\mathcal{U}$ by 0 for $t > \tilde{T}$ is necessarily optimal for problem $\mathcal{R}_S$ (since it is admissible for Problem $\mathcal{R}_S$). Moreover, it possesses the form (5.2) and satisfies Property A. Finally, the optimal value $\Psi_n(\tilde{u}_n^\star)$ converges to $\Psi_\infty(\tilde{u}_\infty^\star)$, which is the optimal value for Problem $\mathcal{R}_S$. □

Remark 4. *Note that we have only shown that any optimal control for Problem $\mathcal{P}_S$ in horizon $T < \infty$ is of the form* (5.2) *and satisfies Property A (Proposition 4). Proposition 5 shows that there exists a control of this form that is optimal for Problem $\mathcal{P}_S$ in horizon $T = +\infty$, and that this control is also an optimal control for Problem $\mathcal{R}_S$. For all we know, there could be other optimal controls for Problems $\mathcal{P}_S$ (with $T = +\infty$) and $\mathcal{R}_S$ which are not of this form.*

Remark 5. *We have not directly tackled Problem $\mathcal{R}_S$ in infinite horizon, because of the lack of transversality conditions. Instead, we have considered the finite horizon Problem $\mathcal{P}_S$, for which we have exploited the transversality conditions and characterized the structure of optimal solutions, which allows us to pass at the limit for Problem $\mathcal{R}_S$ thanks to Proposition 5. This legitimizes us to approach Problem $\mathcal{R}_S$ by Problem $\mathcal{P}_S$ with a large enough T, giving credit to the numerical simulations.*

Remark 6. *Consider the framework of Proposition 5. Let us make some final observations.*

- *If there is a single intervention, this occurs in one of the following situations:*
  - *(i)* $K_c = 0$, *meaning that the entire budget is used before T_c;*
  - *(ii)* $K_c = K$, *meaning that the entire budget is used after T_c; or*
  - *(iii)* $K_c \in (0, K)$ *and the equality* $t_1^\star + (K - K_c)/\bar{u} = t_2^\star = T_c$ *holds, meaning that the single intervention takes place over a time interval that contains T_c.*

  In all these cases, the optimal control of Problem $\mathcal{R}_S$ keeps the simple form (2.11). *From the inclusion* (4.1) *(which remains valid in the case of a piecewise constant β), one can see that this optimal control of the form* (2.11) *remains admissible for Problem Q_S. Thus, Problems $\mathcal{R}_S$ and Q_S share the same solution.*

- *If $K_c \in (0, K)$ and the two interventions are disjoint (i.e., $t_1^\star + (K - K_c)/\bar{u} \neq t_2^\star$), then the optimal control has the form* (5.2). *In this case, one can see that this control is not admissible for Problem Q_S (when considering the length $\delta = K/\bar{u}$). Therefore, Problems $\mathcal{R}_S$ and Q_S do not share the same solution. More precisely, the value of the cost function in Problem $\mathcal{R}_S$ is strictly larger than that in Problem Q_S.*

In the following Section, we illustrate all possible structures of the optimal control of Problem $\mathcal{P}_S$ that may occur.

6. Numerical illustrations

In this section, we provide a couple of examples* that illustrate the structure of the optimal control derived in both cases: constant and piecewise constant transmission rate β, as stated in Propositions 1, 2, 3, 4. We solve both optimal control problems using the `Julia` package `OptimalControl.jl` (we refer to [21] for more details). The numerical resolution involves reformulating the optimal control problem as a finite-dimensional optimization problem (also referred to as the direct method), which is then solved using the `IPOPT` solver with a relative precision tolerance set to 10^{-8}. We emphasize that we fix a finite time horizon $T > 0$ (large enough) and numerically solve Problem $\mathcal{P}_S$ (not Problem $\mathcal{R}_S$).

In what follows, we implicitly assume that the time unit is the day, and thus do not mention the units of the budget and various dates (Accordingly, the quantities T, T_c, K below are expressed in days, while β, γ are measured in days^{-1}).

6.1. Case of constant β

In this subsection, we numerically solve the optimal control problem in a finite horizon (Problem $\mathcal{P}_S$) for the dynamics (2.2) under the constraint $C(T) \geq 0$ and $u(t) \in [0, \bar{u}]$, with $\bar{u} < 1$. The transmission rate β is constant in time. We present two examples to illustrate the optimal strategy described in Propositions 1 and 2. In both cases, we use the same set of parameters:

T	γ	$S(0)$	$I(0)$
300	0.2	0.999	0.001

In a first example, we consider a transmission rate $\beta = 0.6$, a control bound $\bar{u} = 1$, and a budget $K = 28$. This latter means that a complete stop of the transmission is possible (as depicted in Figure 1). One can observe that, in this particular case, the intervention time $t^\star \approx 17.81$ satisfies $S(t^\star) = S_h$ and the state variable S remains constant throughout the intervention (as highlighted in Proposition 2).

In a second example, we consider a larger transmission rate $\beta = 0.9$, a weaker control bound $\bar{u} = 0.5$, and a smaller budget $K = 10$. Having $\bar{u} < 1$ means that (only) a partial stop of the transmission is possible (as depicted in Figure 2a). Moreover, the intervention time $t^\star \approx 8.50$ realizes the maximum of the mapping $t_c \mapsto \log S(t_c) - \log S(t_c + \delta)$ as predicted in Proposition 2 (Figure 2b).

*The scripts for reproducing the numerical experiments are available at: `https://anasxbouali.github.io/SIRcontrol.jl/`.

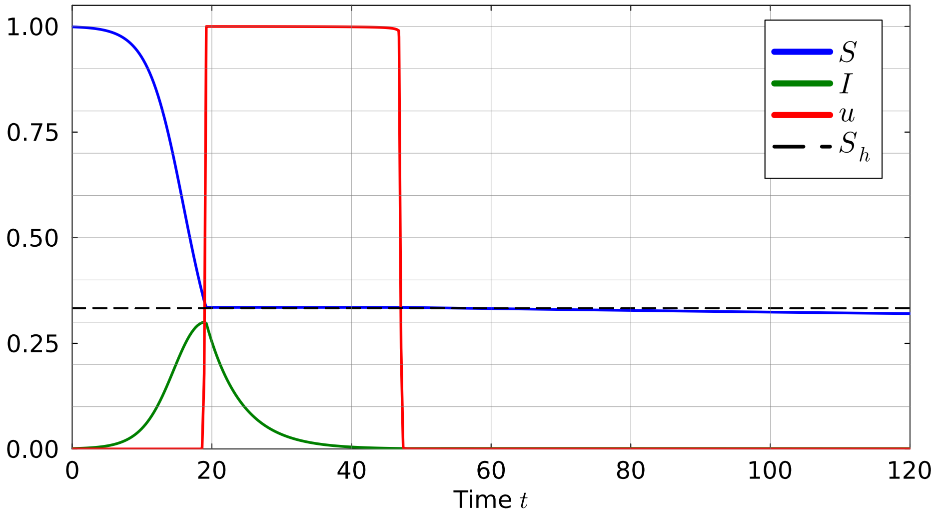


Figure 1. Numerical results for $\beta = 0.6$, $\bar{u} = 1$, $K = 28$.

6.2. *Case of piecewise constant* β

In this subsection, we solve again the optimal control problem in finite horizon (Problem $\mathcal{P}_S$) for the dynamics (2.2) under the constraint $C(T) \geq 0$ and $u(t) \in [0, \bar{u}]$, with $\bar{u} < 1$, but here with a parameter β that is time varying of the form (5.1). We present four numerical examples to illustrate the optimal strategy described in Propositions 3 and 4. We emphasize that the same set of parameters is used for all examples, namely:

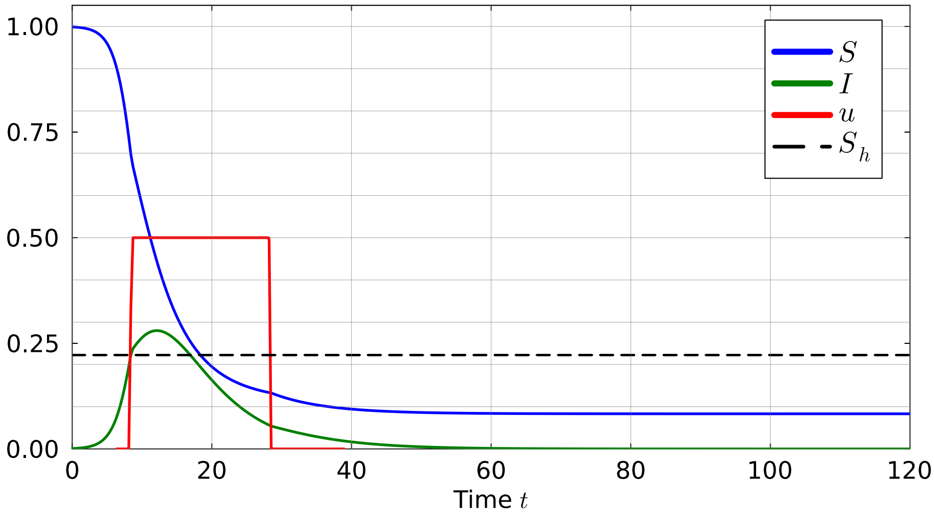
T	T_c	γ	K	$\bar{u}$	$S(0)$	$I(0)$
700	50	0.15	35	0.8	0.999	0.001

To use the numerical toolbox, it is necessary to provide a smooth dynamics. Since β is discontinuous at T_c, we will use an *augmentation procedure* (see [22, 23]), which consists in parameterizing the state and control variables on the two intervals $[0, T_c]$ and $[T_c, T]$ on a common time interval $[0, 1]$, while ensuring the continuity of the state variables. The new parameterization is as follows :

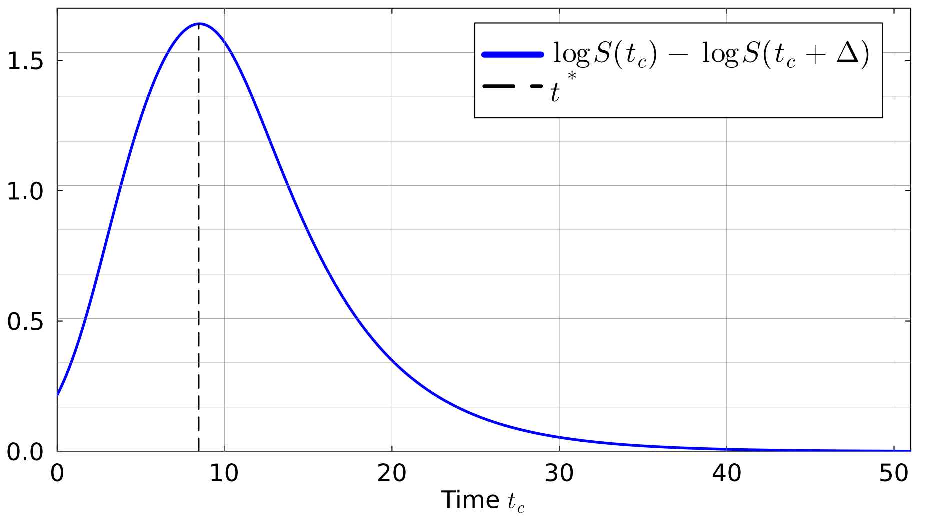
$$\begin{cases} (S_1(s), I_1(s), C_1(s)) = (S, I, C)(sT_c), \\ (S_2(s), I_2(s), C_2(s)) = (S, I, C)(s(T - T_c) + T_c), \\ u_1(s) = u(sT_c), \quad u_2(s) = u(s(T - T_c) + T_c), \end{cases} \qquad s \in [0, 1].$$

Hence, the (augmented) control system is given by

$$\begin{cases} \dot{S}_1 = -T_c(1 - u_1)\beta_1 S_1 I_1, \\ \dot{I}_1 = T_c((1 - u_1)\beta_1 S_1 I_1 - \gamma I_1), \\ \dot{C}_1 = -T_c u_1, \\ \dot{S}_2 = -(T - T_c)(1 - u_2)\beta_2 S_2 I_2, \\ \dot{I}_2 = (T - T_c)((1 - u_2)\beta_2 S_2 I_2 - \gamma I_2), \\ \dot{C}_2 = -(T - T_c)u_2, \end{cases} \qquad s \in [0, 1]. \tag{6.1}$$



(a) States variables (S, I) and optimal control $u^{\star}$.



(b) Optimal intervention time $t^{\star}$.

Figure 2. Numerical Results for $\beta = 0.9$, $\bar{u} = 0.5$, $K = 10$.

Furthermore, the initial conditions write as follows

$$S_1(0) = S_0, \quad I_1(0) = I_0, \quad C_1(0) = K,$$

and to ensure continuity of the state variables, we impose the two-boundaries constraints:

$$S_2(0) = S_1(1), \quad I_2(0) = I_1(1), \quad C_2(0) = C_1(1),$$

the terminal condition being

$$C_1(1) \geq 0, \quad C_2(1) \geq 0.$$

Thus, this procedure allows to reduce Problem $\mathcal{P}_S$ (with piecewise constant β) to an equivalent optimal control problem with smooth dynamics (given above) that writes

$$\sup_{(u_1(\cdot),u_2(\cdot))\in\mathcal{U}} S_2(1)$$

with $u_1(s), u_2(s) \in [0, \bar{u}]$ for a.e. $s \in [0, 1]$. Finally, to recover the original state variables (S, I, C) and the control function u on $[0, T]$, we use the inverse change of variables:

$$(S(t), I(t), C(t)) = \begin{cases} (S_1, I_1, C_1)(\frac{t}{T_c}), & t \in [0, T_c], \\ (S_2, I_2, C_2)(\frac{t-T_c}{T-T_c}), & t \in [T_c, T], \end{cases}$$

$$u(t) = \begin{cases} u_1(\frac{t}{T_c}), & t \in [0, T_c], \\ u_2(\frac{t-T_c}{T-T_c}), & t \in [T_c, T]. \end{cases}$$

Our goal is to showcase various situations that may arise by changing the values of β_1 and β_2. We begin with the first two examples, which illustrate cases where $\beta_1 > \beta_2$ (with one and two intervention intervals in the optimal solution). In a first example, we consider $\beta(t) = 0.4$ for $t \in [0, T_c)$ and $\beta(t) = 0.2$ for $t \in [T_c, T]$ (as depicted in Figure 3). This example highlights that, even when the transmission rate decreases, it may be optimal to split the intervention into two phases: one within the interval $[0, T_c]$ and the other over $[T_c, T]$. More precisely, the first phase takes place over the interval $[t_1^\star, t_1^\star + \frac{K-K_c}{\bar{u}}]$ with $t_1^\star \approx 19.7$ and $K_c \approx 10.76$. Notice that $t_1^\star + \frac{K-K_c}{\bar{u}} = T_c$ thanks to Proposition 4. The second phase occurs over $[t_2^\star, t_2^\star + \frac{K_c}{\bar{u}}]$, where $t_2^\star \approx 55.6 > T_c$ (which is consistent with Proposition 3).

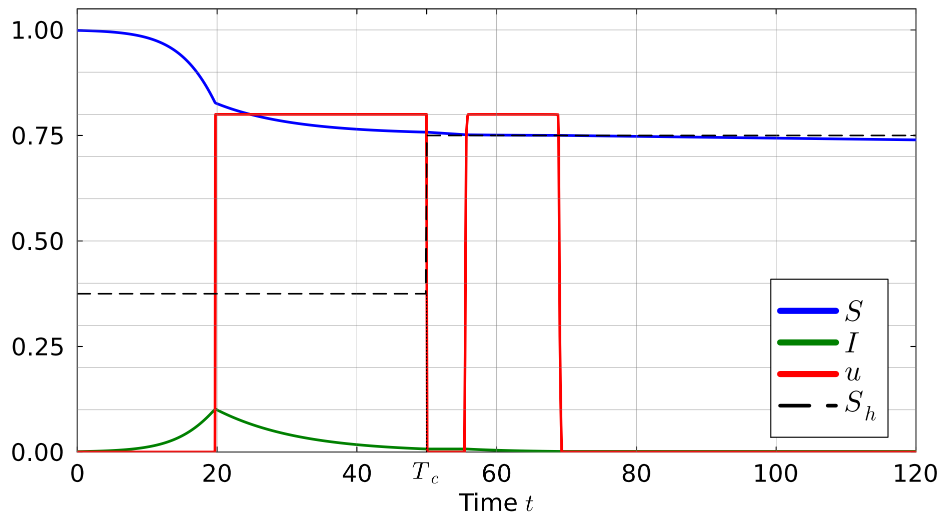


Figure 3. Numerical results for for β that changes from 0.4 to 0.2.

In a second example, we consider $\beta(t) = 0.7$ for $t \in [0, T_c)$ and $\beta(t) = 0.4$ for $t \in [T_c, T]$ (as depicted in Figure 4). This example illustrates that, when the transmission rate decreases, it may be optimal to have a single intervention interval that contains T_c (which can be seen as two interventions over $[0, T_c]$ and $[T_c, T]$; this is still consistent with Propositions 3 and 4). More precisely, the intervention takes

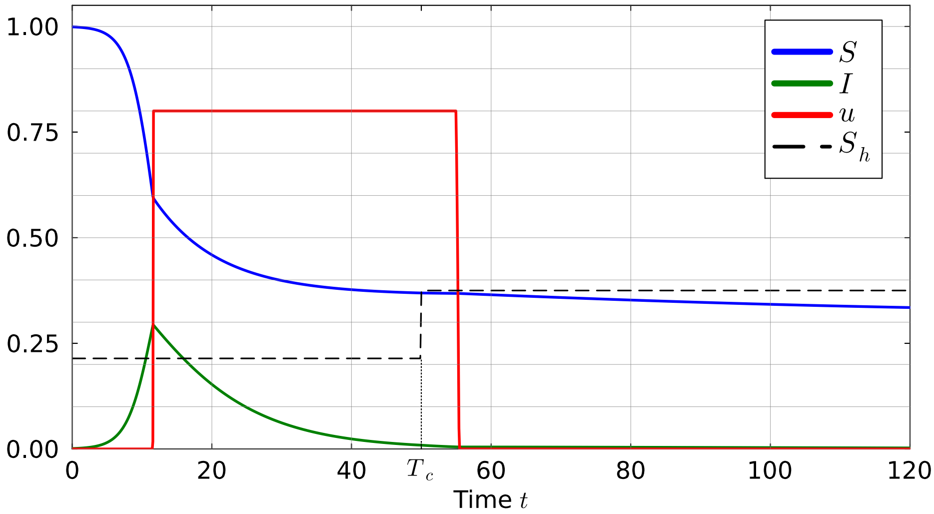


Figure 4. Numerical results for β that changes from 0.7 to 0.4.

place over the interval $[t_1^\star, t_1^\star + \frac{K-K_c}{\bar{u}}]$, with $t_1^\star \approx 11.6$, $K_c \approx 4.28$, and $t_1^\star + \frac{K-K_c}{\bar{u}} = T_c$. The second phase occurs over $[t_2^\star, t_2^\star + \frac{K_c}{\bar{u}}]$, where $t_2^\star = T_c$ (which is consistent with Proposition 3 and 4).

In contrast to the previous examples, the remaining examples vary T_c, illustrating cases where $\beta_1 < \beta_2$ and the optimal solution includes either one or two intervention intervals. In the third example, we consider $\beta(t) = 0.4$ for $t \in [0, T_c)$ and $\beta(t) = 0.8$ for $t \in [T_c, T]$ (as depicted in Figure 5). This example highlights that, when the transmission rate increases, it may be optimal to split the intervention into two phases: one within the interval $[0, T_c]$ and the other over $[T_c, T]$. More precisely, the first phase takes place over the interval $[t_1^\star, t_1^\star + \frac{K-K_c}{\bar{u}}]$ with $t_1^\star \approx 32$ and $K_c \approx 29.08$. The second phase occurs over $[t_2^\star, t_2^\star + \frac{K_c}{\bar{u}}]$, where $t_2^\star = T_c$ (which is consistent with Proposition 3 and 4).

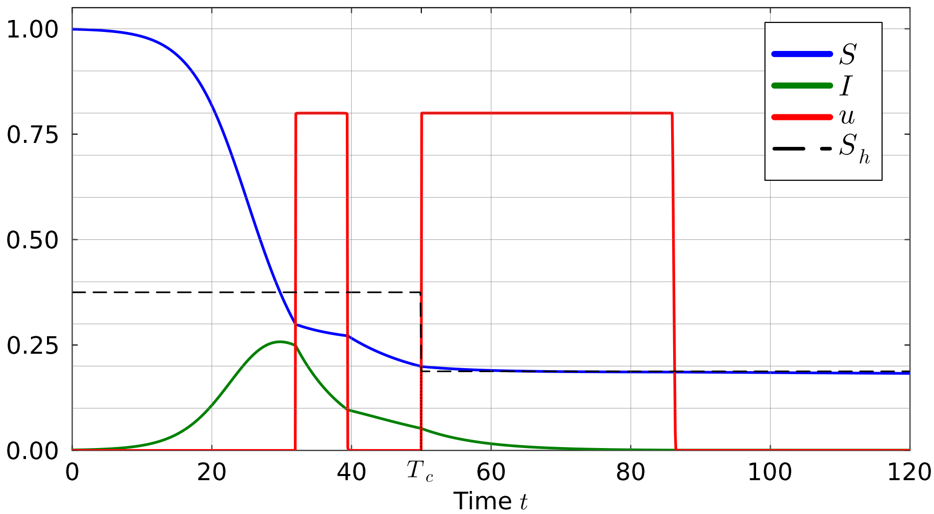


Figure 5. Numerical results for β that changes from 0.4 to 0.8.

In the fourth example, we consider $\beta(t) = 0.2$ for $t \in [0, T_c)$ and $\beta(t) = 0.8$ for $t \in [T_c, T]$ (as depicted in Figure 6). This example highlights that, when the transmission rate increases, it may be optimal to have a single intervention over $[T_c, T]$. More precisely, this intervention takes place over the interval $[t_2^\star, t_2^\star + \frac{K}{\bar{u}}]$ with $t_2^\star \approx 58.1 > T_c$ (which is consistent with Proposition 3 and 4 with $K_c = K$).

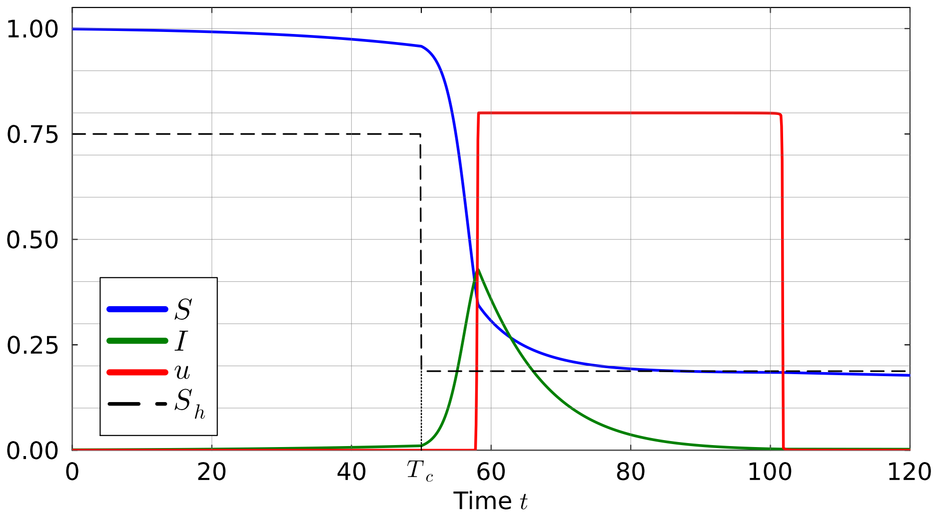


Figure 6. Numerical results for for β that changes from 0.2 to 0.8.

7. Conclusion

In this work, we studied the problem of minimizing the final size of an epidemic obeying SIR model by a lockdown policy with a constraint on the lockdown duration (Problem $\mathcal{Q}_S$). We first considered the case where the transmission rate is constant in time. In order to understand whether there is a benefit in splitting the intervention, we considered a more general problem by introducing a bound on the measure of the support of u (Problem $\mathcal{R}_S$). The main difficulty, namely the infinite horizon, has been circumvented by considering a finite horizon problem (Problem $\mathcal{P}_S$) where we introduced an L^1 budget constraint, thereby finding the structure of the optimal controls with the Maximum Principle, and applying a limiting procedure to obtain optimal solutions to Problem $\mathcal{P}_S$ with an infinite horizon. We found that the optimal control is unique and consists in an 'all or nothing' strategy on a single time interval. These three problems are in fact equivalent, since the optimal solution of the most general (Problem $\mathcal{P}_S$) is admissible for the least one (Problem $\mathcal{Q}_S$).

In an attempt to generalize this result, we considered a second case where the transmission rate jumps at a known time from a known value to another. We found that optimal strategies are still 'all or nothing', but may be spread over two distinct time intervals. Then, roblems $\mathcal{Q}_S$ and $\mathcal{R}_S$ are no longer equivalent in this context.

Future works could investigate more complex situations when β is piecewise constant with more than one change, or when β is periodic.

- If β has several jumps w.r.t a time partition $0 < T_c^1 < \cdots < T_c^N$, the Hybrid Maximum Principle still applies on each interval where β is constant. In that case, one may expect at most one intervention per interval, leading to at most $N + 1$ interventions in total. The main challenge is that the argument used in the single-jump case (Property A) relies on comparing two constant values of the Hamiltonian over $[0, T_c)$ and $[T_c, T]$, together with the transversality conditions.

With multiple jumps, the Hamiltonian takes on several constant values on more than two intervals, which cannot be easily tackled in the same way.

- For a periodic β, the times of jumps are not bounded, while the approach we deployed in this paper is based on an approximation with a finite horizon, which took into consideration that β does nochange after a (large) finite time. Therefore, this technique needs to be revisited for the periodic case.
- However, one can try to find a bound on the number of interventions independent of N, possibly depending on the data K, γ, $\|\beta\|_\infty$,... . If such a bound is found, one can look for controls in the compact set of bang-bang controls that satisfy $\|u\|_{L^1} \leq K$, which have a number of interventions fewer than the said bound. This would allow us to consider problems in which $N \to \infty$, thus allowing to consider the periodic β case, or to pass to the limit and consider time-continuous β by a density argument.

Acknowledgments

This work has received the support of the French National Research Agency (ANR) under the NOCIME project (ANR-23-CE48-0004-03). P.-A.B and A.R. thank Professors T. Britton and L. Leskelä for fruitful discussions.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. P.-A. Bliman, M. Duprez, Y. Privat, N. Vauchelet, Optimal immunity control and final size minimization by social distancing for the SIR epidemic model, *J. Optimiz. Theory App*, **189** (2021), 408–436. https://doi.org/10.1007/s10957-021-01830-1
2. D. I. Ketcheson, Optimal control of an SIR epidemic through finite-time non-pharmaceutical intervention, *J. Math. Biol.*, **83** (2021), 1–21. https://doi.org/10.1007/s00285-021-01628-9
3. P.-A. Bliman, M. Duprez, How best can finite-time social distancing reduce epidemic final size?, *J. Theor. Biol.*, **511** (2020), 110557. https://doi.org/10.1016/j.jtbi.2020.110557
4. R. Balderrama, J. Peressutti, J. P. Pinasco, F. Vazquez, C. S. de la Vega, Optimal control for a SIR epidemic model with limited quarantine, *Sci. Rep.*, **12** (2022), 12583. https://doi.org/10.1038/s41598-022-16619-z
5. T. Britton, L. Leskelä, Optimal intervention strategies for minimizing total incidence during an epidemic, *SIAM J. App. Math.*, **83** (2023), 354–373. https://doi.org/10.1137/22M1504433

6. P.-A. Bliman, A. Rapaport, On the problem of minimizing the epidemic final size for SIR model by social distancing, in: *22nd IFAC World Congress*, *IFAC-PapersOnLine*, **56** (2023), 4049–4054. https://doi.org/10.1016/j.ifacol.2023.10.1726
7. F. Di Lauro, I. Z. Kiss, J. C. Miller, Optimal timing of one-shot interventions for epidemic control, *PLoS Comput. Biol.*, **17** (2021), e1008763. https://doi.org/10.1371/journal.pcbi.1008763
8. A. H. González, A. L. Anderson, A. Ferramosca, E. A. Hernandez-Vargas, Dynamic characterization of control SIR-type systems and optimal single-interval control, *arXiv preprint arXiv:2103.11179*. https://doi.org/10.48550/arXiv.2103.11179
9. J. Sereno, A. Anderson, A. Ferramosca, E. A. Hernandez-Vargas, A. H. González, Minimizing the epidemic final size while containing the infected peak prevalence in SIR systems, *Automatica*, **144** (2022), 110496, https://doi.org/10.1016/j.automatica.2022.110496
10. J. M. Greene, E. D. Sontag, Minimizing the infected peak utilizing a single lockdown: A technical result regarding equal peaks, in *2022 American Control Conference (ACC)*. IEEE, 2022, 3640–3647. https://doi.org/10.23919/ACC53348.2022.9867521
11. D. H. Morris, F. W. Rossine, J. B. Plotkin, S. A. Levin, Optimal, near-optimal, and robust epidemic control, *Commun. Phys.*, **4** (2021), 1–8. https://doi.org/10.1038/s42005-021-00570-y
12. E. D. Sontag, An explicit formula for minimizing the infected peak in an SIR epidemic model when using a fixed number of complete lockdowns, *Int. J. RobustNonlinear Control*, (2021), 1–24. https://doi.org/10.1002/rnc.5701
13. E. Molina, A. Rapaport, An optimal feedback control that minimizes the epidemic peak in the SIR model under a budget constraint, *Automatica*, **146** (2022), 110596. https://doi.org/10.1016/j.automatica.2022.110596
14. F. Avram, L. Freddi, D. Goreac, Optimal control of a SIR epidemic with ICU constraints and target objectives, *Appl. Math. Comput.*, **418** (2022), 126816. https://doi.org/10.1016/j.amc.2021.126816
15. L. Miclo, D. Spiro, J. Weibull, Optimal epidemic suppression under an ICU constraint: An analytical solution, *J. Math. Econom.*,(2022), 102669. https://doi.org/10.1016/j.jmateco.2022.102669
16. M. T. Angulo, F. Castaños, R. Moreno-Morton, J. X. Velasco-Hernández, J. A. Moreno, A simple criterion to design optimal non-pharmaceutical interventions for mitigating epidemic outbreaks, *J. R. Soc. Interface*, **18** (2021), 20200803. https://doi.org/10.1098/rsif.2020.0803
17. A. Bressan, B. Piccoli, *Introduction to the mathematical theory of control*, vol. 2 of AIMS Series on Applied Mathematics, American Institute of Mathematical Sciences (AIMS), Springfield, MO, 2007. Available from: `https://www.aimsciences.org/book/AM/volume/27`
18. G. Dal Maso, *An Introduction to* Γ*-Convergence*, no. 8 in Progress in Nonlinear Differential Equations and Their Applications, Springer Science, New York, 1993. https://doi.org/10.1007/978-1-4612-0327-8
19. F. Clarke, *Functional analysis, calculus of variations and optimal control*, First edition edition, Graduate texts in mathematics, Springer, London, 2013. https://doi.org/10.1007/978-1-4471-4820-3
20. M. Bardi, I. Capuzzo-Dolcetta, *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*, Birkhäuser Boston Inc., 1997, http://dx.doi.org/10.1007/978-0-8176-4755-1

21. J.-B. Caillau, O. Cots, J. Gergaud, P. Martinon, T. Cabel, E. Demairy, et al., OptimalControl. jl: A Julia package to model and solve optimal control problems with ODE's, 2024. Available from: `https://github.com/control-toolbox/OptimalControl.jl`

22. T. Bayen, A. Bouali, L. Bourdin, The hybrid maximum principle for optimal control problems with spatially heterogeneous dynamics is a consequence of a Pontryagin Maximum Principle for $\mathrm{L}^1_\square$-local solutions, *SIAM J. Control Optimiz.*, **62** (2024), 2412–2432. https://doi.org/10.1137/23M155311X

23. T. Bayen, A. Bouali, L. Bourdin, O. Cots, Loss control regions in optimal control problems, *J. Differ. Equat.*, **405** (2024), 359–397. https://doi.org/10.1016/j.jde.2024.06.016