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Research article

On models of shared resource competition, coexistence and traveling waves

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Abstract: We investigate a two-species competition model in which both populations exploit a common standing resource. The dynamics are governed by a system of nonlinear differential equations admitting three equilibrium classes: extinction, competitive exclusion, and coexistence. Analytical conditions on the biological parameters ensuring the existence and asymptotic stability of these equilibria are derived, with particular emphasis on the coexistence equilibrium, representing the stable persistence of both species under shared-resource competition. In the corresponding reaction-diffusion model posed on an unbounded spatial domain, we further examine the stability of the coexistence equilibrium via traveling wavefronts. Using the upper–lower solution method, we establish the existence of traveling wave solutions connecting the extinction or single-dominance states to the coexistence state for a continuum of wave speeds exceeding a biologically determined minimal value, which depends explicitly on equilibrium magnitudes and other key parameters. Numerical simulations are provided to corroborate the theoretical results and to illustrate dynamic transitions from dominance to stable coexistence.

Keywords: reaction-diffusion systems; competition with shared resource; asymptotic stability; traveling wave flowing to coexistence state

1. Introduction

While competing for shared resources such as food and space, multi-species population distributions, shaped by habitat selection and migration, often involve random dispersal represented as diffusive movement. In recent decades, considerable attention has been devoted to reaction–diffusion models of multi-species competition, typically based on Lotka-Volterra interactions [1, 2] and extended models including mating interference [3], as well as Allee effects [4]. Within these frameworks, interspecies competition has been analyzed with respect to conditions for coexistence versus competitive exclusion, the asymptotic stability of equilibria, and the emergence of traveling wavefronts. The following is a representative sample of the literature selected from a broader collection of works relevant to the study of reaction–diffusion competition systems, consumer-resource dynamics, and interference-based population models: [5–9].

Specifically, in a class of differential equation models describing multi-species competition for shared resources, many researchers have examined how interspecific and intraspecific interactions, mediated through explicit resource dynamics and spatial movements, can shape coexistence, competitive exclusion, and spatial pattern formation in the ecological system [1, 2, 10, 11]. These models often incorporate nonlinear functional responses such as the Beddington-DeAngelis or Holling type II forms [12–14] to represent consumer interference and saturation effects, thereby capturing more realistic ecological mechanisms underlying population growth and resource utilization. In this paper, we analyze a model of two diffusive populations competing for a common standing resource under these pressures and impacts. Our study examines how parameters such as mortality rate, metabolic efficiency, and intrinsic harvest time influence possible outcomes, including extinction, competitive exclusion, and coexistence. Our approach involves deriving steady-state solutions, performing stability analysis, and establishing parameter conditions that permit coexistence. Crucially, to demonstrate the global stability of the coexistence state, we establish the existence of traveling wavefronts connecting trivial or semi-trivial equilibria to the coexistence equilibrium.

There are many different factors that impact the growth or decline of a population. The variables implemented in our model are defined as follows:

- $u_1(t)$ and $u_2(t)$, or $u_1(x,t)$ and $u_2(x,t)$: total size or local density of two populations with respect to time;
- R: constant amount of available resources;
- r_i: metabolic efficiency rate for population i, $i = 1, 2$;
- μ_i: uniform mortality rate for population i, $i = 1, 2$;
- a_i and b_i: intraspecific and interspecific competition pressure on members of population i, $i = 1, 2$;
- h_i: the intrinsic (natural) per area harvest time for population i, $i = 1, 2$.

Building on the ecological context described above, we formulate a model for two populations competing for a common standing resource. Each population converts accumulated resources into per capita growth at a metabolic efficiency r_i, and sustains a uniform mortality rate μ_i. The mortality rate accounts for natural causes such as aging and disease, while metabolic efficiency reflects the ability to transform consumed resources into usable energy, defined as the energy intake per unit body weight required to maintain current mass. We further incorporate h_i, representing the intrinsic time required for individuals to hunt or gather available resources, and the influences of the biological parameters a_i and b_i representing conspecific and heterospecific competition pressures. This model is subsequently extended to describe two diffusive population densities distributed over an unbounded spatial habitat, competing for a shared common resource.

Motivated by previous studies [12–14] on population movements under density-dependent interference while competing for a common standing resource R , we consider the dynamics of two regionally co-occurring populations with population density functions $u_i(x,t)$ $(i = 1, 2)$. The temporal and spatial evolution of each population is governed by a fitness function $S_i(R, u_1, u_2)$ $(i = 1, 2)$, which depends on both the local resource level and the densities of the competing species. This function defines an adaptive landscape of resource availability that responds dynamically to changes in current population states. Following the Beddington–DeAngelis formulation of consumer interference [15, 16], we assume that the time required for an individual to collect available resources at a location increases linearly with the abundance of both conspecific and heterospecific competitors. Consequently, S_i represents the per capita rate at which individuals of population i acquire resources in the presence of competitive interactions. This rate, which serves as a measure of local fitness, is defined as the ratio between the available resource and the time spent in competition and harvesting. Formally, we express the resource accrual rate as

$$S_1(u_1,u_2) = \frac{R}{a_1u_1 + b_1u_2 + h_1},\ S_2(u_1,u_2) = \frac{R}{a_2u_2 + b_2u_1 + h_2} \tag{1.1}$$

The presence of populations with competition pressures measured by a_i and b_i ($i = 1, 2$) capture the strength of direct interference competition and the essential behavioral mechanism that can promote or hinder resource-mediated coexistence.

With the fitness functions given in (1.1), we first consider the dynamics of total population functions $u_1(t)$ and $u_2(t)$ governed by the following system of first-order ordinary differential equations:

$$\frac{du_1}{dt} = u_1\left(r_1S_1(u_1,u_2) - \mu_1\right),\ \ \frac{du_2}{dt} = u_2\left(r_2S_2(u_1,u_2) - \mu_2\right) \tag{1.2}$$

The above system admits four classes of nonnegative constant steady states: the extinction state (0, 0), the u_2-dominance state $(0,\ \bar{u}_2)$, the u_1-dominance state $(\bar{u}_1,\ 0)$, and the coexistence state $(u_1^*,\ u_2^*)$. In Section 2, we analyze the existence conditions and asymptotic stability of each equilibrium, with particular emphasis on the coexistence state. These results reveal how various biological parameters govern the long-term outcomes of the system; determining whether the two species persist, one competitively excludes the other, or both coexist while competing for the shared resource.

In Section 3, we further extend the model to a reaction-diffusion system (3.1) to incorporate the diffusion effects of the two competing populations in an unbounded habitat. The long-term persistence of both species and global stability of the coexistence state is characterized by the existence of traveling wave solutions that flow from extinction or competitive-exclusion states to the coexistence state with a large range of wave speeds. Under assumptions **(H1–H4)** given in Section 2, which ensure the presence of all steady states and local stability of the coexistence state in the ordinary differential equation system (2.1), we derive balanced conditions on the ecological parameters (including the single-dominance equilibria magnitudes $\bar{u}_i$ and the diffusion rates D_i) that guarantee traveling wave dynamics mediating transitions from extinction or single-species dominance to coexistence. Using the method of upper–lower solutions, we prove the existence of positive, monotone, traveling wave solutions $(u_1(x+ct), u_2(x+ct))$ of (3.1) for each wave speed above a biologically determined minimal speed, which depends on all ecological parameters including steady-state magnitudes and diffusion rates, satisfying

$$(u_1(-\infty),\ u_2(-\infty)) \in \{(0,0),\ (0,\bar{u}_2),\ (\bar{u}_1,0)\},\quad (u_1(+\infty),\ u_2(+\infty)) = (u_1^*,\ u_2^*).$$

Finally, in Section 4, we present two numerical examples to illustrate the validity of some theoretical results established in Section 3. The first example corresponds to the setting of Theorem 3.5 and demonstrates a traveling wave solution of (3.1) with limits: $(u_1(-\infty), u_2(-\infty)) = (0, baru_2)$

and $(u_1(+\infty), u_2(+\infty)) = (u_1^*, u_2^*)$. The second example, corresponding to Theorem 3.6, demonstrates a traveling wave solution of (3.1) with limits: $(u_1(-\infty), u_2(-\infty)) = (\bar{u}_1, 0)$ and $(u_1(+\infty), u_2(+\infty)) = (u_1^*, u_2^*)$. These examples numerically confirm the global stability of the coexistence equilibrium in model (3.1) and the ecological transition from single-species dominance to permanence.

2. Asymptotic stability of the steady states

In this section, we study the ordinary differential equation model (1.2) for two species competing for a shared resource:

$$\frac{du_1}{dt} = u_1\left(\frac{r_1 R}{a_1 u_1 + b_1 u_2 + h_1} - \mu_1\right), \frac{du_2}{dt} = u_2\left(\frac{r_2 R}{a_2 u_2 + b_2 u_1 + h_2} - \mu_2\right) \tag{2.1}$$

Let $M_i = \max\{\, u_i(0),\ (r_i R - \mu_i h_i)/a_i\mu_i \,\}$, $i = 1, 2$. Then, the comparison argument [17] implies that $0 \le u_i(t) \le M_i$ on $[0, \infty)$. Because of the symmetry in the model, we assume that $a_1 a_2 > b_1 b_2$, which indicates that conspecific competition has relatively stronger impact on each species. Under this assumption, we can solve for the following nonnegative equilibrium states and obtain the conditions for their presence:

- The extinction state: $(0, 0)$.
- The single-dominance state for survival of u_1-species only: $(\bar{u}_1, 0) = \left(\frac{r_1 R - \mu_1 h_1}{a_1 \mu_1}, 0\right)$, it is present if $r_1 R > \mu_1 h_1$.
- The single-dominance state for survival of u_2-species only: $(0, \bar{u}_2) = \left(0, \frac{r_2 R - \mu_2 h_2}{a_2 \mu_2}\right)$, it is present if $r_2 R > \mu_2 h_2$.
- The coexistence state for survival of both species:

$$(u_1^*, u_2^*) = \left(\frac{\mu_2 a_2(r_1 R - \mu_1 h_1) - \mu_1 b_1(r_2 R - \mu_2 h_2)}{\mu_1\mu_2(a_1 a_2 - b_1 b_2)}, \frac{\mu_1 a_1(r_2 R - \mu_2 h_2) - \mu_2 b_2(r_1 R - \mu_1 h_1)}{\mu_1\mu_2(a_1 a_2 - b_1 b_2)}\right), \tag{2.2}$$

the coexistence state is present if

$$\frac{a_2}{b_1} > \frac{\mu_1(r_2 R - \mu_2 h_2)}{\mu_2(r_1 R - \mu_1 h_1)} > \frac{b_2}{a_1}. \tag{2.3}$$

Throughout the rest of this paper, unless otherwise stated, we impose the following assumptions to guarantee the existence of all four equilibria.

- **(H1)** $a_1 a_2 > b_1 b_2$.
- **(H2)** $r_1 R > \mu_1 h_1$.
- **(H3)** $r_2 R > \mu_2 h_2$.
- **(H4)** $\dfrac{a_2}{b_1} > \dfrac{\mu_1(r_2 R - \mu_2 h_2)}{\mu_2(r_1 R - \mu_1 h_1)} > \dfrac{b_2}{a_1}$.

We now investigate the asymptotic stability of the equilibrium states for extinction, competitive exclusion, and coexistence through the standard method of linearization. For the reaction functions:

$$f_1(u_1, u_2) = u_1\left(\frac{r_1 R}{a_1 u_1 + b_1 u_2 + h_1} - \mu_1\right),\ f_2(u_1, u_2) = \frac{du_2}{dt} = u_2\left(\frac{r_2 R}{a_2 u_2 + b_2 u_1 + h_2} - \mu_2\right),$$

the Jacobian matrix is

$$J(u_1,u_2)=\begin{pmatrix} \frac{r_1R(b_1u_2+h_1)}{(a_1u_1+b_1u_2+h_1)^2}-\mu_1 & \frac{-r_1Rb_1u_1}{(a_1u_1+b_1u_2+h_1)^2} \\ \frac{-r_2Rb_2u_2}{(a_2u_2+b_2u_1+h_2)^2} & \frac{r_2R(b_2u_1+h_2)}{(a_2u_2+b_2u_1+h_2)^2}-\mu_2 \end{pmatrix}. \tag{2.4}$$

The Jacobian matrix for the extinction state $(0,0)$ given by

$$J(0,0)=\begin{pmatrix} \frac{r_1R}{h_1}-\mu_1 & 0 \\ 0 & \frac{r_2R}{h_2}-\mu_2 \end{pmatrix}, \tag{2.5}$$

and has eigenvalues:

$$\lambda_1=\frac{r_1R}{h_1}-\mu_1,\ \lambda_2=\frac{r_2R}{h_2}-\mu_2.$$

Therefore, λ_1 or $\lambda_2>0$ when **(H2)** or **(H3)** holds. The extinction state $(0,0)$ is only asymptotically stable when other steady states are not present.

Next, we examine the Jacobian matrix for the u_1-dominate state $(\bar{u}_1,0)$:

$$J(\bar{u}_1,0)=\begin{pmatrix} \frac{-\mu_1(r_1R-\mu_1h_1)}{r_1R} & \frac{-\mu_1b_1(r_1R-\mu_1h_1)}{a_1r_1R} \\ 0 & \frac{a_1\mu_1(r_2R-\mu_2h_2)-b_2\mu_2(r_1R-\mu_1h_1)}{b_2(r_1R-\mu_1h_1)+a_1\mu_1h_2} \end{pmatrix}. \tag{2.6}$$

The Jacobian matrix $J(\bar{u}_1,0)$ has eigenvalues:

$$\lambda_1=\frac{-\mu_1(r_1R-\mu_1h_1)}{r_1R},\ \lambda_2=\frac{a_1\mu_1(r_2R-\mu_2h_2)-b_2\mu_2(r_1R-\mu_1h_1)}{b_2(r_1R-\mu_1h_1)+a_1\mu_1h_2}.$$

One can see that $\lambda_1<0$ when **(H2)** holds for the presence of $(\bar{u}_1,0)$, but $\lambda_2>0$ when **(H4)** holds while the coexistence state is present. Therefore, the u_1-dominance state $(\bar{u}_1,0)$ is only asymptotically stable when **(H2)** holds and $\frac{b_2}{a_1}>\frac{\mu_1(r_2R-\mu_2h_2)}{\mu_2(r_1R-\mu_1h_1)}$, which is the case that $(\bar{u}_1,0)$ is present and the coexistence state (u_1^*,u_2^*) is not.

Similarly, the u_2-dominance state $(0,\bar{u}_2)$ is only asymptotically stable when **(H3)** holds and $\frac{a_2}{b_1}<\frac{\mu_1(r_2R-\mu_2h_2)}{\mu_2(r_1R-\mu_1h_1)}$. It is unstable when the coexistence state (u_1^*,u_2^*) is present.

If only one single-dominance state is present, for example when $\mu_1h_1<r_1R$ and $\mu_2h_2>r_2R$, then $(\bar{u}_1,0)$ is the only existing single-dominance state and is asymptotically stable.

We now assume that all conditions **(H1–H4)** hold and look into the asymptotic stability of the coexistence state (u_1^*,u_2^*). It can be shown that

$$J(u_1^*,u_2^*)=\begin{pmatrix} \frac{-a_1\mu_1^2u_1^*}{r_1R} & \frac{-b_1\mu_1^2u_1^*}{r_1R} \\ \frac{-b_2\mu_2^2u_2^*}{r_2R} & \frac{-a_2\mu_2^2u_2^*}{r_2R} \end{pmatrix}. \tag{2.7}$$

The eigenvalues λ_1 and λ_2 for $J(u_1^*,u_2^*)$ satisfy

$$\lambda_1+\lambda_2=\left(\frac{-a_1\mu_1^2u_1^*}{r_1R}\right)+\left(\frac{-a_2\mu_2^2u_2^*}{r_2R}\right)<0$$

which is the trace of $J(u_1^*,u_2^*)$, and

$$\lambda_1\lambda_2=(a_1a_2-b_1b_2)\frac{\mu_1^2\mu_2^2u_1^*u_2^*}{r_1r_2R^2}>0$$

which is the determinant of $J(u_1^*,u_2^*)$. The Routh–Hurwitz stability criterion implies that when **(H1–H4)** hold, the coexistence state (u_1^*,u_2^*) is present and asymptotically stable.

3. The reaction-diffusion system and wavefront solutions

We now consider the following Cauchy problem, where $u_1(x,t)$ and $u_2(x,t)$ are density functions of the competing species, D_1 and D_2 are their diffusion rates, $(x,t) \in \mathbf{R} \times \mathbf{R}^+$, and

$$\begin{cases} \dfrac{\partial u_1}{\partial t} - D_1 \dfrac{\partial^2 u_1}{\partial x^2} = u_1(x,t)\left(\dfrac{r_1 R}{a_1 u_1(x,t) + b_1 u_2(x,t) + h_1} - \mu_1\right), \\ \dfrac{\partial u_2}{\partial t} - D_2 \dfrac{\partial^2 u_2}{\partial x^2} = u_2(x,t)\left(\dfrac{r_2 R}{a_2 u_2(x,t) + b_2 u_1(x,t) + h_2} - \mu_2\right) \end{cases} \tag{3.1}$$

Setting $\xi = x + ct$, $c > 0$, we investigate the traveling wave solutions $(u_1(\xi), u_2(\xi))$ of (3.1) flowing from the extinction or a single-dominance state to the coexistence state. The wavefront equation connecting the extinction equilibrium $(0,0)$ and the coexistence equilibrium (u_1^*, u_2^*) is as follows:

$$\begin{cases} D_1(u_1)_{\xi\xi} - c(u_1)_\xi + u_1(\xi)\left(\dfrac{r_1 R}{a_1 u_1(\xi) + b_1 u_2(\xi) + h_1} - \mu_1\right) = 0, \\ D_2(u_2)_{\xi\xi} - c(u_2)_\xi + u_2(\xi)\left(\dfrac{r_2 R}{a_2 u_2(\xi) + b_2 u_1(\xi) + h_2} - \mu_2\right) = 0, \\ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}(-\infty) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}(+\infty) = \begin{pmatrix} u_1^* \\ u_2^* \end{pmatrix}. \end{cases} \tag{3.2}$$

We notice that in (3.2), the boundary conditions at $-\infty$ and $+\infty$ are given as equilibrium states, the reaction functions are differential, hence they satisfy the Lipschitz condition in any bounded set, and is mixed quasi-monotone [17] for $u_1,\ u_2 \geq 0$. For the existence of traveling wave solutions of (3.2) through the method of upper-lower solutions [17], we refer to a series of existence-comparison results in [18–20] concerning the general n-dimensional reaction-diffusion system

$$\frac{\partial \mathbf{u}}{\partial t} = D\Delta\mathbf{u} + \mathbf{f}(\mathbf{u}) \quad -\infty < x < \infty, \quad t > 0, \tag{3.3}$$

where $\mathbf{u} = (u_1, \ldots, u_n)^T$, $\mathbf{f} = (f_1, \ldots, f_n)^T$, and D =diag$(d_1, \ldots, d_n)$ with $d_i > 0$ for $i = 1, \ldots, n$. A wavefront solution for (3.3) is a solution in the form $\mathbf{u}(x,t) = \mathbf{w}(x + ct)$ for some $c > 0$, where $\mathbf{w}$ satisfies

$$\begin{cases} D\mathbf{w}''(s) - c\mathbf{w}'(s) + \mathbf{f}(\mathbf{w}(s)) = \mathbf{0} \quad -\infty < s < \infty, \\ \lim_{s\to-\infty} \mathbf{w}(s) = \mathbf{u}^-, \quad \lim_{s\to\infty} \mathbf{w}(s) = \mathbf{u}^+ \end{cases} \tag{3.4}$$

for some $\mathbf{u}^-, \mathbf{u}^+ \in \mathbb{R}^n$. The following definition is for a pair of coupled upper and lower solutions for the wavefront system (3.4).

Definition 3.1. Upper and lower solutions.

A pair of bounded functions $\tilde{\mathbf{w}} \equiv (\tilde{w}_1, \ldots, \tilde{w}_n)$ and $\hat{\mathbf{w}} \equiv (\hat{w}_1, \ldots, \hat{w}_n)$ are coupled upper and lower solutions for (3.4) if $\tilde{\mathbf{w}} \geq \hat{\mathbf{w}}$,

$$\begin{aligned} d_i\tilde{w}_i'' - c\tilde{w}_i' + f_i\left(\tilde{w}_i, [\tilde{\mathbf{w}}]_{k_i}, [\hat{\mathbf{w}}]_{\hat{k}_i}\right) \leq 0, \\ d_i\hat{w}_i'' - c\hat{w}_i' + f_i\left(\hat{w}_i, [\hat{\mathbf{w}}]_{k_i}, [\tilde{\mathbf{w}}]_{\hat{k}_i}\right) \geq 0, \end{aligned} \tag{3.5}$$

where k_i and $\hat{k}_i$ are subsets of positive integers such that

$$k_i \cup \hat{k}_i = \{1, \ldots, n\} \setminus \{i\}, \tag{3.6}$$

for each $i = 1, \ldots, n$, and $\lim_{t\to-\infty} \tilde{\mathbf{w}}(t)$*, and* $\lim_{t\to\infty} \hat{\mathbf{w}}(t)$ *both exist with*

$$\lim_{t\to-\infty} \tilde{\mathbf{w}}(t) \geq \mathbf{u}^- \geq \lim_{t\to-\infty} \hat{\mathbf{w}}(t), \quad \lim_{t\to\infty} \tilde{\mathbf{w}}(t) \geq \mathbf{u}^+ \geq \lim_{t\to\infty} \hat{\mathbf{w}}(t). \tag{3.7}$$

Now, the following hypotheses **(H)** on the reaction functions ensures the existence-comparison result given in our next lemma (Lemma 3.2).

(H):

1) (Equilibrium states) $\mathbf{f}(\mathbf{u}^-) = \mathbf{f}(\mathbf{u}^+) = \mathbf{0}$. Furthermore, $\mathbf{u}^-$ is the only zero of $\mathbf{f}$ between $\lim_{t\to-\infty} \hat{\mathbf{w}}(t)$ and $\lim_{t\to-\infty} \tilde{\mathbf{w}}(t)$, and $\mathbf{u}^+$ is the only zero of $\mathbf{f}$ between $\lim_{t\to\infty} \hat{\mathbf{w}}(t)$ and $\lim_{t\to\infty} \tilde{\mathbf{w}}(t)$.
2) (Mixed quasi-monotonicity) For each i, there are subsets of positive integers, k_i and $\hat{k}_i$, such that (3.6) holds and the function
$$f_i(\mathbf{u}) = f_i\left(u_i, [\mathbf{u}]_{k_i}, [\mathbf{u}]_{\hat{k}_i}\right)$$
is monotone non-decreasing in $[\mathbf{u}]_{k_i}$ and monotone non-increasing in $[\mathbf{u}]_{\hat{k}_i}$. Also, there is $\beta \geq 0$ such that $f_i(\mathbf{u}) + \beta u_i$ is nondecreasing in u_i for each i. Without loss of generality, we assume that $\beta > 0$.
3) (Lipschitz condition) $\mathbf{f}$ satisfies the Lipschitz condition in any bounded set S of $\mathbb{R}^n$. That is, there is $L > 0$ such that $|\mathbf{f}(\mathbf{u}) - \mathbf{f}(\mathbf{v})| \leq L|\mathbf{u} - \mathbf{v}|$ for all $\mathbf{u}, \mathbf{v} \in S$.

Now we can employ the following lemma on existence-comparison results (see, for example, Theorem 2.6 in [18]) for traveling wave solutions.

Lemma 3.2. *Let the hypothesis **(H)** hold. Suppose that the upper-lower solution pair $\tilde{\mathbf{w}}, \hat{\mathbf{w}} \in C^2(\mathbb{R})$ satisfy (3.5) and (3.7). Then, there is a solution $\mathbf{w}$ to (3.4) that satisfies $\hat{\mathbf{w}} \leq \mathbf{w} \leq \tilde{\mathbf{w}}$ in $\mathbb{R}$.*

For the mixed quasi-monotone system (3.2), a pair of $C^2(R) \times C^2(R)$ functions $(\tilde{U}(\xi), \tilde{V}(\xi))$ and $(\hat{U}(\xi), \hat{V}(\xi))$ are defined by Definition 3.1 as coupled upper-lower solutions [18] if

$$\begin{cases} D_1\tilde{U}_{\xi\xi} - c\tilde{U}_\xi + \tilde{U}(\xi)\left(\dfrac{r_1R}{a_1\tilde{U}(\xi) + b_1\hat{V}(\xi) + h_1} - \mu_1\right) \leq 0, \\[2ex] D_2\tilde{V}_{\xi\xi} - c\tilde{V}_\xi + \tilde{V}(\xi)\left(\dfrac{r_2R}{a_2\tilde{V}(\xi) + b_2\hat{U}(\xi) + h_2} - \mu_2\right) \leq 0, \\[2ex] D_1\hat{U}_{\xi\xi} - c\hat{U}_\xi + \hat{U}(\xi)\left(\dfrac{r_1R}{a_1\hat{U}(\xi) + b_1\tilde{V}(\xi) + h_1} - \mu_1\right) \geq 0, \\[2ex] D_2\hat{V}_{\xi\xi} - c\hat{V}_\xi + \hat{V}(\xi)\left(\dfrac{r_2R}{a_2\hat{V}(\xi) + b_2\hat{U}(\xi) + h_2} - \mu_2\right) \geq 0. \\[2ex] \begin{pmatrix} \tilde{U} \\ \tilde{V} \end{pmatrix}(-\infty) \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \geq \begin{pmatrix} \hat{U} \\ \hat{V} \end{pmatrix}(-\infty), \ \begin{pmatrix} \tilde{U} \\ \tilde{V} \end{pmatrix}(+\infty) \geq \begin{pmatrix} u_1^* \\ u_2^* \end{pmatrix} \geq \begin{pmatrix} \hat{U} \\ \hat{V} \end{pmatrix}(+\infty). \end{cases} \tag{3.8}$$

To establish the existence result given in Lemma 3.2 for the traveling wave system (3.2), we will construct a pair of upper-lower solutions by applying a well-known result [21,22] on the solution of the following KPP equation with limiting conditions. For a constant $K > 0$ and a differentiable function $F(y)$ with $F(0) = F(K) = 0$ and $F(y) > 0$ in $(0, K)$, we consider the boundary value problem:

$$\begin{cases} Y_{\xi\xi} - cY_\xi + F(Y) = 0, \\ Y(-\infty) = 0,\ Y(+\infty) = K. \end{cases} \tag{3.9}$$

Lemma 3.3. *Corresponding to every $c > 2\sqrt{a}$ where $a = F'(0) > 0$, there is a unique (up to a translation of the origin) solution Y for the boundary value problem (3.9). Such solution satisfies $Y'(\xi) > 0$ for $\xi \in R$ and has the following asymptotic behaviors at infinities.*

For the wave solution $Y(\xi)$ with speed $c > 2\sqrt{a}$, we have

$$\begin{cases} Y(\xi) = Ae^{\frac{c-\sqrt{c^2-4a}}{2}\xi} + o(e^{\frac{c-\sqrt{c^2-4a}}{2}\xi}),\ as\ \xi \to -\infty; \\ Y(\xi) = 1 - Be^{\frac{c-\sqrt{c^2+4a}}{2}\xi} + o(e^{\frac{c-\sqrt{c^2+4a}}{2}\xi}),\ as\ \xi \to +\infty. \end{cases} \tag{3.10}$$

where $A,\ B > 0$ are constants.

We denote $K_1 = \frac{r_1 R}{\mu_1} - h_1$, and $K_2 = \frac{r_2 R}{\mu_2} - h_2$. Our assumptions **(H1)–(H4)** ensure that $K_1, K_2 > 0$. For each $c \geq 2\sqrt{D_i(\frac{r_i R}{h_i} - \mu_i)}$, we let $Y^{(i)}$ $(i = 1, 2)$ be the unique (up to a translation of the origin) monotone increasing solution of the following KPP equation:

$$\begin{cases} Y^{(i)}_{\xi\xi} - \dfrac{c}{D_i}Y^{(i)}_\xi + \dfrac{Y^{(i)}(\xi)}{D_i}\left(\dfrac{r_i R}{Y^{(i)}(\xi) + h_i} - \mu_i\right) = 0, \\ Y^{(i)}(-\infty) = 0,\ Y^{(i)}(+\infty) = K_i. \end{cases} \tag{3.11}$$

In the above problem, for $i = 1, 2$, the function $F_i(Y^{(i)}) = \frac{Y^{(i)}}{D_i}\left(\frac{r_i R}{Y^{(i)}+h_i} - \mu_i\right)$ has $F_i'(0) = \frac{1}{D_i}\left(\frac{r_i R}{h_i} - \mu_i\right) > 0$. By Lemma 3.3, for $\frac{c}{D_i} \geq 2\sqrt{F_i'(0)}$ or $c \geq 2\sqrt{D_i(\frac{r_i R}{h_i} - \mu_i)}$, the wavefront solution $Y^{(i)}$ exists. It can be verified that $(\tilde{U}, \tilde{V}) = (Y^{(1)}/a_1, Y^{(2)}/a_2)$ is an upper solution coupled with any nonnegative and differential lower solution $(\hat{U}, \hat{V})$, since the differential inequalities for upper solutions in (3.8) are satisfied, and $\tilde{U}(+\infty) = K_i/a_i = \bar{u}_i > u_i^*$ for $i = 1, 2$.

For construction of the lower solution in problem (3.2), we observe that $(0, 0) \leq (\tilde{U}, \tilde{V}) \leq (K_1/a_1, K_2/a_2)$. Now, let $H_1 = \frac{b_1 K_2}{a_2} + h_1$, $H_2 = \frac{b_2 K_1}{a_1} + h_2$, $l_1 = \frac{r_1 R}{\mu_1} - H_1$ and $l_2 = \frac{r_2 R}{\mu_2} - H_2$. Assumptions **(H1)–(H4)** ensure that $0 < l_i < K_i$ $(i = 1, 2)$. For each $c \geq 2\sqrt{D_i(\frac{r_i R}{H_i} - \mu_i)}$, we let $Z^{(i)}$ $(i = 1, 2)$ be the unique (up to a translation of the origin) monotone increasing solution of the following KPP equation:

$$\begin{cases} Z^{(i)}_{\xi\xi} - \dfrac{c}{D_i}Z^{(i)}_\xi + \dfrac{Z^{(i)}(\xi)}{D_i}\left(\dfrac{r_i R}{Z^{(i)}(\xi) + H_i} - \mu_i\right) = 0, \\ Z^{(i)}(-\infty) = 0,\ Z^{(i)}(+\infty) = l_i. \end{cases} \tag{3.12}$$

For function $G_i(Z^{(i)}) = \frac{Z^{(i)}}{D_i}\left(\frac{r_iR}{Z^{(i)}+H_i} - \mu_i\right)$, we have $G_i'(0) = \frac{1}{D_i}\left(\frac{r_iR}{H_i} - \mu_i\right)$. We notice that for $(i, j) = (1, 2)$ or $(2, 1)$, conditions **(H1–H4)** imply that

$$G_i'(0) = \frac{1}{D_i}\frac{a_j\mu_j(r_iR - \mu_ih_i) - \mu_ib_i(r_jR - \mu_jh_j)}{b_i(r_jR - \mu_jh_j) + a_j\mu_jh_i} > 0.$$

Again, by Lemma 3.3, for $\frac{c}{D_i} \geq 2\sqrt{G_i'(0)}$ or $c \geq 2\sqrt{D_i(\frac{r_iR}{H_i} - \mu_i)}$, the wavefront solution $Z^{(i)}$ $(i = 1, 2)$ exists. It can also be verified that $(\hat{U}, \hat{V}) = (Z^{(1)}/a_1, Z^{(2)}/a_2)$ is a lower solution of (3.2) coupled with the upper solution $(\tilde{U}, \tilde{V}) = (Y^{(1)}/a_1, Y^{(2)}/a_2)$. By Lemma 3.2 and the fact that $H_i > h_i$ $(i = 1, 2)$, we have the following theorem

Theorem 3.4. *If conditions **(H1–H4)** hold, then for every wave speed c with*

$$c \geq \max\left\{2\sqrt{D_1\left(\frac{r_1R}{h_1} - \mu_1\right)},\ 2\sqrt{D_2\left(\frac{r_2R}{h_2} - \mu_2\right)}\right\}, \tag{3.13}$$

there exists a wavefront solution $(u_1(x + ct), u_2(x + ct))$ *for (3.2) connecting the extinction state* $(0, 0)$ *and the coexistence state* (u_1^*, u_2^*).

Next, we look into the existence of traveling wave solutions connecting a single-dominance state with the coexistence state. Because of the similarity, we only prove the existence of wavefront solutions flowing from the u_2-dominance state $(0, \bar{u}_2)$ to the coexistence state (u_1^*, u_2^*). The wavefront equation for a solution connecting $(0, \bar{u}_2)$ and (u_1^*, u_2^*) is as follows:

$$\begin{cases} D_1(u_1)_{\xi\xi} - c(u_1)_\xi + u_1(\xi)\left(\dfrac{r_1R}{a_1u_1(\xi) + b_1u_2(\xi) + h_1} - \mu_1\right) = 0, \\ D_2(u_2)_{\xi\xi} - c(u_2)_\xi + u_2(\xi)\left(\dfrac{r_2R}{a_2u_2(\xi) + b_2u_1(\xi) + h_2} - \mu_2\right) = 0, \\ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}(-\infty) = \begin{pmatrix} 0 \\ \bar{u}_2 \end{pmatrix},\ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}(+\infty) = \begin{pmatrix} u_1^* \\ u_2^* \end{pmatrix}. \end{cases} \tag{3.14}$$

Using a transformation $U = u_1$, $V = \bar{u}_2 - u_2$, and noting that $\bar{u}_2 - u_2^* = \frac{b_2}{a_2}u_1^*$, the wavefront equations connecting the equilibria in (3.14) are then changed as quasi-monotone nondecreasing in the following system:

$$\begin{cases} D_1U_{\xi\xi} - cU_\xi + U(\xi)\left(\dfrac{r_1R}{a_1U(\xi) - b_1V(\xi) + b_1\bar{u}_2 + h_1} - \mu_1\right) = 0, \\ D_2V_{\xi\xi} - cV_\xi + (\bar{u}_2 - V(\xi))\left(\mu_2 - \dfrac{r_2R}{b_2U(\xi) - a_2V(\xi) + a_2\bar{u}_2 + h_2}\right) = 0, \\ \begin{pmatrix} U \\ V \end{pmatrix}(-\infty) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},\ \begin{pmatrix} U \\ V \end{pmatrix}(+\infty) = \begin{pmatrix} u_1^* \\ \frac{b_2}{a_2}u_1^* \end{pmatrix}. \end{cases} \tag{3.15}$$

By Definition 3.1, a pair of $C^2(R)\times C^2(R)$ functions $(\tilde{U}(\xi), \tilde{V}(\xi))$ and $(\hat{U}(\xi), \hat{V}(\xi))$ are coupled upper-lower solutions of the above system if they satisfy

$$\begin{cases} D_1\tilde{U}_{\xi\xi} - c\tilde{U}_\xi + \tilde{U}(\xi)\left(\dfrac{r_1R}{a_1\tilde{U}(\xi) - b_1\tilde{V}(\xi) + b_1\bar{u}_2 + h_1} - \mu_1\right) \le 0, \\ D_2\tilde{V}_{\xi\xi} - c\tilde{V}_\xi + \left(\bar{u}_2 - \tilde{V}(\xi)\right)\left(\mu_2 - \dfrac{r_2R}{b_2\tilde{U}(\xi) - a_2\tilde{V}(\xi) + a_2\bar{u}_2 + h_2}\right) \le 0, \\ D_1\hat{U}_{\xi\xi} - c\hat{U}_\xi + \hat{U}(\xi)\left(\dfrac{r_1R}{a_1\hat{U}(\xi) - b_1\hat{V}(\xi) + b_1\bar{u}_2 + h_1} - \mu_1\right) \ge 0, \\ D_2\hat{V}_{\xi\xi} - c\hat{V}_\xi + \left(\bar{u}_2 - \hat{V}(\xi)\right)\left(\mu_2 - \dfrac{r_2R}{b_2\hat{U}(\xi) - a_2\hat{V}(\xi) + a_2\bar{u}_2 + h_2}\right) \ge 0, \\ \begin{pmatrix} \tilde{U} \\ \tilde{V} \end{pmatrix}(-\infty) \ge \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ge \begin{pmatrix} \hat{U} \\ \hat{V} \end{pmatrix}(-\infty), \ \begin{pmatrix} \tilde{U} \\ \tilde{V} \end{pmatrix}(+\infty) \ge \begin{pmatrix} u_1^* \\ \frac{b_2}{a_2}u_1^* \end{pmatrix} \ge \begin{pmatrix} \hat{U} \\ \hat{V} \end{pmatrix}(+\infty). \end{cases} \tag{3.16}$$

We denote $\overline{D} = \max\{ D_1, D_2 \}$, $\underline{D} = \min\{ D_1, D_2 \}$ and $L = \frac{r_1R}{a_1\mu_1} - \frac{b_1}{a_1}\bar{u}_2 + \frac{h_1}{a_1} = \frac{a_1a_2-b_1b_2}{a_1a_2}u_1^*$. The conditions **(H1–H4)** ensure that $L > 0$. For each $c \ge 2\overline{D}\sqrt{\frac{a_1\mu_1L}{\underline{D}(b_1\bar{u}_2+h_1)}}$, we let W be the unique (up to a translation of the origin) monotone increasing solution of the following KPP equation:

$$\begin{cases} W_{\xi\xi} - \dfrac{c}{\overline{D}}W_\xi + \dfrac{W(\xi)}{\underline{D}}\left(\dfrac{r_1R}{a_1W(\xi) + b_1\bar{u}_2 + h_1} - \mu_1\right) = 0, \\ W(-\infty) = 0, \ W(+\infty) = L. \end{cases} \tag{3.17}$$

In the above KPP equation, $F(W) = \frac{W}{\underline{D}}\left(\frac{r_1R}{a_1W+b_1\bar{u}_2+h_1} - \mu_1\right)$ has $F'(0) = \frac{1}{\underline{D}}\left(\frac{r_1R}{b_1\bar{u}_2+h_1} - \mu_1\right) > 0$. By Lemma 3.3, for $\frac{c}{\overline{D}} \ge 2\sqrt{F'(0)}$ or

$$\begin{aligned} c &\ge \frac{2\overline{D}}{\sqrt{\underline{D}}}\sqrt{\frac{r_1R}{b_1\bar{u}_2 + h_1} - \mu_1} = 2\overline{D}\sqrt{\frac{a_1\mu_1L}{\underline{D}(b_1\bar{u}_2 + h_1)}} \\ &= 2\overline{D}\sqrt{\frac{\mu_2a_2(r_1R - \mu_1h_1) - \mu_1b_1(r_2R - \mu_2h_2)}{\underline{D}b_1(r_2R - \mu_2h_2) + \underline{D}a_2\mu_2h_1}}, \end{aligned} \tag{3.18}$$

the wavefront solution $W(\xi)$ exists and is strictly increasing ($W'(\xi) > 0$). We set our coupled upper solution as

$$(\tilde{U}, \tilde{V}) = \left(\frac{a_1a_2}{a_1a_2 - b_1b_2}W, \frac{a_1b_2}{a_1a_2 - b_1b_2}W\right),$$

The boundary inequalities in (3.16) are satisfied since $\tilde{U}(+\infty) = u_1^*$ and $\tilde{V}(+\infty) = \frac{b_2}{a_2}u_1^*$. From the fact that $a_1\tilde{U} - b_1\tilde{V} = a_1W$ and $b_2\tilde{U} - a_2\tilde{V} = 0$, we can verify the following differential inequalities for the upper solutions in (3.16):

$$
\begin{aligned}
&\tilde{U}_{\xi\xi} - \frac{c}{D_1}\tilde{U}_\xi + \frac{\tilde{U}(\xi)}{D_1}\left(\frac{r_1 R}{a_1\tilde{U}(\xi) - b_1\tilde{V}(\xi) + b_1\bar{u}_2 + h_1} - \mu_1\right) \\
&\leq \frac{a_1 a_2}{a_1 a_2 - b_1 b_2}\left[W_{\xi\xi} - \frac{c}{\overline{D}}W_\xi + \frac{W(\xi)}{\underline{D}}\left(\frac{r_1 R}{a_1 W(\xi) + b_1\bar{u}_2 + h_1} - \mu_1\right)\right] = 0.
\end{aligned}
\tag{3.19}
$$

Noting that $\frac{r_2 R}{a_2\bar{u}_2 + h_2} = \mu_2$, we can also obtain that

$$
\begin{aligned}
&\tilde{V}_{\xi\xi} - \frac{c}{D_2}\tilde{V}_\xi + \frac{1}{D_2}\left(\bar{u}_2 - \tilde{V}(\xi)\right)\left(\mu_2 - \frac{r_2 R}{b_2\tilde{U}(\xi) - a_2\tilde{U}(\xi) + a_2\bar{u}_2 + h_2}\right) \\
&\leq \frac{a_1 b_2}{a_1 a_2 - b_1 b_2}\left[W_{\xi\xi} - \frac{c}{\overline{D}}W_\xi\right] + \frac{1}{D_2}\left(\bar{u}_2 - \tilde{V}(\xi)\right)\left(\mu_2 - \frac{r_2 R}{a_2\bar{u}_2 + h_2}\right) \\
&= -\frac{a_1 b_2 W(\xi)}{\underline{D}(a_1 a_2 - b_1 b_2)}\left[\frac{r_1 R}{a_1 W(\xi) + b_1\bar{u}_2 + h_1} - \mu_1\right] \leq 0.
\end{aligned}
\tag{3.20}
$$

In order to construct the lower solutions in (3.16), for each $c \geq 2\underline{D}\sqrt{\frac{a_1\mu_1 L}{\overline{D}(b_1\bar{u}_2 + h_1)}}$, we let Z be the unique (up to a translation of the origin) monotone increasing solution of the following KPP equation:

$$
\begin{cases}
Z_{\xi\xi} - \frac{c}{\underline{D}}Z_\xi + \frac{Z(\xi)}{\overline{D}}\left(\frac{r_1 R}{a_1 Z(\xi) + b_1\bar{u}_2 + h_1} - \mu_1\right) = 0, \\
Z(-\infty) = 0,\ Z(+\infty) = L.
\end{cases}
\tag{3.21}
$$

By the comparison argument, $Z(\xi) \leq W(\xi)$. We shall verify that $(\hat{U}, \hat{V}) = (Z, \beta Z)$ is a coupled lower solution for some $0 < \beta < 1$. We see that for $0 < \beta < a_1/b_1$,

$$
\begin{aligned}
&\hat{U}_{\xi\xi} - \frac{c}{D_1}\hat{U}_\xi + \frac{\hat{U}(\xi)}{D_1}\left(\frac{r_1 R}{a_1\hat{U}(\xi) - b_1\hat{V}(\xi) + b_1\bar{u}_2 + h_1} - \mu_1\right) \\
&\geq \left(Z_{\xi\xi} - \frac{c}{\underline{D}}Z_\xi\right) + \frac{Z(\xi)}{\overline{D}}\left(\frac{r_1 R}{(a_1 - \beta b_1)Z(\xi) + b_1\bar{u}_2 + h_1} - \mu_1\right) \\
&= \frac{Z(\xi)}{\overline{D}}\left(\frac{r_1 R}{(a_1 - \beta b_1)Z(\xi) + b_1\bar{u}_2 + h_1} - \frac{r_1 R}{a_1 Z(\xi) + b_1\bar{u}_2 + h_1}\right) \geq 0.
\end{aligned}
\tag{3.22}
$$

Also, the constructed lower solution $\hat{V} = \beta Z$ satisfies:

$$
\begin{aligned}
&\hat{V}_{\xi\xi} - \frac{c}{D_2}\hat{V}_\xi + \frac{\left(\bar{u}_2 - \hat{V}(\xi)\right)}{D_2}\left(\mu_2 - \frac{r_2R}{b_2\hat{U}(\xi) - a_2\hat{V}(\xi) + a_2\bar{u}_2 + h_2}\right)\\
&\geq \beta\left(Z_{\xi\xi} - \frac{c}{\underline{D}}Z_\xi\right) + \frac{(\bar{u}_2 - \beta Z(\xi))}{D_2}\left(\mu_2 - \frac{r_2R}{(b_2 - a_2\beta)Z(\xi) + a_2\bar{u}_2 + h_2}\right)\\
&\geq -\frac{\beta Z(\xi)}{\overline{D}}\left(\frac{r_1R}{a_1Z(\xi) + b_1\bar{u}_2 + h_1} - \mu_1\right) + \frac{\mu_2\,(\bar{u}_2 - \beta Z(\xi))}{\overline{D}}\left(1 - \frac{r_2R}{r_2R + \mu_2(b_2 - a_2\beta)Z(\xi)}\right)\\
&= \frac{1}{\overline{D}}\left(\mu_2\,(\bar{u}_2 - \beta Z(\xi))\left(1 - \frac{r_2R}{r_2R + \mu_2(b_2 - a_2\beta)Z(\xi)}\right) - \beta Z(\xi)\left(\frac{r_1R}{a_1Z(\xi) + b_1\bar{u}_2 + h_1} - \mu_1\right)\right).
\end{aligned}
\tag{3.23}
$$

We let

$$
F(Z) = \mu_2\,(\bar{u}_2 - \beta Z)\left(1 - \frac{r_2R}{r_2R + \mu_2(b_2 - a_2\beta)Z}\right) - \beta Z\left(\frac{r_1R}{a_1Z + b_1\bar{u}_2 + h_1} - \mu_1\right),
$$

and notice that

$$
\begin{aligned}
F'(Z) = {}&\mu_2\,(\bar{u}_2 - \beta Z)\,\frac{r_2R\mu_2(b_2 - a_2\beta)}{(r_2R + \mu_2(b_2 - a_2\beta)Z)^2} - \beta\mu_2\left(1 - \frac{r_2R}{r_2R + \mu_2(b_2 - a_2\beta)Z}\right)\\
&-\beta\left(\frac{r_1R}{a_1Z + b_1\bar{u}_2 + h_1} - \mu_1\right) + \frac{\beta a_1r_1RZ}{(a_1Z + b_1\bar{u}_2 + h_1)^2},
\end{aligned}
$$

we can conclude that $F'(Z) > 0$ for small enough $\beta > 0$ and $0 < Z < L$. Since $F(0) = 0$, then with some $0 < \beta < 1$, we have $F(Z) \geq 0$ for all $Z \in (0, L)$. (3.22) and (3.23) imply that $(\hat{U}, \hat{V}) = (Z, \beta Z)$ is a lower solution satisfying (3.16) and coupled with $(\tilde{U}, \tilde{V}) = (\frac{a_1a_2}{a_1a_2 - b_1b_2}W, \frac{a_1b_2}{a_1a_2 - b_1b_2}W)$.

Theorem 3.5. *If conditions **(H1–H4)** hold, then for every wave speed*

$$
c \geq 2\overline{D}\sqrt{\frac{\mu_2a_2(r_1R - \mu_1h_1) - \mu_1b_1(r_2R - \mu_2h_2)}{\underline{D}b_1(r_2R - \mu_2h_2) + \underline{D}a_2\mu_2h_1}}, \tag{3.24}
$$

there exists a wavefront solution $(u_1(x + ct), u_2(x + ct))$ for (3.14) connecting the single-dominate state $(0, \bar{u}_2)$ and the coexistence state (u_1^*, u_2^*).

Similarly, we can prove the existence of traveling wave solutions connecting the single-dominance state $(\bar{u}_1, 0)$ and the coexistence state (u_1^*, u_2^*). The wavefront equation for a solution flowing from $(\bar{u}_1, 0)$ to (u_1^*, u_2^*) is as follows:

$$
\begin{cases}
D_1(u_1)_{\xi\xi} - c(u_1)_\xi + u_1(\xi)\left(\dfrac{r_1R}{a_1u_1(\xi) + b_1u_2(\xi) + h_1} - \mu_1\right) = 0,\\[2ex]
D_2(u_2)_{\xi\xi} - c(u_2)_\xi + u_2(\xi)\left(\dfrac{r_2R}{a_2u_2(\xi) + b_2u_1(\xi) + h_2} - \mu_2\right) = 0,\\[2ex]
\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}(-\infty) = \begin{pmatrix} \bar{u}_1 \\ 0 \end{pmatrix},\ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}(+\infty) = \begin{pmatrix} u_1^* \\ u_2^* \end{pmatrix}.
\end{cases}
\tag{3.25}
$$

This final existence result for wavefront solutions which further demonstrates the global stability of the coexistence state (u_1^*, u_2^*) is stated as the following theorem.

Theorem 3.6. *If conditions **(H1–H4)** hold, then for every wave speed*

$$c \geq 2\overline{D}\sqrt{\frac{\mu_1 a_1(r_2 R - \mu_2 h_2) - \mu_2 b_2(r_1 R - \mu_1 h_1)}{\underline{D} b_2(r_1 R - \mu_1 h_1) + \underline{D} a_1 \mu_1 h_2}}, \tag{3.26}$$

there exists a wavefront solution $(u_1(x+ct), u_2(x+ct))$ for (3.25) connecting the single-dominate state $(\bar{u}_1, 0)$ and the coexistence state (u_1^*, u_2^*).

4. Numerical simulations

At the end of this work, we present two numerical examples illustrating Theorems 3.5 and 3.6, concerning traveling wave solutions that connect the single-dominant states $(0, \bar{u}_2)$ or $(\bar{u}_1, 0)$ to the coexistence state (u_1^*, u_2^*). According to Section 2, when conditions **(H1–H4)** hold, the steady states $(0,0)$, $(\bar{u}_1, 0)$, $(0, \bar{u}_2)$, and (u_1^*, u_2^*) are all present. Moreover, the coexistence state (u_1^*, u_2^*) is asymptotically stable. After discretizing the differential equation system (3.1) into finite-difference form, numerical solutions can be obtained using the monotone iterative scheme developed in several earlier studies (see, for example, [23–25]). Similar numerical methods have been used to demonstrate and analyze traveling wave solutions in a variety of reaction-diffusion models (see, for examples, [3,4,18]).

In the reaction-diffusion system (3.1), we choose the following set of biological parameters: $D_1 = 0.21$, $D_2 = 0.12$, $R = 10.0$, $r_1 = 0.6$, $r_2 = 0.5$, $\mu_1 = \mu_2 = 0.25$, $h_1 = 0.4$, $h_2 = 0.3$, $a_1 = 0.4$, $b_1 = 0.2$, $a_2 = 0.2$ and $b_2 = 0.3$, which makes conditions **(H1–H4)** hold. There exist the following steady states: extinction state $(0,0)$, two single-dominance states: $(\bar{u}_1, 0) = (59, 0)$ and $(0, \bar{u}_2) = (0, 98.5)$, and the coexistence state $(u_1^*, u_2^*) = (39, 40)$.

Example 4.1. We display a traveling wavefront flowing from the $\bar{u}_2$-dominance state $(0, 98.5)$ to the coexistence state $(39, 40)$.

When conditions **(H1–H4)** hold, Theorem 3.5 indicates that for any wave speed c with

$$c \geq 2\overline{D}\sqrt{\frac{\mu_2 a_2(r_1 R - \mu_1 h_1) - \mu_1 b_1(r_2 R - \mu_2 h_2)}{\underline{D} b_1(r_2 R - \mu_2 h_2) + \underline{D} a_2 \mu_2 h_1}},$$

the system (3.14) has a traveling wave solution $(u(x+ct), v(x+ct))$ flowing from $(0, \bar{u}_2) = (0, 98.5)$ to the coexistence state $(u_1^*, u_2^*) = (39, 40)$. Using the above set of biological parameters and fixing the initial function (u_0, v_0) for (3.1) as a small perturbation of $(0, 98.5)$, we demonstrate one of the traveling wavefronts and its spreading speed for global attraction of the coexistence state shifting from dominance of u_2-species in Figure 1.

Example 4.2. Here we consider a traveling wavefront flowing from the $\bar{u}_1$-dominance state $(59, 0)$ to the coexistence state $(39, 40)$.

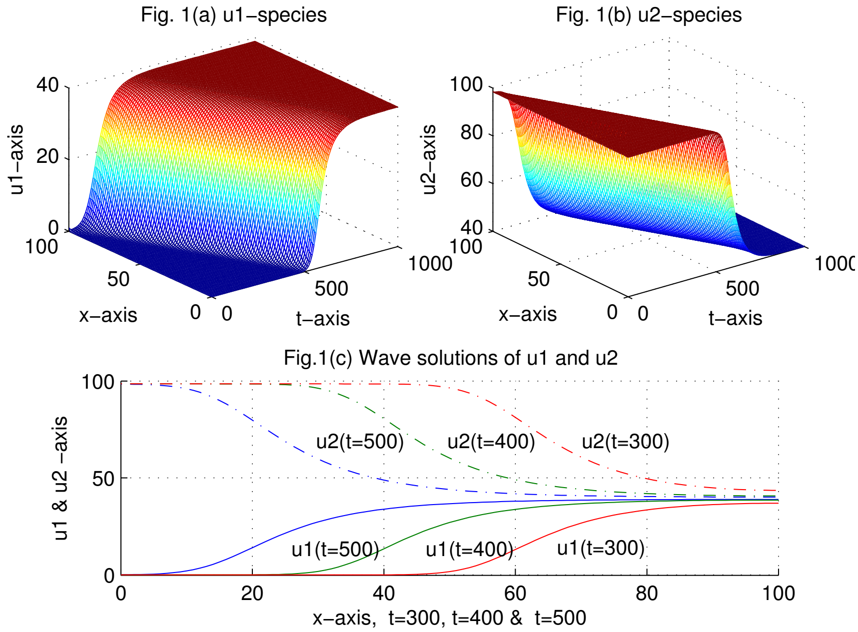

Figure 1. Traveling wavefront connecting $(0, \bar{u}_2)$ to (u_1^*, u_2^*), coexistence of both species.

When conditions **(H1–H4)** hold, Theorem 3.6 indicates that for any wave speed c with

$$c \geq 2\overline{D}\sqrt{\frac{\mu_1 a_1(r_2R - \mu_2 h_2) - \mu_2 b_2(r_1 R - \mu_1 h_1)}{\underline{D}b_2(r_1R - \mu_1 h_1) + \underline{D}a_1\mu_1 h_2}},$$

system (3.25) has a traveling wave solution $(u(x + ct), v(x + ct))$ flowing from $(\bar{u}_1, 0) = (59, 0)$ to the coexistence state $(u_1^*, u_2^*) = (39, 40)$. Using the above set of biological parameters and fixing the initial function (u_0, v_0) for (3.1) as a small perturbation of $(59, 0)$, we demonstrate one of the traveling wavefronts and its spreading speed for global attraction of the coexistence state shifting from dominance of u_1-species in Figure 2.

5. Conclusions

In this article, we establish and analyze a reaction-diffusion model (3.1) describing two populations competing for a shared common resource under Beddington–DeAngelis type fitness functions. We show that when all types of steady states (extinction, single-species dominance, and coexistence) are present in the corresponding ordinary differential equation system (2.1), the coexistence state (u_1^*, u_2^*) given in (2.2) is (locally) asymptotically stable, whereas the other equilibria are unstable. Under the same set of ecological assumptions **(H1–H4)**, we extend the analysis to the reaction–diffusion system (3.1) with diffusion rates D_i $(i = 1, 2)$ and establish a global coexistence result by proving the existence of traveling wave solutions connecting the unstable equilibria to the coexistence state. Although this coexistence result holds for all positive values of the diffusion rates, Theorems 3.4–3.6 reveal that the diffusion coefficients D_i $(i = 1, 2)$ influence our estimated range of admissible wave speeds for the traveling waves.

In reference to condition (3.13) in Theorem 3.4, the reaction-diffusion system (3.1) admits traveling

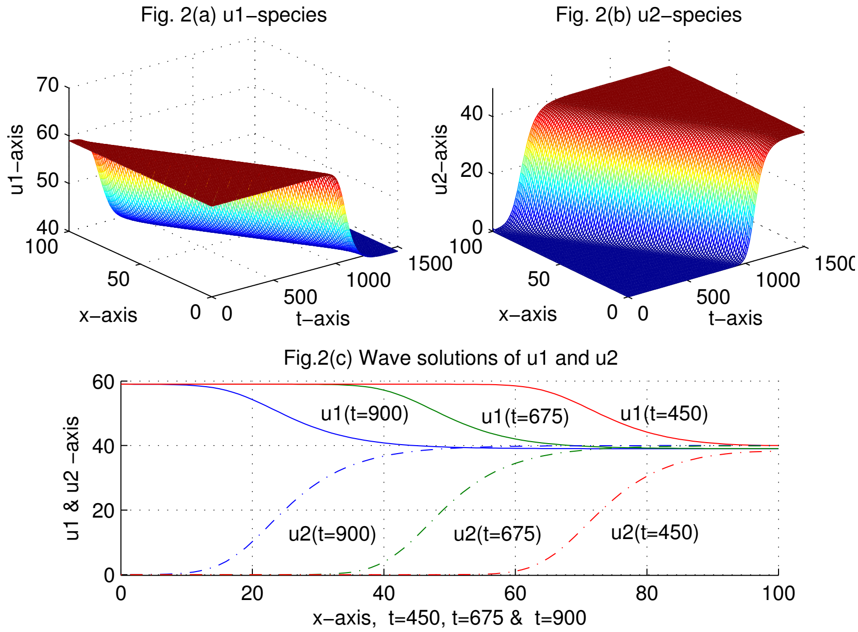

Figure 2. Traveling wavefront connecting $(\bar{u}_1, 0)$ to (u_1^*, u_2^*), coexistence of both species.

wave solutions $(u_1(x + ct), u_2(x + ct))$ connecting the extinction equilibrium $(0, 0)$ to the coexistence equilibrium (u_1^*, u_2^*) for all wave speeds satisfying

$$c \geq \max\left\{ 2\sqrt{D_1\left(\frac{r_1 R}{h_1} - \mu_1\right)},\ 2\sqrt{D_2\left(\frac{r_2 R}{h_2} - \mu_2\right)} \right\}.$$

This condition indicates that the minimal admissible wave speed increases proportionally to the square root of the diffusion coefficients. Moreover, a larger product of the metabolic efficiency rate and the available common resource, $r_i R$, leads to a higher minimal wave speed. In contrast, an increase in either the uniform mortality rate μ_i or the intrinsic harvest rate h_i results in a decrease in the minimal wave speed.

Similarly, condition (3.24) in Theorem 3.5 and its interpretation (3.18) imply that the minimal wave speed for traveling wave solutions connecting the single-species dominance equilibrium $(0, \bar{u}_2)$ to the coexistence equilibrium (u_1^*, u_2^*) is given by

$$c \geq \frac{2\overline{D}}{\sqrt{\underline{D}}} \sqrt{\frac{r_1 R}{b_1 \bar{u}_2 + h_1} - \mu_1}.$$

In addition to the influences of D_i, $r_i R$, μ_i, and h_i discussed above, this expression shows that a larger product of the interspecific competition coefficient and the magnitude of the opposite dominant species, $b_i \bar{u}_j$, reduces the minimal wave speed.

In model (3.1), we assume that the level of available resource R remains constant throughout the infinite spatial domain. This assumption allows the system to admit multiple spatially homogeneous steady states and facilitates the analytical derivation of our main results in Theorems 3.4–3.6, which establish the existence of traveling wave solutions and the global attraction of the coexistence equilibrium. If the available resource R varies in space or time, i.e., $R = R(x,t)$, the presence of a nonconstant resource distribution generally destroys the spatial homogeneity of steady states and significantly complicates the analysis of existence, multiplicity, and stability of coexistence states. In particular, if the common resource is spatially heterogeneous but bounded, $0 < \underline{R} \le R(x) \le \overline{R}$ in the infinite spatial domain, then the reaction–diffusion system extended from (3.1) may admit spatially dependent steady states, including coexistence equilibria. By replacing R in the corresponding ordinary differential equation system (2.1) with $\overline{R}$ or $\underline{R}$, one can obtain constant equilibria that serve as upper and lower solutions for the spatially dependent coexistence state. The asymptotic stability of these constant solutions obtained in their simplified equations further provides insight into the permanence of the ecological system. Moreover, the methodology developed for constructing upper and lower solutions of resource-population wavefronts can be adapted to investigate the emergence of spatiotemporal patterns in the extended reaction-diffusion system, the characterization of which would require additional analytical and numerical study.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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