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Research article

A meta-population model of malaria with asymptomatic cases, transmission blocking drugs, migration and screening

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Abstract: We consider a two-Patch malaria model, where the individuals can freely move between the patches. We assume that one site has better resources to fight the disease, such as screening facilities and the availability of transmission-blocking drugs (TBDs) that offer full, though waning, immunity and non-infectivity. Moreover, individuals moving to this site are screened at the entry points, and the authorities can either refuse entry to infected individuals or allow them in but immediately administer a TBD. However, an illegal entry into this Patch is also possible. We provide a qualitative analysis of the model, focusing on the emergence of endemic equilibria and the occurrence of backward bifurcations. Furthermore, we comprehensively analyse the model with low migration rates using recent refinements of the regular perturbation theory. We conclude the paper with numerical simulations that show, in particular, that malaria can be better controlled by allowing the entry of detected cases and treating them in the better-resourced site rather than deporting the identified infectives and risking them entering the site illegally.

Keywords: transmission-blocking drugs; meta-populations; migrations; mathematical modelling; regular perturbations; asymptotic analysis

1. Introduction

Malaria remains one of the world's most devastating infectious diseases, with millions of cases and thousands of deaths reported annually. According to the latest World Malaria Report by the Word

Health Organization (WHO), there were an estimated 263 million cases of malaria and 597,000 deaths from malaria worldwide in 2023. This represents about 11 million more cases in 2023 than in 2022 and nearly the same number of deaths [1]. Around 60% of malaria clinical cases, and about 80% of malaria deaths, occur in sub-Saharan Africa [2], constituting a major barrier to social and economic development in the region, including South Africa.

Mathematical modelling plays a crucial role in understanding malaria's transmission dynamics and informing control strategies. This can be performed in a single region or across several regions, where the movement of populations is integrated through meta-population modelling. Many malaria models use the meta-population approach. For instance, in [3], the authors used a meta-population model to alert the Chinese government to prioritise the risk of importing malaria from neighbouring countries, particularly Myanmar, to prevent the re-emergence of malaria. Another mathematical model was developed in [4] to examine how cross-border mobility between Nepal and India affects Nepal's goal of eliminating malaria by 2026. The model suggests that elimination of malaria is achievable with strategies that reduce cross-border mobility, ensure complete transmission protection abroad, or implement strict border screening. Agusto et al. examined the impact of human mobility between Botswana and Zimbabwe on malaria dynamics, finding that interventions in Zimbabwe can significantly affect malaria elimination efforts in Botswana due to differences in population size and transmission rates [5]. Another paper [6] considers migrations including seasonal migrations in Ghana. A stochastic meta-population model proposed in [7] explored the transmission dynamics of malaria in two bordering regions, one in South Africa and the other in Mozambique.

In South Africa, the National Malaria Elimination Plan, launched in 2012, initially set a target to eliminate malaria by 2018. However, after not meeting this goal, the target was extended, with the country now aiming for regional elimination of malaria by 2025, focusing on the most affected provinces [8, 9]. KwaZulu-Natal, one of South Africa's three malaria-endemic provinces, is close to malaria eliminating, reporting fewer than 100 locally acquired cases annually since 2010. However, despite the sustained implementation of essential interventions, including annual indoor residual spraying, screening using rapid diagnostic tests, and treatment with effective artemisinin-based combination therapy, low-level focal transmission persists in the province. A study confirmed that 97% (65/67) of detected malaria carriers had been identified as asymptomatic Mozambican nationals transiting through the informal border market from Mozambique to economic hubs within South Africa [10]. This challenge may be further exacerbated by significant human migration caused by the recent civil unrest in neighbouring Mozambique. According to [11], without the imported cases, KwaZulu-Natal would have been malaria-free for at least the last 7 years of the decade 2008–2018. Hence, to eradicate malaria in KwaZulu-Natal, a special effort must be made in screening and treatment to prevent or significantly limit the entry of infected individuals into the country.

This study aimed to assess the impact of the border and in-Patch screening and treatment using transmission-blocking drugs (TBD) when there is cross-border movement of asymptomatic cases. The primary objective was to determine whether it is more effective to allow infected individuals to enter the country and treat them immediately or to deny them entry, thereby risking their illegal entry into South Africa and contributing to uncontrolled spread of the disease.

The structure of the paper is as follows. In Section 2, we introduce the model. In Section 3, we present the qualitative analysis of the model and compute its basic reproduction number. In particular, we prove the global stability of the disease-free equilibrium in the case of isolated patches. Section 4

is devoted to the bifurcation analysis. First, in Subsection 4.1, we explore the existence of an endemic equilibrium for the cases of no migration. These results, combined with the global stability of disease-free equilibrium, form the basis of Subsection 4.2, where we study the stability of disease-free equilibrium and the possibility of the occurrence of backward bifurcation for low migration rates using the recently developed tools of the regular perturbation theory, which are proved in Appendix A. Then, in Subsection 4.3, we present a complete study of bifurcation scenarios when the migration from Patch 2 to Patch 1 can be neglected. Finally, Section 5 presents numerical results to illustrate the impact of our control parameters.

2. Model formulation

We consider the development of malaria in two patches: The first experiencing severe consequences from the disease, with a large number of infected individuals and high mortality rates, and the second being endemic but with relatively low numbers of infections and mortality rates. The endemicity is largely due to, among other factors, asymptomatic individuals migrating from the first Patch. This configuration is similar to the situation observed between Mozambique and South Africa, particularly in regions like KwaZulu-Natal. We suppose that malaria-stricken individuals cannot migrate between patches. We use k_{ij} to denote the constant migration rate from Patch j to Patch i, $(i, j \in \{1, 2\})$. We assume that there is no change in the disease status during migration. The individuals in Patch i are subdivided into five compartments, comprising susceptible S_i^h, infectious symptomatic I_i^h, protected P_i^h, undetected A_i^u, and detected A_i^d asymptomatic individuals (who are also infective). We assume there are no detected asymptomatic individuals in Patch 1.

Moreover, in each Patch, we have a population of mosquitoes subdivided into two compartments: susceptible S_i^v and infectious I_i^v. We assume that mosquitoes do not migrate between the patches.

Let us describe the dynamics of individuals in each compartment of the isolated Patch i. The entry into the susceptible compartment S_i^h occurs by recruiting new individuals at a constant rate Λ_i or by the protected individuals P_i^h losing their protection at the a rate ϑ_i. Individuals exit this compartment either by infection with the infection force

$$\lambda_i = \frac{a_i \beta_i^h I_i^v}{N_i^h}$$

or by natural death at the rate μ_i. Thus,

$$\frac{dS_i^h}{dt} = \Lambda_i + \vartheta_i P_i^h - (\lambda_i + \mu_i) S_i^h. \tag{2.1}$$

We assume that upon infection, an individual can become immediately symptomatically sick and infective with a probability p_i, or asymptomatic (and infective but undetected) with a probability $(1 - p_i)$. Thus, individuals enter the infectious compartment I_i^h at the rate $p_i\lambda_i$, and leave either due to a treatment at the rate ω_i, or following natural death or death due to the disease at the rate δ_i. Thus,

$$\frac{dI_i^h}{dt} = p_i \lambda_i S_i^h - (\omega_i + \mu_i + \delta_i) I_i^h. \tag{2.2}$$

The protected compartment P_i^h consists of individuals who were successfully treated and are noninfectious and immune. We assume that there is no blanket roll-out of the TBDs, so individuals

enter P_i^h after a successful treatment of infectious symptomatic or detected asymptomatic patients at a rate $c_i\omega_i$, where c_i is the probability of a successful treatment, and leave it either by the loss of protection or by natural death. Therefore, we have

$$\frac{dP_i^h}{dt} = c_i\omega_i(I_i^h + A_i^d) - (\vartheta_i + \mu_i)P_i^h. \tag{2.3}$$

The compartment of undetected asymptomatic individuals increases either through the infection of susceptibles or an unnoticed failure of the treatment and decreases following the detection of malaria in a patient at a rate of α_i or by natural death. Thus,

$$\frac{dA_i^u}{dt} = (1 - p_i)\lambda_i S_i^h + (1 - c_i)\omega_i I_i^h - (\alpha_i + \mu_i)A_i^u. \tag{2.4}$$

The class of detected asymptomatic individuals increases with the detection of infection in undetected asymptomatic individuals and decreases with treatment or natural death. We assumed that testing of asymptomatic individuals is only available in Patch 2, so the class A_1^d is empty. Hence,

$$\frac{dA_2^d}{dt} = \alpha_2 A_2^u - (c_2\omega_2 + \mu_2)A_2^d. \tag{2.5}$$

The total population of humans in Patch i is given by

$$N_i^h = S_i^h + P_i^h + I_i^h + A_i^u + A_i^d. \tag{2.6}$$

The class S_i^v of susceptible mosquitoes in Patch i increases at the constant rate Π_i and decreases by infection at the rate

$$\phi_i = \frac{a_i\beta_i^v(I_i^h + \zeta_i^u A_i^u + \zeta_i^d A_i^d)}{N_i^h}.$$

Further, susceptible and infectious mosquitoes die at a natural death rate ν_i. Thus,

$$\begin{aligned} \frac{dS_i^v}{dt} &= \Pi_i - (\phi_i + \nu_i)S_i^v, \\ \frac{dI_i^v}{dt} &= \phi_i S_i^v - \nu_i I_i^v. \end{aligned} \tag{2.7}$$

The total population of mosquitoes in the Patch i is given by

$$N_i^v = S_i^v + I_i^v. \tag{2.8}$$

After describing the dynamics in each Patch, we now consider the migration of populations between the patches. We assume that the leaders of Patch 1 do not attempt to detect asymptomatic cases coming from Patch 2, while the leaders of Patch 2 implement measures to detect asymptomatic individuals coming from Patch 1 at the border. We recall that no symptomatic individuals can migrate. In this context, within the non-infective compartments S_i^h and P_i^h, there will be outflows due to migration at the rates k_{ji} and new arrivals from Patch j at rates k_{ij}. Furthermore, undetected asymptomatic individuals from Patch 2, A_2^u, arrive at Patch 1 at a rate k_{12}, while those from Patch 1, A_1^u, leave at a rate k_{21} and arrive at the border of Patch 2. The disease is detected in a fraction α_1 of arrivals, and a fraction

θ of them is allowed to enter Patch 2. The remaining fraction $1-\alpha_1$ can be either undetected at the border post due to, e.g., faulty equipment, or slip through the border illegally. Consequently, the rate of new arrivals in A_2^d is $\theta\alpha_1 k_{21}A_1^u$, while the undetected asymptomatic individuals from Patch 1 will enter the undetected compartment A_2^u at the rate $(1-\alpha_1)k_{21}A_1^u$. On the other hand, the individuals denied the entry will return to the A_1^u at the rate $(1-\theta)\alpha_1 k_{21}A_1^u$, which combined with the emigration rate $-k_{21}A_1^u$ and natural death, gives the last term in the third equation of (2.9).

We emphasise that the described model is significantly simplified. For instance, we only considered two patches, in contrast to several papers considering multiple-Patch models, such as, e.g., [12, 13], or [14, Example 8.41]. This is because our primary interest is understanding the border screening mechanism and its impact, for which two patches were deemed to be sufficient. Including more patches would complicate the analysis (apart from the low migration case), reducing the analysis to simulations only, like in [13]. Another significant simplification is neglecting the possibility of mosquito migrations, either between geographically close adjacent patches or passively, e.g., via cargo vehicles. Here, we followed the literature, such as [12, 15], but we note that low rates of mosquito migration can be approached using the regular perturbation method employed in this paper. A model allowing the mosquito migration to play a bigger role would be significantly more complex; see general models of this type in [16], which are outside the scope of the analysis presented in this paper.

The flow chart of the model is shown in Figure 1, leading to the system (2.9). A summary of the variables and parameters is given in Tables 1 and 2.

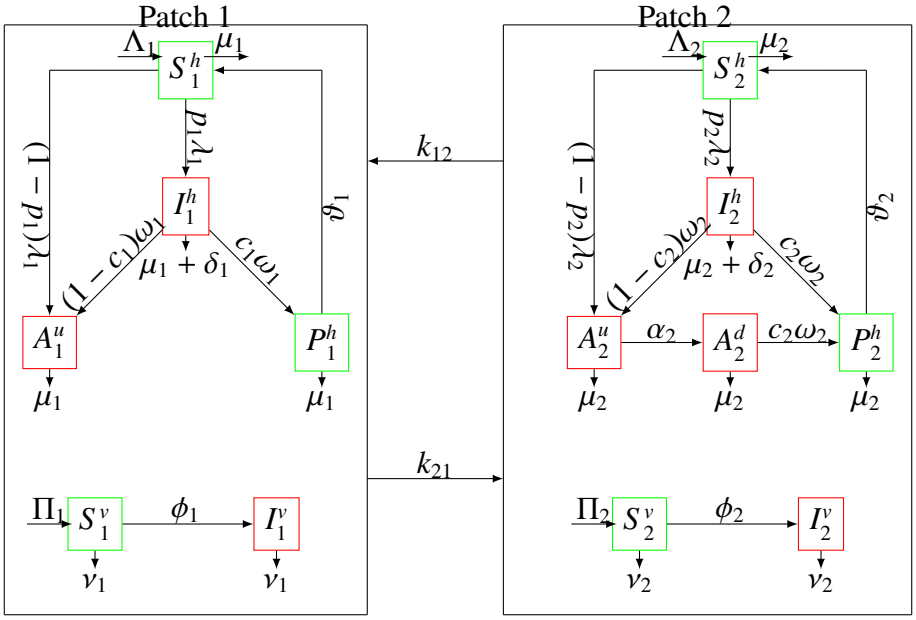

Figure 1. Flow chart showing the multi-Patch malaria transmission dynamics between human and mosquito populations.

Table 1. State variables and their description.

Variables	Description	Quasi-dimension
S_i^h	Susceptible humans in the Patch i	H
I_i^h	Infectious humans in the Patch i	H
A_i^u	Asymptomatic humans undetected in the Patch i	H
A_i^d	Asymptomatic humans detected in the Patch i	H
P_i^h	Protected (successfully treated and noninfective) humans in Patch i	H
S_i^v	Susceptible mosquitoes in the Patch i	V
I_i^v	Infectious mosquitoes in the Patch i	V

Table 2. Parameters, their description, and corresponding quasi-dimensions. We use the notation H for the dimension of the number of humans, V for the number of vectors, and T for time.

Parameters	Description	Quasi-dimension
Λ_i	Constant recruitment rate of susceptible humans in Patch i	T^{-1}
Π_i	Constant recruitment rate of susceptible mosquitoes in Patch i	T^{-1}
β_i^h	Probability of transmission from an infectious mosquito to susceptible humans during a bite in Patch i	Dimensionless
β_i^v	Probability of transmission from an infectious humans to susceptible mosquitoes during bite in Patch i	Dimensionless
a_i	Average biting rate of mosquitoes on humans in Patch i	$H(VT)^{-1}$
λ_i	Force of infection from infectious mosquitoes to susceptible humans in Patch i	T^{-1}
ϕ_i	Force of infection from infectious humans to susceptible mosquitoes in Patch i	T^{-1}
ω_i	Constant rate of treatment of infectious humans with TBD in Patch i	T^{-1}
c_i	Probability that the TBD drug confers total protection in Patch i	Dimensionless
α_1	Detection rate of asymptomatic individuals at the border of Patch 2	T^{-1}
α_2	Detection rate of asymptomatic malaria-infected individuals in Patch 2	T^{-1}
θ	Proportion of individuals testing positive and admitted to Patch 2	Dimensionless
ζ_i^u	Reduction of the infectivity of undetected asymptomatic humans in Patch i to vectors	Dimensionless
ζ_i^d	Reduction of the infectivity of detected asymptomatic humans in Patch i to mosquitoes	Dimensionless
μ_i	Natural death rate of humans in Patch i	T^{-1}
δ_i	Disease induced death rate of infectious humans in Patch i	T^{-1}
ν_i	Natural death rate of mosquitoes in Patch i	T^{-1}
ϑ_i	Rate of loss of protection to become susceptible in Patch i	T^{-1}
k_{12}	Migration rate of humans from Patch 2 to Patch 1	T^{-1}
k_{21}	The rate of individuals from Patch 1 arriving at the Patch 2 border	T^{-1}

$$
\begin{cases}
\dfrac{dS_i^h}{dt} = \Lambda_i + \vartheta_i P_i^h + k_{ij} S_j^h - \left(\lambda_i + k_{ji} + \mu_i\right) S_i^h, \quad 1 \le i, j \le 2, \\
\dfrac{dI_i^h}{dt} = p_i \lambda_i S_i^h - (\omega_i + \mu_i + \delta_i)\, I_i^h, \quad 1 \le i \le 2, \\
\dfrac{dA_1^u}{dt} = (1-p_1)\lambda_1 S_1^h + (1-c_1)\omega_1 I_1^h + k_{12} A_2^u - \left[(\theta\alpha_1 + (1-\alpha_1))k_{21} + \mu_1\right] A_1^u, \\
\dfrac{dA_2^u}{dt} = (1-p_2)\lambda_2 S_2^h + (1-c_2)\omega_2 I_2^h + (1-\alpha_1) k_{21} A_1^u - (\alpha_2 + k_{12} + \mu_2)\, A_2^u, \\
\dfrac{dA_2^d}{dt} = \theta\alpha_1 k_{21} A_1^u + \alpha_2 A_2^u - (c_2\omega_2 + \mu_2)\, A_2^d, \\
\dfrac{dP_1^h}{dt} = c_1\omega_1 I_1^h + k_{12} P_2^h - (\vartheta_1 + k_{21} + \mu_1)\, P_1^h, \\
\dfrac{dP_2^h}{dt} = c_2\omega_2 (I_2^h + A_2^d) + k_{21} P_1^h - (\vartheta_2 + k_{12} + \mu_2)\, P_2^h, \\
\dfrac{dS_i^v}{dt} = \Pi_i - (\phi_i + \nu_i)\, S_i^v, \quad 1 \le i \le 2, \\
\dfrac{dI_i^v}{dt} = \phi_i S_i^v - \nu_i I_i^v, \quad 1 \le i \le 2,
\end{cases}
\tag{2.9}
$$

with $k_{11} = k_{22} = 0$ and the initial conditions

$$
S_i^h(0) > 0, \quad I_i^h(0) \ge 0, \quad A_i^u(0) \ge 0, \quad A_2^d(0) \ge 0, \quad P_i^h(0) \ge 0, \quad S_i^v(0) > 0, \quad I_i^v(0) \ge 0, \quad \text{for } 1 \le i \le 2.
$$

Let $S = (S_1^h, P_1^h, S_1^v, S_2^h, P_2^h, S_2^v)^T \in \mathbb{R}_+^{*6}$ and $I = (I_1^h, A_1^u, I_1^v, I_2^h, A_2^u, A_2^d, I_2^v)^T \in \mathbb{R}_+^{*7}$, where $\mathbb{R}_+^{*n} = \mathbb{R}_+^n \setminus \{\mathbf{0}\} = [0, \infty)^n \setminus \{\mathbf{0}\}$. We can then rewrite Eq (2.9) in the matrix form as

$$
\begin{aligned}
\frac{dS}{dt} &= \psi_s(S, I), \\
\frac{dI}{dt} &= \psi_i(S, I),
\end{aligned}
\tag{2.10}
$$

where

$$
\psi_s(S, I) = \begin{pmatrix}
\Lambda_1 + \vartheta_1 P_1^h + k_{12} S_2^h - (\lambda_1 + k_{21} + \mu_1) S_1^h \\
c_1\omega_1 I_1^h + k_{12} P_2^h - (\vartheta_1 + k_{21} + \mu_1) P_1^h \\
\Pi_1 - (\phi_1 + \nu_1) S_1^v \\
\Lambda_2 + \vartheta_2 P_2^h + k_{21} S_1^h - (\lambda_2 + k_{12} + \mu_2) S_2^h \\
c_2\omega_2 I_2^h + k_{21} P_1^h - (\vartheta_2 + k_{12} + \mu_2) P_2^h \\
\Pi_2 - (\phi_2 + \nu_2) S_2^v
\end{pmatrix},
\tag{2.11}
$$

and

$$\psi_i(S,I) = \begin{pmatrix} p_1\lambda_1 S_1^h - (\omega_1+\mu_1+\delta_1)\, I_1^h \\ (1-p_1)\lambda_1 S_1^h + (1-c_1)\omega_1 I_1^h + k_{12}A_2^u - ((\theta\alpha_1+(1-\alpha_1))k_{21}+\mu_1)\, A_1^u \\ \phi_1 S_1^v - \nu_1 I_1^v \\ p_2\lambda_2 S_2^h - (\omega_2+\mu_2+\delta_2)\, I_2^h \\ (1-p_2)\lambda_2 S_2^h + (1-c_2)\omega_2 I_2^h + (1-\alpha_1)k_{21}A_1^u - (\alpha_2+k_{12}+\mu_2)\, A_2^u \\ \theta\alpha_1 k_{21}A_1^u + \alpha_2 A_2^u - (c_2\omega_2+\mu_2)\, A_2^d \\ \phi_2 S_2^v - \nu_2 I_2^v \end{pmatrix}. \tag{2.12}$$

3. Model analysis

Theorem 3.1. *Let $\Omega = \mathbb{R}_+^{*6} \times \mathbb{R}_+^{*7}$. For any initial condition $(S(0), I(0)) \in \Omega$, the system (2.9) has a unique globally defined solution $(S(t), I(t))$ which remains in Ω for all $t \geq 0$. Moreover, the total human, $\left(N^h(t)\right)$, and mosquito, $(N^v(t))$, populations are bounded for all $t \geq 0$.*

Proof. The function $\psi = (\psi_s, \psi_i)$ is C^1 in Ω, so the system (2.9) admits a unique local solution. In the ensemble $(S_1^h, P_1^h, S_1^v, S_2^h, P_2^h, S_2^v, I_1^h, A_1^u, I_1^v, I_2^h, A_2^u, A_2^d, I_2^v) \in \mathbb{R}_+^{*13}$, let one of the entries equals 0. Then the corresponding row on the right-hand side of Eq (2.9) is non-negative, hence, by, e.g., [14, Theorem B.21], Ω is forward-invariant. Further, after some calculations, the equation of the total population is given by

$$\frac{dN^h}{dt} = \sum_{i=1}^{2}(\Lambda_i - \mu_i N_i^h - \delta_i I_i^h).$$

Knowing that $I_i^h \leq N_i^h$, we have

$$\sum_{i=1}^{2}\Lambda_i - \sum_{i=1}^{2}(\mu_i+\delta_i)N_i^h \leq \frac{dN^h}{dt} \leq \sum_{i=1}^{2}\Lambda_i - \sum_{i=1}^{2}\mu_i N_i^h.$$

Thus,

$$\sum_{i=1}^{2}\Lambda_i - \max_{1\leq i\leq 2}\{\mu_i+\delta_i\}N^h \leq \frac{dN^h}{dt} \leq \sum_{i=1}^{2}\Lambda_i - \min_{1\leq i\leq 2}\{\mu_i\}N^h.$$

We conclude that for all $t \geq 0$,

$$\min\left\{\frac{\sum_{i=1}^{2}\Lambda_i}{\max_{1\leq i\leq 2}\{\mu_i+\delta_i\}}, N^h(0)\right\} \leq N^h(t) \leq \max\left\{\frac{\sum_{i=1}^{2}\Lambda_i}{\min_{1\leq i\leq 2}\{\mu_i\}}, N^h(0)\right\}. \tag{3.1}$$

Similarly, for all $t \geq 0$,

$$\min\left\{\frac{\sum_{i=1}^{2}\Pi_i}{\max_{1\leq i\leq 2}\{\nu_i\}}, N^v(0)\right\} \leq N^v(t) \leq \max\left\{\frac{\sum_{i=1}^{2}\Pi_i}{\min_{1\leq i\leq 2}\{\nu_i\}}, N^v(0)\right\}. \tag{3.2}$$

This ends the proof of the result.

3.1. Disease-free equilibrium point

A disease-free equilibrium (DFE) is an equilibrium solution, (S^*, I^*) of the system $\psi_s(S, 0) = \psi_i(S, 0) = 0$. Let

$$G_s = \begin{pmatrix} k_{21} + \mu_1 & -k_{12} \\ -k_{21} & k_{12} + \mu_2 \end{pmatrix}, \quad G_p = \begin{pmatrix} (\vartheta_1 + k_{21} + \mu_1) & -k_{12} \\ -k_{21} & (\vartheta_2 + k_{12} + \mu_2) \end{pmatrix}, \quad G_v = \begin{pmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{pmatrix},$$

$\Lambda = (\Lambda_1, \Lambda_2)^T$ and $\Pi = (\Pi_1, \Pi_2)^T$, $X = (S_1^h, S_2^h)^T$, $Y = (P_1^h, P_2^h)^T$, and $Z = (S_1^v, S_2^v)^T$.

Theorem 3.2. *The unique disease-free equilibrium of* (2.10) *is given by* $(S^*, 0)$*, where* $S^* = (S_1^{h*}, 0, S_2^{h*}, 0, S_1^{v*}, S_2^{v*})^T$ *with* $X^* = G_s^{-1}\Lambda$*,* $Y^* = (0, 0)^T$ *and* $Z^* = G_v^{-1}\Pi$.

Proof. At the disease-free equilibrium, the system: $\psi_s(S^*, 0) = \psi_i(S^*, 0) = 0$ is reduced to the following system

$$\begin{cases} \Lambda - G_s X^* = 0, \\ G_p Y^* = 0, \\ \Pi - G_v Z^* = 0. \end{cases} \tag{3.3}$$

It is easy to see that $Z^* = G_v^{-1}\Pi = \left(\dfrac{\Pi_1}{\nu_1}, \dfrac{\Pi_2}{\nu_2}\right)^T > 0$ and $det(G_p) \neq 0$, so that $Y^* = (0, 0)^T$.

We also observe that G_s has negative off-diagonal entries and positive column sums. It follows that G_s is a non-singular M-matrix: hence G_s^{-1} is positive, [17]. Therefore, we conclude that $X^* = G_s^{-1}\Lambda > 0$.

3.2. Basic reproduction number

Following the next-generation matrix method, (e.g., [18]), we write $\psi_i(S, I) = \mathcal{F} - \mathcal{V}$, where

$$\mathcal{F} = \begin{pmatrix} p_1\lambda_1 S_1^h \\ (1 - p_1)\lambda_1 S_1^h \\ \phi_1 S_1^v \\ p_2\lambda_2 S_2^h \\ (1 - p_2)\lambda_2 S_2^h \\ 0 \\ \phi_2 S_2^v \end{pmatrix} \tag{3.4}$$

and

$$\mathcal{V} = \begin{pmatrix} (\omega_1 + \mu_1 + \delta_1)\, I_1^h \\ -(1 - c_1)\omega_1 I_1^h - k_{12}A_2^u + ((\theta\alpha_1 + (1 - \alpha_1))k_{21} + \mu_1)\, A_1^u, \\ \nu_1 I_1^v \\ (\omega_2 + \mu_2 + \delta_2)\, I_2^h \\ -(1 - c_2)\omega_2 I_2^h - (1 - \alpha_1)k_{21}A_1^u + (\alpha_2 + k_{12} + \mu_2)\, A_2^u \\ -\theta\alpha_1 k_{21}A_1^u - \alpha_2 A_2^u + (c_2\omega_2 + \mu_2)\, A_2^d \\ \nu_2 I_2^v \end{pmatrix}. \tag{3.5}$$

To simplify the expressions, we set $Q_1 = \omega_1 + \mu_1 + \delta_1$, $Q_2 = (1 - c_1)\omega_1$, $Q_3 = \theta\alpha_1 + (1 - \alpha_1)$, $Q_4 = \omega_2 + \mu_2 + \delta_2$, $Q_5 = 1 - \alpha_1$, $Q_6 = (1 - c_2)\omega_2$, $Q_7 = \alpha_2 + \mu_2$, and $Q_8 = c_2\omega_2 + \mu_2$. The Jacobi matrices of F and V evaluated at $(S^*, 0)$ are given by

$$F = \begin{pmatrix} 0 & 0 & a_1p_1\beta_1^h & 0 & 0 & 0 & 0 \\ 0 & 0 & a_1\beta_1^h(1-p_1) & 0 & 0 & 0 & 0 \\ a_1\beta_1^v\frac{S_1^v}{S_1^h} & a_1\beta_1^v\xi_1^u\frac{S_1^v}{S_1^h} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_2p_2\beta_2^h \\ 0 & 0 & 0 & 0 & 0 & 0 & a_2\beta_2^h(1-p_2) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_2\beta_2^v\frac{S_2^v}{S_2^h} & a_2\xi_2^u\beta_2^v\frac{S_2^v}{S_2^h} & a_2\xi_2^d\beta_2^v\frac{S_2^v}{S_2^h} & 0 \end{pmatrix},$$

and

$$V = \begin{pmatrix} Q_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -Q_2 & (Q_3k_{21} + \mu_1) & 0 & 0 & -k_{12} & 0 & 0 \\ 0 & 0 & \nu_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Q_4 & 0 & 0 & 0 \\ 0 & -Q_5k_{21} & 0 & -Q_6 & (Q_7 + k_{12}) & 0 & 0 \\ 0 & -\theta\alpha_1k_{21} & 0 & 0 & -\alpha_2 & Q_8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \nu_2 \end{pmatrix}.$$

Let us write the matrix FV^{-1} in block form as $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$. The element (l, s) of FV^{-1} is interpreted as the expected number of new infections in the compartment l generated by the infectious mosquito or human originally introduced into the compartment s [12]. Hence, A is the matrix of the expected numbers of new infections in Patch 1, D is the matrix of the expected new infections in Patch 2, B is the matrix of the expected new infections due to migration from Patch 2 to Patch 1, and D is the matrix of the expected new infections due to migration from Patch 1 to Patch 2.

Using Maple, we find that

$$FV^{-1} = \begin{pmatrix} 0 & 0 & a_{13} & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{23} & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & 0 & k_{12}b_{31} & k_{12}b_{32} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & d_{14} \\ 0 & 0 & 0 & 0 & 0 & 0 & d_{24} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ k_{21}c_{41} & k_{21}c_{42} & 0 & d_{41} & d_{42} & d_{43} & 0 \end{pmatrix}$$

with $a_{13} = \dfrac{a_1p_1\beta_1^h}{\nu_1}$, $a_{23} = \dfrac{a_1(1-p_1)\beta_1^h}{\nu_1}$, $a_{31} = a_1\beta_1^vS_1^{v*}\frac{G+(Q_7+k_{12})Q_2\xi_1^u}{GQ_1S_1^{h*}}$, $a_{32} = a_1\beta_1^v\xi_1^uS_1^{v*}\frac{(Q_7+k_{12})}{GS_1^{h*}}$,

$b_{31} = a_1\beta_1^vQ_6\xi_1^u\dfrac{S_1^{v*}}{GQ_4S_1^{h*}}$, $b_{32} = a_1\beta_1^v\xi_1^u\frac{S_1^{v*}}{GS_1^{h*}}$, $c_{41} = a_2\beta_2^vS_2^{v*}Q_2\frac{\alpha_2Q_5\xi_2^d+Q_5Q_8\xi_2^u+\theta\alpha_1\xi_2^d(Q_7+k_{12})}{GQ_1Q_8S_2^{h*}}$,

$c_{42} = a_2\beta_2^vS_2^{v*}\frac{\alpha_2Q_5\xi_2^d+Q_5Q_8\xi_2^u+\theta\alpha_1\xi_2^d(Q_7+k_{12})}{GQ_8S_{h_2}}$, $d_{14} = \dfrac{a_2p_2\beta_2^h}{\nu_2}$, $d_{24} = \dfrac{a_2(1-p_2)\beta_2^h}{\nu_2}$,

$$d_{41} = a_2\beta_2^v S_2^{v*}\frac{GQ_8 + Q_6\left[(Q_3k_{21}+\mu_1)(\alpha_2\xi_2^d + Q_8\xi_2^u) + \theta\alpha_1\xi_2^d k_{12}k_{21}\right]}{GQ_4Q_8S_2^{h*}},$$

$$d_{42} = a_2\beta_2^v S_2^{v*}\frac{(Q_3k_{21}+\mu_1)(\alpha_2\xi_2^d + Q_8\xi_2^u) + \theta\alpha_1\xi_2^d k_{12}k_{21}}{GQ_8S_2^{h*}},\ d_{43} = a_2\xi_2^d S_2^{v*}\frac{\beta_2^v}{Q_8S_2^{h*}},$$

where $G = Q_3Q_7k_{21} + \theta\alpha_1k_{12}k_{21} + (Q_7 + k_{12})\mu_1$.

The characteristic polynomial of FV^{-1} is given by

$$X^3\Big(X^4 - ((a_{13}a_{31} + a_{23}a_{32}) + (d_{14}d_{41} + d_{24}d_{42}))X^2 + ((a_{13}a_{31} + a_{23}a_{32})(d_{14}d_{41} + d_{24}d_{42}) - k_{12}k_{21}(b_{31}d_{14} + b_{32}d_{24})(a_{13}c_{41} + a_{23}c_{42}))) = 0.$$

We have a triple root $X = 0$, and the other roots can be obtained by letting $X = \Gamma^2$ and solving the resulting quadratic equation. This yields the basic reproduction number

$$\mathcal{R}_c^2 = \frac{1}{2}\Big(((a_{13}a_{31} + a_{23}a_{32}) + (d_{14}d_{41} + d_{24}d_{42})) + \sqrt{(a_{13}a_{31} + a_{23}a_{32} - d_{14}d_{41} - d_{24}d_{42})^2 + 4k_{12}k_{21}(b_{31}d_{14} + b_{32}d_{24})(a_{13}c_{41} + a_{23}c_{42})}\Big),$$

which can be written as

$$\mathcal{R}_c^2 = \frac{1}{2}\left(\left(\mathcal{R}_{c,1}^2 + \mathcal{R}_{c,2}^2\right) + \sqrt{\left(\mathcal{R}_{c,1}^2 - \mathcal{R}_{c,2}^2\right)^2 + 4 \times k_{12}\mathcal{R}_{c,12}^2 \times k_{21}\mathcal{R}_{c,21}^2}\right),$$

where

$$\mathcal{R}_{c,1}^2 = a_{13}a_{31} + a_{23}a_{32},\ \mathcal{R}_{c,2}^2 = d_{14}d_{41} + d_{24}d_{42},\ \mathcal{R}_{c,12}^2 = b_{31}d_{14} + b_{32}d_{24} \text{ and } \mathcal{R}_{c,21}^2 = a_{13}c_{41} + a_{23}c_{42}.$$

3.2.1. System of isolated patches

As a preliminary step, we consider the case when there are no migrations, that is, $k_{12} = k_{21} = 0$; hence the matrices B and C are zero matrices. Consequently, we obtain

$$\mathcal{R}_{c,1}^2 = \frac{a_1^2\beta_1^h\beta_1^v S_1^{v*}}{\nu_1 S_1^{h*}}\left(\frac{p_1}{Q_1}\left(1 + \frac{Q_2}{\mu_1}\xi_1^u\right) + \frac{(1-p_1)}{\mu_1}\xi_1^u\right), \tag{3.6}$$

$$\mathcal{R}_{c,2}^2 = \frac{a_2^2\beta_2^h\beta_2^v S_2^{v*}}{\nu_2 S_2^{h*}}\left(\frac{p_2}{Q_4}\left(1 + \frac{Q_6}{Q_7}\xi_2^u + \frac{Q_6}{Q_7Q_8}\alpha_2\xi_2^d\right) + \frac{(1-p_2)}{Q_7}\left(\xi_2^u + \frac{\alpha_2}{Q_8}\xi_2^d\right)\right) \tag{3.7}$$

and

$$\mathcal{R}_c^2 = \max\{\mathcal{R}_{c,1}^2, \mathcal{R}_{c,2}^2\}. \tag{3.8}$$

Remark 3.1.

1) *Since* $\mathcal{R}_c^2 > 1$ *does not imply* $\mathcal{R}_{c,1}^2 > 1$ *and* $\mathcal{R}_{c,2}^2 > 1$, $\mathcal{R}_c^2 > 1$ *does not mean that the disease persists in both patches. However, if* $\mathcal{R}_c^2 < 1$, *then both* $\mathcal{R}_{c,i}^2$ *values are less than* 1, *and the disease-free equilibrium of each Patch is locally asymptotically stable. Note that the disease-free equilibrium of each Patch is unstable if* $\min\mathcal{R}_{c,i}^2 > 1$.

2) *The control reproduction number $\mathcal{R}^2_{c,i}$ in each Patch is a decreasing function of the probability of total protection c_i.*

3) *The control reproduction number $\mathcal{R}^2_{c,2}$ of the isolated Patch 2 is the decreasing function of the testing rate α_2.*

4) *The analysis of the basic reproduction number shows that any control measure in Patch 2 will also impact Patch 1. We present numerical simulations of our model for different values of the control parameters to see if the applied control measure will bring more benefits for Patch 2.*

3.2.2. Global stability of DFE in the case of isolated patches

In this subsection, we show the global asymptotic stability of the disease-free equilibrium using the approach outlined in [19] (see also [20, 21]). In the case considered, we have two separate but slightly different evolutions in non-communicating patches. To shorten the notation, we continue using the form (2.9), that is, treating it as a single system but with the understanding that we consider two non communicating systems. First, we specify Eqs (3.1) and (3.2) for the current case. We have, for $i = 1, 2$

$$\min\left\{\frac{\Lambda_i}{\mu_i+\delta_i}, N_i^h(0)\right\} \le N_i^h(t) \le \max\left\{\frac{\Lambda_i}{\mu_i}, N_i^h(0)\right\}, \tag{3.9}$$

and

$$\min\left\{\frac{\Pi_i}{\nu_i}, N_i^v(0)\right\} \le N_i^v(t) \le \max\left\{\frac{\Pi_i}{\nu_i}, N_i^v(0)\right\}. \tag{3.10}$$

We must check that Eq (2.9) satisfies the assumptions of [19, Theorem 4.3]. For this, we write the system in the form required by that theorem, introducing the state variables $(x, y)^T$, where $x = (S_1^h, P_1^h, S_1^v, S_2^h, P_2^h, S_2^v)^T$ and $y = (A_1^u, I_1^h, I_1^v, A_2^u, A_2^d, I_2^h, I_2^v)^T$, corresponding to the non-infected and infected human and mosquito populations, respectively. We then rewrite Eq (2.9) as

$$\begin{cases} x' &= A_1(x,0)(x-x^*) &+ A_{12}(x,y)y, \\ y' &= &+ A_2(x,y)y, \end{cases} \tag{3.11}$$

where

$$\boldsymbol{A_1(x,0)} = \begin{pmatrix} \boldsymbol{-\mu_1} & \boldsymbol{\vartheta_1} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{-(\vartheta_1+\mu_1)} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{-\nu_1} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{-\mu_2} & \boldsymbol{\vartheta_2} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{-(\vartheta_2+\mu_2)} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{-\nu_2} \end{pmatrix}, \tag{3.12}$$

$$A_{12}(x,y)=\begin{pmatrix} 0 & 0 & -\frac{a_1\beta_1^h S_1^h}{N_1^h} & 0 & 0 & 0 & 0\\ 0 & c_1\omega_1 & 0 & 0 & 0 & 0 & 0\\ -\zeta_1^u\frac{a_1\beta_1^v S_1^v}{N_1^h} & -\frac{a_1\beta_1^v S_1^v}{N_1^h} & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{a_2\beta_2^h S_2^h}{N_2^h}\\ 0 & 0 & 0 & 0 & c_2\omega_2 & c_2\omega_2 & 0\\ 0 & 0 & 0 & -\zeta_2^u\frac{a_2\beta_2^v S_2^v}{N_2^h} & -\zeta_2^d\frac{a_2\beta_2^v S_2^v}{N_2^h} & -\frac{a_2\beta_2^v S_2^v}{N_2^h} & 0 \end{pmatrix}, \tag{3.13}$$

and

$$A_2(x,y)=\begin{pmatrix} -\mu_1 & Q_2 & (1-p_1)\frac{a_1\beta_1^h S_1^h}{N_1^h} & 0 & 0 & 0 & 0\\ 0 & -Q_1 & p_1\frac{a_1\beta_1^h S_1^h}{N_1^h} & 0 & 0 & 0 & 0\\ \zeta_1^u\frac{a_1\beta_1^v S_1^v}{N_1^h} & \frac{a_1\beta_1^v S_1^v}{N_1^h} & -\nu_1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -(Q_7+k_{12}) & 0 & Q_6 & (1-p_2)\frac{a_2\beta_2^h S_2^h}{N_2^h}\\ 0 & 0 & 0 & \alpha_2 & -Q_8 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & -Q_4 & p_1\frac{a_2\beta_2^h S_2^h}{N_2^h}\\ 0 & 0 & 0 & \zeta_2^u\frac{a_2\beta_2^v S_2^v}{N_2^h} & \zeta_2^d\frac{a_2\beta_2^v S_2^v}{N_2^h} & \frac{a_2\beta_2^v S_2^v}{N_2^h} & -\nu_2 \end{pmatrix}. \tag{3.14}$$

Let us show that the assumptions $\mathbf{H}_1$ to $\mathbf{H}_5$ of [19, Theorem 4.3] are satisfied.

$\mathbf{H}_1$: Let us define $\Gamma_\epsilon=\Gamma_{1,\epsilon}\times\Gamma_{2,\epsilon}$ as a subset of Ω with $\Gamma_{1,\epsilon}$ and $\Gamma_{2,\epsilon}$ defined as follows:

$$\begin{aligned}\Gamma_{1,\epsilon}&=\left\{\left(S_1^h,P_1^h,S_1^v,A_1^u,I_1^h,I_1^v\right)\in\mathbb{R}_+^6:\frac{\Lambda_1}{\mu_1+\delta_1}-\epsilon\le N_1^h\le\frac{\Lambda_1}{\mu_1}+\epsilon,\ N_1^v\le\frac{\Pi_1}{\nu_1}+\epsilon\right\},\\ \Gamma_{2,\epsilon}&=\left\{\left(S_2^h,P_2^h,S_2^v,A_2^u,A_2^d,I_2^h,I_2^v\right)\in\mathbb{R}_+^7:\frac{\Lambda_2}{\mu_2+\delta_2}-\epsilon\le N_2^h\le\frac{\Lambda_2}{\mu_2}+\epsilon,\ N_2^v\le\frac{\Pi_2}{\nu_2}+\epsilon\right\},\end{aligned} \tag{3.15}$$

with some small $\epsilon>0$. The sets $\Gamma_{i,\epsilon}$, $i=1,2$, are compact and, by Eqs (3.9) and (3.10), invariant and absorbing. Hence, Eq (3.11) is dissipative on Ω and, therefore, if we prove that the DFE for

Eq (3.11) is globally asymptotically stable (GAS) on Γ_ϵ, it is also GAS on Ω, as any trajectory originating outside Γ_ϵ must enter the interior of Γ_ϵ and thus will be attracted to the DFE.

$\mathbf{H}_2$: The DFE $\boldsymbol{x}^*$ of the subsystem $\boldsymbol{x}' = A_1(\boldsymbol{x}, 0)(\boldsymbol{x} - \boldsymbol{x}^*)$ of Eq (3.11) is globally asymptotically stable on Γ_ϵ.

$\mathbf{H}_3$: $A_2(\boldsymbol{x}, \boldsymbol{y})$ is a Metzler matrix whose nonzero diagonal blocks represent the isolated patches. This matrix is reducible, but, recalling that we look at each Patch separately, we see in Figure 2 that each diagonal block is irreducible.

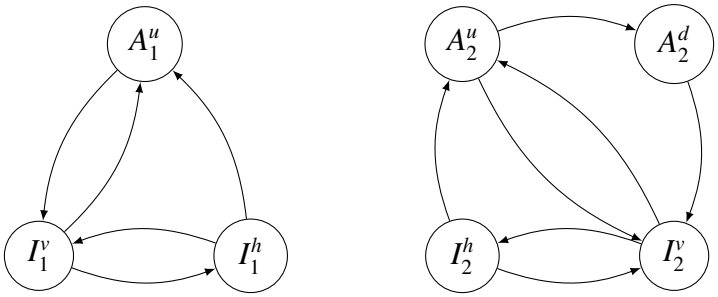

Figure 2. Digraphs associated with the matrix $A_2(x, y)$.

$\mathbf{H}_4$: We have $\frac{S_i^h}{N_i^h} \leq 1, i = 1, 2$, over Ω. On the other hand, we only have

$$\frac{S_i^v}{N_i^h} \leq \frac{(\Pi_i + \epsilon\nu_i)(\mu_i + \delta_i)}{\nu_i(\Lambda_i - \epsilon(\mu_i + \delta_i))} =: \hat{\xi}_{i,\epsilon}$$

on $\Gamma_{i,\epsilon}$. Moreover, clearly, $\frac{S_i^v}{N_i^h} = \hat{\xi}_{i,\epsilon}$ is realised on $\Gamma_{i,\epsilon}$ but it does not have to be on the disease-free manifold. Thus, we consider an arbitrary $\xi_{i,\epsilon} = \hat{\xi}_{i,\epsilon} + \epsilon$ so that we have

$$\frac{S_i^v}{N_i^h} < \xi_{i,\epsilon}$$

on $\Gamma_{i,\epsilon}$, that is, the upper bound is not achieved on $\Gamma_{i,\epsilon}$. Thus, the upper-bound matrix

$$\bar{A}_{2,\epsilon} = \begin{pmatrix} L_\epsilon & 0 \\ 0 & P_\epsilon \end{pmatrix}$$

$$= \begin{pmatrix} -\mu_1 & Q_2 & (1-p_1)a_1\beta_1^h & 0 & 0 & 0 & 0 \\ 0 & -Q_1 & p_1 a_1 \beta_1^h & 0 & 0 & 0 & 0 \\ \zeta_1^u a_1 \beta_1^v \xi_{1,\epsilon} & a_1\beta_1^v\xi_{1,\epsilon} & -\nu_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -(Q_7 + k_{12}) & 0 & Q_6 & (1-p_2)a_2\beta_2^h \\ 0 & 0 & 0 & \alpha_2 & -Q_8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -Q_4 & p_2 a_2 \beta_2^h \\ 0 & 0 & 0 & \zeta_2^u a_2 \beta_2^v \xi_{2,\epsilon} & \zeta_2^d a_2 \beta_2^v \xi_{2,\epsilon} & a_2\beta_2^v\xi_{2,\epsilon} & -\nu_2 \end{pmatrix}$$

satisfies assumption $\mathbf{H}_4$ of [19, Theorem 4.3].

$\mathbf{H}_5$: The stability of the $\bar{A}_{2,\epsilon}$ matrix is determined by the stability of the matrices L_ϵ and P_ϵ representing, respectively, Patches 1 and 2. Matrix L_ϵ is a Metzler matrix, so, by decomposing it into block matrices and using [19, Proposition 3.1], we find that L_ϵ is Metzler stable if and only if

$$\mathcal{R}_{c,1}^2 < \frac{1}{\xi_{1,\epsilon}} \frac{S_1^{v*}}{S_1^{h*}} = \frac{1}{\xi_{1,\epsilon}} \frac{\Pi_1 \mu_1}{\Lambda_1 \nu_1}. \tag{3.16}$$

Now,

$$\xi_{1,\epsilon} \to \frac{\Pi_1(\mu_1 + \delta_1)}{\Lambda_1 \nu_1}, \quad \epsilon \to 0.$$

Therefore, if

$$\mathcal{R}_{c,1}^2 < \frac{\mu_1}{\mu_1 + \delta_1},$$

then (3.16) is satisfied if we take a sufficiently small positive ϵ. In a similar way, we find that if

$$\mathcal{R}_{c,2}^2 < \frac{\mu_2}{\mu_2 + \delta_2},$$

then P_ϵ is Metzler stable if ϵ is small enough. Hence, if $\mathcal{R}_c^2 = \max\left\{\mathcal{R}_{c,1}^2, \mathcal{R}_{c,2}^2\right\} < \min\left\{\frac{\mu_1}{\mu_1 + \delta_1}, \frac{\mu_2}{\mu_2 + \delta_2}\right\}$, then, for some $\epsilon > 0$, $\bar{A}_{2,\epsilon}$ is Metzler stable and assumption $\mathbf{H}_5$ of [19, Theorem 4.3] is satisfied.

We can, therefore, state the following theorem.

Theorem 3.3 (Global stability of DFE)**.** *In the case of isolated patches, the DFE of the model* (2.9) *is globally asymptotically stable in* $\mathbb{R}_+^{*13}$ *when* $\mathcal{R}_c^2 < \min\left\{\frac{\mu_1}{\mu_1 + \delta_1}, \frac{\mu_2}{\mu_2 + \delta_2}\right\}$.

4. Bifurcation analysis

4.1. Case of isolated patches

In this section, we continue analysing the case of isolated patches, $k_{12} = k_{21} = 0$, and we look at the existence of a bifurcation in the isolated Patch 2. We follow the approach of [22].

The endemic equilibrium of the isolated Patch 2 is the solution of the following system:

$$\begin{cases} \Lambda_2 + \vartheta_2 P_2^{h\bullet} - \left(\lambda_2^\bullet + \mu_2\right) S_2^{h\bullet} = 0, \\ p_2 \lambda_2^\bullet S_2^{h\bullet} - (\omega_2 + \mu_2 + \delta_2) I_2^{h\bullet} = 0, \\ (1 - p_2)\lambda_2^\bullet S_2^{h\bullet} + (1 - c_2)\omega_2 I_2^{h\bullet} - (\alpha_2 + \mu_2) A_2^{u\bullet} = 0, \\ \alpha_2 A_2^{u\bullet} - (c_2\omega_2 + \mu_2) A_2^{d\bullet} = 0, \\ c_2\omega_2(I_2^{h\bullet} + A_2^{d\bullet}) - (\vartheta_2 + \mu_2) P_2^{h\bullet} = 0, \\ \Pi_2 - \left(\phi_2^\bullet + \nu_2\right) S_2^{v\bullet} = 0, \\ \phi_2^\bullet - \nu_2 I_2^{v\bullet} = 0. \end{cases} \tag{4.1}$$

Let us set

$$Q_9 = \vartheta_2 + \mu_2, \quad K_1 = \frac{Q_6 + \frac{(1-p_2)Q_4}{p_2}}{Q_7}, \quad K_2 = \frac{\left(Q_6 + \frac{P_2Q_4}{p_2}\right)\alpha_2}{Q_7Q_8}, \quad K_3 = \frac{\left(1 + \frac{\left(Q_6+\frac{P_2Q_4}{p_2}\right)\alpha_2}{Q_7Q_8}\right)\alpha_2c_2}{Q_9},$$
$$K_4 = 1 + \zeta_2^u K_1 + \zeta_2^d K_2, \quad \mathcal{T} = \frac{a_2p_2\delta_2\beta_2^h N_2^{v*}}{\Lambda_2 Q_4}.$$

To find the endemic equilibria (non-zero infectious states), we will express all the state variables in terms of $I_2^{h\bullet}$ and then discuss the existence of an endemic equilibrium in terms of the existence of the number of positive values of $I_2^{h\bullet}$. After some algebra, we obtain

$$N_2^{h\bullet} = \frac{\Lambda_2 - \delta_2 I_2^{h\bullet}}{\mu_2}, \quad I_2^{v\bullet} = \frac{a_2\beta_2^v N_2^{v*} K_4 I_2^{h\bullet}}{a_2\beta_2^v K_4 I_2^{h\bullet} + \nu_2 N_2^{h\bullet}}, \quad A_2^{u\bullet} = K_1 I_2^{h\bullet}, \quad A_2^{d\bullet} = K_2 I_2^{h\bullet}, \quad P_2^{h\bullet} = K_3 I_2^{h\bullet}. \tag{4.2}$$

Replacing $S_2^{h\bullet}$ by $N_2^{h\bullet} - P_2^{h\bullet} - I_2^{h\bullet} - A_2^{d\bullet} - A_2^{u\bullet}$, from the second equation of (4.1), we obtain

$$I_2^{h\bullet}\left[q_2\left(I_2^{h\bullet}\right)^2 + q_1 I_2^{h\bullet} + q_0\right] = 0, \tag{4.3}$$

where

$$q_2 = -\frac{\delta_2\nu_2\Lambda_2 Q_4^2}{a p_2\beta_2^h N_2^{v*}}(\mathcal{R}_{c,2}^2 - \mathcal{T}),$$
$$q_1 = a_2\beta_2^h p_2\left[a_2p_2\mu_2^2\beta_2^v K_4 N_2^{v*}(K_1 + K_2 + K_3) + (\mu_2 + \delta_2)\right] + \Lambda_2 Q_4\left(\mathcal{R}_{c,2}^2 - 2\mathcal{T}\right),$$
$$q_0 = \nu_2\Lambda_2^2 Q_4\left(1 - \mathcal{R}_{c,2}^2\right).$$

From Eq (4.3), we obtain the disease-free equilibrium $I_2^{h\bullet} = 0$, or

$$q_2\left(I_2^{h\bullet}\right)^2 + q_1 I_2^{h\bullet} + q_0 = 0. \tag{4.4}$$

We are interested in the number of positive solutions of Eq (4.4) that correspond to the number of endemic equilibria of our model.

We begin with the case of $\delta_2 = 0$. Then $N_2^{h\bullet}$ and all components of the endemic equilibria are positive, and we have the following result.

Theorem 4.1. *If $\delta_2 = 0$, then Eq* (4.4) *has*

1) *No positive solution if $\mathcal{R}_{c,2}^2 < 1$;*
2) *A unique positive solution if $\mathcal{R}_{c,2}^2 > 1$.*

Next, consider the case of $\delta_2 \neq 0$, and let $I_h^{\#} = \frac{\Lambda_2}{\delta_2}$. To make sure that $N_2^{h\bullet} = \frac{\delta_2(I_h^{\#} - I_2^{h\bullet})}{\mu_2}$ is non-negative, we must find $I_2^{h\bullet} \in \left[0, I_h^{\#}\right]$. To do this, we define $x^{\bullet} = \frac{I_h^{\#}}{I_2^{h\bullet}} - 1$ and look for a non-negative

solution $x^\bullet$ of the resulting equation, which will guarantee that $I_2^{h\bullet} \in \left[0, I_h^\#\right]$, and hence $N_2^{h\bullet}$ is positive. Thus, replacing $I_2^{h\bullet}$ by $\dfrac{I_h^\#}{x^\bullet + 1}$ in the Eq (4.4), we obtain

$$a\,(x^\bullet)^2 + bx^\bullet + c = 0, \tag{4.5}$$

where

$$\begin{cases} a = \delta_2 q_0 = \delta_2 \nu_2 \Lambda_2^2 Q_4 \left(1 - \mathcal{R}_{c,2}^2\right), \\ b = 2q_0 + q_1 I_h^\# = a_2 \mu_2 \beta_2^v \Lambda_2 K_5 \left[a_2 \beta_2^h p_2 N_v^* \left(\mu_2(1 + K_1 + K_2 + K_3) - \delta_2\right) + \Lambda_2 Q_4\right], \\ c = I_h^\# \left(q_2 I_h^\# + q_1\right) + q_0 = (1 + K_1 + K_2 + K_3)\, K_5 \Lambda_2 N_v^* a^2 \beta_h \beta_v \mu_2^2 p_2. \end{cases}$$

Remark 4.1. *The sign of b depends on δ_2. We define δ_2^* to be the critical value such that $b > 0$ if $\delta_2 < \delta_2^*$, and $b < 0$ otherwise.*

The following result summarises the different cases.

Theorem 4.2. *Equation* (4.5) *has*

1) *One positive solution if $\mathcal{R}_{0,2}^2 > 1$;*
2) *One double positive solution if*
   a. *$\mathcal{R}_{0,2}^2 < 1$, $b^2 - 4ac = 0$, and $\delta_2 > \delta_2^*$;*
   b. *$\mathcal{R}_{0,2}^2 > 1$, $b^2 - 4ac = 0$, and $\delta_2 < \delta_2^*$;*
3) *Two positive solutions if $\mathcal{R}_{0,2}^2 < 1$, $\delta_2 > \delta_2^*$, and $b^2 - 4ac > 0$;*
4) *No positive solution otherwise.*

Remark 4.2.

1) *Under the assumptions of Theorem 4.2, Point 3), one of which is high disease-induced mortality, a backward bifurcation occurs. This means that there is an endemic equilibrium for $\mathcal{R}_{0,2}^2 < 1$.*

2) *It is mentioned in [12] that if the number of individuals moving from one Patch to another is very small, then if a backward bifurcation appears in an isolated Patch, it can occur in the multi-Patch model. We shall provide a more detailed description of this situation below.*

3) *The result of Patch 1 is similar that of Patch 2.*

4.2. Analysis of the low migration rates

As mentioned above, [12] commented on the case of low migrations. Here, we show that, the dynamics of (2.9) with low migration rates k_{ij} is indeed the same as that in the case of isolated patches.

Thus, we consider (2.9) with k_{ij} replaced by $\epsilon\hat{k}_{ij}$, written in the form

$$\begin{cases} \dfrac{d\boldsymbol{S}^h}{dt} &= \boldsymbol{f}^S(\boldsymbol{S}^h, \boldsymbol{P}^h, \boldsymbol{I}^v) + \epsilon\widehat{\mathcal{K}^S}\boldsymbol{S}^h, \\ \dfrac{d\boldsymbol{A}}{dt} &= \boldsymbol{f}^A(\boldsymbol{S}^h, \boldsymbol{A}, \boldsymbol{I}^h, \boldsymbol{I}^v) + \epsilon\widehat{\mathcal{K}^A}\boldsymbol{A}, \\ \dfrac{d\boldsymbol{P}}{dt} &= \boldsymbol{f}^P(\boldsymbol{I}^h, \boldsymbol{A}, \boldsymbol{P}) + \epsilon\widehat{\mathcal{K}^P}\boldsymbol{P}, \\ \dfrac{d\boldsymbol{I}^h}{dt} &= \boldsymbol{f}^I(\boldsymbol{S}^h, \boldsymbol{I}^h), \\ \dfrac{d\boldsymbol{S}^v}{dt} &= \boldsymbol{g}^S(\boldsymbol{I}^h, \boldsymbol{A}, \boldsymbol{S}^v), \\ \dfrac{d\boldsymbol{I}^v}{dt} &= \boldsymbol{g}^I(\boldsymbol{I}^h, \boldsymbol{A}, \boldsymbol{S}^v, \boldsymbol{I}^v), \end{cases} \tag{4.6}$$

where $\boldsymbol{S}^h = (S_1^h, S_2^h)$, $\boldsymbol{I}^h = (I_1^h, I_2^h)$, $\boldsymbol{A} = (A_1^u, A_2^u, A_2^d)$, $\boldsymbol{P}^h = (P_1^h, P_2^h)$, $\boldsymbol{S}^v = (S_1^v, S_2^v)$, and $\boldsymbol{I}^v = (I_1^v, I_2^v)$,

$$\widehat{\mathcal{K}^S} = \widehat{\mathcal{K}^P} = \begin{pmatrix} -\hat{k}_{12} & \hat{k}_{21} \\ \hat{k}_{12} & -\hat{k}_{21} \end{pmatrix}, \quad \widehat{\mathcal{K}^A} = \begin{pmatrix} -(\theta\alpha_1 + (1-\alpha_1))\hat{k}_{12} & \hat{k}_{21} & 0 \\ (1-\alpha_1)\hat{k}_{12} & -\hat{k}_{21} & 0 \\ \theta\alpha_1\hat{k}_{21} & 0 & 0 \end{pmatrix}.$$

The "hatted" variables $\hat{k}_{ij}, i, j = 1, 2$, are supposed to be of the same magnitude as the parameters of the leading mechanisms driving the process.

The necessary tools for the analysis of (4.6) are recalled and further developed in Appendix. We will specify these results for the current context, focusing on the stability and the basin of attraction of the disease-free equilibrium and the existence of backward bifurcation in (4.6), based on the corresponding results in the case of isolated patches.

To specify Proposition A.2 to (2.9) (with the notation of (2.10)) and (4.6), we define the following modifications of the sets $\Gamma_{1,\epsilon}$ and $\Gamma_{2,\epsilon}$ in (3.15)

$$\Gamma_{1,\rho} = \left\{ \left(S_1^h, P_1^h, S_1^v, A_1^u, I_1^h, I_1^v\right) \in \mathbb{R}_+^6 : \rho \le N_1^h \le \frac{1}{\rho},\ N_1^v \le \frac{1}{\rho} \right\},$$

and

$$\Gamma_{2,\rho} = \left\{ \left(S_2^h, P_2^h, S_2^v, A_2^u, A_2^d, I_2^h, I_2^v\right) \in \mathbb{R}_+^7 : \rho \le N_2^h \le \frac{1}{\rho},\ N_2^v \le \frac{1}{\rho} \right\}.$$

Since for any $\rho > 0$, the set $\Gamma_{1,\rho} \times \Gamma_{2,\rho}$ is a compact subset of the basin of attraction of the DFE $(\boldsymbol{S}_0^*, \boldsymbol{0})$ of (2.10), we obtain the following corrolary

Corollary 4.1. *Assume that the assumptions of Theorem 3.3 hold particulary,*

$$\mathcal{R}_c^2 < min\left\{ \frac{\mu_1}{\mu_1 + \delta_1}.\frac{\mu_2}{\mu_2 + \delta_2} \right\},$$

Let ϵ be defined as in (4.6)*. Then, for any sufficiently small ϵ, the unique disease-free equilibrium $(\boldsymbol{S}_\epsilon^*, \boldsymbol{0})$ of* (2.10)*, given by Theorem 3.2, is $O(\epsilon)$-close to the disease-free equilibrium $(\boldsymbol{S}_0^*, \boldsymbol{0})$ in the*

case of isolated patches case, where, ()see (2.10)*),* $\boldsymbol{S}_0^* = \left(S_{01}^{h*}, \mathbf{0}, S_{02}^{h*}, 0, S_{01}^{v*}, S_{02}^{v*}\right)^T$*, with* $\left(S_{01}^{h*}, S_{02}^{h*}\right)^T = \left(\frac{\Lambda_1}{\mu_1}, \frac{\Lambda_2}{\mu_1}\right)^T$ *and* $\left(S_{01}^{v*}, S_{02}^{v*}\right)^T = \left(\frac{\Pi_1}{\nu_1}, \frac{\Pi_2}{\nu_2}\right)^T$*. Moreover,* $(\boldsymbol{S}_\epsilon^*, \mathbf{0})$ *is locally asymptotically stable, and for any* $\rho > 0$*, there an* ϵ_0 *such that for any* $\epsilon < \epsilon_0$*,* $\Gamma_{1,\rho} \times \Gamma_{2,\rho}$ *is in the domain of attraction of* $(\boldsymbol{S}_\epsilon^*, \mathbf{0})$*.*

Next, we apply the regular perturbation theorem to endemic equilibria. For instance, under the assumptions of Theorem 4.2, Point 3, we see that a backward bifurcation occurs, and we have two positive solutions to (4.1) in Patch 2; a corresponding analysis can be carried out for Patch 1.

Corollary 4.2. *Assume that the assumptions of Theorem 4.2, Point 3, are satisfied in, say, Patch 2; that is, we have two positive solutions* (4.4)*,* $I_{21}^{h\bullet}$ *and* $I_{22}^{h\bullet}$*. Then, for sufficiently small* ϵ*, there are endemic equilibria* $I_{21,\epsilon}^{h\bullet}$ *and* $I_{22,\epsilon}^{h\bullet}$ *of* (4.6)*. Moreover, if* $I_{21}^{h\bullet}$ *and* $I_{22}^{h\bullet}$ *are hyperbolic, then so are* $I_{21,\epsilon}^{h\bullet}$ *and* $I_{22,\epsilon}^{h\bullet}$ *and, respectively, they have the same stability properties. Moreover, any compact subset of the domain of attraction of the stable endemic equilibrium of* $I_{2i}^{h\bullet}$ *is in the domain of attraction* $I_{2i,\epsilon}^{h\bullet}$ *for a sufficiently small* ϵ*.*

Proof. For the endemic equilibria of (4.6), we solve (4.1) (coupled with the corresponding system in Patch 1 with $O(\epsilon)$ terms on the right-hand side (precisely, apart from the second (for I_2^h) and the last two (for S_2^v and I_2^v) equations. Since mosquitoes do not migrate, the expression for N_2^{v*} in (4.2) does not change. Components $N_2^{h\bullet}, A_2^{u\bullet}, A_2^{d\bullet}$, and $P_2^{h\bullet}$ are found from linear equations, and thus we have

$$N_2^{h\bullet} = \frac{\Lambda_2 - \delta_2 I_2^{h\bullet}}{\mu_2} + O(\epsilon), \quad A_2^{u\bullet} = K_1 I_2^{h\bullet} + O(\epsilon), \quad A_2^{d\bullet} = K_2 I_2^{h\bullet} + O(\epsilon), \quad P_2^{h\bullet} = K_3 I_2^{h\bullet} + O(\epsilon),$$

where $O(\epsilon)$ depends linearly on $I_2^{h\bullet}$ and $I_1^{h\bullet}$. Then

$$I_2^{v\bullet} = \frac{a_2 \beta_2^v N_2^{v*} K_4 I_2^{h\bullet} + O(\epsilon)}{a_2 \beta_2^v K_4 I_2^{h\bullet} + O(\epsilon) + \nu_2 N_2^{h\bullet}},$$

with $O(\epsilon)$ as above, and

$$I_2^{h\bullet}\left[q_2\left(I_2^{h\bullet}\right)^2 + q_1 I_2^{h\bullet} + q_0\right] + O(\epsilon) = 0, \tag{4.7}$$

where the $O(\epsilon)$ term is cubic in $I_2^{h\bullet}$ (and linear in $I_1^{h\bullet}$). Under the assumptions of Theorem 4.2, Point 3, there are two isolated positive solutions to (4.7) with $\epsilon = 0$ and thus two isolated positive solutions to (4.7) for a small ϵ. The remaining part of the theorem follows from Proposition A.2.

4.3. Case of one-way migration (from Patch 1 to Patch 2)

In this section, we suppose that $k_{12} = 0$; that is, the migration from Patch 2 to Patch 1 is negligible. This is often the case when we neglect local cross-border traffic and consider economically driven migration from less-developed to better-developed regions. Setting $k_{12} = 0$ in this case can be justified by the regular perturbation theory, as in Section 4.2. The equations for the endemic equilibrium, in this case, can be written as

$$\begin{cases} \Lambda_1 + \vartheta_1 P_1^{h\bullet} - \left(\lambda_1^\bullet + k_{21} + \mu_1\right) S_1^{h\bullet} = 0, \\ p_1 \lambda_1^\bullet S_1^{h\bullet} - (\omega_1 + \mu_1 + \delta_1)\, I_1^{h\bullet} = 0, \\ (1-p_1)\lambda_1^\bullet S_1^{h\bullet} + (1-c_1)\omega_1 I_1^{h\bullet} - \left[(\theta\alpha_1 + (1-\alpha_1))k_{21} + \mu_1\right] A_1^{u\bullet} = 0, \\ c_1\omega_1 I_1^{h\bullet} - (\vartheta_1 + k_{21} + \mu_1)\, P_1^{h\bullet} = 0, \\ \Pi_1 - \left(\phi_1^\bullet + \nu_1\right) S_1^{v\bullet} = 0, \\ \phi_1^\bullet S_1^{v\bullet} - \nu_1 I_1^{v\bullet} = 0, \end{cases} \tag{4.8}$$

and

$$\begin{cases} \Lambda_2 + \vartheta_2 P_2^{h\bullet} + k_{21} S_1^{h\bullet} - \left(\lambda_2^\bullet + \mu_2\right) S_2^{h\bullet} = 0, \\ p_2 \lambda_2^\bullet S_2^{h\bullet} - (\omega_2 + \mu_2 + \delta_2)\, I_2^{h\bullet} = 0, \\ (1-p_2)\lambda_2^\bullet S_2^{h\bullet} + (1-c_2)\omega_2 I_2^{h\bullet} + (1-\alpha_1)k_{21} A_1^{u\bullet} - (\alpha_2 + \mu_2)\, A_2^{u\bullet} = 0, \\ \alpha_2 A_2^{u\bullet} + \theta\alpha_1 k_{21} A_1^{u\bullet} - (c_2\omega_2 + \mu_2)\, A_2^{d\bullet} = 0, \\ c_2\omega_2(I_2^{h\bullet} + A_2^{d\bullet}) + k_{21} P_1^{h\bullet} - (\vartheta_2 + \mu_2)\, P_2^{h\bullet} = 0, \\ \Pi_2 - \left(\phi_2^\bullet + \nu_2\right) S_2^{v\bullet} = 0, \\ \phi_2^\bullet S_2^{v\bullet} - \nu_2 I_2^{v\bullet} = 0. \end{cases} \tag{4.9}$$

Using (4.8), we first determine $N_1^{h\bullet},\ P_1^{h\bullet},\ A_1^{u\bullet},\ I_1^{h\bullet},\ S_1^{v\bullet}$ and, $I_1^{v\bullet}$ and then use them as constants in (4.9) to determine $N_2^{h\bullet},\ P_2^{h\bullet},\ A_2^{u\bullet},\ A_2^{d\bullet},\ I_2^{h\bullet},\ S_2^{v\bullet}$ and $I_2^{v\bullet}$. Let

$$Z_1 = \frac{Q_2 + \dfrac{1-p_1}{p_1} Q_1}{Q_3 k_{21} + \mu_1}, \quad Z_2 = \frac{c_1\omega_1}{\vartheta_1 + k_{21} + \mu_1}, \quad Z_3 = 1 + \zeta_1^u Z_1, \quad Z_4 = (1 + (1-\theta)\alpha_1 Z_1)\, k_{21},$$
$$I_1^{v\bullet} = \frac{a_1\beta_1^v N_1^{v*} Z_3 I_1^{h\bullet}}{a_1\beta_1^v Z_3 I_1^{h\bullet} + \nu_1 N_1^{h\bullet}}, \quad A_1^{u\bullet} = Z_1 I_1^{h\bullet}, \quad P_1^{h\bullet} = Z_2 I_1^{h\bullet}.$$

Replacing $S_1^{h\bullet}$ by $N_1^{h\bullet} - P_1^{h\bullet} - I_1^{h\bullet} - A_1^{u\bullet}$ in the second equation of (4.8), we obtain

$$I_1^{h\bullet} = \frac{\nu_1 Q_1 N_1^{h\bullet} N_1^{h*} \left(\mathcal{R}_{c,1}^2 - \dfrac{N_1^{h\bullet}}{N_1^{h*}}\right)}{\left[a_1\beta_1^h p_1 N_1^{v\bullet}(1 + Z_1 + Z_2) + N_1^{h\bullet} Q_1\right] a_1\beta_1^v Z_3}. \tag{4.10}$$

Then, the first equation of (4.8) can be written as

$$\Lambda_1 - (\mu_1 + k_{21}) N_1^{h\bullet} - [\delta_1 - (1 + (1-\theta)\alpha_1 Z_1) k_{21}]\, I_1^{h\bullet} = 0,$$

and we observe that

$$\delta_1 - Z_4 = 0 \Leftrightarrow \alpha_1 = \alpha_1^* := \frac{(1-p_1)(k_{21} + \mu_1)(\delta_1 - k_{21})}{k_{21}(1-\theta)\left[(1-p_1)Q_1 + p_1 Q_2 + \delta_1 p_1\right] + k_{21}^2 p_1 \theta}.$$

We then have the following proposition

Proposition 4.3. *If $\delta_1 - Z_4 = 0$ (i.e., $\alpha_1 = \alpha_1^*$), then $N_1^{h\bullet} = N_1^{h*}$ and the system* (4.8) *has*

1) *A unique positive solution if $\mathcal{R}_{c,1}^2 > 1$;*
2) *No positive solution if $\mathcal{R}_{c,1}^2 \in [0, 1]$.*

Now, we assume that $Z_4 - \delta_1 \neq 0$. Using the same approach as in [22], we set $N_1^{h\bullet} = \dfrac{N_1^{h*}}{x^\bullet + 1}$ and, after some calculations, we obtain

$$a\,(x^\bullet)^2 + bx^\bullet + c = 0, \tag{4.11}$$

where

$$\begin{aligned}
a &= Z_3 a_1^2 \beta_1^h \beta_1^v p_1 N_1^{v*} N_1^{h*} (1 + Z_1 + Z_2)(k_{21} + \mu_1),\\
b &= \Big[a_1 \beta_1^v Z_3 (\mu_1 + k_{21}) \Big(N_1^{h*} Q_1 + N_1^{v*} a_1 \beta_1^h p_1 (1 + Z_1 + Z_2)\Big) + N_1^{h*} Q_1 \nu_1 \mathcal{R}_{c,1}^2 (Z_4 - \delta_1)\Big] N_1^{h*},\\
c &= N_1^{h*2} Q_1 \nu_1 \Big(\mathcal{R}_{c,1}^2 - 1\Big)(Z_4 - \delta_1).
\end{aligned}$$

We then have the following proposition.

Proposition 4.4.

1) *If $Z_4 - \delta_1 < 0$ (i.e., $\alpha_1 < \alpha_1^*$), then the system* (4.8) *has*

   a. One positive solution if $\mathcal{R}_{c,1}^2 > 1$;
   b. Two positive solutions if $\mathcal{R}_{c,1}^2 \leq 1$, $b^2 - 4ac > 0$ and $b < 0$;
   c. No solution otherwise.

2) *If $Z_4 - \delta_1 > 0$ (i.e., $\alpha_1 > \alpha_1^*$), then the system* (4.8) *has*

   a. Two positive solutions if $\mathcal{R}_{c,1}^2 \geq 1$, $b^2 - 4ac > 0$ and $b < 0$;
   b. One positive solution if $\mathcal{R}_{c,1}^2 < 1$;
   c. No solution otherwise.

Remark 4.3.

1) *If the migration rate k_{21} is low, so that it is less than δ_1, then α_1^* is positive. In this case, depending on the choice of α_1 and θ, we can have one of the cases of Proposition 4.4.*
2) *Otherwise, α_1^* is negative and will always be less than α_1. Therefore we have the second case of Proposition 4.4.*

For the system (4.9), let

$$\begin{aligned}
&K_1 = \frac{Q_6 + \dfrac{1 - p_2}{p_2} Q_4}{Q_7}, \quad W_1 = \frac{k_{21} Q_5 Z_1}{Q_7}, \quad W_2 = \frac{\alpha_2 K_1}{Q_8}, \quad W_3 = \frac{\alpha_2 W_1 + \theta \alpha_1 k_{21} Z_1}{Q_8},\\
&W_4 = \frac{c_2 \omega_2 (1 + W_2)}{Q_9}, W_5 = \frac{c_2 \omega_2 W_3 + k_{21} Z_2}{Q_9}, \quad W_6 = 1 + \zeta_2^u K_1 + \zeta_2^d W_2, \quad W_7 = \zeta_2^u W_1 + \zeta_2^d W_3.
\end{aligned}$$

We then have

$$I_2^{v\bullet} = \frac{a_2 \beta_2^v N_2^{v*} \left(W_6 I_2^{h\bullet} + W_7 I_1^{h\bullet}\right)}{a_2 \beta_2^v \left(W_6 I_2^{h\bullet} + W_7 I_1^{h\bullet}\right) + \nu_2 N_2^{h\bullet}}, \; A_2^{u\bullet} = K_1 I_2^{h\bullet} + W_1 I_1^{h\bullet}, \; A_2^{d\bullet} = W_2 I_2^{h\bullet} + W_3 I_1^{h\bullet}, \; P_2^{h\bullet} = W_4 I_2^{h\bullet} + W_5 I_1^{h\bullet},$$

and the total population is given by

$$N_2^{h\bullet} = \frac{\Lambda_2 + k_{21}\left[S_1^{h\bullet} + ((1-\alpha_1) + \theta\alpha_1)A_1^{u\bullet} + P_1^{h\bullet}\right] - \delta_2 I_2^{h\bullet}}{\mu_2} = \frac{\Lambda_2 + k_{21}\Gamma_2 - \delta_2 I_2^{h\bullet}}{\mu_2}. \tag{4.12}$$

If $\delta_2 = 0$, then $N_2^{h\bullet} = \dfrac{\Lambda_2 + k_{21}\Gamma_2}{\mu_2} > 0$ and $I_2^{h\bullet}$ is the positive solution of the equation

$$q_0 + q_1 I_2^{h\bullet} + q_2\left(I_2^{h\bullet}\right)^2 = 0, \tag{4.13}$$

with

$$\begin{aligned}
q_0 &= \left(N_2^{h\bullet} - I_1^{h\bullet}(W_1 + W_3 + W_5)\right) I_1^{h\bullet} N_2^{v\bullet} W_7 a_2^2 \beta_2^h \beta_2^v p_2,\\
q_1 &= a_2\beta_2^v I_1^{h\bullet}\left[a_2\beta_2^h p_2 N_2^{v\bullet}\left(W_6(W_1 + W_3 + W_5) + W_7(1 + K_1 + W_2 + W_4)\right) + N_2^{h\bullet} Q_4\nu_2\right]\\
&\quad + N_2^{h2\bullet} Q_4\nu_2\left(1 - \mathcal{R}_{c,2}^2\right),\\
q_2 &= a_2\beta_2^v W_6\left(a_2 p_2\beta_2^h N_2^{v\bullet}(1 + K_1 + W2 + W_4) + Q_4 N_2^{h\bullet}\right).
\end{aligned}$$

We then have the following proposition.

Proposition 4.5. *The system* (4.9) *has*

1) *A unique positive solution if* $q_0 < 0$*;*
2) *Two positive solutions if* $q_0 > 0$*,* $q_1^2 - 4q_0q_2 > 0$*, and* $q_1 < 0$*;*
3) *No positive solution otherwise.*

If, on the other hand, $\delta_2 \neq 0$, then, defining $x^\bullet$ by $I_2^{h\bullet} = \dfrac{\Lambda_2 + k_{21}\Gamma_2}{\delta_2(x^\bullet + 1)}$, we are looking for a positive $x^\bullet$, which will also ensure that $N_2^{h\bullet}$ is positive. After some calculations, we obtain the

$$\psi(x^\bullet) := b_0 + b_1 x^\bullet + b_2\left(x^\bullet\right)^2 + b_3\left(x^\bullet\right)^3 = 0, \tag{4.14}$$

where

$$\begin{aligned}
b_0 &= a_2^2\beta_2^h\beta_2^v\mu_2^2 p_2 N_2^{v*}\left(\delta_2 I_1^{h\bullet} W_7 + W_6(\Lambda_2 + k_{21}\Gamma_2)\right)\left[(\Lambda_2 + k_{21}\Gamma_2)(1 + K_1 + W_2 + W_4) + \delta_2 I_1^{h\bullet}\hat{W}\right],\\
b_1 &= a_2\beta_2^v\mu_2\left[a_2\beta_2^h\delta_2^2 p_2 N_2^{v*} W_7 I_1^{h\bullet}\left(3\mu_2 I_1^{h\bullet}\hat{W} - (\Lambda_2 + k_{21}\Gamma_2)\right)\right.\\
&\quad + a_2\beta_2^h\delta_2 p_2 N_2^{v*} W_6(\Lambda_2 + k_{21}\Gamma_2)\left(2\mu_2 I_1^{h\bullet}\hat{W} - (\Lambda_2 + k_{21}\Gamma_2)\right)\\
&\quad \left. + a_2\beta_2^h\mu_2 W_6(\Lambda_2 + k_{21}\Gamma_2)^2 N_2^{v*}(1 + K_1 + W_2 + W_4) + Q_4(\Lambda_2 + k_{21}\Gamma_2)^2\left(W_6(\Lambda_2 + k_{21}\Gamma_2) + \delta_2 W_7 I_1^{h\bullet}\right)\right],\\
b_2 &= a_2^2\beta_2^h\beta_2^v\mu_2^2\delta_2^2 p_2 N_2^{v*} W_7 I_1^{h\bullet}\left(3\mu_2 I_1^{h\bullet}\hat{W} - 2(\Lambda_2 + k_{21}\Gamma_2)\right)\\
&\quad + a_2\beta_2^v\mu_2\delta_2(\Lambda_2 + k_{21}\Gamma_2)\left[a_2\beta_2^h p_2 N_2^{v*}\mu_2\left(W_6\hat{W} + W_7(1 + K_1 + W_2 + W_4)\right) + Q_4 W_7(\Lambda_2 + k_{21}\Gamma_2)\right] I_1^{h\bullet}\\
&\quad + \left(1 - \mathcal{R}_{c,2}^2\right)(\Gamma_2 k_{21} + \Lambda_2)^3 Q_4\delta_2\nu_2,\\
b_3 &= \mathcal{T} I_1^{h\bullet} N_2^{v\bullet} W_7 a_2^2\beta_2^h\beta_2^v\delta_2^2\mu_2 p_2,
\end{aligned}$$

and $\hat{W} = W_1 + W_3 + W_5$, $\mathcal{T} = \mu_2 I_1^{h\bullet}\hat{W} - (\Lambda_2 + k_{21}\Gamma_2)$.

Proposition 4.6. *If $b_3 = 0$, then* (4.14) *has one simple positive solution if $b_2 < 0$ but no positive solutions if $b_2 \geq 0$.*

Proof. The result follows immediately from the fact that the condition $b_3 = 0$ implies $b_1 > 0$.

When $b_3 \neq 0$, Cardano's method [23] provides explicit conditions for the existence of one, two, or three distinct reals solutions of (4.14); see https://onlinelibrary.wiley.com/doi/pdf/10.1002/9781118635360.app1 for a concise summary. The discriminant of (4.14) is

$$D = q^2 + 4\left(\frac{p}{3}\right)^3,$$

where

$$p = \frac{3b_1b_3 - b_2^2}{3b_3^2} \text{ and } q = \frac{27b_0b_3^2 + 2b_2^3 - 9b_1b_2b_3}{27b_3^3}.$$

In this case,

1) If $D < 0$, then Eq (4.14) has three real solutions $x_0^{*-} < x_1^{*-} < x_2^{*-}$, given by:

$$\begin{cases} x_0^{*-} = 2\sqrt{\frac{-p}{3}}\cos\left(\frac{\arccos\left(\frac{3q}{2p}\sqrt{-\frac{3}{p}}\right)+2\pi}{3}\right) - \frac{b_2}{3b_3} \\ x_1^{*-} = 2\sqrt{\frac{-p}{3}}\cos\left(\frac{\arccos\left(\frac{3q}{2p}\sqrt{-\frac{3}{p}}\right)+4\pi}{3}\right) - \frac{b_2}{3b_3} \\ x_2^{*-} = 2\sqrt{\frac{-p}{3}}\cos\left(\frac{\arccos\left(\frac{3q}{2p}\sqrt{-\frac{3}{p}}\right)}{3}\right) - \frac{b_2}{3b_3}. \end{cases} \tag{4.15}$$

2) If $D > 0$, then Eq (4.14) has two complex conjugate solutions and one real solution

$$x_3^{*+} = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - \frac{b_2}{3b_3}. \tag{4.16}$$

3) If $D = 0$, then Eq (4.14) has one simple solution and one double solution,

$$\begin{cases} x_0^{*0} = 2\sqrt[3]{\frac{-q}{2}} - \frac{b_2}{3b_3} \\ x_1^{*0} = x_2^{*0} = -\sqrt[3]{\frac{-q}{2}} - \frac{b_2}{3b_3}. \end{cases} \tag{4.17}$$

By exploring the properties of $\psi(x)$, we obtain the following results.

Proposition 4.7. *Assume that $b_3 > 0$. In this case*

1) *If $b_2 \geq 0$, then Eq* (4.14) *has no positive solutions.*
2) *If $b_2 < 0$, then we have the following cases:*
   a. *If $D < 0$, then Eq (4.14) has two simple positive solutions $x_1^{*-} < x_2^{*-}$ given by* (4.15)*;*
   b. *If $D = 0$, then Eq (4.14) has one double positive solution $x_1^{*0} = x_2^{*0}$ given by* (4.17)*;*
   c. *If $D > 0$, then Eq (4.14) has no positive solutions.*

Proof. If $b_3 > 0$, then $b_1 > 0$ and we have the following cases.

1) If $b_2 \geq 0$, then, by $b_3 > 0, b_1 > 0$ and $b_0 > 0$, Eq (4.14) has no positive solutions.
2) If $b_2 < 0$, then there are the following possibilities.

a. If $D < 0$, then Eq (4.14) has three real solutions $x_0^{*-} < x_1^{*-} < x_2^{*-}$ given by (4.15). Moreover, $D < 0$ implies that $p^3 < \frac{-27}{4}q^2 \leq 0$, and therefore $p = 3b_1b_3 - b_2^2 < 0$, implying that $\psi'(x) = 3b_3x^2 + 2b_2x + b_1$ has two real roots $x_1^{\#}$ and $x_2^{\#}$ such that

$$0 < x_1^{\#} = \frac{-b_2}{3b_3} - \sqrt{\frac{-p}{3}} < x_2^{\#} = \frac{-b_2}{3b_3} + \sqrt{\frac{-p}{3}}. \tag{4.18}$$

After some simplifications, we obtain

$$\begin{cases} \psi\left(x_1^{\#}\right) = b_3\left(q + 2\left(\frac{-p}{3}\right)^{\frac{3}{2}}\right), \\ \psi\left(x_2^{\#}\right) = b_3\left(q - 2\left(\frac{-p}{3}\right)^{\frac{3}{2}}\right). \end{cases} \tag{4.19}$$

Since $D < 0$, then $-2\left(\frac{-p}{3}\right)^{\frac{3}{2}} < q < 2\left(\frac{-p}{3}\right)^{\frac{3}{2}}$, which implies that $\psi\left(x_1^{\#}\right) > 0$ and $\psi\left(x_2^{\#}\right) < 0$. Moreover, we have

$$\begin{cases} \psi''\left(x_1^{\#}\right) = -6b_3\sqrt{\frac{-p}{3}} < 0, \\ \psi''\left(x_2^{\#}\right) = 6b_3\sqrt{\frac{-p}{3}} > 0. \end{cases}$$

Hence, $\psi(x)$ has a local maximum at $x_1^{\#}$ and a local minimum at $x_2^{\#}$, implying that

$$x_0^{*-} \leq x_1^{\#} \leq x_1^{*-} \leq x_2^{\#} \leq x_2^{*-}.$$

Since $b_3 > 0$ and $b_0 \geq 0$, then $\lim_{x\to-\infty}\psi(x) = -\infty$ and $\psi(0) \geq 0$, implying that $x_0^{*-} \leq 0 \leq x_1^{\#} \leq x_1^{*-} \leq x_2^{\#} \leq x_2^{*-}$. Thus, Eq (4.14) has two simple positive solutions, $x_1^{*-} < x_2^{*-}$, given by (4.15).

b. If $D = 0$, then Eq (4.14) has one simple real solution x_0^{*0} and one double solution $x_1^{*0} = x_2^{*0}$, given by (4.17). The condition $D = 0$ also implies that $p^3 = \frac{-27}{4}q^2 \leq 0$, leading to $p \leq 0$. Therefore, by using $q = \frac{27b_0b_3^2 - b_2^3 + 3(b_2^3 - 3b_1b_2b_3)}{27b_3^3} > 0$, we obtain $q > 0$. This, together with $p^3 = \frac{-27}{4}q^2$, leads to $p < 0$. Hence, by following the same argument as in **2)a.**, we find that $\psi(x)$ has a local maximum at $x_1^{\#}$ and a local minimum at $x_2^{\#}$, where $x_1^{\#}$ and $x_2^{\#}$ are given in (4.18). Moreover, from $D = 0$ and given that $q > 0$, we get $q = 2\left(\frac{-p}{3}\right)^{\frac{3}{2}}$, implying that $\psi\left(x_2^{\#}\right) = 0$ and $\psi\left(x_1^{\#}\right) = 4b_3\left(\frac{-p}{3}\right)^{\frac{3}{2}} > 0$. Furthermore,

$$\begin{cases} x_2^{\#} - x_1^{*0} = \sqrt{\frac{-p}{3}} + \sqrt[3]{\frac{-q}{2}} = 0, \\ x_0^{*0} - x_1^{*0} = 3\sqrt[3]{\frac{-q}{2}} = -3\sqrt{\frac{-p}{3}} \leq 0 \end{cases}$$

implying that $x_0^{*0} < x_1^{*0} = x_2^{*0} = x_1^{\#}$.
Since $b_3 > 0$ and $b_0 \geq 0$, then $\lim_{x\to-\infty}\psi(x) = -\infty$ and $\psi(0) \geq 0$, implying that $x_0^{*0} < 0 < x_1^{*0} = x_2^{*0}$. Thus, Eq (4.14) has one double positive solution $x_1^{*0} = x_2^{*0}$, given by (4.17).

c. If $D > 0$, then Eq (4.14) has two complex conjugate solutions and one real solution which is negative because $b_3 > 0$ and $b_0 > 0$.

Proposition 4.8. *Let $b_3 < 0$.*

1) *If $b_1 > 0$, or $b_1 < 0$ and $b_2 < 0$, then the Eq* (4.14) *has one simple positive solution.*
2) *If $b_1 < 0$ and $b_2 > 0$, then we have the following cases.*
   a. If $D < 0$, then Eq (4.14) has three positive solutions $x_0^{*-} < x_1^{*-} < x_2^{*-}$, given by Eq (4.15).
   b. If $D = 0$, then
      i. If $p < 0$, then
         - *If $q = 2\left(\frac{-p}{3}\right)^{\frac{3}{2}}$, then Eq (4.14) has three positive solutions $x_0^{*-} < x_1^{*-} < x_2^{*-}$ given by* (4.15)*;*
         - *If $q = -2\left(\frac{-p}{3}\right)^{\frac{3}{2}}$, then Eq (4.14) has no positive solutions.*

      ii. If $p = 0$, then Eq (4.14) has one triple positive solution, given by $\dfrac{-b_2}{b_3}$.
   c. If $D > 0$, then Eq (4.14) has one simple positive solution x_3^{*+}, given by Eq (4.16).

Proof. Let $b_3 < 0$. We have the following cases.

1) If $b_1 > 0$, or $b_1 < 0$ and $b_2 < 0$, then, by $b_3 < 0$ and $b_0 > 0$, the number of sign changes between consecutive nonzero coefficients in Eq (4.14) is equal to one. Thus, by Descartes' rule of signs, Eq (4.14) has one positive solution.
2) If $b_1 < 0$ and $b_2 > 0$, then Eq (4.14) has three sign changes between its consecutive nonzero coefficients, implying that it has three or one positive roots. To determine the exact number of such roots, we utilise Cardano's method.
   a. If $D < 0$, then Eq (4.14) has three real solutions $x_0^{*-} < x_1^{*-} < x_2^{*-}$, given by Eq (4.15). By following the same argument as in the proof of Item **2)a.** of Proposition 4.8. Noting that $b_3 < 0$, we find that $\psi(x)$ has a local minimum at $x_1^{\#}$ and a local maximum at $x_2^{\#}$ with $\psi\left(x_1^{\#}\right) < 0$ and $\psi\left(x_2^{\#}\right) > 0$. Therefore,

$$x_0^{*-} \leq x_1^{\#} \leq x_1^{*-} \leq x_2^{\#} \leq x_2^{*-}.$$

   Since $b_0 > 0$, then $\psi(0) > 0$, implying that $0 \leq x_0^{*-} \leq x_1^{\#} \leq x_1^{*-} \leq x_2^{\#} \leq x_2^{*-}$. Thus, x_0^{*-}, x_1^{*-} and x_2^{*-} are positive.
   b. If $D = 0$, then Eq (4.14) has one simple real solution x_0^{*0} and one double solution $x_1^{*0} = x_2^{*0}$ given by Eq (4.17). The condition $D = 0$ also implies that $p^3 = \frac{-27}{4}q^2 \leq 0$.
      i. If $p < 0$, then, by the same argument as in **2)a.**, $\psi(x)$ has a local minimum at $x_1^{\#}$ and a local maximum at $x_2^{\#}$ given in Eq (4.18), where $0 < x_1^{\#} < x_2^{\#}$.
         - If $q = 2\left(\frac{-p}{3}\right)^{\frac{3}{2}}$, then, from Eq (4.19), we have $\psi\left(x_2^{\#}\right) = 0$ and $\psi\left(x_1^{\#}\right) = 4\left(\frac{-p}{3}\right)^{\frac{3}{2}} > 0$. Furthermore,

$$\begin{cases} x_2^{\#} - x_1^{*0} = \sqrt{\frac{-p}{3}} + \sqrt[3]{\frac{-q}{2}} = 0, \\ x_0^{*0} - x_1^{*0} = 3\sqrt[3]{\frac{-q}{2}}. \end{cases} \tag{4.20}$$

Since $q > 0$, then $x_0^{*0} - x_1^{*0} < 0$, implying that $x_0^{*0} < x_1^{*0} = x_2^{*0} = x_1^{\#}$. Next, since $b_0 \geq 0$ and $b_3 < 0$, we have $\psi(0) \geq 0$ and $\lim_{x\to+\infty} \psi(x) = -\infty$, implying that $0 < x_0^{*0} < x_1^{*0} = x_2^{*0}$. Thus, Eq (4.14) has three simple positive solutions.

- If $q = -2\left(\frac{-p}{3}\right)^{\frac{3}{2}}$, then, from Eq (4.19), we have $\psi\left(x_1^{\#}\right) = 0$ and $\psi\left(x_2^{\#}\right) = -4\left(\frac{-p}{3}\right)^{\frac{3}{2}} < 0$. Moreover, using Eq (4.20) and the fact that $q < 0$, we obtain $x_1^{*0} = x_2^{*0} = x_1^{\#} < x_0^{*0}$. Furthermore, by $b_0 \geq 0$, $x_1^{*0} = x_2^{*0} = x_1^{\#} < x_0^{*0} < 0$. Thus, Eq (4.14) has no positive solutions.

ii. If $p = 0$, then we also have $q = 0$, implying that

$$\psi(x) = b_3\left(x + \frac{b_2}{3b_3}\right)^3.$$

Therefore, the function $\psi(x)$ has a triple root $\frac{-b_2}{3b_3}$, which is positive because $b_3 < 0$ and $b_2 > 0$.

c. If $D > 0$, then Eq (4.14) has two complex conjugate solutions and one real solution x_3^{*+} given by Eq (4.16), which is positive because $b_3 < 0$ and $b_0 > 0$.

5. Numerical simulations

In this section, we evaluate the impact of detecting asymptomatic cases at the border and within Patch 2. To do this, we assign two values to the parameters α_1 and α_2, respectively, corresponding to a low detection rate (below 0.5) and a high detection rate (above 0.5). This gives us four possible scenarios. In each case, we will plot the numerical solutions of I_1^h and I_2^h for different values of θ namely $\theta = 0,\ 0.2,\ 0.4,\ 0.6,\ 0.8$, and 1. We recall that θ is the proportion of individuals who tested positive but were admitted to Patch 2. The other parameters are given in Table 3.

First, we consider the case with low detection of asymptomatic individuals at the border and in Patch 2 (see Figure 3).

Next, we illustrate the case with a low detection rate of asymptomatic individuals at the border but a high rate within Patch 2 (see Figure 4).

In the Figure 5, we illustrate the case with a high detection rate of asymptomatic individuals at the border, but a low rate within Patch 2.

Finally, we present the results of simulations in the case of high detection rates of asymptomatic individuals both at the border and within Patch 2 (see (Figure 6)).

The simulation results above can be summarised as follows:

1) The control parameters α_1, α_2, and θ mostly impact the epidemiological situation in Patch 2. This means that the decision-makers in Patch 2 can control malaria there, even if those in Patch 1 do not participate.
2) With a high detection rate of asymptomatic individuals at the border (α_1 around 0.9) and a proportion of people admitted to Patch 2 of around 50% (θ around 0.5), malaria can be eradicated there, even if the detection rate within the Patch is low ($\alpha_2 \leq 0.5$).

Table 3. Parameters, baseline values, and references.

Parameters	Baseline value or range	Reference
$\Lambda_i,\ i = 1, 2$	2000	Assumed
$\Pi_i,\ i = 1, 2$	3000	Assumed
$\beta_i^h,\ i = 1, 2$	0.24	[24]
$\beta_i^v,\ i = 1, 2$	0.022	[24]
a_1	10	Assumed
a_2	5	Assumed
$\omega_i,\ i = 1, 2$	3.5×10^{-3}	[24, 25]
c_1	0.1	Assumed
c_2	0.8	Assumed
ζ_u	0.05	[26]
ζ_d	0.00002	Assumed
μ_1	$\frac{1}{62 \times 365}$	Assumed
μ_2	$\frac{1}{65 \times 365}$	Assumed
δ_1	1.28×1^{-5}	[27]
δ_2	1.28×1^{-7}	Assumed
$\nu_i,\ i = 1, 2$	0.047	[24]
$\vartheta_i,\ i = 1, 2$	0.001	[27]
k_{12}	0.01	Assumed
k_{21}	0.02	Assumed

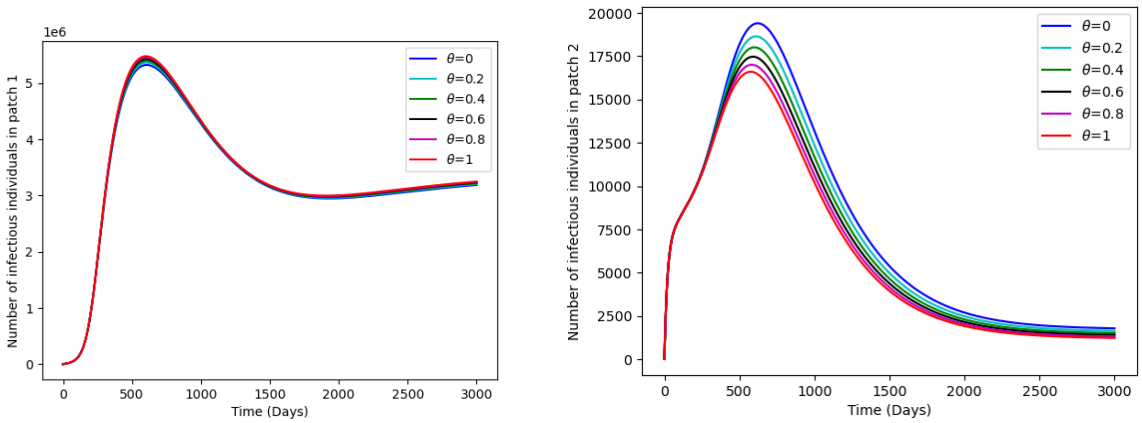

Figure 3. Evolution of I_1^h (left) and I_2^h (right) for $\alpha_1 = 0.3$ and $\alpha_2 = 0.3$.

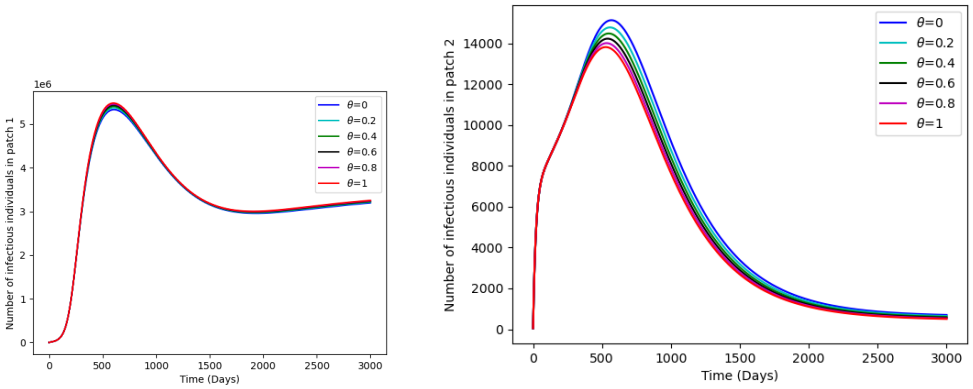

Figure 4. Evolution of I_1^h (left) and I_2^h (right) for $\alpha_1 = 0.3$ and $\alpha_2 = 0.9$.

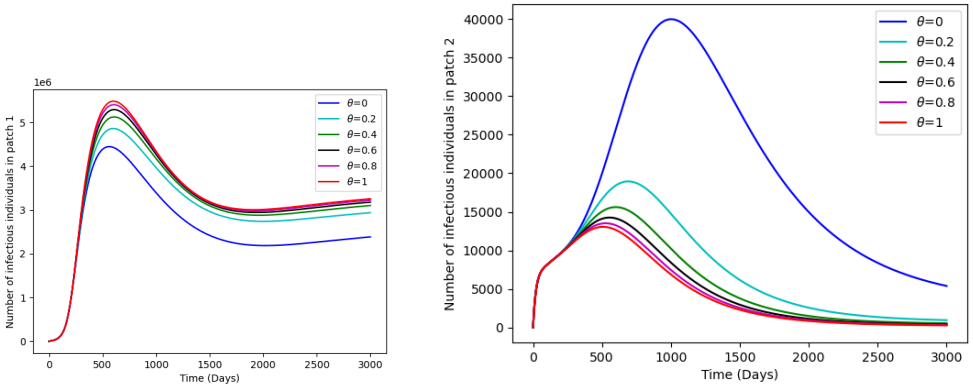

Figure 5. Evolution of I_1^h (left) and I_2^h (right) for $\alpha_1 = 0.9$ and $\alpha_2 = 0.3$.

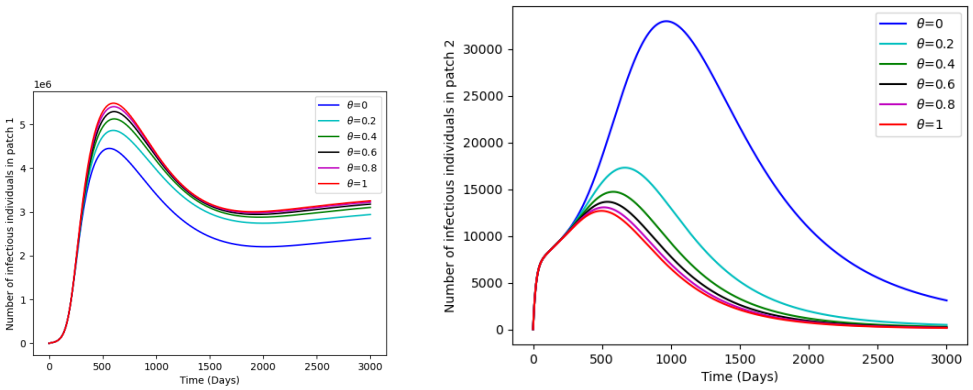

Figure 6. Evolution of I_1^h (left) and I_2^h (right) for $\alpha_1 = 0.9$ and $\alpha_2 = 0.9$.

6. Conclusions

We studied a model of malaria's evolution in two adjacent sites, with humans able to migrate between them. One site is better resourced than the other to fight the disease, with available screening facilities and TBDs that offer full, though potentially waning, immunity and make the successfully treated individuals non-infective. We considered the situation when the individuals moving to this site are screened at the entry points, and upon identifying an infected individual, the authorities can either refuse entry or allow the individual in but immediately administer a TBD. We provided a qualitative analysis of the model, finding, in particular, the basic reproduction number $\mathcal{R}_c$. As the first step, we show that, in the case of isolated patches, the disease-free equilibrium is globally stable, and we provide a complete analysis of the structure of bifurcations, identifying, in particular, the conditions for the occurrence of backward bifurcations. On the basis of these results and recent refinements of the regular perturbation theory, we provide a comprehensive analysis of the model with low migration rates, proving, in particular, the existence of backward bifurcation for certain ranges of parameters. Furthermore, using Cardano's formulae, we provide a complete bifurcation analysis of the case when the migrations from Patch 2 to Patch 1 can be neglected. We conclude the paper with numerical simulations that show, in particular, that malaria can be better controlled by allowing the entry of detected cases and treating them in the better-resourced site rather than deporting the identified infective individuals and risking that they will enter the site illegally.

We emphasise that this is the first study of this problem, and we focused on the impact on the mechanism of border screening and the potential compulsory treatment of infected migrants. We acknowledge that, as such, our study has several limitations. We focused on only two patches, whereas the possibility that KwaZulu-Natal is a transient province in the migration to Gauteng, the economic hub of South Africa, could alter the picture. Further, we ignored several aspects, such as the age structure in each Patch, which may significantly impact the findings. We also neglected possible migrations of mosquitoes, both active and passive, e.g., through cargo vehicles. As far as the conclusion is concerned, it must be emphasised that our results simply state that if we deny entry to infected asymptomatic individuals into the country, we risk that, due to porous borders, they will enter illegally and, staying undetected, may, under certain conditions, cause malaria to become endemic in the region. However, this scenario must be examined against the availability of resources and be further studied from the optimal control point of view to determine their most cost-effective utilisation.

Use of AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of Interest

The authors declare no conflict of interest.

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Appendix

A. Regular perturbation method

Before we proceed with this problem, let us recall the basic results from [28, 29] (see also [30, Theorem 5.3], where the result is proven under weaker differentiability assumptions but with less explicit constants). The general results are valid in non-autonomous cases; here, due to the applications we have in mind, we restrict ourselves to autonomous problems. Thus, for a function $\boldsymbol{f} : D \times I \to \mathbb{R}^n$, where $D \subset \mathbb{R}^n$ is an open set and $I = (-\epsilon', \epsilon')$ for some $\epsilon' > 0$, we consider the problems

$$\boldsymbol{x}_\epsilon' = \boldsymbol{f}(\boldsymbol{x}_\epsilon, \epsilon), \quad \boldsymbol{x}_\epsilon(0) = \mathring{\boldsymbol{x}}_\epsilon \in D, \tag{A.1a}$$

and

$$\boldsymbol{x}_0' = \boldsymbol{f}(\boldsymbol{x}_0, 0), \quad \boldsymbol{x}_0(0) = \mathring{\boldsymbol{x}}_0 \in D, \tag{A.1b}$$

where $\epsilon > 0$ is a small parameter. In the formulation of [28, Theorem 4.1], we then have

Theorem A.1. *Assume that $\boldsymbol{f}$ is a continuous function with respect to all variables, and C^2 with respect to $\boldsymbol{x}$ and C^1 with respect to ϵ, with bounded derivatives, in $D \times [0, \epsilon']$. Let $\boldsymbol{x}_0^* \in \Omega$ be an equilibrium of* (A.1b)*, and the spectral bound of the Jacobian of $\boldsymbol{f}$ at $\boldsymbol{x}_0^*$ satisfies*

$$s(\mathcal{J}_f(\boldsymbol{x}_0^*)) < 0. \tag{A.2}$$

Further, let $\mathring{\boldsymbol{x}}_\epsilon - \mathring{\boldsymbol{x}}_0 = O(\epsilon)$ as $\epsilon \to 0^+$ and $\mathring{\boldsymbol{x}}_0 \in \Omega_{\boldsymbol{x}_0^*}$, which is a basin of attraction of $\boldsymbol{x}_0^*$. In this case,

$$\|\boldsymbol{x}_\epsilon(t, \mathring{\boldsymbol{x}}_\epsilon) - \boldsymbol{x}_0(t, \mathring{\boldsymbol{x}}_0)\| = O(\epsilon), \tag{A.3}$$

uniformly in $t \in [0, \infty)$. More precisely, there are constants α, K and C, independent of $\epsilon \in [0, \epsilon']$, and $t \in [0, \infty)$, such that

$$\|\boldsymbol{x}_\epsilon(t, \mathring{\boldsymbol{x}}_\epsilon) - \boldsymbol{x}_0(t, \mathring{\boldsymbol{x}}_0)\| \le K e^{-\alpha t} \|\mathring{\boldsymbol{x}}_\epsilon - \mathring{\boldsymbol{x}}_0\| + \epsilon C, \quad t \in [0, \infty). \tag{A.4}$$

Remark A.1. *The constant α is determined by the spectral bound of $\boldsymbol{f}$ in some neighbourhood of $(\boldsymbol{x}_0^*, 0)$, while K and C depend on the suprema of the relevant derivatives of $\boldsymbol{f}$ with respect to $\boldsymbol{x}$ and ϵ. Moreover, C depends on the time $T_{\mathring{\boldsymbol{x}}_0}$ needed by $\boldsymbol{x}(t, \mathring{\boldsymbol{x}}_0)$ to reach a prescribed neighbourhood of $\boldsymbol{x}_0^*$, see [28, Eq (43)] combined with [30, Eq (5.53)].*

Remark A.2. *We emphasise that the validity of* (A.3) *depends $\mathring{\boldsymbol{x}}_\epsilon$ being sufficiently close to $\mathring{\boldsymbol{x}}_0$. For instance, if there are multiple equilibria in D and $\mathring{\boldsymbol{x}}_\mathbf{0}$ is in the domain of attraction of one of them, then* (A.3) *is valid as long as $\mathring{\boldsymbol{x}}_\epsilon$ does not fall into the domain of attraction of the other; see Corollary A.2 below.*

We recall the following known result, which will be needed later.

Corollary A.1. *Let $\boldsymbol{x}_0^*$ be an asymptotically stable hyperbolic equilibrium of* (A.1b)*, where $\boldsymbol{f}$ satisfies the assumptions of Theorem A.1. Then, the basin of attraction of $\boldsymbol{x}_0^*$ is an open set.*

Proof. Let $\Omega_{\boldsymbol{x}_0^*}$ be the basin of attraction of $\boldsymbol{x}_0^*$ and let $\mathring{\boldsymbol{x}} \in \Omega_{\boldsymbol{x}_0^*}$, that is, $\boldsymbol{x}_0(t, \mathring{\boldsymbol{x}}) \to \boldsymbol{x}_0^*$ as $t \to \infty$. From [14, Theorem 6.34] (see also the proof of Lemma A.2), $\Omega_{\boldsymbol{x}_0^*}$ contains an open neighbourhood $V_{\boldsymbol{x}_0^*}$ of $\boldsymbol{x}_0^*$. Theorem A.1 with $\boldsymbol{f}(\boldsymbol{x}, \epsilon) \equiv \boldsymbol{f}(\boldsymbol{x}, 0)$ for $\epsilon \in [0, \epsilon_0]$ in (A.1a) implies that that the solution $\boldsymbol{x}(t, \mathring{\boldsymbol{x}}_\epsilon)$ to (A.1a) stays $O(\epsilon)$-close to $\boldsymbol{x}(t, \mathring{\boldsymbol{x}}_0)$ if ϵ is sufficiently small. Hence, $\boldsymbol{x}(t, \mathring{\boldsymbol{x}}_\epsilon)$ must enter for $V_{\boldsymbol{x}_0^*}$ and thus it will converge to $\boldsymbol{x}_0^*$ as $t \to \infty$, showing that $\mathring{\boldsymbol{x}}_\epsilon \in \Omega_{\boldsymbol{x}_0^*}$.

The results above, however, do not address the existence of equilibria of (A.1a) and their stability. The answer to the first question is relatively straightforward.

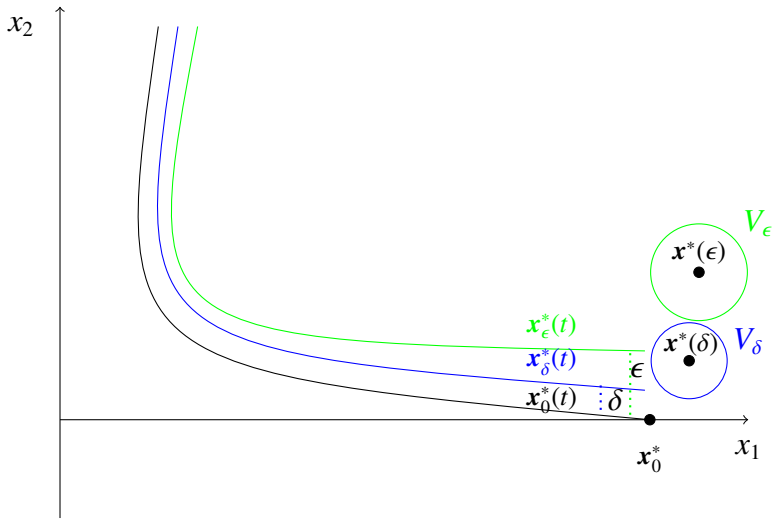

Figure 7. The solutions $\boldsymbol{x}_\epsilon(t)$ and $\boldsymbol{x}_\delta(t)$ to (A.1a) stay, respectively, ϵ- and δ-close to the limit solution $\boldsymbol{x}_0(t)$ to (A.1b) but may miss the basins of attractions V_ϵ and V_δ of, respectively, the respective equilibria $\boldsymbol{x}^*(\epsilon)$ and $\boldsymbol{x}^*(\delta)$ of (A.1b).

Lemma A.1. *Assume that $(\boldsymbol{x}, \epsilon) \mapsto \boldsymbol{f}(\boldsymbol{x}, \epsilon)$ is a C^1-function in some neighbourhood of $(\boldsymbol{x}_0^*, 0)$ and $\boldsymbol{f}(\boldsymbol{x}_0^*, 0) = \boldsymbol{0}$. If* (A.2) *is satisfied, then there is an interval $I = (-\epsilon_0, \epsilon_0)$, a C^1 function $I \ni \epsilon \mapsto \boldsymbol{x}^*(\epsilon)$, and $\kappa' > 0$ such that $\boldsymbol{x}^*(0) = \boldsymbol{x}_0^*$, $\boldsymbol{f}(\boldsymbol{x}^*(\epsilon), \epsilon) \equiv 0$, and $s(\mathcal{J}_{f(\cdot,\epsilon)}(\boldsymbol{x}^*(\epsilon))) < -\kappa'$ for $\epsilon \in I$.*

Proof. The existence of $\boldsymbol{x}^*(\epsilon)$ follows directly from the implicit function theorem, as (A.2) ensures that $\mathcal{J}_f(\boldsymbol{x}^*)$ is invertible. The statement for $s(\mathcal{J}_{f(\cdot,\epsilon)}(\boldsymbol{x}^*(\epsilon)))$ is a consequence of the continuity of the eigenvalues with respect to the parameter [31, Section 3.1.2].

Let us reflect on what we proved. We know that arbitrarily close to a hyperbolic equilibrium of (A.1b), there are hyperbolic equilibria $\boldsymbol{x}^*(\epsilon)$ of (A.1a) which converge to $\boldsymbol{x}_0^*$ as $\epsilon \to 0$. At the same time, the solutions to (A.1a) stay at an $O(\epsilon)$ distance from $\boldsymbol{x}_0^*$ as $t \to \infty$. It may happen, however, that the basins of attraction of $\boldsymbol{x}^*(\epsilon)$ may shrink as $\epsilon \to 0$, so that the solutions $\boldsymbol{x}_\epsilon(t)$ do not enter them and thus $\boldsymbol{x}_\epsilon$ may be not attracted to $\boldsymbol{x}^*(\epsilon)$ if it starts far away from it, staying, however, at an $O(\epsilon)$ distance

from it for large time; see Figure 7. To show that this is not the case, we shall show that the basins of attraction of $\boldsymbol{x}^*(\epsilon)$ do not shrink to 0 as $\epsilon \to 0$.

Lemma A.2. *Let the assumptions of Lemma A.1 be satisfied and let V_ϵ to denote the basin of attraction of $\boldsymbol{x}^*(\epsilon)$. Then we have $\rho > 0$ such that the ball $B(\boldsymbol{x}^*(\epsilon), \rho) \subset V_\epsilon$ for any sufficiently small ϵ.*

Proof. We begin with moving the equilibria $\boldsymbol{x}^*(\epsilon)$ to $\mathbf{0}$ by introducing $\boldsymbol{z}_\epsilon = \boldsymbol{x}_\epsilon - \boldsymbol{x}^*(\epsilon)$ so that

$$\boldsymbol{z}_\epsilon' = \boldsymbol{f}(\boldsymbol{z}_\epsilon + \boldsymbol{x}^*(\epsilon), \epsilon) =: \boldsymbol{F}(\boldsymbol{z}_\epsilon, \epsilon), \quad \boldsymbol{z}_\epsilon(0) = \mathring{\boldsymbol{z}}_\epsilon = \mathring{\boldsymbol{x}}_\epsilon - \boldsymbol{x}^*(\epsilon). \tag{A.5}$$

We then linearise the problem as

$$\boldsymbol{z}_\epsilon' = \mathcal{A}_\epsilon \boldsymbol{z}_\epsilon + \boldsymbol{g}_\epsilon(\boldsymbol{z}_\epsilon), \tag{A.6}$$

where $\mathcal{A}_\epsilon = \mathcal{J}_{\boldsymbol{F}}((\mathbf{0}, \epsilon))$ and $\boldsymbol{g}_\epsilon$ is the reminder. Thus

$$\boldsymbol{z}_\epsilon(t) = e^{t\mathcal{A}_\epsilon}\mathring{\boldsymbol{z}}_\epsilon + \int_0^t e^{(t-s)\mathcal{A}_\epsilon}\boldsymbol{g}_\epsilon(\boldsymbol{z}_\epsilon(s))ds.$$

Now, since $\boldsymbol{F}(\mathbf{0}, 0) = \boldsymbol{f}(\boldsymbol{x}_0^*, 0)$, we can write $\mathcal{A}_\epsilon = \mathcal{A}_0 + \mathcal{B}_\epsilon = \mathcal{J}_f(\boldsymbol{x}_0^*) + \mathcal{J}_{\boldsymbol{F}}((\mathbf{0}, \epsilon)) - \mathcal{J}_{\boldsymbol{F}}((\mathbf{0}, 0))$, where, since $\boldsymbol{f}$ is continuously differentiable, $\|\mathcal{B}_\epsilon\| \to 0$ as $\epsilon \to 0$.

By assumption, $s(\mathcal{A}_0) < 0$, hence, by [14, Section 5.3.8] implies that for any $\alpha' \in (0, -s(\mathcal{A}))$ we have $K \geq 1$ such that

$$\|e^{t\mathcal{A}_0}\mathring{\boldsymbol{z}}\| \leq Ke^{-\alpha' t}\|\mathring{\boldsymbol{z}}\|, \quad t \geq 0,\ \mathring{\boldsymbol{z}} \in \mathbb{R}^n.$$

Since we deal with matrices, which can be identified with bounded linear operators, we can use the bounded perturbation theorem [32, Chapter 3, Theorem 1.1], to get

$$\|e^{t\mathcal{A}_\epsilon}\mathring{\boldsymbol{z}}\| \leq Ke^{-(\alpha' - K\|\mathcal{B}_\epsilon\|)t}\|\mathring{\boldsymbol{z}}\|, \quad t \geq 0,\ \mathring{\boldsymbol{z}} \in \mathbb{R}^n.$$

Since $\|\mathcal{B}_\epsilon\| \to 0$ as $\epsilon \to 0$, for any given $\alpha \in (0, \alpha')$, $\epsilon' > 0$ exists such that for all $0 < \epsilon < \epsilon'$

$$\alpha < \alpha' - K\|\mathcal{B}_\epsilon\|.$$

Thus, for any $\epsilon \in [0, \epsilon')$,

$$\|e^{t\mathcal{A}_\epsilon}\mathring{\boldsymbol{z}}\| \leq Ke^{-\alpha t}\|\mathring{\boldsymbol{z}}\|, \quad t \geq 0,\ \mathring{\boldsymbol{z}} \in \mathbb{R}^n,$$

where K and α are independent of ϵ.

Next, consider $\boldsymbol{g}_\epsilon$. Using the formula for the remainder of the Taylor series, for the ith component $\boldsymbol{g}_\epsilon$, we have

$$g_{\epsilon,i}(\boldsymbol{z}) = \int_0^1 (1-\sigma) \sum_{j,k=1}^{n} \partial^2_{x_j x_k} F_i(\sigma \boldsymbol{z}, \epsilon)) z_j z_k d\sigma. \tag{A.7}$$

Using the assumption of Corollary A.1 that $\boldsymbol{f}$ (and hence $\boldsymbol{F}$) has second-order derivatives bounded with respect to all variables, we find that there is M, independent of ϵ, such that

$$\|\boldsymbol{g}_\epsilon(\boldsymbol{z})\| \leq M\|\boldsymbol{z}\|^2$$

in some fixed ball $B(\mathbf{0}, \kappa)$. Thus, for any sufficiently small $\eta > 0$, there is $\delta = \frac{\eta}{M}$, independent of ϵ, such that

$$\|\boldsymbol{g}_\epsilon(\boldsymbol{z})\| \leq \eta\|\boldsymbol{z}\| \tag{A.8}$$

for $\|z\| \leq \delta$. We then follow the proof of the theorem on linearised stability; see, e.g., [14, Theorem 6.34] and show that if we take $\|\mathring{z}_\epsilon\| < \delta/K \leq \delta$, where δ was fixed to ensure that $\eta < \alpha/2K$, then

$$\|z_\epsilon(t)\| \leq K\|\mathring{z}_\epsilon\|e^{-\frac{\alpha t}{2}} \tag{A.9}$$

for all $t \geq 0$. Consequently, $z_\epsilon(t)$ converges exponentially to 0 and hence, for a sufficiently small ϵ, $\boldsymbol{x}_\epsilon(t)$ converges to $\boldsymbol{y}^*(\epsilon)$ for any initial condition satisfying $\|\mathring{\boldsymbol{x}}_\epsilon - \boldsymbol{x}^*(\epsilon)\| = \|\mathring{z}_\epsilon\| \leq \frac{\delta}{K}$, where neither δ nor K depend on ϵ. Thus, we can use the balls $B\left(\boldsymbol{x}^*(\epsilon), \frac{\delta}{K}\right)$ in the statement of the proposition.

We can summarise the above results as follows:

Proposition A.2. *Consider the problem* (A.1a) *and assume that the assumptions of Theorem A.1 are satisfied. Let $\boldsymbol{x}_0^*$ be a unique equilibrium of* (A.1a)*, that is globally exponentially stable in Ω. Then, $\epsilon_0 > 0$ exists such that for any $\epsilon \in (-\epsilon_0, \epsilon_0)$, there is a unique asymptotically stable equilibrium $\boldsymbol{x}^*(\epsilon)$ of* (A.1a) *such that $\boldsymbol{x}^*(\epsilon) \to \boldsymbol{x}^*$ as $\epsilon \to 0$, and for any $\mathring{\boldsymbol{x}} \in \Omega$, ϵ_1 exists such for any $0 < \epsilon < \epsilon_1$, $\boldsymbol{x}^*(\epsilon)$ attracts the solutions $\boldsymbol{x}_\epsilon(t, \mathring{\boldsymbol{x}})$.*

Proof. Corollary A.1 ensures that if $\mathring{\boldsymbol{x}} \in \Omega$ and $\boldsymbol{x}(t, \mathring{\boldsymbol{x}})$ is the solution to (A.1b), converging to $\boldsymbol{x}_0^*$ as $t \to \infty$, then the solution $\boldsymbol{x}_\epsilon(t, \mathring{\boldsymbol{x}})$ satisfies

$$\|\boldsymbol{x}(t, \mathring{\boldsymbol{x}}) - \boldsymbol{x}_\epsilon(t, \mathring{\boldsymbol{x}})\| \leq C\epsilon$$

for all $t \in [0, \infty)$, where C depends on $\mathring{\boldsymbol{x}}$; see Remark A.1. Since $\boldsymbol{x}^*(\epsilon) \to \boldsymbol{x}_0^*$, there is $\epsilon' > 0$ exists such that for all $0 < \epsilon < \epsilon'$, $\boldsymbol{x}_0^* \in B\left(\boldsymbol{x}^*(\epsilon), \frac{\rho}{2}\right)$, where ρ was defined in Proposition A.2. In this case,

$$\|z - \boldsymbol{x}^*(\epsilon)\| \leq \|z - \boldsymbol{x}_0^*\| + \|\boldsymbol{x}_0^* - \boldsymbol{x}^*(\epsilon)\|$$

implies that $B\left(\boldsymbol{x}_0^*, \frac{\rho}{2}\right) \subset B(\boldsymbol{x}^*(\epsilon), \rho)$. The estimate

$$\|\boldsymbol{x}_\epsilon(t, \mathring{\boldsymbol{x}}) - \boldsymbol{x}_0^*\| \leq \|\boldsymbol{x}_\epsilon(t, \mathring{\boldsymbol{x}}) - \boldsymbol{x}(t, \mathring{\boldsymbol{x}})\| + \|\boldsymbol{x}(t, \mathring{\boldsymbol{x}}) - \boldsymbol{x}_0^*\|$$

shows that by selecting t_1 to be sufficiently large for $\|\boldsymbol{x}(t, \mathring{\boldsymbol{x}}) - \boldsymbol{x}_0^*\| < \rho/4$ for $t \geq t_1$ and ϵ'' to be sufficiently small for $C\epsilon < \rho/4$ for all $0 < \epsilon < \epsilon''$ (and thus dependent on $\mathring{\boldsymbol{x}}$), we obtain

$$\|\boldsymbol{x}_\epsilon(t, \mathring{\boldsymbol{x}}) - \boldsymbol{x}_0^*\| < \frac{\rho}{2}$$

for $0 < \epsilon < \min\{\epsilon', \epsilon''\}$ and $t > t_1$. Thus, $x_\epsilon(t, \mathring{\boldsymbol{x}}) \in B\left(\boldsymbol{x}_0^*, \frac{\rho}{2}\right) \subset B(\boldsymbol{x}^*(\epsilon), \rho)$ and thus it is in the basin of attraction of $\boldsymbol{x}^*(\epsilon)$.

The lack of uniformity of the estimates with respect to the initial values shows that, in general, we cannot claim that (A.1a) with a fixed ϵ has the same dynamical properties as (A.1b). This can be seen in the following example.

Example A.3. *The equation*

$$x'_\epsilon = r(x_\epsilon - \epsilon)(1 - (x_\epsilon - \epsilon)), \quad x_\epsilon(0) = \mathring{x}, \tag{A.10}$$

is a regular perturbation of the logistic equation

$$x' = rx(1-x), \quad x_\epsilon(0) = \mathring{x}. \tag{A.11}$$

Equation (A.11) *has* $x^* = 1$ *as the asymptotically stable equilibrium and* $\bar{x}^* = 0$ *as an unstable one, with* $(0, \infty)$ *being the domain of attraction of* x^*. *Similarly,* $x^*_\epsilon = 1+\epsilon$ *and* $\hat{x}^*_e$ *are, respectively, the stable and unstable equilibria of* (A.10), *and* (ϵ, ∞) *is the domain of attraction of* x^*_ϵ. *Clearly,* $\mathring{x} \in (0, \infty)$ *belongs to the domain of attraction of* x^*_ϵ *only if* $\epsilon < \mathring{x}$.

Corollary A.2. *Let* $\boldsymbol{x}^*_0, \boldsymbol{x}^*(\epsilon)$ *be the asymptotically stable hyperbolic equilibria to, respectively,* (A.1b), *and* (A.1a) *and let* $\Omega_{\boldsymbol{x}^*_0}$ *and* $\Omega_{\boldsymbol{x}^*(\epsilon)}$ *be their respective basins of attraction. Then*

a) Let $\Gamma \subset \Omega_{\boldsymbol{x}^*_0}$ *and use* $T_{\mathring{\boldsymbol{x}},\eta}$ *to denote the time needed by* $\boldsymbol{x}(t, \mathring{\boldsymbol{x}})$ *to reach the ball* $B(\boldsymbol{x}^*_0, \eta)$. *If, for any* $\eta > 0$, $\sup_{\mathring{\boldsymbol{x}}\in\Gamma} T_{\mathring{\boldsymbol{x}},\eta} < +\infty$, *then* ϵ_Γ *exists such that for any* $0 < \epsilon < \epsilon_\Gamma$, $\Gamma \subset \Omega_{\boldsymbol{x}^*(\epsilon)}$.

b) For any compact set $\Delta \subset \Omega_{\boldsymbol{x}^*_0}$, $\epsilon_\Delta > 0$ *exists such that for any* $0 < \epsilon < \epsilon_\Delta$, $\Delta \subset \Omega_{\boldsymbol{x}^*(\epsilon)}$.

Proof. Statement a) follows directly from the proof of Proposition A.2. To prove b), let $\mathring{\boldsymbol{x}} \in \Delta$. The only $\mathring{\boldsymbol{x}}$-dependent constant in (A.4) is C, which depends on $T_{\mathring{\boldsymbol{x}}}$, the time needed by $\boldsymbol{x}(t, \mathring{\boldsymbol{x}})$ to reach a fixed open ball $B(\boldsymbol{x}^*_0, \eta)$ in which it will stay for $t \geq T_{\mathring{\boldsymbol{x}}}$ and where one can use the exponential stability of $\boldsymbol{x}^*_0$, [28, Theorem 4.1]. Since $\boldsymbol{\xi} \mapsto \boldsymbol{x}(T_{\mathring{\boldsymbol{x}}}, \boldsymbol{\xi})$ is a diffeomorphism, there is a neighbourhood $U_{\mathring{\boldsymbol{x}}}$ of $\mathring{\boldsymbol{x}}$ transported onto a neighbourhood $U_{\boldsymbol{x}(T_{\mathring{\boldsymbol{x}}},\mathring{\boldsymbol{x}})}$ of $\boldsymbol{x}(T_{\mathring{\boldsymbol{x}}}$ satisfying $U_{\boldsymbol{x}(T_{\mathring{\boldsymbol{x}}},\mathring{\boldsymbol{x}})} \subset B(\boldsymbol{x}^*_0, \eta)$. By the compactness, we can select a finite number of $U_{\mathring{\boldsymbol{x}}_i}, i = 1, \ldots, p$, covering Δ, and thus there is a finite time T such that $\boldsymbol{x}(t, \mathring{\boldsymbol{x}}) \in B(\boldsymbol{x}^*_0, \eta)$ for $t \geq T$ and any $\mathring{\boldsymbol{x}} \in \Delta$. This means that we select K, α, and C in (A.4) to be independent of $\mathring{\boldsymbol{x}} \in \Delta$ and thus a universal ϵ_Δ, such that, arguing as in the proof of the previous proposition, for any $\epsilon < \epsilon_\Delta$, we have $\|\boldsymbol{x}_\epsilon(t, \mathring{\boldsymbol{x}}_0) - \boldsymbol{x}^*_0\| < \frac{\rho}{2}$. This ensures that for any $\mathring{\boldsymbol{x}}_0 \in \Delta$, $\boldsymbol{x}_\epsilon(t, \mathring{\boldsymbol{x}}_0) \to \boldsymbol{x}^*(\epsilon)$ as $t \to \infty$.