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Research article

Dynamics of stochastic diffusive coral reef ecosystems with Lévy noise

Zaitang Huang[1,2,*], **Zhiye Zhong**[2], **Yousu Huang**[2] **and Yumei Lu**[1]

[1] School of Mathematics and Statistics, Nanning Normal University, Nanning 530023, China
[2] Center for Applied Mathematics of Guangxi, Nanning Normal University, Nanning 530023, China

* **Correspondence:** Email: zaitanghuang@163.com.

Abstract: This paper was mainly concerned with the asymptotic dynamics of stochastic diffusive coral reef ecosystems with Lévy noise. First, we proved the well-posedness and energy estimates of solution. Second, under some suitable conditions, we proved the existence and uniqueness of weak pullback mean random attractors and invariant measures. Finally, a large deviation principle result for solutions of stochastic diffusive coral reef ecosystems with Lévy noise was obtained by a variational formula for positive functionals of a Poisson random measure and the method of weak convergence. Interestingly, this showed the effect of Lévy noise which can stabilize or destabilize systems, which was significantly different from the classical Brownian motion process.

Keywords: stochastic diffusive coral reef model; existence results; weak attractors; invariant measures; large deviation result

1. Introduction

The deteriorating health of the world's coral reefs has attracted widespread attention, especially in terms of global biodiversity, ecosystem functioning, and the livelihoods of millions of people living in tropical coastal areas [1]. In addition to providing safe habitats for various fish and other animals, it also has important ecosystem functions, including biodiversity maintenance, carbon cycling, coastal protection, and tourism economy.

However, the degradation of coral reefs is due to a combination of human activities (e.g., overfishing, pollution, excessive tourism development) and natural disturbances (e.g., abnormal seawater temperature, typhoons, ocean acidification) [2–5]. For example, several diseases can affect corals, such as black belt and disease, black spot disease, white belt disease, white plague, white spot disease, and yellow belt disease [6]. Various studies have indicated that corallivores are associated with specific diseases directly or indirectly [7]. In the analysis of the recovery trajectory, Cornell found limited evidence that the Caribbean ecosystem is recovering [8]. One of the reasons for this lack of recovery is the depletion of the herbivorous sea urchin Diadema antillarum caused by disease in 1983, which led to a rapid decline in the natural control ability of algal populations [9]. The cause of coral disease outbreak is still unclear and complex. Climate change, anthropogenic pollution, eutrophication, overfishing, and sedimentation are the main driving factors of coral diseases. These factors have had a serious impact on coral reef ecosystems, leading to coral bleaching, death, and ecosystem collapse. Furthermore, several bacteria responsible for coral diseases include pathogenic bacteria, cyanobacterial-dominated microbial consortiums, ciliates, and parasites, and the disease is rarely mentioned as a possible contributing factor to coral extinction [10]. Nowadays, many scientists are trying to learn more about the causes of coral disease, especially discovering the pathogens involved.

Moreover, the disease can also drive corals to extinction directly or indirectly. This does not seem surprising, but until today it has not been proven. Despite abundant evidence, the lack of a comprehensive mathematical model to simulate the interactions between corallivores and diseases hinders our understanding of coral ecosystem dynamics. To explore potential threats to coral reefs through the development of spatial eco-epidemiological mathematical models, Rani and Roy [11] formulated a diffusive coral reef ecosystems as follows:

$$\begin{cases} x_t = d_1\triangle x + r_1x(1-\rho_1x) - \chi xy - \frac{\alpha_1xz}{\theta+x} - \varrho_1x, u \in \mathbb{O},\ t>0, \\ y_t = d_2\triangle y + \chi xy - \frac{\beta_1yz}{\theta+y} - \varrho_2y, u \in \mathbb{O},\ t>0, \\ z_t = d_3\triangle z + r_2z(1-\rho_2z) + \frac{\alpha_2xz}{\theta+x} + \frac{\beta_2yz}{\theta+y} - \delta_1zw - \varrho_3z, u \in \mathbb{O},\ t>0, \\ w_t = d_4\triangle w + r_3w(1-\rho_3w) + \delta_2zw - \varrho_4w, u \in \mathbb{O},\ t>0, \end{cases} \tag{1.1}$$

where $\partial_\nu x = \partial_\nu y = \partial_\nu z = \partial_\nu w = 0, u \in \partial\mathbb{O}, t>0$, and $x(u,0) = u_0(u) \geq 0, y(u,0) = v_0(u) \geq 0,\ z(u,0) = v_0(u) \geq 0,\ w(u,0) = w_0(u) \geq 0$.

The variables, functions, and parameters have the following biological meanings:

• Ω is a bounded domain in $\mathfrak{R}^4$ with the smooth boundary $\partial\mathbb{O}$;

• ν is the outward unit normal vector;

• x, y, z, w are the density of susceptible corals population, infected corals population, Crown-of-thorns starfish population, and Humphead wrasse, respectively;

• d_1, d_2, d_3, d_4 are diffusion coefficients of susceptible corals population, infected corals population, Crown-of-thorns starfish population and Humphead wrasse, respectively;

• r_1, r_2, r_3 are the intrinsic growth rate of x population, intrinsic growth rate of star fish, and intrinsic growth rate of Humphead wrasse, respectively;

• $\frac{1}{\rho_1}, \frac{1}{\rho_2}, \frac{1}{\rho_3}$ are the capacity of the environment to carry coral population, starfish population, and Humphead wrasse, respectively;

• θ is indices of how much the environment protects the corals population, lowering the maximum predating rate;

• χ is average contact rate of x and y;

• α_1, α_2 are predation rate of corals by starfish and biomass conversion rates of x to z, respectively;

• β_1, β_2 are predation rate of starfish over infected corals and biomass conversion rates of y to z, respectively;

• δ_1, δ_2 are predation rate of Humphead wrasse over starfish and biomass conversion rates of z to w, respectively;

• $\varrho_1, \varrho_2, \varrho_3, \varrho_4$ are harvesting rate of susceptible coral, infected coral, starfish and Humphead wrasse, respectively.

The coral reef models perturbed by the Gaussian white noise have been considered extensively by many authors [12–14]. However, such models cannot describe the phenomenon that the population may suffer sudden catastrophic shocks in nature, mainly because these models are usually based on statistical data and linear predictions, and catastrophic events are often sudden and difficult to predict. For example, sudden climate change, high temperatures, extreme cold, droughts, floods, storms, giant waves, tsunamis, earthquakes, and so on, could have lasting and profound impacts on natural ecosystems [15–17]. Therefore, how to use the discontinuous stochastic process to model these abrupt nature phenomenon in ecosystems is a challenging problem [18–24].

Due to the dual impacts of climate change and human activities, such as rising sea temperatures, hurricanes, earthquakes, seawater acidification, marine pollution, etc., these uncertain factors may affect the dynamic properties of macroalgae and the corals ecosystem. In the paper, we only consider that the stochastic diffusive coral reef model with Lévy noise can be described by

$$\begin{cases} dx = \Big(d_1\triangle x + r_1x(1-\rho_1x) - \chi xy - \frac{\alpha_1xz}{\theta+x} - \varrho_1x\Big)dt + \sigma_1(x,t)dB_t + \int_{\mathbb{Z}} g_1(x,\xi)\widetilde{N}(dt,d\xi), \\ dy = \Big(d_2\triangle y + \chi xy - \frac{\beta_1yz}{\theta+y} - \varrho_2y\Big)dt + \sigma_2(y,t)dB_t + \int_{\mathbb{Z}} g_2(y,\xi)\widetilde{N}(dt,d\xi), \\ dz = \Big(d_3\triangle z + r_2z(1-\rho_2z) + \frac{\alpha_2xz}{\theta+x} + \frac{\beta_2yz}{\theta+y} - \delta_1zw - \varrho_3z\Big)dt + \sigma_3(z,t)dB_t + \int_{\mathbb{Z}} g_3(z,\xi)\widetilde{N}(dt,d\xi), \\ dw = (d_4\triangle w + r_3w(1-\rho_3w) + \delta_2zw - \varrho_4w)dt + \sigma_4(x,t)dB_t + \int_{\mathbb{Z}} g_4(w,\xi)\widetilde{N}(dt,d\xi), \end{cases} \tag{1.2}$$

where B_t are independent space time white noises. N is a Poisson measure induced by a stationary $\mathcal{F}_t$-Poisson point process on $(s,T]\times\mathbb{Z}$ with a σ-finite intensity measure $L_{T-s}\times\lambda$, L_{T-s} is the Lebesgue measure on $(s,T]$, λ is a σ-finite measure on a measurable space $\mathbb{Z}$, and $\widetilde{N}(dt,\xi) := N(dt,\xi) - \lambda(d\xi)dt$ is the compensated Poisson random measure. Assume B_t and $\widetilde{N}(dt,\xi)$ are independent.

We are very interested in the dynamics of the stochastic diffusive coral reef model with Lévy noise equation (1.2). To investigate the ergodicity, quasi-ergodicity, quasi-stationary distributions, persistence and extinction, weak pullback mean random attractors, stochastic bifurcation, stochastic chaos, and large deviation principle for a stochastic infinite dimensional dynamical system is important, but quite a challenging task in general [18–26]. There is an extensive literature dealing with the dynamics of the deterministic coral reef model [1–12, 27–29]. For the stochastic diffusive coral reef model with Lévy noise, to the best of the author's knowledge, comparably little progress has been made by now. In this paper, the global existence and uniqueness of solutions, weak attractors, and invariant measures of the stochastic diffusive coral reef model with Lévy noise are investigated. The large deviation result is derived.

The organization of the paper is as follows. In Section 2, we present some energy estimates and existence results to be used in a subsequent section. In Section 3, the existence and uniqueness of weak pullback mean random attractors for the equations are established by defining a mean random dynamical system. In Section 4, the uniqueness of this invariant measure is presented, which ensures the ergodicity of the problem. In Section 5, a large deviation principle result for solutions of stochastic diffusive coral reef model with small Lévy noise and Brownian motion is obtained by a variational formula for positive functionals of a Poisson random measure and Brownian motion.

2. Existence results

Throughout the paper, for the Sobolev space $\mathbb{H}^1(\mathbb{O})$, it follows, from Poincare's inequality and the boundedness of $\mathbb{O}$, that the norm $(\int_{\mathbb{O}} |\nabla X|^2 du)^{1/2}$ is equivalent to the predefined $\mathbb{H}^1(\mathbb{O})$ norm. Let $(\Omega, \mathcal{F}, P)$ be the complete probability space with a filtration $\{\mathcal{F}\}_{t>0}$ satisfying the usual conditions, i.e., $\{\mathcal{F}\}_{t>0}$ is an increasing right continuous family of sub-σ -algebras of $\mathcal{F}$ that contains all P-null sets. The independent Wiener process B_t is defined on $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t>0}, P)$, where B_t is an $\mathbb{L}^2(\mathbb{O})$-valued process. Let Q be the covariance operator of the Wiener process $B(t)$ with the assumption that it is strictly positive, symmetric, and a trace class operator on $H = \mathbb{L}^2(\mathbb{O})$, such that $H_0 = Q^{1/2}H$ is a Hilbert space with inner product

$$(\phi, \psi)_0 = (\phi Q^{-1/2}, \psi Q^{-1/2}) \text{ for } \phi, \psi \in H_0.$$

Let $\mathcal{L}_Q$ be the space of linear operators S such that $SQ^{1/2}$ is a Hilbert-Schmidt operator from H to H with the norm $|S|_{\mathcal{L}_Q}$. Since Q is trace class, the identity mapping from H_0 to $\mathbb{L}^2(\mathbb{O})$ is a Hilbert-Schmidt operator. For a locally compact Polish space $\mathbb{Z}$, let N be Poisson random measures defined on $[0, T]\times\mathbb{Z}$, independent of B_t. Then, let $\widetilde{N}$ be the compensated Poisson measure with the compensator λ. Let

$$\mathbb{H}^2_\lambda([0, T] \times \mathbb{Z}, H) = \Big\{h : [0, T] \times \mathbb{Z} \to H : h \text{ is measurable and } \int^T \int_{\mathbb{Z}} E(|h(t, \xi, \omega)|^2_H)\lambda(d\xi)dt < \infty\Big\}.$$

Let $\mathbb{H}^1(\mathbb{O}, \mathfrak{R}^4_+)$ be the Sobolev space $\mathbb{H}^1(\mathbb{O})$ of all $\mathfrak{R}^4_+$-valued functions, where $\mathfrak{R}^4_+$ denotes the first quadrant of $\mathfrak{R}^4$. Let us denote the Lebesgue space $H = \mathcal{L}^4(\mathbb{O})$ and the Sobolev space $V = \mathbb{H}(\mathbb{O})$ with the norms defined, respectively, as follows. For $X \in H$ and $v \in V$,

$$|X|^2_H = |X|^2 = \int_\Omega |X|^2(u)du, \quad |v|^2_V = |v|^2 + |\nabla v|^2.$$

Similarly, the Sobolev space $\mathbb{H}^2(\mathbb{O})$ is equipped with

$$|u|^2_{\mathbb{H}^2} = |u|^2 + |\nabla u|^2 + |\triangle u|^2.$$

For $X = (x, y, z, w)$, the abstract formulation of stochastic diffusive coral reef ecosystems with Lévy noise is given by

$$dX(t) = AX(t)dt + f(X(t))dt + \sigma(X(t))dB_t + \int_{\mathbb{Z}} g(X(t), \xi)\widetilde{N}(dt, d\xi). \tag{2.1}$$

The stochastic diffusive coral reef ecosystems with Lévy noise perturbed by a small parameter ε is also given by

$$dX^\varepsilon(t) = AX^\varepsilon(t)dt + f(X^\varepsilon(t))dt + \sqrt{\varepsilon}\sigma(t, X^\varepsilon(t))dB_t + \varepsilon\int_{\mathbb{Z}} g(X^\varepsilon(t), \xi)\widetilde{N}(dt, d\xi). \tag{2.2}$$

where $X^\varepsilon = (x^\varepsilon, y^\varepsilon, z^\varepsilon, w^\varepsilon)$.

Here

• The linear operator A and nonlinear functional f are given by

$$A = \begin{pmatrix} d_1\Delta + r_1 - \varrho_1 & 0 & 0 & 0 \\ 0 & d_2\Delta - \varrho_2 & 0 & 0 \\ 0 & 0 & d_3\Delta + r_2 - \varrho_3 & 0 \\ 0 & 0 & 0 & d_4\Delta + r_3 - \varrho_4 \end{pmatrix},$$

$$f(X(t)) = \begin{pmatrix} -\rho_1 r_1 x^2 - \chi xy - \frac{\alpha_1 xz}{\theta+x} \\ \chi xy - \frac{\beta_1 yz}{\theta+y} \\ -\rho_2 r_2 z^2 + \frac{\alpha_2 xz}{\theta+x} + \frac{\beta_2 yz}{\theta+y} - \delta_1 zw \\ -\rho_3 r_3 w^2 + \delta_2 zw \end{pmatrix}.$$

• The functions $\sigma(X) = (\sigma_1(x), \sigma_2(y), \sigma_3(z), \sigma_4(w))$, $g(X,\xi) = (g_1(x,\xi), g_2(y,\xi), g_3(z,\xi), g_4(w,\xi))$ are noise coefficients subject to conditions stated later.

• B_t is the independent Wiener process; $\widetilde{N}(dt, d\xi)$ is the compensated Poisson random measure.

The space $D([0,T];\mathbb{H}^1)$ denotes the space of càdlàg functions from $[0,T]$ to $\mathbb{H}^1$. The assumptions on the noise coefficients σ and g are as follows.

The functions $\sigma \in C([0,T] \times V, \mathcal{L}_Q(H_0, H))$ and $g \in \mathbb{H}^2_\lambda([0,T] \times \mathbb{Z}, H)$ satisfy:

• $(\mathcal{H}_1)$ For all $t \in [0,T]$, there exists a constant $\mathcal{K}_1 > 0$ such that for all $X \in V$,

$$|\sigma(t,X)|^2_{\mathcal{L}_Q} + \int_{\mathbb{Z}} |g(X,\xi)|^2 \lambda(d\xi) \leq \mathcal{K}_1(1 + |\nabla X|^2).$$

• $(\mathcal{H}_2)$ For all $t \in [0,T]$, there exists a constant $\mathcal{K}_2 > 0$ such that for all $X, Y \in V$,

$$|\sigma(t,X) - \sigma(t,Y)|^2_{\mathcal{L}_Q} + \int_{\mathbb{Z}} |g(X,\xi) - g(Y,\xi)|^2 d(\xi)dt \leq \mathcal{K}_2(|\nabla(X - Y)|^2).$$

Lemma 2.1. The following coercive inequality holds for all $X = (x,y,z,w) \in \mathbb{H}^1(\mathbb{O}, \mathfrak{R}^4_+)$:

$$(AX, X) \leq -d\|\nabla X\|^2_{\mathbb{L}^2} + (r_1 - \varrho_1)\|x\|^2_{\mathbb{L}^2} - \varrho_2\|y\|^2_{\mathbb{L}^2} + (r_2 - \varrho_3)\|z\|^2_{\mathbb{L}^2} + (r_3 - \varrho_4)\|w\|^2_{\mathbb{L}^2},$$

where $d = d_1 \wedge d_2 \wedge d_3 \wedge d_4$.

Proof. The proof follows at once from the definition of the linear operator A.

Lemma 2.2. For $\mathbb{H}^1(\mathbb{O}, \mathfrak{R}^4_+)$, where $X = (x_1, y_1, z_1, w_1)$, $Y = (x_2, y_2, z_2, w_2)$ the following hold

(i) Boundedness:

$$(f(X), X) \leq \frac{\chi}{2}\|x_1\|^2_{\mathbb{L}^2} + \frac{\chi}{2}\|y_1\|^2_{\mathbb{L}^4} + \left(\alpha_2 + \beta_2 + \frac{\delta_2}{2}\right)\|z_1\|^2_{\mathbb{L}^2} + \frac{\delta_2}{2}\|w_1\|^2_{\mathbb{L}^4}.$$

(ii) Lipschitz continuity: when $\Psi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4) := X - Y$

$$
\begin{aligned}
&2(f(X)-f(Y),X-Y) \le \Lambda_1\|\nabla\Psi\|_{\mathbb{L}^2}^2 + \Lambda_2\|\Psi\|_{\mathbb{L}^2}^2\\
&+\left[\frac{4\rho_1 r_1}{d}\|x_1\|_{\mathbb{L}^2}^2 + \left(\frac{4\rho_1 r_1}{d}+\frac{4}{d\chi}\right)\|x_2\|_{\mathbb{L}^2}^2 + \frac{8}{d\chi}\|y_1\|_{\mathbb{L}^2}^2\right]\|\varphi_1\|_{\mathbb{L}^2}^2\\
&+\frac{4}{d\chi}\left[\|y_1\|_{\mathbb{L}^2}^2 + 2\|x_2\|_{\mathbb{L}^2}^2\right]\|\varphi_2\|_{\mathbb{L}^2}^2\\
&+\left[\frac{4\rho_2 r_2}{d}\|z_1\|_{\mathbb{L}^2}^2 + \left(\frac{4\rho_2 r_2}{d}+\frac{4}{d\delta_1}\right)\|z_2\|_{\mathbb{L}^2}^2 + \left(\frac{4}{d\delta_1}+\frac{4}{d\delta_2}\right)\|w_1\|_{\mathbb{L}^2}^2\right]\|\varphi_3\|_{\mathbb{L}^2}^2\\
&+\left[\left(\frac{4}{d\delta_1}+\frac{4}{d\delta_2}\right)\|z_2\|_{\mathbb{L}^2}^2 + \left(\frac{4\rho_3 r_3}{d}+\frac{4}{d\delta_2}\right)\|w_1\|_{\mathbb{L}^2}^2 + \frac{4\rho_3 r_3}{d}\|w_2\|_{\mathbb{L}^2}^2\right]\|\varphi_4\|_{\mathbb{L}^2}^2,
\end{aligned}
$$

where

$$
\begin{aligned}
\Lambda_1 &= \max\left\{\frac{d\rho_1 r_1 + 2d\chi}{4} + \frac{d\alpha_1}{4\theta}, \frac{d\chi}{2} + \frac{d\beta_1}{4\theta}, \frac{d\rho_2 r_2 + d\delta_1 + d\delta_2 + 2d\alpha_2 + 2d\beta_2}{4},\right.\\
&\qquad \left.\frac{d\rho_3 r_3 + d\delta_1 + d\delta_2}{4}\right\},\\
\Lambda_2 &= \max\left\{\frac{4\beta_1}{d\theta} + 2\beta_1 + 2\alpha_2, \frac{4\beta_1}{d\theta} + 2\beta_1 + 2\beta_2, \frac{8\alpha_2}{d} + \frac{\alpha_2}{\theta} + \frac{8\beta_2}{d} + \frac{\beta_2}{\theta} + 2\alpha_1 + 2\beta_1\right\}.
\end{aligned}
$$

Proof. Let $X=(x,y,z,w)$. Consider

$$
\begin{aligned}
(f(X),X) &= \left(-\rho_1 r_1 x^2 - \chi xy - \frac{\alpha_1 xz}{\theta + x}, x\right) + \left(\chi xy - \frac{\beta_1 yz}{\theta + y}, y\right)\\
&+ \left(-\rho_2 r_2 z^2 + \frac{\alpha_2 xz}{\theta + x} + \frac{\beta_2 yz}{\theta + y} - \delta_1 zw, z\right) + \left(-\rho_3 r_3 w^2 + \delta_2 zw, w\right)\\
&\le \left(-\chi xy - \frac{\alpha_1 xz}{\theta + x}, x\right) + \left(\chi xy - \frac{\beta_1 yz}{\theta + y}, y\right)\\
&+ \left(\frac{\alpha_2 xz}{\theta + x} + \frac{\beta_2 yz}{\theta + y} - \delta_1 zw, z\right) + (\delta_2 zw, w)\\
&\le (\chi xy, y) + \left(\frac{\alpha_2 xz}{\theta + x} + \frac{\beta_2 yz}{\theta + y}, z\right) + (\delta_2 zw, w)\\
&\le \chi\int_{\mathbb{O}} |x(u,t)||y(u,t)|^2 du + (\alpha_2+\beta_2)\int_{\mathbb{O}} |z(u,t)|^2 du + \delta_2\int_{\mathbb{O}} |z(u,t)||w(u,t)|^2 du\\
&\le \chi\|x(u,t)\|_{\mathbb{L}^2}\|y(u,t)\|_{\mathbb{L}^4} + (\alpha_2+\beta_2)\|z(u,t)\|_{\mathbb{L}^2}^2 + \delta_2\|z(u,t)\|_{\mathbb{L}^2}\|w(u,t)\|_{\mathbb{L}^4}\\
&\le \frac{\chi}{2}\|x(u,t)\|_{\mathbb{L}^2}^2 + \frac{\chi}{2}\|y(u,t)\|_{\mathbb{L}^4}^2 + \left(\alpha_2+\beta_2+\frac{\delta_2}{2}\right)\|z(u,t)\|_{\mathbb{L}^2}^2 + \frac{\delta_2}{2}\|w(u,t)\|_{\mathbb{L}^4}^2\\
&\le \frac{\chi}{2}\|x(u,t)\|_{\mathbb{L}^2}^2 + \frac{\chi}{4}\sqrt{2}\|y(u,t)\|_{\mathbb{L}^2}\|\nabla y(u,t)\|_{\mathbb{L}^2}\\
&+ \left(\alpha_2+\beta_2+\frac{\delta_2}{2}\right)\|z(u,t)\|_{\mathbb{L}^2}^2 + \frac{\delta_2}{4}\sqrt{2}\|w(u,t)\|_{\mathbb{L}^2}\|\nabla w(u,t)\|_{\mathbb{L}^2}\\
&\le \frac{\chi}{2}\|x(u,t)\|_{\mathbb{L}^2}^2 + \frac{\chi\eta_1}{16}\|\nabla y(u,t)\|_{\mathbb{L}^2}^2 + \frac{\chi}{\eta_1}\|y(u,t)\|_{\mathbb{L}^2}^2
\end{aligned}
$$

$$+ \quad \left(\alpha_2 + \beta_2 + \frac{\delta_2}{2}\right)\|z(u,t)\|^2_{\mathbb{L}^2} + \frac{\delta_2\eta_2}{16}\|\nabla w(u,t)\|^2_{\mathbb{L}^2} + \frac{\delta_2}{\eta_2}\|w(u,t)\|^2_{\mathbb{L}^2}, \tag{2.3}$$

where η_1 and η_2 are positive constant. Combining the above estimates, we obtain (ii). In order to prove the Lipschitz continuity of $f(\cdot)$, let us first consider

$$\begin{aligned}
2(f(X) - f(Y), \Psi) &= -2\rho_1 r_1\left(x_1^2 - x_2^2, \varphi_1\right) - 2\chi\left(x_1y_1 - x_2y_2, \varphi_1\right) - 2\alpha_1\left(\frac{x_1z_1}{\theta + x_1} - \frac{x_2z_2}{\theta + x_2}, \varphi_1\right) \\
&+ 2\chi\left(x_1y_1 - x_2y_2, \varphi_2\right) - 2\beta_1\left(\frac{y_1z_1}{\theta + y_1} - \frac{y_2z_2}{\theta + y_2}, \varphi_2\right) \\
&- 2\rho_2 r_2\left(z_1^2 - z_2^2, \varphi_3\right) + 2\alpha_2\left(\frac{x_1z_1}{\theta + x_1} - \frac{x_2z_2}{\theta + x_2}, \varphi_3\right) \\
&+ 2\beta_2\left(\frac{y_1z_1}{\theta + y_1} - \frac{y_2z_2}{\theta + y_2}, \varphi_3\right) - 2\delta_1\left(z_1w_1 - z_2w_2, \varphi_3\right) \\
&- 2\rho_3 r_3\left(w_1^2 - w_2^2, \varphi_4\right) + 2\delta_2\left(z_1w_1 - z_2w_2, \varphi_4\right).
\end{aligned} \tag{2.4}$$

By means of applying the Hölder's inequality followed by the Ladyzhenskaya and Young inequalities as

$$\begin{aligned}
-2\rho_1 r_1\left(x_1^2 - x_2^2, \varphi_1\right) &\le 2\rho_1 r_1\|\varphi_1\|^2_{\mathbb{L}^4}\|x_1 + x_2\|_{\mathbb{L}^2} \le \sqrt{2}\rho_1 r_1\|\varphi_1\|_{\mathbb{L}^2}\|\nabla\varphi_1\|_{\mathbb{L}^2}\|x_1 + x_2\|_{\mathbb{L}^2} \\
&\le \frac{d\rho_1 r_1}{4}\|\nabla\varphi_1\|^2_{\mathbb{L}^2} + \frac{4\rho_1 r_1}{d}\|\varphi_1\|^2_{\mathbb{L}^2}\left(\|x_1\|^2_{\mathbb{L}^2} + \|x_2\|^2_{\mathbb{L}^2}\right),
\end{aligned} \tag{2.5}$$

similarly,

$$-2\rho_2 r_2\left(z_1^2 - z_2^2, \varphi_3\right) \le \frac{d\rho_2 r_2}{4}\|\nabla\varphi_3\|^2_{\mathbb{L}^2} + \frac{4\rho_2 r_2}{d}\|\varphi_3\|^2_{\mathbb{L}^2}\left(\|z_1\|^2_{\mathbb{L}^2} + \|z_2\|^2_{\mathbb{L}^2}\right), \tag{2.6}$$

$$-2\rho_3 r_3\left(w_1^2 - w_2^2, \varphi_4\right) \le \frac{d\rho_3 r_3}{4}\|\nabla\varphi_4\|^2_{\mathbb{L}^2} + \frac{4\rho_3 r_3}{d}\|\varphi_4\|^2_{\mathbb{L}^2}\left(\|w_1\|^2_{\mathbb{L}^2} + \|w_2\|^2_{\mathbb{L}^2}\right). \tag{2.7}$$

By means of applying the Hölder's inequality, we have

$$\begin{aligned}
-2\chi\left(x_1y_1 - x_2y_2, \varphi_1\right) &\le 2\chi\|y_1\|_{\mathbb{L}^2}\|x_1 - x_2\|_{\mathbb{L}^4} + \chi\|x_2\|_{\mathbb{L}^2}\left(\|y_1 - y_2\|_{\mathbb{L}^4} + \|\varphi_1\|_{\mathbb{L}^4}\right) \\
&\le 2\chi\left(\|y_1\|_{\mathbb{L}^2} + \|x_2\|_{\mathbb{L}^2}\right)\|\varphi_1\|_{\mathbb{L}^4} + \chi\|x_2\|_{\mathbb{L}^2}\|\varphi_2\|_{\mathbb{L}^4} \\
&\le \sqrt{2}\chi\left(\|y_1\|_{\mathbb{L}^2} + \|x_2\|_{\mathbb{L}^2}\right)\|\varphi_1\|_{\mathbb{L}^2}\|\nabla\varphi_1\|_{\mathbb{L}^2} + \sqrt{2}\chi\|x_2\|_{\mathbb{L}^2}\|\varphi_2\|_{\mathbb{L}^2}\|\nabla\varphi_2\|_{\mathbb{L}^2} \\
&\le \frac{d\chi}{4}\|\nabla\varphi_1\|^2_{\mathbb{L}^2} + \frac{4}{d\chi}\left(\|y_1\|^2_{\mathbb{L}^2} + \|x_2\|^2_{\mathbb{L}^2}\right)\|\varphi_1\|^2_{\mathbb{L}^2} \\
&+ \frac{d\chi}{4}\|\nabla\varphi_2\|^2_{\mathbb{L}^2} + \frac{4}{d\chi}\|x_2\|^2_{\mathbb{L}^2}\|\varphi_2\|^2_{\mathbb{L}^2},
\end{aligned} \tag{2.8}$$

similarly,

$$
\begin{aligned}
2\chi\,(x_1y_1 - x_2y_2, \varphi_2) &\leq \frac{d\chi}{4}\|\nabla\varphi_2\|^2_{\mathbb{L}^2} + \frac{4}{d\chi}\left(\|y_1\|^2_{\mathbb{L}^2} + \|x_2\|^2_{\mathbb{L}^2}\right)\|\varphi_2\|^2_{\mathbb{L}^2} \\
&+ \frac{d\chi}{4}\|\nabla\varphi_1\|^2_{\mathbb{L}^2} + \frac{4}{d\chi}\|y_1\|^2_{\mathbb{L}^2}\|\varphi_1\|^2_{\mathbb{L}^2}, \qquad (2.9)\\
-2\delta_1\,(z_1w_1 - z_2w_2, \varphi_3) &\leq \frac{d\delta_1}{4}\|\nabla\varphi_3\|^2_{\mathbb{L}^2} + \frac{4}{d\delta_1}\left(\|w_1\|^2_{\mathbb{L}^2} + \|z_2\|^2_{\mathbb{L}^2}\right)\|\varphi_3\|^2_{\mathbb{L}^2} \\
&+ \frac{d\delta_1}{4}\|\nabla\varphi_4\|^2_{\mathbb{L}^2} + \frac{4}{d\delta_1}\|z_2\|^2_{\mathbb{L}^2}\|\varphi_4\|^2_{\mathbb{L}^2}, \qquad (2.10)\\
2\delta_2\,(z_1w_1 - z_2w_2, \varphi_4) &\leq \frac{d\delta_2}{4}\|\nabla\varphi_4\|^2_{\mathbb{L}^2} + \frac{4}{d\delta_2}\left(\|w_1\|^2_{\mathbb{L}^2} + \|z_2\|^2_{\mathbb{L}^2}\right)\|\varphi_4\|^2_{\mathbb{L}^2} \\
&+ \frac{d\delta_2}{4}\|\nabla\varphi_3\|^2_{\mathbb{L}^2} + \frac{4}{d\delta_2}\|w_1\|^2_{\mathbb{L}^2}\|\varphi_3\|^2_{\mathbb{L}^2}. \qquad (2.11)
\end{aligned}
$$

By utilizing the algebraic identity, we get

$$
\begin{aligned}
\left|\frac{x_1z_1}{\theta + x_1} - \frac{x_2z_2}{\theta + x_2}\right| &= \left|\frac{\theta(x_1 - x_2) + (z_1 - z_2)(\theta x_2 - x_1x_2)}{(\theta + x_1)(\theta + x_2)}\right| \\
&\leq \frac{\theta\,|x_1 - x_2| + |z_1 - z_2|\,|\theta x_2 - x_1x_2|}{|(\theta + x_1)(\theta + x_2)|} \\
&\leq \frac{1}{\theta}\,|x_1 - x_2| + 2\,|z_1 - z_2|\,,
\end{aligned}
$$

$$
\begin{aligned}
-2\alpha_1\left(\frac{x_1z_1}{\theta + x_1} - \frac{x_2z_2}{\theta + x_2}, \varphi_1\right) &\leq 2\alpha_1\left(\left|\frac{x_1z_1}{\theta + x_1} - \frac{x_2z_2}{\theta + x_2}\right|, \varphi_1\right) \\
&\leq 2\alpha_1\left(\frac{1}{\theta}\,|x_1 - x_2| + 2\,|z_1 - z_2|\,, \varphi_1\right) \\
&\leq \frac{2\alpha_1}{\theta}\,\|\varphi_1\|^2_{\mathbb{L}^4} + 4\alpha_1\,\|z_1 - z_2\|_{\mathbb{L}^2}\,\|\varphi_1\|_{\mathbb{L}^2} \\
&\leq \frac{\sqrt{2}\alpha_1}{\theta}\,\|\varphi_1\|_{\mathbb{L}^2}\,\|\nabla\varphi_1\|_{\mathbb{L}^2} + 4\alpha_1\,\|z_1 - z_2\|_{\mathbb{L}^2}\,\|\varphi_1\|_{\mathbb{L}^2} \qquad (2.12)\\
&\leq \frac{d\alpha_1}{4\theta}\,\|\nabla\varphi_1\|^2_{\mathbb{L}^2} + \left(\frac{4\alpha_1}{d\theta} + 2\alpha_1\right)\|\varphi_1\|^2_{\mathbb{L}^2} + 2\alpha_1\,\|\varphi_3\|^2_{\mathbb{L}^2}\,,
\end{aligned}
$$

similarly,

$$
-2\beta_1\left(\frac{y_1z_1}{\theta + y_1} - \frac{y_2z_2}{\theta + y_2}, \varphi_2\right) \leq \frac{d\beta_1}{4\theta}\,\|\nabla\varphi_2\|^2_{\mathbb{L}^2} + \left(\frac{4\beta_1}{d\theta} + 2\beta_1\right)\|\varphi_2\|^2_{\mathbb{L}^2} + 2\beta_1\,\|\varphi_3\|^2_{\mathbb{L}^2}\,, \qquad (2.13)
$$

$$
2\alpha_2\left(\frac{x_1z_1}{\theta + x_1} - \frac{x_2z_2}{\theta + x_2}, \varphi_3\right) \leq \frac{d\alpha_2}{2}\,\|\nabla\varphi_3\|^2_{\mathbb{L}^2} + \left(\frac{8\alpha_2}{d} + \frac{\alpha_2}{\theta}\right)\|\varphi_3\|^2_{\mathbb{L}^2} + 2\alpha_2\,\|\varphi_1\|^2_{\mathbb{L}^2}\,, \qquad (2.14)
$$

$$
2\beta_2\left(\frac{y_1z_1}{\theta + y_1} - \frac{y_2z_2}{\theta + y_2}, \varphi_3\right) \leq \frac{d\beta_2}{2}\,\|\nabla\varphi_3\|^2_{\mathbb{L}^2} + \left(\frac{8\beta_2}{d} + \frac{\beta_2}{\theta}\right)\|\varphi_3\|^2_{\mathbb{L}^2} + 2\beta_2\,\|\varphi_2\|^2_{\mathbb{L}^2}\,. \qquad (2.15)
$$

From the estimates equations (2.5)–(2.15), we have

$$
\begin{aligned}
&2(f(X)-f(Y),X-Y)\leq \Lambda_1\|\nabla\Psi\|_{\mathbb{L}^2}^2+\Lambda_2\|\Psi\|_{\mathbb{L}^2}^2\\
&+\left[\frac{4\rho_1r_1}{d}\|x_1\|_{\mathbb{L}^2}^2+\left(\frac{4\rho_1r_1}{d}+\frac{4}{d\chi}\right)\|x_2\|_{\mathbb{L}^2}^2+\frac{8}{d\chi}\|y_1\|_{\mathbb{L}^2}^2\right]\|\varphi_1\|_{\mathbb{L}^2}^2\\
&+\frac{4}{d\chi}\left[\|y_1\|_{\mathbb{L}^2}^2+2\|x_2\|_{\mathbb{L}^2}^2\right]\|\varphi_2\|_{\mathbb{L}^2}^2\\
&+\left[\frac{4\rho_2r_2}{d}\|z_1\|_{\mathbb{L}^2}^2+\left(\frac{4\rho_2r_2}{d}+\frac{4}{d\delta_1}\right)\|z_2\|_{\mathbb{L}^2}^2+\left(\frac{4}{d\delta_1}+\frac{4}{d\delta_2}\right)\|w_1\|_{\mathbb{L}^2}^2\right]\|\varphi_3\|_{\mathbb{L}^2}^2\\
&+\left[\left(\frac{4}{d\delta_1}+\frac{4}{d\delta_2}\right)\|z_2\|_{\mathbb{L}^2}^2+\left(\frac{4\rho_3r_3}{d}+\frac{4}{d\delta_2}\right)\|w_1\|_{\mathbb{L}^2}^2+\frac{4\rho_3r_3}{d}\|w_2\|_{\mathbb{L}^2}^2\right]\|\varphi_4\|_{\mathbb{L}^2}^2, \qquad (2.16)
\end{aligned}
$$

where

$$
\begin{aligned}
\Lambda_1 &= \max\left\{\frac{d\rho_1r_1+2d\chi}{4}+\frac{d\alpha_1}{4\theta},\frac{d\chi}{2}+\frac{d\beta_1}{4\theta},\frac{d\rho_2r_2+d\delta_1+d\delta_2+2d\alpha_2+2d\beta_2}{4},\right.\\
&\left.\frac{d\rho_3r_3+d\delta_1+d\delta_2}{4}\right\},
\end{aligned}
$$

$$
\Lambda_2 = \max\left\{\frac{4\beta_1}{d\theta}+2\beta_1+2\alpha_2,\frac{4\beta_1}{d\theta}+2\beta_1+2\beta_2,\frac{8\alpha_2}{d}+\frac{\alpha_2}{\theta}+\frac{8\beta_2}{d}+\frac{\beta_2}{\theta}+2\alpha_1+2\beta_1\right\}.
$$

By using Galerkin approximation, it is easy to prove the existence. Let a complete orthonormal basis of the space H be $\{\phi_n\}_{n\geq1}$ such that $\phi_n\in D(A)$. For any $n\geq1$, let $H_n=\mathrm{span}(\phi_1,\phi_2,\dots,\phi_n)$ and $P_n:H\to H_n$ be an orthogonal projection onto H_n, which contracts H and V norms. Let $W_n=P_nW,\sigma_n=P_n\sigma$, and $g_n=P_ng$. Then, for $\phi\in H_n$, consider the equation in H_n

$$
(dX_n,\phi)=(AX_n+f(X_n),\phi)dt+(\sigma(X_n)dB_n,\phi)+\int_{\mathbb{Z}}(g_n(X_n,\xi),\phi)\widetilde{N}(dt,d\xi), \qquad (2.17)
$$

with $X_n(0)=P_nX_0$. Due to the Lipschitz property satisfied by the coefficients, it is easy to see that system (2.17) ensures well-posedness, then there exists a maximal solution to system (2.17), namely, a stopping time $\tau_n\leq T$ such that for $t<\tau_n$, system (2.17) holds, and for $t\uparrow\tau_n<T,|X_n(t)|\to\infty$. We now prove $\tau_n=T$ and estimate X_n for all n. For $N>0$, take

$$
\tau_N=\inf\{t:|X(t)|+|\nabla X(t)|\geq N\}\wedge T.
$$

Then on $\{\tau_n=T\},X_n\in\mathcal{D}([0,T],H_n)$ a.s.

Proposition 2.3 (Energy estimate). Let X_n be the unique solution of the system of stochastic ordinary differential equation (2.17) with $X_0\in\mathcal{D}([0,T],H_n)$. Then, there exists a unique solution $X_n\in\mathcal{D}([0,T],H_n)$ satisfying

$$
\begin{aligned}
&\mathbb{E}\sup_{0\leq s\leq t\wedge T}|X_n(s)|^{2p}+\hat{\alpha}\mathbb{E}\left(\int_0^T\|\nabla X_n\|^2ds\right)^p\leq C\left(1+\mathbb{E}|X_0|^{2p}\right),\\
&\mathbb{E}\sup_{0\leq s\leq T}|\nabla X_n(s)|^{2p}+\hat{\alpha}\mathbb{E}\left(\int_0^T\|\Delta X_n\|^2|\nabla X_n|^{2(p-1)}ds\right)^p\leq C\left(1+\mathbb{E}|X_0|^{2p}\right),
\end{aligned}
$$

where C is an appropriate constant.

Proof. Let us define a sequence of stopping times τ_N by

$$\tau_N = \inf\{t : |X(t)| + |\nabla X(t)| \geq N\},$$

for $N \in \mathbb{N}$. Applying the finite dimensional Itô$'s$ formula to the process $|X_n(\cdot)|^2$, we obtain

$$\begin{aligned} |X_n(t \wedge \tau_N)|^2 &= |P_n X_0|^2 + 2\int_0^{t\wedge\tau_N} (AX_n + f(X_n)(s), X_n(s))ds \\ &+ \int_0^{t\wedge\tau_N} |\sigma_n(X_n(s)|^2_{\mathcal{L}_Q} ds + 2\int_0^{t\wedge\tau_N} (\sigma_n(X_n(s))dB_n, X_n(s)) \\ &+ \int_0^{t\wedge\tau_N}\int_{\mathbb{Z}} |g_n(X_n,\xi)| N(dt, d\xi) + 2\int_0^{t\wedge\tau_N}\int_{\mathbb{Z}} (g_n(X_n,\xi), X_n)\widetilde{N}(dt, d\xi). \end{aligned} \tag{2.18}$$

Applying Itô$'s$ formula for $|X_n(\cdot)|^{2p}$, we get

$$\begin{aligned} |X_n(t \wedge \tau_N)|^{2p} &= |P_n X_0|^{2p} + 2p\int_0^{t\wedge\tau_N} (AX_n + f(X_n)(s), X_n(s))|X_n(s)|^{2(p-1)} ds \\ &+ p\int_0^{t\wedge\tau_N} |\sigma_n(X_n(s)|^2_{\mathcal{L}_Q}|X_n(s)|^{2(p-1)} ds \\ &+ 2p(p-1)\int_0^{t\wedge\tau_N} |\Pi_n\sigma_n(X_n(s)X_n(s)|^2_{\mathcal{L}_Q}|X_n(s)|^{2(p-2)} ds \\ &+ 2p\int_0^{t\wedge\tau_N} (\sigma_n(X_n(s))dB_n, X_n(s))|X_n(s)|^{2(p-1)} \\ &+ \int_0^{t\wedge\tau_N}\int_{\mathbb{Z}} [|X_n(s-) + g_n(X_n(s-),\xi)|^{2p} - |X_n(s-)|^{2p}]\lambda(d\xi)ds \\ &- 2p\int_0^{t\wedge\tau_N}\int_{\mathbb{Z}} (g_n(X_n(s-),\xi), X_n(s-))|X_n(s-)|^{2(p-1)}]\lambda(d\xi)ds \\ &+ \int_0^{t\wedge\tau_N}\int_{\mathbb{Z}} [|X_n(s-) + g_n(X_n(s-),\xi)|^{2p} - |X_n(s-)|^{2p}]\widetilde{N}(dt, d\xi). \end{aligned} \tag{2.19}$$

Applying Lemmas 2.1 and 2.2, we have

$$\begin{aligned} I_1 &= 2p\int_0^{t\wedge\tau_N} (AX_n + f(X_n)(s), X_n(s))|X_n(s)|^{2(p-1)} ds \\ &\leq 2p\int_0^{t\wedge\tau_N} \Big[-d\|\nabla X_n\|^2 + (r_1 - \varrho_1)\|x_n\|^2 - \varrho_2\|y_n\|^2 \\ &\quad + (r_2 - \varrho_3)\|z_n\|^2 + (r_3 - \varrho_4)\|w_n\|^2\Big] |X_n(s)|^{2(p-1)} ds \\ &+ 2p\int_0^{t\wedge\tau_N} \left[\frac{\mathcal{X}}{2}\|x_n\|^2_{\mathbb{L}^2} + \frac{\mathcal{X}}{2}\|y_n\|^2_{\mathbb{L}^4} + \left(\alpha_2 + \beta_2 + \frac{\delta_2}{2}\right)\|z_n\|^2_{\mathbb{L}^2} + \frac{\delta_2}{2}\|w_n\|^2_{\mathbb{L}^4}\right] |X_n(s)|^{2(p-1)} ds \\ &\leq 2p\int_0^{t\wedge\tau_N} \Big[-d\|\nabla X_n\|^2 + (r_1 - \varrho_1)\|x_n\|^2 - \varrho_2\|y_n\|^2 \\ &\quad + (r_2 - \varrho_3)\|z_n\|^2 + (r_3 - \varrho_4)\|w_n\|^2\Big] |X_n(s)|^{2(p-1)} ds \end{aligned}$$

$$
\begin{aligned}
&+ \quad 2p\int_0^{t\wedge\tau_N}\Big[\frac{\chi}{2}\|x_n(u,t)\|^2_{\mathbb{L}^2}+\frac{\chi\eta_1}{16}\|\nabla y_n(u,t)\|^2_{\mathbb{L}^2}+\frac{\chi}{\eta_1}\|y_n(u,t)\|^2_{\mathbb{L}^2}\\
&+ \quad \Big(\alpha_2+\beta_2+\frac{\delta_2}{2}\Big)\|z_n(u,t)\|^2_{\mathbb{L}^2}+\frac{\delta_2\eta_2}{16}\|\nabla w_n(u,t)\|^2_{\mathbb{L}^2}+\frac{\delta_2}{\eta_2}\|w_n(u,t)\|^2_{\mathbb{L}^2}\Big]|X_n(s)|^{2(p-1)}ds\\
&\leq \quad 2p\int_0^{t\wedge\tau_N}\Big[-d\|\nabla X_n\|^2+\Big(r_1-\varrho_1+\frac{\chi}{2}\Big)\|x_n(u,t)\|^2_{\mathbb{L}^2}+\Big(\frac{\chi}{\eta_1}-\varrho_2\Big)\|y_n(u,t)\|^2_{\mathbb{L}^2}\Big]|X_n(s)|^{2(p-1)}ds\\
&+ \quad 2p\int_0^{t\wedge\tau_N}\Big[\frac{\chi\eta_1}{16}\|\nabla y_n(u,t)\|^2_{\mathbb{L}^2}+\Big(r_2-\varrho_3+\alpha_2+\beta_2+\frac{\delta_2}{2}\Big)\|z_n(u,t)\|^2_{\mathbb{L}^2}\\
&+ \quad \frac{\delta_2\eta_2}{16}\|\nabla w_n(u,t)\|^2_{\mathbb{L}^2}+\Big(r_3-\varrho_4+\frac{\delta_2}{\eta_2}\Big)\|w_n(u,t)\|^2_{\mathbb{L}^2}\Big]|X_n(s)|^{2(p-1)}ds\\
&\leq \quad 2p\int_0^{t\wedge\tau_N}\Big[-\hat{d}\|\nabla X_n\|^2-c_1\|X_n\|^2_{\mathbb{L}^2}\Big]|X_n(s)|^{2(p-1)}ds, \qquad (2.20)
\end{aligned}
$$

where

$$
\begin{aligned}
&\hat{d}=d-\max\Big\{\frac{\chi\eta_1}{16},\frac{\delta_2\eta_2}{16}\Big\}\geq 0,\\
&c_1=\inf\Big\{\varrho_1-r_1-\frac{\chi}{2},\varrho_2-\frac{\chi}{\eta_1},\varrho_3-r_2-\alpha_2-\beta_2-\frac{\delta_2}{2},\varrho_4-r_3-\frac{\delta_2}{\eta_2}\Big\}.
\end{aligned}
$$

$$
\begin{aligned}
I_2 \quad &= \quad p\int_0^{t\wedge\tau_N}\Big[|\sigma_n(X_n(s)|^2_{\mathcal{L}_Q}+2(p-1)|\Pi_n\sigma_n(X_n(s)|^2_{\mathcal{L}_Q}\Big]|X_n(s)|^{2(p-1)}ds\\
&\leq \quad 2p(p-1)\mathcal{K}_1\int_0^{t\wedge\tau_N}\Big[1+|\nabla X_n(s)|^2\Big]|X_n(s)|^{2(p-1)}ds. \qquad (2.21)
\end{aligned}
$$

Using Taylor's formula and the Cauchy-Schwarz inequality, we obtain

$$
\begin{aligned}
I_3 \quad &= \quad \int_0^{t\wedge\tau_N}\int_{\mathbb{Z}}[|X_n(s-)+g_n(X_n(s-),\xi)|^{2p}-|X_n(s-)|^{2p}]N(dt,d\xi)\\
&+ \quad 2p\int_0^{t\wedge\tau_N}\int_{\mathbb{Z}}[(g_n(X_n(s-),\xi),X_n(s-))|X_n(s-)|^{2(p-1)}]\widetilde{N}(dt,d\xi)\\
&- \quad 2p\int_0^{t\wedge\tau_N}\int_{\mathbb{Z}}[(g_n(X_n(s-),\xi),X_n(s-))|X_n(s-)|^{2(p-1)}]N(dt,d\xi)\\
&\leq \quad 2p(2p-1)\int_0^{t\wedge\tau_N}\int_{\mathbb{Z}}|X_n(s-)|^{2(p-1)}|g_n(X_n(s-)|^2N(dt,d\xi)\\
&+ \quad 2p\int_0^{t\wedge\tau_N}\int_{\mathbb{Z}}[(g_n(X_n(s-),\xi),X_n(s-))|X_n(s-)|^{2(p-1)}]\widetilde{N}(dt,d\xi). \qquad (2.22)
\end{aligned}
$$

Taking expectation, we get

$$
\begin{aligned}
\mathbb{E}\sup_{0\le s\le t\wedge\tau_N}|X_n(s)|^{2p} &+ 2p\hat{d}\mathbb{E}\int_0^{t\wedge\tau_N}\|\nabla X_n\|^2|X_n(s)|^{2(p-1)}ds\\
&\le \mathbb{E}|P_nX_0|^{2p}+2pc_1\mathbb{E}\int_0^{t\wedge\tau_N}|X_n(s)|^{2p}ds\\
&+ 3p(p-1)\mathcal{K}_1\mathbb{E}\int_0^{t\wedge\tau_N}\left[1+|\nabla X_n(s)|^2\right]|X_n(s)|^{2(p-1)}ds\\
&+ \mathbb{E}\sup_{0\le s\le t\wedge\tau_N}J_1(s)+\mathbb{E}\sup_{0\le s\le t\wedge\tau_N}J_2(s), \qquad (2.23)
\end{aligned}
$$

where

$$
\begin{aligned}
J_1 &= 2p\int_0^{t\wedge\tau_N}(\sigma_n(X_n(s))dB_n, X_n(s))|X_n(s)|^{2(p-1)},\\
J_2 &= 2p\int_0^{t\wedge\tau_N}\int_{\mathbb{Z}}[(g_n(X_n(s-),\xi),X_n(s-))|X_n(s-)|^{2(p-1)}]\widetilde{N}(dt,d\xi).
\end{aligned}
$$

By Burkholder-Davis-Gundy inequality and Young's inequality, we have

$$
\begin{aligned}
\mathbb{E}\sup_{0\le s\le t\wedge\tau_N}J_1(s) &\le 2pC_1\mathbb{E}\left\{\int_0^{t\wedge\tau_N}|\sigma_n(X_n(s))|^2_{\mathcal{L}_Q}|X_n(s)|^{4p-2}ds\right\}^{1/2} \qquad (2.24)\\
&\le \eta_2\mathbb{E}\sup_{0\le s\le t\wedge\tau_N}|X_n(s)|^{2p}+\frac{p^2C_1^2\mathcal{K}_1}{\eta_2}\mathbb{E}\int_0^{t\wedge\tau_N}\left[1+|\nabla X_n(s)|^2\right]|X_n(s)|^{2(p-1)}ds
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{E}\sup_{0\le s\le t\wedge\tau_N}J_2(s) &\le 2pC_2\mathbb{E}\left\{\int_0^{t\wedge\tau_N}|X_n(s)|^{4p-2}\int_{\mathbb{Z}}|g_n(X_n(s-),\xi)|^2d(\xi)ds\right\}^{1/2} \qquad (2.25)\\
&\le \eta_3\mathbb{E}\sup_{0\le s\le t\wedge\tau_N}|X_n(s)|^{2p}+\frac{p^2C_2^2\mathcal{K}_1}{\eta_3}\mathbb{E}\int_0^{t\wedge\tau_N}\left[1+|\nabla X_n(s)|^2\right]|X_n(s)|^{2(p-1)}ds
\end{aligned}
$$

Substituting Eqs (2.24) and (2.25) in Eq (2.23), we find

$$
\begin{aligned}
\mathbb{E}\sup_{0\le s\le t\wedge\tau_N}|X_n(s)|^{2p} &+ 2p\hat{d}\mathbb{E}\int_0^{t\wedge\tau_N}\|\nabla X\|^2|X_n(s)|^{2(p-1)}ds\\
&\le \mathbb{E}|P_nX_0|^{2p}+2pc_1\mathbb{E}\int_0^{t\wedge\tau_N}|X_n(s)|^{2p}ds\\
&+ 3p(p-1)\mathcal{K}_1\mathbb{E}\int_0^{t\wedge\tau_N}\left[1+|\nabla X_n(s)|^2\right]|X_n(s)|^{2(p-1)}ds\\
&+ (\eta_2+\eta_3)\mathbb{E}\sup_{0\le s\le t\wedge\tau_N}|X_n(s)|^{2p} \qquad (2.26)\\
&+ \left(\frac{p^2C_1^2\mathcal{K}_1}{\eta_2}+\frac{p^2C_3^2\mathcal{K}_1}{\eta_3}\right)\mathbb{E}\int_0^{t\wedge\tau_N}\left[1+|\nabla X_n(s)|^2\right]|X_n(s)|^{2(p-1)}ds.
\end{aligned}
$$

Comparing with Lemma 3.9 [30], we assign

$$X(t) = \sup_{0\le s\le t\wedge\tau_N} |X_n(s)|^{2p}, \quad Y(t) = \int_0^{t\wedge\tau_N} \|\nabla X\|^2 |X_n(s)|^{2(p-1)} ds,$$
$$Z(t) = |X_0|^{2p} + 3p(p-1)\mathcal{K}_1 \mathbb{E}\int_0^t |X_n(s)|^{2(p-1)} ds, \quad I(t) = \sup_{0\le s\le t\wedge\tau_N} \{J_1(s) + J_2(s)\},$$
$$\varphi(t) = 2pc_1, \quad \int_0^T \varphi(t) ds = 2pc_1 T, \quad \hat{\alpha} = 2p\hat{d} - 3p(p-1)\mathcal{K}_1, \quad \hat{\beta} = \eta_2 + \eta_3,$$
$$\hat{\gamma} = \frac{p^2 C_1^2 \mathcal{K}_1}{\eta_2} + \frac{p^2 C_3^2 \mathcal{K}_1}{\eta_3}, \quad \tilde{C} = \left(\frac{p^2 C_1^2 \mathcal{K}_1}{\eta_2} + \frac{p^2 C_3^2 \mathcal{K}_1}{\eta_3}\right) \int_0^{\tau_N} X_n(s)|^{2(p-1)} ds.$$

Then, we have, for $t \in [0, T]$,

$$\sup_n \mathbb{E}\left(\sup_{0\le s\le t\wedge\tau_N} |X_n(s)|^{2p} + \hat{\alpha} \int_0^{t\wedge\tau_N} \|\nabla X\|^2 |X_n(s)|^{2(p-1)} ds \right) \le C(P, T, \mathcal{K}_1). \tag{2.27}$$

Applying Itô's formula for $|\nabla X_n|^{2p}$, we get

$$|\nabla X_n|^{2p} = |P_n \nabla X_0|^{2p} + I_4 + I_5 + I_6 + J_3 + J_4, \tag{2.28}$$

where

$$\begin{aligned}
I_4 &= 2p\int_0^{t\wedge\tau_N} (AX_n(s) + f(X_n(s)), \nabla X_n) |\nabla X_n(s)|^{2(p-1)} ds, \\
I_5 &= p\int_0^{t\wedge\tau_N} |\sigma_n(X_n(s)|^2_{\mathcal{L}_Q} |\nabla X_n(s)|^{2(p-1)} ds \\
&\quad + 2p(p-1) \int_0^{t\wedge\tau_N} |\Pi_n \sigma_n(X_n(s)) X_n(s)|^2_{\mathcal{L}_Q} |\nabla X_n(s)|^{2(p-2)} ds, \\
I_6 &= \int_0^{t\wedge\tau_N} \int_{\mathbb{Z}} [|\nabla X_n(s-) + g_n(X_n(s-), \xi)|^{2p} - |\nabla X_n(s-)|^{2p}] \lambda(d\xi) ds \\
&\quad - 2p \int_0^{t\wedge\tau_N} \int_{\mathbb{Z}} (g_n(X_n(s-), \xi), \nabla X_n(s-) |\nabla X_n(s-)|^{2(p-1)}] \lambda(d\xi) ds \\
&\quad + \int_0^{t\wedge\tau_N} \int_{\mathbb{Z}} [|\nabla X_n(s-) + g_n(X_n(s-), \xi)|^{2p} - |\nabla X_n(s-)|^{2p}] \widetilde{N}(dt, d\xi), \\
J_3 &= 2p\int_0^{t\wedge\tau_N} (\sigma_n(X_n(s)) dB_n, \nabla X_n) |\nabla X_n(s)|^{2(p-1)}, \\
J_4 &= 2p\int_0^{t\wedge\tau_N} \int_{\mathbb{Z}} (g_n(X_n(s-), \xi), \nabla X_n(s-) |\nabla X_n(s-)|^{2(p-1)}] \widetilde{N}(dt, d\xi).
\end{aligned}$$

By reducing as before, we get

$$\begin{aligned}
\mathbb{E}\sup_{0\le s\le t\wedge\tau_N}|\nabla X_n(s)|^{2p} &+ 2p\hat{d}\mathbb{E}\int_0^{t\wedge\tau_N}\|\Delta X_n\|^2|\nabla X_n(s)|^{2(p-1)}ds\\
&\le \mathbb{E}|P_n\nabla X_0|^{2p}-2pc_1\mathbb{E}\int_0^{t\wedge\tau_N}|\nabla X_n(s)|^{2p}ds\\
&+ 3p(p-1)\mathcal{K}_1\mathbb{E}\int_0^{t\wedge\tau_N}\left[1+|\nabla X_n(s)|^2\right]|\nabla X_n(s)|^{2(p-1)}ds\\
&+ \mathbb{E}\sup_{0\le s\le t\wedge\tau_N}\{J_3+J_4\}. \qquad (2.29)
\end{aligned}$$

Using a similar computation, we can get

$$\mathbb{E}\left(\sup_{0\le s\le t\wedge\tau_N}|\nabla X_n(s)|^{2p}\right)+\hat{\alpha}\mathbb{E}\int_0^{t\wedge\tau_N}\|\Delta X_n\|^2|\nabla X_n|^{2(p-1)}ds\le C. \qquad (2.30)$$

Here $\tau_N\to\tau_n$ when $N\to+\infty$, and for $\{\tau_n<T\}$, $\sup_{0\le s\le\tau_n}|X_n(s)|\to+\infty$. Hence $P\{\tau_n<T\}=0$, and so for large N, $\tau_N=T$ and $X_n(s)\in\mathcal{D}([0,T],H_n)$. Hence, the proof.

Theorem 2.4. Assuming $\mathcal{H}_1$-$\mathcal{H}_2$ hold, let$E|X_0|^2<\infty, \exists\varepsilon_0>0, \forall\varepsilon\in[0,\varepsilon_0]$. Then, there exists a pathwise unique weak solution X for the stochastic diffusive coral reef ecosystems with Lévy noise equation (1.2) in $\mathbb{X}=\mathcal{D}([0,T];\mathbb{H}^1)\cap\mathbb{L}^2((0,T);\mathbb{H}^2)$ with $X^\varepsilon(0)=X_0\in H$ such that

$$\begin{aligned}
\mathbb{E}\sup_{0\le s\le t\wedge T}|X^\varepsilon(s)|^2+\hat{\alpha}\mathbb{E}\left(\int_0^T\|\nabla X^\varepsilon\|^2ds\right)&\le C\left(1+\mathbb{E}|X_0|^2\right),\\
\mathbb{E}\sup_{0\le s\le T}|\nabla X^\varepsilon(s)|^2+\hat{\alpha}\mathbb{E}\left(\int_0^T\|\Delta X^\varepsilon\|^2ds\right)&\le C\left(1+\mathbb{E}|X_0|^2\right),
\end{aligned}$$

where C is an appropriate constant.

Proof. Let $\mathbb{O}_T=[0,T]\times\mathbb{O}$ and $F(X)=AX+f(X)$. From Proposition 2.3, we can conclude that there exist a subsequence $\{X_n^\varepsilon\}$ and processes $X^\varepsilon\in\mathcal{L}^2(\mathbb{O}_T,\mathbb{H}^2)\cap\mathcal{L}^4(\mathbb{O},\mathcal{D}([0,T],\mathbb{H}^1))$, $F^\varepsilon\in\mathcal{L}^2(\mathbb{O}_T,V')$, $S^\varepsilon\in\mathcal{L}^2(\mathbb{O}_T,\mathcal{L}_Q)$, and $G^\varepsilon\in\mathbb{H}^2_\lambda([0,T]\times\mathbb{Z};H)$ such that

$$\begin{cases}
X_n^\varepsilon\to X^\varepsilon \text{ weakly in } \mathcal{L}^2(\mathbb{O}_T,\mathbb{H}^2);\\
X_n^\varepsilon \text{ is weak star converging to } X^\varepsilon \text{ in } \mathcal{L}^4(\mathbb{O},\mathcal{D}([0,T],\mathbb{H}^1));\\
F(X_n^\varepsilon)\to F^\varepsilon \text{ in } \mathcal{L}^2(\mathbb{O}_T,V');\\
\sigma_n(X_n^\varepsilon)\to S^\varepsilon \text{ in } \mathcal{L}^2(\mathbb{O}_T,\mathcal{L}_Q);\\
g_n(X_n^\varepsilon)\to G^\varepsilon \text{ in } \mathbb{H}^2_\lambda([0,T]\times\mathbb{Z};H).
\end{cases}$$

$X_n^\varepsilon\to X^\varepsilon$ weakly in $\mathcal{L}^2(\mathbb{O}_T,\mathbb{H}^2)$ and X_n^ε is weak star converging to X^ε in $\mathcal{L}^4(\mathbb{O},\mathcal{D}([0,T],\mathbb{H}^1))$, which holds as a direct consequence of Proposition 2.3. To prove $F(X_n^\varepsilon)\to F(X^\varepsilon)$ in $\mathcal{L}^2(\mathbb{O}_T,V')$, let $\psi\in\mathcal{L}^2(\mathbb{O}_T,\mathbb{H}^2)$. Then, we get

$$\begin{aligned}
\mathbb{E}\int_0^T(F(X_n^\varepsilon(s)),\psi(s))ds&=\int_0^T(A(X_n^\varepsilon(s)),\psi(s))+(f(X_n^\varepsilon(s)),\psi(s))ds\\
\le\ \int_0^T-\left[d-\frac{\chi\eta_1}{16}-\frac{\delta_2}{16\eta_2}\right]&(\nabla X_n^\varepsilon(s),\nabla\psi(s))ds+\int_0^T\Lambda|X_n(s)||\psi(s)|ds, \qquad (2.31)
\end{aligned}$$

where

$$\Lambda = (r_1 - \varrho_1) + (r_2 - \varrho_3) + (r_3 - \varrho_4) + \frac{\chi}{2} + \alpha_2 + \beta_2 + \frac{\delta_2}{2} + \frac{\chi}{\eta_1} + \frac{\delta_2}{\eta_2}.$$

By $X_n^\varepsilon \to X^\varepsilon$ weakly in $\mathcal{L}^2(\mathbb{O}_T, \mathbb{H}^2)$, we can prove $F(X_n^\varepsilon) \to F(X^\varepsilon)$ in $\mathcal{L}^2(\mathbb{O}_T, V')$.

From $\mathcal{H}_1$, we have

$$\mathbb{E}\int_0^T |\sigma(t, X^\varepsilon)|^2_{\mathcal{L}_Q} ds + \mathbb{E}\int_0^T \int_{\mathbb{Z}} |g(X^\varepsilon, \xi)|^2 \lambda(d\xi) ds \le \mathcal{K}_1 \mathbb{E}\int_0^T (1 + |\nabla X^\varepsilon|^2) ds < \infty.$$

It implies that $\sigma_n(X_n^\varepsilon) \to \sigma(X^\varepsilon)$ in $\mathcal{L}^2(\mathbb{O}_T, \mathcal{L}_Q)$ and $g_n(X_n^\varepsilon) \to g(X^\varepsilon)$ in $\mathbb{H}^2_\lambda([0,T]\times\mathbb{Z}; H)$.

Since $n \to \infty$, $P_n X_0 = X_n^\varepsilon(0) \to X_0$ in H, it is known that X^ε satisfies the equation

$$X^\varepsilon(t) = X_0 + \int_0^t F^\varepsilon(s) ds + \sqrt{\varepsilon}\int_0^t S^\varepsilon(s) dB_s + \varepsilon \int_0^T \int_{\mathbb{Z}} G^\varepsilon(s) \widetilde{N}(ds, d\xi). \tag{2.32}$$

It remains to prove that $F^\varepsilon(s) = F^\varepsilon(X^\varepsilon(s)), S^\varepsilon(s) = \sigma(s, X^\varepsilon(s))$ and $G^\varepsilon(s) = g(X^\varepsilon(s), \xi)$. For $\psi \in \mathcal{D}(\mathbb{O}_T, \mathbb{H}^1)$, let

$$r(t) = \int_0^t \left[c_1 + c_2\left(|X_n^\varepsilon(s)|^2 + |\psi(s)|^2\right)\right] ds \quad \text{for all } t \in [0, T]. \tag{2.33}$$

By Fatou's lemma, we have

$$\{e^{-r(T)}|X^\varepsilon(T)|^2\} \le \liminf_n \mathbb{E}\{e^{-r(T)}|X_n^\varepsilon(T)|^2\}. \tag{2.34}$$

Applying Ito's formula to $e^{-r(T)}|X^\varepsilon(T)|^2$, we can obtain that

$$\begin{aligned} e^{-r(T)}|X_n^\varepsilon(T)|^2 &= |P_n X_0|^2 + \int_0^T e^{-r(s)}\{-r'(s)(X_n^\varepsilon(s)), X_n^\varepsilon(s))) + 2(F(X_n^\varepsilon(s))), X_n^\varepsilon(s)))\} ds \\ &+ \int_0^T e^{-r(s)}\varepsilon|\sigma(s, X_n^\varepsilon(s))|^2_{\mathcal{L}_Q} ds + \int_0^T \varepsilon \int_{\mathbb{Z}} e^{-r(s)}|g(X_n^\varepsilon(s), \xi)|^2 N(dt, d\xi) \\ &+ 2\sqrt{\varepsilon}\int_0^T e^{-r(s)}(\sigma(s, X_n^\varepsilon(s)) dB_n, X_n^\varepsilon(s)) \\ &+ 2\varepsilon \int_{\mathbb{Z}} e^{-r(s)}(g(X_n^\varepsilon(s), \xi), X_n^\varepsilon(s))\widetilde{N}(dt, d\xi). \end{aligned} \tag{2.35}$$

Therefore, we get

$$
\begin{aligned}
\mathbb{E}\left[e^{-r(T)}|X_n^{\varepsilon}(T)|^2\right] &\leq \mathbb{E}|P_nX_0|^2 \\
&+ \mathbb{E}\int_0^T e^{-r(s)}\left\{-r'(s)(X_n^{\varepsilon}(s)), X_n^{\varepsilon}(s))) + 2(F(X_n^{\varepsilon}(s))), X_n^{\varepsilon}(s)))\right\}ds \\
&+ \varepsilon\mathbb{E}\int_0^T e^{-r(s)}|\sigma(s, X_n^{\varepsilon}(s))|^2_{\mathcal{L}_Q}ds \\
&+ \varepsilon\mathbb{E}\int_0^T\int_{\mathbb{Z}} e^{-r(s)}|g(X_n^{\varepsilon}(s), \xi)|^2\lambda(d\xi)ds. \qquad (2.36)
\end{aligned}
$$

Similarly, applying Ito's formula to $e^{-r(T)}|X^{\varepsilon}(T)|^2$, we also get

$$
\begin{aligned}
\mathbb{E}\left[e^{-r(T)}|X^{\varepsilon}(T)|^2\right] &\leq \mathbb{E}|X_0|^2 \\
&+ \mathbb{E}\int_0^T e^{-r(s)}\left\{-r'(s)(X^{\varepsilon}(s)), X^{\varepsilon}(s))) + 2(F(X^{\varepsilon}(s))), X^{\varepsilon}(s)))\right\}ds \\
&+ \varepsilon\mathbb{E}\int_0^T e^{-r(s)}|S^{\varepsilon}(s))|^2_{\mathcal{L}_Q}ds + \varepsilon\mathbb{E}\int_0^T\int_{\mathbb{Z}} e^{-r(s)}|G^{\varepsilon}(s), \xi)|^2\lambda(d\xi)ds. \qquad (2.37)
\end{aligned}
$$

From Eq (2.34), we have

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^T e^{-r(s)}\left\{-r'(s)(X^{\varepsilon}(s)), X^{\varepsilon}(s))) + 2(F(X^{\varepsilon}(s))), X^{\varepsilon}(s)))\right\}ds\right. \\
&+ \left.\varepsilon\mathbb{E}\int_0^T e^{-r(s)}|S^{\varepsilon}(s)|^2_{\mathcal{L}_Q}ds + \varepsilon\mathbb{E}\int_0^T\int_{\mathbb{Z}} e^{-r(s)}|G^{\varepsilon}(s), \xi)|^2\lambda(d\xi)ds\right] \\
&\leq \liminf_n \mathbb{E}\left[\int_0^T e^{-r(s)}\left\{-r'(s)(X^{\varepsilon}(s)), X^{\varepsilon}(s))) + 2(F(X^{\varepsilon}(s))), X^{\varepsilon}(s)))\right\}ds\right. \\
&+ \left.\varepsilon\int_0^T e^{-r(s)}|S^{\varepsilon}(s)|^2_{\mathcal{L}_Q}ds + \varepsilon\int_0^T\int_{\mathbb{Z}} e^{-r(s)}|G^{\varepsilon}(s), \xi)|^2\lambda(d\xi)ds\right]. \qquad (2.38)
\end{aligned}
$$

Using properties of A and f, we can obtain

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^T e^{-r(s)}\left\{-r'(s)|X_n^{\varepsilon}(s)) - \psi(s)|^2 + 2(F(X_n^{\varepsilon}(s)) - F(\psi), X_n^{\varepsilon}(s) - \psi(s))\right\}ds\right. \\
&\left.+\,\varepsilon\int_0^T e^{-r(s)}\left\{|\sigma(s, X_n^{\varepsilon}(s) - \sigma(s, \psi(s)|^2_{\mathcal{L}_Q}ds + \int_{\mathbb{Z}} e^{-r(s)}|g(X_n^{\varepsilon}(s), \xi) - g(\psi(s), \xi)|^2\lambda(d\xi)\right\}ds\right] \\
&< 0. \qquad (2.39)
\end{aligned}
$$

From Eqs (2.38) and (2.39), applying limit, we get

$$
\begin{aligned}
& \mathbb{E}\Bigg[\int_0^T e^{-r(s)}\{-r'(s)(X^\varepsilon(s)), X^\varepsilon(s))) + 2(F(X^\varepsilon(s))), X^\varepsilon(s)))\}\,ds \\
+ \;& \varepsilon\mathbb{E}\int_0^T e^{-r(s)}|S^\varepsilon(s)|^2_{\mathcal{L}_Q} ds + \varepsilon\mathbb{E}\int_0^T\int_{\mathbb{Z}} e^{-r(s)}|G^\varepsilon(s)|^2\lambda(d\xi)ds\Bigg] \\
\le \;& \mathbb{E}\Bigg[\int_0^T e^{-r(s)}\{2(F^\varepsilon(s),\psi(s)) + 2(F(\psi(s)), X^\varepsilon(s)-\psi(s))\}\,ds \\
- \;& \int_0^T e^{-r(s)}r'(s)(2X^\varepsilon(s)) - \psi(s),\psi(s)) \\
+ \;& \varepsilon\int_0^T e^{-r(s)}(2S^\varepsilon(s) - \sigma(s,\psi(s)),\sigma(s,\psi(s)))ds \\
+ \;& \varepsilon\int_0^T\int_{\mathbb{Z}} e^{-r(s)}(2G^\varepsilon(s) - g(\psi(s),\xi), g(\psi(s),\xi))\lambda(d\xi)ds\Bigg].
\end{aligned}
\tag{2.40}
$$

From Eq (2.38), rearranging the terms, then we have

$$
\begin{aligned}
& \mathbb{E}\Bigg[\int_0^T e^{-r(s)}\left\{-r'(s)|X_n^\varepsilon(s)) - \psi(s)|^2 + 2(F(X_n^\varepsilon(s)) - F(\psi), X_n^\varepsilon(s) - \psi(s))\right\}ds \\
& + \varepsilon\int_0^T e^{-r(s)}\left\{|S^\varepsilon(s) - \sigma(s,\psi(s)|^2_{\mathcal{L}_Q} ds + \int_{\mathbb{Z}} e^{-r(s)}|G^\varepsilon(s) - g(\psi(s),\xi)|^2\lambda(d\xi)\right\}ds\Bigg] \\
& < 0.
\end{aligned}
\tag{2.41}
$$

Taking $\psi = X^\varepsilon(s)$ in the above inequality, we get $S^\varepsilon(t) = \sigma(t, X^\varepsilon(s))$ and $G^\varepsilon(t) - g(X^\varepsilon(t),\xi)$. For some $\mu > 0$ and $\overline{\psi} \in \mathbb{L}^\infty(\mathbb{O}_T, H)$, let $\psi = X^\varepsilon - \mu\overline{\psi}$, and we get

$$
\mathbb{E}\left[\int_0^T e^{-r(s)}\left\{2\mu(F^\varepsilon(s) - F(\psi(s)),\overline{\psi}) - \mu^2 r'(s)|\mu\overline{\psi}|^2\right\}ds\right] \le 0. \tag{2.42}
$$

By Lemma 2.2, we get

$$
(F(\psi(s)) - F(X^\varepsilon(s)),\overline{\psi}) \le \mu^2\left[-\Lambda_1|\nabla\overline{\psi}|^2 + \Lambda_2|\overline{\psi}|^2 + \Lambda_3\left(|\psi(s)|^2 + |X^\varepsilon(s)|^2\right)|\overline{\psi}|^2\right]. \tag{2.43}
$$

Then dividing Eq (2.42) by μ and letting $\mu \to 0$,

$$
\mathbb{E}\left[\int_0^T e^{-r(s)}\left\{2(F^\varepsilon(s) - F(X^\varepsilon(s)),\overline{\psi})\right\}ds\right] \le 0. \tag{2.44}
$$

Due to the arbitrariness of $\overline{\psi}$, $F^\varepsilon(s) = F(X^\varepsilon(s))$. Hence, $X^\varepsilon(s)$ satisfies

$$
X^\varepsilon(t) = X_0^\varepsilon + \int_0^t F(X^\varepsilon(s))ds + \sqrt{\varepsilon}\int_0^t \sigma(X^\varepsilon(s))dB_s + \varepsilon\int_0^t\int_{\mathbb{Z}} g(X^\varepsilon(s),\xi)\widetilde{N}(ds,d\xi). \tag{2.45}
$$

Then, the existence of the solution is proved.

To prove uniqueness, consider $\psi^\varepsilon \in \mathbb{X}$. Let $\tau_N = \widetilde{\tau}_N \wedge \overline{\tau}_N \to T$ as $N \to \infty$, where $\widetilde{\tau}_N = \inf\{t : |X^\varepsilon(t)| + |\nabla X^\varepsilon(t)| \ge N\}$ and $\overline{\tau}_N = \inf\{t : |\psi^\varepsilon(t)| + |\nabla\psi^\varepsilon(t)| \ge N\}$. Then, $\vartheta = X^\varepsilon(t) - \psi^\varepsilon(t)$ satisfies

$$
\begin{aligned}
d\vartheta(t) &= [F(X^{\varepsilon}(s)) - F(\psi^{\varepsilon}(s))]dt + \sqrt{\varepsilon}[\sigma(X^{\varepsilon}(s)) - \sigma(\psi^{\varepsilon}(s))]dB_s \\
&+ \varepsilon \int_{\mathbb{Z}} [g(X^{\varepsilon}(s), \xi) - g(\psi^{\varepsilon}(s), \xi)]\widetilde{N}(ds, d\xi).
\end{aligned} \tag{2.46}
$$

For $a = \Lambda_2$, let $\rho'(t) = a\left(|\psi^{\varepsilon}(s)|^2 + |X^{\varepsilon}(s)|^2\right)$. Ito's formula in Lemma 4 [23] for $\phi_1 = X^{\varepsilon}(s), \phi_2 = \psi^{\varepsilon}(s)$ and $\mathcal{H}_2$ gives

$$
\begin{aligned}
e^{-\rho(t\wedge\tau_N)}|\vartheta(t \wedge \tau_N)|^2 &\leq I(t \wedge \tau_N) + \int_0^{t\wedge\tau_N} e^{-\rho(s)}(\varepsilon\mathcal{K}_2 - \Lambda_1)|\nabla\vartheta(s)|^2 ds \\
&+ \int_0^{t\wedge\tau_N} e^{-\rho(s)}\left[\Lambda_2 + \Lambda_3(|X^{\varepsilon}(s)|^2 + |\psi^{\varepsilon}(s)|^2) - \rho'(s)\right]|\vartheta(s)|^2 ds,
\end{aligned} \tag{2.47}
$$

where

$$
\begin{aligned}
I(t \wedge \tau_N) &= 2\sqrt{\varepsilon}\int_0^{t\wedge\tau_N} e^{-\rho(s)}([\sigma(X^{\varepsilon}(s)) - \sigma(\psi^{\varepsilon}(s))]dB_s, \vartheta(s)) \\
&+ 2\varepsilon \int_0^{t\wedge\tau_N}\int_{\mathbb{Z}} (g(X^{\varepsilon}(s), \xi) - g(\psi^{\varepsilon}(s), \xi), \vartheta(s))\, \widetilde{N}(ds, d\xi).
\end{aligned} \tag{2.48}
$$

Therefore, we have

$$
\mathbb{E}\sup_{0\leq s\leq t\wedge\tau_N} I(s) \leq \Xi\mathbb{E}\sup_{0\leq s\leq t\wedge\tau_N}\left[e^{-\rho(s)}|\vartheta(s)|^2\right] + \frac{C_1(\varepsilon+\varepsilon^2)\mathcal{K}_2}{\Xi}\mathbb{E}\int_0^{t\wedge\tau_N} e^{-\rho(s)}|\nabla\vartheta(s)|^2 ds. \tag{2.49}
$$

Therefore by using Lemma 3.9 [23], with $Z(t) = 0$ and $\widetilde{C} = 0$, we get

$$
\mathbb{E}\sup_{0\leq t\leq T}\left[e^{-\rho(t\wedge\tau_N)}|\vartheta(t \wedge \tau_N)|^2\right] = 0.
$$

Hence $|\vartheta(s)|^2 = 0$ for all $t \in [0, T]$, since $\tau_N \to T$ as $N \to \infty$.

3. Random attractors

In the section, we study the existence and uniqueness of weak mean random attractors for the stochastic diffusive coral reef ecosystems with Lévy noise equation (1.2).

Let $\mathcal{B} = \{\mathcal{B}(s) \subset L^2(\Omega, \mathcal{F}_s, \mathbb{H}) : s \in \mathfrak{R}\}$ be a family of nonempty bounded sets such that

$$
\lim_{\tau\to-\infty} e^{\Gamma t}\|\mathcal{B}(s)\|^2_{L^2(\Omega,\mathcal{F}_s,\mathbb{H})} = 0, \tag{3.1}
$$

where $\Gamma + \eta_2 + \eta_3 \geq 2c_1$ and $\|O\|_{L^2(\Omega,\mathcal{F}_s,\mathbb{H})} = \sup_{u\in O}\|X\|_{L^2(\Omega,\mathcal{F}_s,\mathbb{H})}$ for a subset O in $L^2(\Omega, \mathcal{F}_s, \mathbb{H})$. We will use $\mathcal{D}$ to denote the collection of all families of nonempty bounded sets satisfying Eq (3.1).

We will first derive uniform estimates on the solutions of the stochastic diffusive coral reef ecosystems with Lévy noise equation (1.2), then we construct a $\mathcal{D}$-pullback absorbing set for the system Φ, where $\Phi(t, s)$ is a mapping from $L^2(\Omega, \mathcal{F}_s, \mathbb{H})$ to $L^2(\Omega, \mathcal{F}_{t+s}, \mathbb{H})$ defined by

$\Phi(t,s)u_0 = \Phi(t+s,s,u_0)$, where $u_0 \in L^2(\Omega,\mathcal{F}_s,\mathbb{H})$. The uniqueness of solution to the stochastic diffusive coral reef ecosystems with Lévy noise equation (1.2) implies that $\Phi(t+\tau,s) = \Phi(t,\tau+s)\circ\Phi(\tau,s)$ for every $t,s>0$ and $\tau\in\mathfrak{R}$. This cocycle Φ is called the mean random dynamical system generated by the stochastic diffusive coral reef ecosystems with Lévy noise equation (1.2) on $L^2(\Omega,\mathcal{F},\mathbb{H})$.

Lemma 3.1. Suppose $\mathcal{H}_1$ and $\mathcal{H}_2$ hold. In addition, assume

$$2\hat{d} - \frac{2\mathcal{K}_1}{\Gamma} - \frac{p^2C_1^2\mathcal{K}_1}{\Gamma\eta_2} - \frac{p^2C_3^2\mathcal{K}_1}{\Gamma\eta_3} > 0, \quad \Gamma+\eta_2+\eta_3 \geq 2c_1 > \eta_2+\eta_3.$$

Then there exists $T = T(\tau,\mathcal{B})$ such that the solution u of stochastic diffusive coral reef ecosystems with Lévy noise equation (1.2) satisfies

$$\mathbb{E}\left[|X(t)|^2\right] \leq \left(\frac{2\mathcal{K}_1}{\Gamma} + \frac{p^2C_1^2\mathcal{K}_1}{\Gamma\eta_2} + \frac{p^2C_3^2\mathcal{K}_1}{\Gamma\eta_3}\right)|\mathbb{O}|\left[1 + \frac{\Gamma+\eta_2+\eta_3-2c_1}{2c_1-\eta_2-\eta_3}\right] + 1, \quad \forall X_0\in\mathcal{B}(t),$$

for every $\tau\in\mathfrak{R}$, $\mathcal{B}\in\mathcal{D}$, and $t\geq T$.

Proof. Applying Itô's formula to $e^{\Gamma t}|X(t,X_0)|^2$, we get, for $t\geq 0$,

$$\begin{aligned} e^{\Gamma t}|X(t)|^2(t) &= |X_0|^2 + \Gamma\int_0^t e^{\Gamma t}|X(s)|^2 ds + 2\int_0^t e^{\Gamma t}(AX + f(X(s)),X(s))ds \\ &\quad + \int_0^t e^{\Gamma t}|\sigma(X(s))|^2_{\mathcal{L}_Q} ds + 2\int_0^t e^{\Gamma t}(\sigma(X(s))dB, X(s)) \\ &\quad + \int_0^t\int_{\mathbb{Z}} e^{\Gamma t}|g(X,\xi)|N(dt,d\xi) + 2\int_0^t\int_{\mathbb{Z}} e^{\Gamma t}(g_n(X,\xi),X)\widetilde{N}(dt,d\xi). \end{aligned} \tag{3.2}$$

Taking supremum and then taking expectation, we have

$$\begin{aligned} \mathbb{E}\sup_{0\leq s\leq t} e^{\Gamma t}|X(s)|^2 &+ 2\hat{d}\mathbb{E}\int_0^t e^{\Gamma s}\|\nabla X\|^2 ds \leq \mathbb{E}|X_0|^2 + (2c_1+\Gamma)\mathbb{E}\int_0^t e^{\Gamma s}|X(s)|^2 ds \\ &+ 2\mathcal{K}_1\mathbb{E}\int_0^t e^{\Gamma s}\left[1+|\nabla X(s)|^2\right]ds + \mathbb{E}\sup_{0\leq s\leq t} e^{\Gamma s}J_1(s) + \mathbb{E}\sup_{0\leq s\leq t} e^{\Gamma t}J_2(s), \end{aligned} \tag{3.3}$$

where

$$J_1 = 2\int_0^t (\sigma(X(s))dB, X(s)), \quad J_2 = 2\int_0^t\int_{\mathbb{Z}}[(g(X(s-),\xi),X(s-))]\widetilde{N}(dt,d\xi).$$

By the Burkholder-Davis-Gundy inequality and Young's inequality, we get

$$\begin{aligned} \mathbb{E}\sup_{0\leq s\leq t} J_1(s) &\leq 2C_1\mathbb{E}\left\{\int_0^t |\sigma(X(s))|^2_{\mathcal{L}_Q}|X(s)|^2 ds\right\}^{1/2} \\ &\leq \eta_2\mathbb{E}\sup_{0\leq s\leq t}|X(s)|^2 + \frac{C_1^2\mathcal{K}_1}{\eta_2}\mathbb{E}\int_0^t\left[1+|\nabla X(s)|^2\right]ds \end{aligned} \tag{3.4}$$

and

$$\begin{aligned}\mathbb{E}\sup_{0\le s\le t} J_2(s) &\le 2C_2\mathbb{E}\left\{\int_0^t |X_n(s)|^2\int_{\mathbb{Z}}|g(X(s-),\xi)|^2 d(\xi)\right\}^{1/2} \\ &\le \eta_3\mathbb{E}\sup_{0\le s\le t}|X(s)|^2+\frac{C_2^2\mathcal{K}_1}{\eta_3}\mathbb{E}\int_0^t\left[1+|\nabla X(s)|^2\right]ds.\end{aligned}\tag{3.5}$$

Substituting Eqs (3.4) and (3.5) in Eq (3.3), we find

$$\begin{aligned}\mathbb{E}\sup_{0\le s\le t} e^{\Gamma t}|X(s)|^2 &+ 2\hat{d}\mathbb{E}\int_0^t e^{\Gamma s}\|\nabla X\|^2 ds\le \mathbb{E}|X_0|^2 \\ &+ (\Gamma+\eta_2+\eta_3-2c_1)\mathbb{E}\int_0^t e^{\Gamma s}|X(s)|^2 ds \\ &+ \left(2\mathcal{K}_1+\frac{p^2C_1^2\mathcal{K}_1}{\eta_2}+\frac{p^2C_3^2\mathcal{K}_1}{\eta_3}\right)\mathbb{E}\int_0^t e^{\Gamma s}\left[1+|\nabla X(s)|^2\right]ds.\end{aligned}\tag{3.6}$$

Therefore, we have

$$\begin{aligned}\mathbb{E}\sup_{0\le s\le t} e^{\Gamma t}|X(s)|^2 &+ \left(2\hat{d}-\frac{2\mathcal{K}_1}{\Gamma}-\frac{p^2C_1^2\mathcal{K}_1}{\Gamma\eta_2}-\frac{p^2C_3^2\mathcal{K}_1}{\Gamma\eta_3}\right)\mathbb{E}\int_0^t e^{\Gamma s}\|\nabla X\|^2 ds\le \mathbb{E}|X_0|^2 \\ &+ (\Gamma+\eta_2+\eta_3-2c_1)\mathbb{E}\int_0^t e^{\Gamma s}|X(s)|^2 ds \\ &+ \frac{1}{\Gamma}\left(2\mathcal{K}_1+\frac{p^2C_1^2\mathcal{K}_1}{\eta_2}+\frac{p^2C_3^2\mathcal{K}_1}{\eta_3}\right)|\mathbb{O}|e^{\Gamma t}.\end{aligned}\tag{3.7}$$

Thus, we get

$$\begin{aligned}\mathbb{E}\left[e^{\Gamma t}|X(t)|^2\right] &\le \mathbb{E}|X_0|^2+(\Gamma+\eta_2+\eta_3-2c_1)\int_0^t\mathbb{E}\left[e^{\Gamma s}|X(s)|^2\right]ds \\ &+ \left(\frac{2\mathcal{K}_1}{\Gamma}+\frac{p^2C_1^2\mathcal{K}_1}{\Gamma\eta_2}+\frac{p^2C_3^2\mathcal{K}_1}{\Gamma\eta_3}\right)|\mathbb{O}|e^{\Gamma t}.\end{aligned}\tag{3.8}$$

By the Gronwall inequality, we get

$$\begin{aligned}\mathbb{E}\left[|X(t)|^2\right] &\le \mathbb{E}|X_0|^2 e^{-(2c_1-\eta_2-\eta_3)} \\ &+ \left(\frac{2\mathcal{K}_1}{\Gamma}+\frac{p^2C_1^2\mathcal{K}_1}{\Gamma\eta_2}+\frac{p^2C_3^2\mathcal{K}_1}{\Gamma\eta_3}\right)|\mathbb{O}|\left[1+\frac{\Gamma+\eta_2+\eta_3-2c_1}{2c_1-\eta_2-\eta_3}\right] \\ &- \left(\frac{2\mathcal{K}_1}{\Gamma}+\frac{p^2C_1^2\mathcal{K}_1}{\Gamma\eta_2}+\frac{p^2C_3^2\mathcal{K}_1}{\Gamma\eta_3}\right)|\mathbb{O}|\frac{\Gamma+\eta_2+\eta_3-2c_1}{2c_1-\eta_2-\eta_3}e^{-(2c_1-\eta_2-\eta_3)}.\end{aligned}\tag{3.9}$$

Let us proceed like the uniqueness of solution in Theorem 2.4. Define $\vartheta = X(t)-\psi(t)$. Similar arguments as Eqs (2.46)–(2.49) imply that we get

$$\mathbb{E}\sup_{0\le t\le T}\left[e^{-\rho(t\wedge\tau_N)}|\vartheta(t\wedge\tau_N)|^2\right]=0.\tag{3.10}$$

Hence, $|\vartheta(s)|^2 = 0$ for all $t \in [0, T]$ since $\tau_N \to T$ as $N \to \infty$, which, together with Eq (3.9), yields

$$\begin{aligned}\mathbb{E}\left[|X(t)|^2\right] &\leq \mathbb{E}|X_0|^2 e^{-(2c_1-\eta_2-\eta_3)} \\ &+ \left(\frac{2\mathcal{K}_1}{\Gamma} + \frac{p^2C_1^2\mathcal{K}_1}{\Gamma\eta_2} + \frac{p^2C_3^2\mathcal{K}_1}{\Gamma\eta_3}\right)|\mathbb{O}|\left[1 + \frac{\Gamma + \eta_2 + \eta_3 - 2c_1}{2c_1 - \eta_2 - \eta_3}\right] \\ &- \left(\frac{2\mathcal{K}_1}{\Gamma} + \frac{p^2C_1^2\mathcal{K}_1}{\Gamma\eta_2} + \frac{p^2C_3^2\mathcal{K}_1}{\Gamma\eta_3}\right)|\mathbb{O}|\frac{\Gamma + \eta_2 + \eta_3 - 2c_1}{2c_1 - \eta_2 - \eta_3} e^{-(2c_1-\eta_2-\eta_3)}. \end{aligned} \tag{3.11}$$

Since $X_0 \in \mathcal{B}$ and $\mathcal{B} \in \mathcal{D}$, one has

$$\mathbb{E}|X_0|^2 e^{-(2c_1-\eta_2-\eta_3)} \leq e^{-(2c_1-\eta_2-\eta_3)}|\mathcal{B}|^2 \to 0$$

and

$$\left(\frac{2\mathcal{K}_1}{\Gamma} + \frac{p^2C_1^2\mathcal{K}_1}{\Gamma\eta_2} + \frac{p^2C_3^2\mathcal{K}_1}{\Gamma\eta_3}\right)|\mathbb{O}|\frac{\Gamma + \eta_2 + \eta_3 - 2c_1}{2c_1 - \eta_2 - \eta_3} e^{-(2c_1-\eta_2-\eta_3)} \leq e^{-(2c_1-\eta_2-\eta_3)}|\mathcal{B}|^2 \to 0$$

as $t \to \infty$, which along with Eq (3.11), concludes the proof.

Lemma 3.2. Assume the conditions of Lemma 3.1 hold, and the mean random dynamical system Φ related to stochastic diffusive coral reef model with Lévy noise equation (1.2) has a weakly compact $\mathcal{D}$-pullback absorbing set $\mathcal{K} = \{\mathcal{K}(s) : s \in \Re\} \in \mathcal{D}$ as follows:

$$\mathcal{K}(s) = \left\{X \in L^2(\Omega, \mathcal{F}_s; \mathbb{H}) : \mathbb{E}X(s) \leq R\right\}, \tag{3.12}$$

where

$$R := \left(\frac{2\mathcal{K}_1}{\Gamma} + \frac{p^2C_1^2\mathcal{K}_1}{\Gamma\eta_2} + \frac{p^2C_3^2\mathcal{K}_1}{\Gamma\eta_3}\right)|\mathbb{O}|\left[1 + \frac{\Gamma + \eta_2 + \eta_3 - 2c_1}{2c_1 - \eta_2 - \eta_3}\right] + 1.$$

Proof. For each $s \in \Re$, from Eq (3.12), it is easy to know that $\mathcal{K}(s)$ is a bounded closed convex subset of $L^2(\Omega, \mathcal{F}_s; \mathbb{H})$. Therefore, it is a weakly compact subset of $L^2(\Omega, \mathcal{F}_s; \mathbb{H})$. Moreover, it follows from Lemma 3.1 that there exists $T = T(s, \mathcal{B}) > 0$ such that $\Phi(s, \mathcal{B}(s)) \subset \mathcal{K}(s)$ for every $s \in \Re$ and $\mathcal{B} = \{\mathcal{B}(t)\} \in \mathcal{D}$, $s \geq T$. On the other hand,

$$\begin{aligned}&\lim_{s\to-\infty} e^{\Gamma s}\|\mathcal{K}(s)\|^2_{L^2(\Omega,\mathcal{F}_s;\mathbb{H})} \\ &\leq \lim_{s\to-\infty} e^{\Gamma s}\left(\left(\frac{2\mathcal{K}_1}{\Gamma} + \frac{p^2C_1^2\mathcal{K}_1}{\Gamma\eta_2} + \frac{p^2C_3^2\mathcal{K}_1}{\Gamma\eta_3}\right)|\mathbb{O}|\left[1 + \frac{\Gamma + \eta_2 + \eta_3 - 2c_1}{2c_1 - \eta_2 - \eta_3}\right] + 1\right) = 0.\end{aligned}$$

Therefore, it has been proven that $\mathcal{K}(s) \in \mathcal{D}$. That is, $\mathcal{K}$ is a weakly compact $\mathcal{D}$-pullback absorbing set for Φ.

Theorem 3.3. Under assumptions of Lemma 3.1. Then, the mean random dynamical system Φ to stochastic diffusive coral reef model with Lévy noise equation (1.2) has a unique weak $\mathcal{D}$-pullback mean random attractor $\mathcal{A} = \{\mathcal{A}(s) : s \in \Re\} \in \mathcal{D}$ in $\mathcal{L}^2(\Omega, \mathcal{F}; \mathbb{H})$ over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\Re}, \mathbb{P})$.

Proof. By Lemma 3.2 and Theorem 2.13 [31], it is easy to see that the existence and uniqueness of the weak $\mathcal{D}$-pullback mean random attractors $\mathcal{A} \in \mathcal{D}$ of Φ to the stochastic diffusive coral reef model with Lévy noise equation (1.2) are immediate consequences.

4. Invariant measures and ergodicity

In the section, we establish the existence of invariant measures and ergodicity to the stochastic diffusive coral reef model with Lévy noise equation (1.2).

Lemma 4.1 Suppose $\mathcal{H}_1$-$\mathcal{H}_2$ hold. In addition, assume

$$\max\left\{\frac{\chi}{2},\frac{\chi}{\eta_1},\alpha_2+\beta_2+\frac{\delta_2}{2},\frac{\delta_2}{\eta_2}\right\}=c_2<\Gamma_1=\inf\{\varrho_1-r_1,\varrho_2,\varrho_3-r_2,\varrho_4-r_3\},\ \ \widehat{d}>\mathcal{K}_1.$$

Then, the solution $X(t,X_0)$ of the stochastic diffusive coral reef model with Lévy noise equation (1.2) satisfies

$$\mathbb{E}|X(t)|^2+(\widehat{d}-\mathcal{K}_1)\mathbb{E}\int_0^t e^{2(\Gamma_1-c_2)(s-t)}|\nabla X(s)|^2ds\le e^{-2(\Gamma_1-c_2)t}\mathbb{E}|X_0|^2+\frac{\mathcal{K}_1|\mathbb{O}|}{2(\Gamma_1-c_2)}$$

and

$$\frac{(\widehat{d}-\mathcal{K}_1)}{t}\mathbb{E}\int_0^t|\nabla X(s)|^2ds\le\frac{\mathbb{E}|X_0|^2}{T_0}+\mathcal{K}_1|\mathbb{O}|,\quad\forall t>0.$$

Proof. Applying Itô's formula to $e^{2\Gamma_1 t}|u(t)|^2$, we get, for $t\ge 0$,

$$\begin{aligned}e^{2(\Gamma_1-c_2)t}\mathbb{E}|X(t)|^2 &= \mathbb{E}|X_0|^2+2\Gamma_1\mathbb{E}\int_0^t e^{2(\Gamma_1-c_2)s}|X(s)|^2ds\\&\quad+2\int_0^t e^{2(\Gamma_1-c_2)s}(AX+f(X(s)),X(s))ds\\&\quad+\mathbb{E}\int_0^t e^{2(\Gamma_1-c_2)s}|\sigma(X(s))|^2_{\mathcal{L}_Q}ds+\mathbb{E}\int_0^t\int_{\mathbb{Z}}e^{2(\Gamma_1-c_2)s}|g(X,\xi)|^2\lambda(d\xi)ds.\end{aligned}\tag{4.1}$$

From Lemmas 2.1 and 2.2, we get

$$\begin{aligned}e^{2(\Gamma_1-c_2)t}\mathbb{E}|X(t)|^2 &+ (\widehat{d}-\mathcal{K}_1)\mathbb{E}\int_0^t e^{2(\Gamma_1-c_2)s}|\nabla X(s)|^2ds\\&\le\mathbb{E}|X_0|^2+2(\Gamma_1-c_2)\mathbb{E}\int_0^t e^{2(\Gamma_1-c_2)s}|X(s)|^2ds\\&+ 2\int_0^t e^{2(\Gamma_1-c_2)t}(c_2-\Gamma_1)|X(s)|^2ds+\frac{\mathcal{K}_1|\mathbb{O}|}{2(\Gamma_1-c_2)}e^{2(\Gamma_1-c_2)t}\\&= \mathbb{E}|X_0|^2+\frac{\mathcal{K}_1|\mathbb{O}|}{2(\Gamma_1-c_2)}e^{2(\Gamma_1-c_2)t}.\end{aligned}\tag{4.2}$$

Thus, we have

$$\mathbb{E}|X(t)|^2+(\widehat{d}-\mathcal{K}_1)\mathbb{E}\int_0^t e^{2(\Gamma_1-c_2)(s-t)}|\nabla X(s)|^2ds\le e^{-2(\Gamma_1-c_2)t}\mathbb{E}|X_0|^2+\frac{\mathcal{K}_1|\mathbb{O}|}{2(\Gamma_1-c_2)}.\tag{4.3}$$

On the other hand, using Itô$'s$ formula to the process $|X(\cdot)|^2$, we obtain

$$\mathbb{E}|X(t)|^2 = \mathbb{E}|X_0|^2 + 2\int_0^t (AX + f(X(s)), X(s))ds \tag{4.4}$$

$$+\mathbb{E}\int_0^t |\sigma(X(s))|^2_{\mathcal{L}_Q} ds + \mathbb{E}\int_0^t \int_{\mathbb{Z}} |g(X,\xi)|^2 \lambda(d\xi)ds. \tag{4.5}$$

From Lemmas 2.1 and 2.2, we get

$$\begin{aligned}\mathbb{E}|X(t)|^2 + (\widehat{d} - \mathcal{K}_1)\mathbb{E}\int_0^t |\nabla X(s)|^2 ds &\le \mathbb{E}|X_0|^2 + 2(c_2 - \Gamma_1)\int_0^t |X(s)|^2 ds + \mathcal{K}_1|\mathbb{O}|t \\ &\le \mathbb{E}|X_0|^2 + \mathcal{K}_1|\mathbb{O}|t, \quad \forall t > 0, \end{aligned} \tag{4.6}$$

which implies

$$\frac{(\widehat{d} - \mathcal{K}_1)}{t}\mathbb{E}\int_0^t |\nabla X(s)|^2 ds \le \frac{\mathbb{E}|X_0|^2}{T_0} + \mathcal{K}_1|\mathbb{O}|, \quad \forall t > T_0. \tag{4.7}$$

This concludes the proof of this Lemma 4.1.

Theorem 4.2. Under the condition of Lemma 4.1, there exists an invariant measure to the stochastic diffusive coral reef model with Lévy noise equation (1.2) on $\mathbb{H}$.

Proof. Using the Chebyshev inequality and Lemma 4.1, we infer for T_0 and $R > 0$,

$$\begin{aligned}\sup_{t\ge T_0} \frac{1}{t}\int_0^t \mathbb{P}\left(\{\|X(s,X_0)\| > R\}\right) ds &\le \sup_{t\ge T_0} \frac{1}{tR^2}\int_0^t \mathbb{E}\|X(s,X_0)\|^2 ds \\ \le \frac{\mathbb{E}|X_0|^2}{T_0R^2(\widehat{d} - \mathcal{K}_1)} + \frac{\mathcal{K}_1|\mathbb{O}|}{4R^2(\widehat{d} - \mathcal{K}_1)}. \end{aligned} \tag{4.8}$$

The above inequality implies, for all $t \ge T_0$ and every $\varepsilon > 0$, that there exists

$$R_0 = \sqrt{\frac{\mathbb{E}|X_0|^2}{T_0R^2(\widehat{d} - \mathcal{K}_1)} + \frac{\mathcal{K}_1|\mathbb{O}|}{4R^2(\widehat{d} - \mathcal{K}_1)}} > 0$$

such that, for any $R \ge R_0$,

$$\begin{aligned}\mu_{t,u_0}(\Gamma) &= \frac{1}{t}\int_0^t P(s,u_0,\Gamma)ds = \frac{1}{t}\int_0^t \mathbb{P}\left(\omega\in\Omega, u(s,u_0)\in\Gamma\right) ds \\ &\ge \frac{1}{t}\int_0^t \mathbb{P}\left(\omega\in\Omega, \|u(s,u_0)\| \le R_0\right) ds \\ &= 1 - \frac{1}{t}\int_0^t \mathbb{P}\left(\omega\in\Omega, \|u(s,u_0)\| > R_0\right) ds = 1-\varepsilon, \quad \Gamma := B(0,R), \end{aligned} \tag{4.9}$$

where $\mathcal{B}(0,R)$ is the ball centered at 0 with radius R in $\mathbb{V}$. Since $\mathbb{V}$ is compactly embedded in $\mathbb{H}$ ($\mathbb{V} \hookrightarrow\hookrightarrow \mathbb{H}$), for every $\varepsilon > 0$, from Eq (4.9), it shows there exists a compact set $\mathbb{J} \in \mathbb{H}$ such that $\mu_{t,u_0}(\mathbb{J}) > 1-\varepsilon$. Hence, the sequence of probability measure μ_{t,u_0} is tight on $\mathbb{H}$.

By the Krylov-Bogoliubov theorem (see [32]), it shows that there exists a sequence $t_n \to \infty$ as $n \to \infty$ such that $\mu_{t,u_0} \to \mu$ weakly as $n \to \infty$. Moreover, for this transition operator P_t, defined by

$(P_t\varphi)(u_0) = E\varphi(u(t,u_0))$ for all $\varphi \in C_b(\mathbb{H})$, μ is an invariant measure. Thus, the proof of this theorem is complete.

Lemma 4.3. Under assumptions of Lemma 4.1, additionally, suppose $\Lambda_3 > 0$. Let $X(t)$ and $Y(t)$ be two solutions of the stochastic diffusive coral reef model with Lévy noise equation (1.2) with initial data $X_0, Y_0 \in L^2(\Omega, \mathbb{H})$, respectively. Then, we have

$$\mathbb{E}\,|X(t) - Y(t)|^2 \leq \mathbb{E}\,|X_0(t) - Y_0(t)|^2\, e^{-\Lambda_3 \int_0^t \left(|X(s)|^2 + |Y(s)|^2\right) ds}.$$

Proof. To prove uniqueness, consider $Y \in \mathbb{X}$. Let $\tau_N = \widetilde{\tau}_N \wedge \overline{\tau}_N \to T$ as $N \to \infty$, where $\widetilde{\tau}_N = \inf\{t : |X(t)| + |\nabla X(t)| \geq N\}$ and $\overline{\tau}_N = \inf\{t : |Y(t)| + |\nabla Y(t)| \geq N\}$ Then, $\Psi = X(t) - Y(t)$ satisfies

$$\begin{aligned} d\Psi(t) &= [F(X(s)) - F(Y(s))]dt + \sqrt{\varepsilon}[\sigma(X(s)) - \sigma(Y(s))]dB_s \\ &+ \varepsilon \int_{\mathbb{Z}} [g(X(s),\xi) - g(Y(s),\xi)]\widetilde{N}(ds, d\xi). \end{aligned} \tag{4.10}$$

Let $\rho'(t) = \Lambda_3 \left(|X(s)|^2 + |Y(s)|^2\right)$. Ito's formula in Lemma 2.4 for $\phi_1 = X(s), \phi_2 = Y(s)$ and condition $\mathcal{H}_2$ gives

$$\begin{aligned} e^{-\rho(t\wedge\tau_N)}|\Psi(t\wedge\tau_N)|^2 &\leq I(t\wedge\tau_N) + \int_0^{t\wedge\tau_N} e^{-\rho(s)}(\varepsilon\mathcal{K}_2 - \Lambda_1)|\nabla\Psi(s)|^2 ds \\ &+ \int_0^{t\wedge\tau_N} e^{-\rho(s)} \left[\Lambda_2 + \Lambda_3(|X(s)|^2 + |Y(s)|^2) - \rho'(s)\right] |\Psi(s)|^2 ds, \end{aligned} \tag{4.11}$$

where

$$\begin{aligned} I(t\wedge\tau_N) &= 2\sqrt{\varepsilon} \int_0^{t\wedge\tau_N} e^{-\rho(s)}([\sigma(X(s)) - \sigma(Y(s))]dB_s, \Psi(s)) \\ &+ 2\varepsilon \int_0^{t\wedge\tau_N} \int_{\mathbb{Z}} (g(X(s),\xi) - g(Y(s),\xi), \Psi(s))\, \widetilde{N}(ds, d\xi). \end{aligned} \tag{4.12}$$

Therefore, we have

$$\mathbb{E} \sup_{0\leq s\leq t\wedge\tau_N} I(s) \leq \Xi\mathbb{E} \sup_{0\leq s\leq t\wedge\tau_N} \left[e^{-\rho(s)}|\Psi(s)|^2\right] + \frac{C_1(\varepsilon + \varepsilon^2)\mathcal{K}_2}{\Xi}\mathbb{E} \int_0^{t\wedge\tau_N} e^{-\rho(s)}|\nabla\Psi(s)|^2 ds. \tag{4.13}$$

Therefore by using Lemma 3.9 [30], with $Z(t) = 0$ and $\widetilde{C} = 0$, we get

$$\mathbb{E} \sup_{0\leq t\leq T} \left[e^{-\rho(t\wedge\tau_N)}|\Psi(t\wedge\tau_N)|^2\right] = 0.$$

Hence $|\Psi(s)|^2 = 0$ for all $t \in [0, T]$, since $\tau_N \to T$ as $N \to \infty$.

Theorem 4.4. Assuming the condition of Lemma 4.3 holds. Then there exists a unique invariant measure μ to the stochastic diffusive coral reef model with Lévy noise equation (1.2) for $\forall X_0 \in L^2(\Omega, \mathbb{H})$. Moreover, the measure μ is ergodic.

Proof. If there exists another invariant measure $\overline{\mu}$ for transition operator $(P_t)_{t\geq 0}$, then, for every $\varphi \in \mathrm{Lip}(\mathbb{H})$ and initial data $X_0, Y_0 \in L^2(\Omega, \mathbb{H})$, there is a Lipschitz function φ with Lipschitz constant L_φ. By means of the definitions of invariant measures, ergodicity and Lemma 4.3, we get

$$
\begin{aligned}
&\left|\int_{\mathbb{H}}\varphi(X_0)\mu(dX_0)-\int_{\mathbb{H}}\varphi(Y_0)\overline{\mu}(dY_0)\right| = \left|\int_{\mathbb{H}}(P_t\varphi)(X_0)\mu(dX_0)-\int_{\mathbb{H}}(P_t\varphi)(Y_0)\overline{\mu}(dY_0)\right| \\
= &\left|\int_{\mathbb{H}}\int_{\mathbb{H}}\left[(P_t\varphi)(X_0)-(P_t\varphi)(Y_0)\right]\mu(dX_0)\overline{\mu}(dY_0)\right| \\
= &\left|\int_{\mathbb{H}}\int_{\mathbb{H}}\left[\mathbb{E}\varphi(X(t,X_0))-\mathbb{E}\varphi(Y(t,Y_0))\right]\mu(dX_0)\overline{\mu}(dY_0)\right| \\
\leq & L_\varphi e^{-L_b t}\int_{\mathbb{H}}\int_{\mathbb{H}}\mathbb{E}\left|X_0(t)-Y_0(t)\right|^2\mu(dX_0)\overline{\mu}(dY_0)\to 0 \text{ as } t\to\infty. \qquad (4.14)
\end{aligned}
$$

Therefore, there exists a unique invariant measure μ for transition operator $(P_t)_{t\geq 0}$. By the density of Lip($\mathbb{H}$) in $C_b(\mathbb{H})$, we know μ is ergodic.

5. Large deviation principle

In the section, we study the large deviation principle for the stochastic reaction-diffusion coral reef model with Lévy noise equation (1.2).

Let $\mathfrak{O}$ be a locally compact Polish space and $\mathfrak{O}_T = [0,T]\times\mathfrak{O}$ corresponding to $\mathfrak{O}$ for finite $T > 0$. Define

$$\mathbb{M}(\mathfrak{O}) = \{\mu \text{ on}(\mathfrak{O},\mathcal{B}(\mathfrak{O})) : \mu(\mathrm{K}) < \infty \text{ for compact } \mathrm{K}\subset\mathfrak{O}\}.$$

Let $\mathcal{M} = \mathbb{M}(\mathfrak{O}_T)$ and P be the probability measure on $(\mathcal{M},\mathcal{B}(\mathcal{M}))$. Then, $\mathcal{B}(\mathcal{M})$ is a Polish space. Let

$$\mathbb{V} = C([0,T],H)\times\mathcal{M},\ \mathcal{G}_t = \sigma\{N(s,\mathbb{Z}) : 0\leq s\leq t, \mathbb{Z}\in\mathcal{B}(\mathfrak{O}_T)\} \text{ for } t>0.$$

Let $\widetilde{P}$ be the probability measure on the space $(\mathbb{V},\mathcal{B}(\mathbb{V}))$, $\{\mathcal{F}_t\}$ be completion of $\mathcal{G}_t$, and $\mathcal{P}$ be predictable σ-field with respect to it. Define the class $\mathcal{A}$ and $L : [0,+\infty)\to[0,+\infty)$ such that

$$
\begin{aligned}
&\mathcal{A} = \{\psi : \mathfrak{O}_T\times\mathbb{V}\to[0,+\infty) : \psi \text{ is } (\mathcal{P}\times\mathcal{B}(\mathfrak{O}))/\mathcal{B}[0,+\infty) - \text{measurable}\}, \\
&l(a) = a\log a - a + 1.
\end{aligned}
$$

Let $\mathcal{X}$ be a locally compact Polish space with $\mathcal{X}_T = [0,T]\times\mathcal{X}$. For $\psi\in\mathcal{A}$, define N^ψ as

$$N^\psi(t,\mathbb{Z}) = \int_{[0,T]\times\mathbb{Z}}\int_0^\infty \mathbf{I}_{[0,\psi(s,z)]}(r)\widetilde{N}(ds,dz)dr,\quad t\in[0,T],\ Z\in\mathcal{B}(\mathcal{X}).$$

For $\psi\in\mathcal{A}$, define

$$\widehat{L_T}(\psi) = \int_0^T\int_{\mathbb{Z}} l(\psi(t,\xi,\omega))\lambda(d\xi)dt.$$

Define $\mathcal{P}_2 = \{\phi : \phi \text{ is } \mathcal{P}/\mathcal{B}(\mathfrak{R}) - \text{measurable}, \int_0^T|\phi(s)|_0^2 ds\}$ and $\mathcal{U}(H) = \mathcal{P}_2\times\mathcal{A}$. For $\phi\in\mathcal{A}$, consider

$$\widehat{L_T}(\psi) = \frac{1}{2}\int_0^T\int_{\mathbb{Z}}|\phi(s)|_0^2 ds.$$

For $N \in \mathbb{N}$, define

$$\widehat{S}_N(\psi) = \left\{\psi : \mathfrak{D}_T \to [0,+\infty) : \widehat{L}_T(\psi) \le N\right\},$$
$$\widehat{S}_N(H_0) = \left\{\phi \in \mathbb{L}^2([0,T]; H_0) : \widehat{L}_T(\phi) \le N\right\}.$$

Define a compact set $\{\lambda_T^g : g \in \widehat{S}_N\}$ in $\mathcal{M}$, where

$$\lambda_T^g = \int_0^T \int_{\mathbb{Z}} g(s,\xi))\lambda(d\xi)dt, \quad \mathbb{Z} \in \mathcal{B}(\mathfrak{D}_T).$$

Let $\mathcal{U} = \mathcal{P}_2(H_0) \times \mathcal{A}$ and $\mathbb{S} = \bigcup_{N\ge 1} S_N$, where $S_N = \widehat{S}_N(H_0) \times \widehat{S}_N$. Here take $\mathcal{U}^N = \{\zeta = (\phi,\varphi) \in \mathcal{U}, \zeta(\omega) \in S_N\}$. Let $\mathcal{X}_0$ be a Polish space.

The stochastic control equation with respect to the stochastic diffusive coral reef ecosystems with Lévy noise equation (2.2) is

$$\begin{aligned} dX^\varepsilon_{\zeta^\varepsilon}(t) &= [AX^\varepsilon_{\zeta^\varepsilon}(t) + f(X^\varepsilon_{\zeta^\varepsilon}(t)) + \sigma(t, X^\varepsilon_{\zeta^\varepsilon}(t))\phi^\varepsilon]dt + \sqrt{\varepsilon}\sigma(t, X^\varepsilon_{\zeta^\varepsilon}(t))dB_t \\ &+ \varepsilon \int_{\mathbb{Z}} g(X^\varepsilon_{\zeta^\varepsilon}(t),\xi)l(\psi^\varepsilon)\lambda(d\xi)dt + \varepsilon \int_{\mathbb{Z}} g(X^\varepsilon_{\zeta^\varepsilon}(t),\xi)\widetilde{N}(dt,d\xi). \end{aligned} \tag{5.1}$$

Lemma 5.1 Assume that $\mathcal{H}_1$ and $\mathcal{H}_2$ hold. There exists a unique strong solution for stochastic control equation (5.1) with $X^\varepsilon_{\zeta^\varepsilon}(0) = X_0$ for $\zeta^\varepsilon \in \mathcal{U}^{\mathfrak{M}}, \mathfrak{M} \in (0,+\infty)$, and $\varepsilon > 0$ satisfying the estimate

$$\mathbb{E} \sup_{0\le s\le T} |X^\varepsilon_{\zeta^\varepsilon}(s)|^2 + \hat{\alpha}\mathbb{E}\left(\int_0^T \|\nabla X^\varepsilon_{\zeta^\varepsilon}(s)\|^2 ds\right) \le \mathrm{K},$$
$$\mathbb{E} \sup_{0\le s\le T} |\nabla X^\varepsilon_{\zeta^\varepsilon}(s)|^2 + \hat{\alpha}\mathbb{E}\left(\int_0^T \|\Delta X^\varepsilon_{\zeta^\varepsilon}(s)\|^2 ds\right) \le \mathrm{K},$$

in $\mathbb{X}$, where K is an appropriate constant.

Proof. The solution of Eq (5.1) can be expressed as $\mathcal{G}^\varepsilon(\sqrt{\varepsilon}B(\cdot) + \int_0^\cdot \phi^\varepsilon ds, \varepsilon N^{\varepsilon^{-1}\psi^\varepsilon})$. This proof process is similar to Theorem 2.4, omitted here.

Lemma 5.2 Assume that $\mathcal{H}_1$ and $\mathcal{H}_2$ hold. For $\zeta \in \mathcal{A}$ and the initial condition $X_\zeta(0) = X_0$, the deterministic controlled equation

$$dX_\zeta(t) = [AX_\zeta(t) + f(X_\zeta(t)) + \sigma(X_\zeta)\phi]dt + \varepsilon \int_{\mathbb{Z}} g(X_\zeta,\xi)l(\psi)\lambda(d\xi)dt. \tag{5.2}$$

exists a unique strong solution in $\mathbb{X}$.

Proof. The solution of deterministic controlled equation (5.2) can be expressed as $G^0(\int_0^\cdot \phi(s)ds, \lambda_T^\psi)$. The proof of Theorem 2.4 consists essentially to prove the weak convergence of the solution of stochastic control diffusive coral reef ecosystems equation (5.1) to the solution of deterministic controlled equation (5.2) as $\varepsilon \to 0$.

Lemma 5.3 Let $\mathfrak{M} > 0$ be a constant, $\zeta \in \mathcal{A}$, and the set

$$K_{\mathfrak{M}} = \left\{X_\zeta \in \mathbb{X} = \mathcal{D}([0,T];\mathbb{H}^1) \cap \mathbb{L}^2((0,T);\mathbb{H}^2)\right\},$$

where X_ζ is the solution of deterministic controlled equation (5.2). Then the set $K_{\mathfrak{M}}$ is compact in $\mathbb{X}$.

Proof. Let $\{X_{\zeta_n}\} \in K_{\mathfrak{M}}$ be the solution of deterministic controlled equation (5.2), where the control $\zeta = (\phi, \psi)$ is replaced by $\zeta_n = (\phi_n, \psi_n)$ for $n \in \mathbb{N}$. Since $S_{\mathfrak{M}}$ is weakly compact, then there exists a subsequence of $\zeta_n \in S_{\mathfrak{M}}$ also denoted by ζ_n, and it converges weakly to ζ. In order to prove $X_{\zeta_n} \to X_\zeta$ weakly, it is enough to prove that $\Upsilon_n = X_{\zeta_n} - X_\zeta$ tends to 0 as $n \to \infty$. The difference Υ_n satisfies the equation

$$
\begin{aligned}
d\Upsilon_n(t) &= \Big[A\Upsilon_n(t) + f(X_{\zeta_n}(t)) - f(X_\zeta) + \sigma(X_{\zeta_n})\phi_n - \sigma(X_\zeta)\phi\Big]dt \\
&+ \varepsilon \int_{\mathbb{Z}} \Big[g(X_{\zeta_n}, \xi)l(\psi_n(t,\xi)) - g(X_\zeta, \xi)l(\psi(t,\xi))\Big]\lambda(d\xi)dt. \qquad (5.3)
\end{aligned}
$$

Taking the inner product with Υ_n and integrating, we get

$$
\begin{aligned}
|\Upsilon_n(t)|^2 &+ 2\Lambda_1 \int_0^t |\nabla\Upsilon_n(s)|^2 ds = 2\int_0^t \Big(f(X_{\zeta_n}(s)) - f(X_\zeta(s)), \Upsilon_n(s)\Big)ds \\
&+ 2\int_0^t \Big(\sigma(X_{\zeta_n}(s))\phi_n(s) - \sigma(X_\zeta(s))\phi(s), \Upsilon_n(s)\Big)ds \\
&+ 2\varepsilon \int_{\mathbb{Z}} \Big(g(X_{\zeta_n}(s), \xi)l(\psi_n(s,\xi)) - g(X_\zeta(s), \xi)l(\psi(s,\xi)), \Upsilon_n(s)\Big)\lambda(d\xi)ds \\
&+ 2\int_0^t (r_1 - \varrho_1)|\Upsilon_{1,n}(s)|^2 + (r_2 - \varrho_3)|\Upsilon_{3,n}(s)|^2 + (r_3 - \varrho_4)|\Upsilon_{4,n}(s)|^2 ds. \qquad (5.4)
\end{aligned}
$$

By the property of the nonlinear operator f, we have

$$
\begin{aligned}
&\int_0^t \Big(f(X_{\zeta_n}(s)) - f(X_\zeta(s)), \Upsilon_n(s)\Big)ds \le \int_0^t \Big[\Lambda_1\|\nabla\Upsilon_n(s)\|_{\mathbb{L}^2}^2 + \Lambda_2\|\Upsilon_n(s)\|_{\mathbb{L}^2}^2 \\
&+\left[\frac{4\rho_1 r_1}{d}\|x_{1,\zeta_n}\|_{\mathbb{L}^2}^2 + \left(\frac{4\rho_1 r_1}{d} + \frac{4}{d\chi}\right)\|x_{2,\zeta}\|_{\mathbb{L}^2}^2 + \frac{8}{d\chi}\|y_{1,\zeta_n}\|_{\mathbb{L}^2}^2\right]\|\Upsilon_{1,n}(s)\|_{\mathbb{L}^2}^2 \\
&+\frac{4}{d\chi}\left[\|y_{1,\zeta_n}\|_{\mathbb{L}^2}^2 + 2\|x_{2,\zeta}\|_{\mathbb{L}^2}^2\right]\|\Upsilon_{2,n}(s)\|_{\mathbb{L}^2}^2 \\
&+\left[\frac{4\rho_2 r_2}{d}\|z_{1,\zeta_n}\|_{\mathbb{L}^2}^2 + \left(\frac{4\rho_2 r_2}{d} + \frac{4}{d\delta_1}\right)\|z_{2,\zeta}\|_{\mathbb{L}^2}^2 + \left(\frac{4}{d\delta_1} + \frac{4}{d\delta_2}\right)\|w_{1,\zeta_n}\|_{\mathbb{L}^2}^2\right]\|\Upsilon_{3,n}(s)\|_{\mathbb{L}^2}^2 \\
&+\left[\left(\frac{4}{d\delta_1} + \frac{4}{d\delta_2}\right)\|z_{2,\zeta}\|_{\mathbb{L}^2}^2 + \left(\frac{4\rho_3 r_3}{d} + \frac{4}{d\delta_2}\right)\|w_{1,\zeta_n}\|_{\mathbb{L}^2}^2 + \frac{4\rho_3 r_3}{d}\|w_{2,\zeta}\|_{\mathbb{L}^2}^2\right]\|\Upsilon_{4,n}(s)\|_{\mathbb{L}^2}^2\Big]ds \\
&\le \int_0^t \Big[\Lambda_1\|\nabla\Upsilon_n(s)\|_{\mathbb{L}^2}^2 + \Lambda_2\|\Upsilon_n(s)\|_{\mathbb{L}^2}^2 + \varrho\left(|X_{\zeta_n}(s)\|_{\mathbb{L}^2}^2 + |X_{\zeta(s)}\|_{\mathbb{L}^2}^2\right)\|\Upsilon_n(s)\|_{\mathbb{L}^2}^2\Big]ds, \qquad (5.5)
\end{aligned}
$$

where

$$
\begin{aligned}
\varrho &= \frac{4\rho_1 r_1}{d} + \frac{4\rho_1 r_1}{d} + \frac{4}{d\chi} + \frac{8}{d\chi} + \frac{4}{d\chi} + \frac{4\rho_2 r_2}{d} + \frac{4\rho_2 r_2}{d} + \frac{4}{d\delta_1} \\
&+\frac{4}{d\delta_1} + \frac{4}{d\delta_2} + \frac{4}{d\delta_1} + \frac{4}{d\delta_2} + \frac{4\rho_3 r_3}{d} + \frac{4}{d\delta_2} + \frac{4\rho_3 r_3}{d}.
\end{aligned}
$$

Appying Young's inequality and $\mathcal{H}_2$, we get

$$
\begin{aligned}
&2\int_0^t \Big(\sigma(X_{\zeta_n}(s))\phi_n(s) - \sigma(X_\zeta(s))\phi(s), \Upsilon_n(s)\Big)\, ds \\
&= 2\int_0^t \Big(\sigma(X_{\zeta_n}(s))\phi_n(s) - \sigma(X_\zeta(s))\phi_n(s) + \sigma(X_\zeta(s))\phi_n(s) - \sigma(X_\zeta(s))\phi(s), \Upsilon_n(s)\Big)\, ds \\
&\le 2\int_0^t |\sigma(X_{\zeta_n}(s))\phi_n(s) - \sigma(X_\zeta(s))|_{\mathcal{L}_Q} |\phi_n(s)||\Upsilon_n(s)| + |\sigma(X_\zeta(s))||\phi_n(s) - \phi(s)||\Upsilon_n(s)| ds \\
&\le \int_0^t \left[\frac{\Lambda_1}{4}|\nabla\Upsilon_n(s)|^2 + \left(\frac{4\mathcal{K}_2}{\Lambda_1}|\phi_n(s)|^2 + 1\right)|\Upsilon_n(s)|^2\right] ds + \int_0^t |\sigma(X_\zeta(s))|^2|\phi_n(s) - \phi(s)|^2 ds.
\end{aligned} \tag{5.6}
$$

Similarly, we have

$$
\begin{aligned}
&2\int_{\mathbb{Z}} \Big(g(X_{\zeta_n}(s),\xi)l(\psi_n(s,\xi)) - g(X_\zeta(s),\xi)l(\psi(s,\xi)), \Upsilon_n(s)\Big)\lambda(d\xi)ds \\
&\le \int_0^t \left[\frac{\Lambda_1}{4}|\nabla\Upsilon_n(s)|^2 + \left(\frac{4\mathcal{K}_2}{\Lambda_1}|l(\psi_n(s,\xi))|^2 + 1\right)|\Upsilon_n(s)|^2\right]\lambda(d\xi)ds \\
&+ \int_0^t |g(X_\zeta(s),\xi)|^2|l(\psi_n(s,\xi)) - l(\psi(s,\xi))|^2\lambda(d\xi)ds.
\end{aligned} \tag{5.7}
$$

From Eqs (5.4)–(5.7), we get

$$
\begin{aligned}
&|\Upsilon_n(t)|^2 + \frac{\Lambda_1}{2}\int_0^t |\nabla\Upsilon_n(s)|^2 ds \le \varrho_0 \int_0^t |\Upsilon_n(s)|^2 ds + \varrho\int_0^t \Big(|X_{\zeta_n}(s)|^2 + |X_{\zeta(s)}|^2\Big)|\Upsilon_n(s)|^2 ds \\
&+ \int_0^t |\sigma(X_\zeta(s))|^2|\phi_n(s) - \phi(s)|^2 ds + \int_0^t |g(X_\zeta(s),\xi)|^2|l(\psi_n(s)) - l(\psi(s))|^2\lambda(d\xi)ds,
\end{aligned} \tag{5.8}
$$

where

$$
\varrho_0 = \frac{4\mathcal{K}_2}{\Lambda_1}|l(\psi_n(s,\xi))|^2 + \frac{4\mathcal{K}_2}{\Lambda_1}|\phi_n(s)|^2 + \Lambda_1 + 2 + 2\max\{r_1 - \varrho_1, r_2 - \varrho_3, r_3 - \varrho_4\}.
$$

Applying Gronwall's inequality as $n \to \infty$, we have

$$
|\Upsilon_n(t)|^2 + \frac{\Lambda_1}{2}\int_0^t |\nabla\Upsilon_n(s)|^2 ds \to 0.
$$

This implies that $K_{\mathfrak{M}}$ is a compact set in $\mathbb{X}$.

Lemma 5.4 Suppose $\{\zeta^\varepsilon, \varepsilon > 0\} \in \mathcal{A}$ converges to ζ in distribution with respect to the weak topology in $\mathcal{A}$, then

$$
\mathcal{G}^\varepsilon\left(\sqrt{\varepsilon}B(\cdot) + \int_0^\cdot \phi^\varepsilon ds, \varepsilon N^{\varepsilon^{-1}\psi^\varepsilon}\right) \to \mathcal{G}^0\left(\int_0^\cdot \phi^\varepsilon ds, \lambda_T^\psi\right)
$$

in distribution in $\mathbb{X}$ as $\varepsilon \to 0$.

Proof. In order to prove that $\Upsilon^\varepsilon = X^\varepsilon_{\zeta^\varepsilon} - X_\zeta$ tends to 0 as $\varepsilon \to 0$, where $X^\varepsilon_{\zeta^\varepsilon}$ and X_ζ are the solutions of the stochastic control diffusive coral reef ecosystems equation (5.1) and deterministic controlled equation (5.2), respectively. The equation satisfied by X_ζ is

$$\begin{aligned} d\Upsilon^\varepsilon(t) &= \left[A\Upsilon^\varepsilon(t) + f(X^\varepsilon_{\zeta^\varepsilon}(t)) - f(X_\zeta) + \sigma(X^\varepsilon_{\zeta^\varepsilon}, t)\phi^\varepsilon - \sigma(X_\zeta, t)\phi\right] dt + \sqrt{\varepsilon}\sigma(X^\varepsilon_{\zeta^\varepsilon}, t)dB_t \\ &+ \int_{\mathbb{Z}} \left[g(X^\varepsilon_{\zeta^\varepsilon}(t), \xi)l(\psi^\varepsilon) - g(X_\zeta, \xi)l(\psi)\right] \lambda(d\xi)dt + \varepsilon \int_{\mathbb{Z}} g(X^\varepsilon_{\zeta^\varepsilon}(t), \xi)\widetilde{N}(dt, d\xi). \end{aligned} \tag{5.9}$$

By Ito's formula, we get

$$\begin{aligned} |\Upsilon^\varepsilon(t)|^2 &= 2\int_0^t \left(\left[A\Upsilon^\varepsilon(s) + f(X^\varepsilon_{\zeta^\varepsilon}(s)) - f(X_\zeta) + \sigma(X^\varepsilon_{\zeta^\varepsilon}, s)\phi^\varepsilon - \sigma(X_\zeta, s)\phi\right], \Upsilon^\varepsilon(s)\right) ds \\ &+ \int_0^t \int_{\mathbb{Z}} \left(\left[g(X^\varepsilon_{\zeta^\varepsilon}(s), \xi)l(\psi^\varepsilon) - g(X_\zeta, \xi)l(\psi)\right], \Upsilon^\varepsilon(s)\right) \lambda(d\xi)ds \\ &+2\sqrt{\varepsilon}\int_0^t \left(\sigma(X^\varepsilon_{\zeta^\varepsilon}, t)dB_s, \Upsilon^\varepsilon(s)\right) ds + 2\varepsilon \int_0^t \int_{\mathbb{Z}} \left(g(X^\varepsilon_{\zeta^\varepsilon}(t), \xi), \Upsilon^\varepsilon(s)\right) \widetilde{N}(ds, d\xi) \\ &+\varepsilon \int_0^t |\sigma(X^\varepsilon_{\zeta^\varepsilon}(s), s)|^2_{\mathcal{L}_Q} ds + \varepsilon \int_0^t \int_{\mathbb{Z}} |g(X^\varepsilon_{\zeta^\varepsilon}(t), \xi)|^2 N(ds, d\xi). \end{aligned} \tag{5.10}$$

Employing the same method as done earlier, we get

$$\begin{aligned} &|\Upsilon^\varepsilon(t)|^2 + \frac{\Lambda_1}{2}\int_0^t |\nabla\Upsilon^\varepsilon(s)|^2 ds \le \varrho_0 \int_0^t |\Upsilon^\varepsilon(s)|^2 ds + \varrho \int_0^t \left(|X_{\zeta_n}(s)|^2 + |X_{\zeta(s)}|^2\right) |\Upsilon^\varepsilon(s)|^2 ds \\ &+ \int_0^t |\sigma(X_\zeta(s))|^2 |\phi^\varepsilon(s) - \phi(s)|^2 ds + \int_0^t \int_{\mathbb{Z}} |g(X_\zeta(s), \xi)|^2 |l(\psi^\varepsilon(s)) - l(\psi(s)|^2)\lambda(d\xi)ds \\ &+2\sqrt{\varepsilon}\int_0^t \left(\sigma(X^\varepsilon_{\zeta^\varepsilon}, s)dB_s, \Upsilon^\varepsilon(s)\right) ds + 2\varepsilon \int_0^t \int_{\mathbb{Z}} \left(g(X^\varepsilon_{\zeta^\varepsilon}(s), \xi), \Upsilon^\varepsilon(s)\right) \widetilde{N}(ds, d\xi) \\ &+\epsilon\mathcal{K}_1 \int_0^t (1 + |\nabla X^\varepsilon_{\zeta^\varepsilon}(s))|^2 ds. \end{aligned} \tag{5.11}$$

Taking expectation and applying the Burkholder-Davis-Gundy inequality, we have

$$\begin{aligned} &\mathbb{E}\sup_{0\le t\le T}\left\{2\sqrt{\varepsilon}\int_0^t \left(\sigma(X^\varepsilon_{\zeta^\varepsilon}, s)dB_s, \Upsilon^\varepsilon(s)\right) ds + 2\varepsilon \int_0^t \int_{\mathbb{Z}} \left(g(X^\varepsilon_{\zeta^\varepsilon}(s), \xi), \Upsilon^\varepsilon(s)\right) \widetilde{N}(ds, d\xi)\right\} \\ &\le \frac{1}{2}\mathbb{E}\left\{\sup_{0\le t\le T} |\Upsilon^\varepsilon(t)|^2\right\} + (\varepsilon^2 + \varepsilon)C\mathcal{K}_1^2 \int_0^t \left(1 + |\nabla X^\varepsilon_{\zeta^\varepsilon}(s)\right)|^2 ds. \end{aligned} \tag{5.12}$$

By appropriate constant ϱ_1, we have

$$
\begin{aligned}
&\frac{1}{2}\mathbb{E}\sup_{0\le t\le T}|\Upsilon^{\varepsilon}(t)|^2+\frac{\Lambda_1}{2}\mathbb{E}\int_0^t|\nabla\Upsilon^{\varepsilon}(s)|^2ds\le\varrho_1\mathbb{E}\int_0^t|\Upsilon^{\varepsilon}(s)|^2ds\\
&+\mathbb{E}\int_0^t|\sigma(X_{\zeta}(s))|^2|\phi^{\varepsilon}(s)-\phi(s)|^2ds+\mathbb{E}\int_0^t\int_{\mathbb{Z}}|g(X_{\zeta}(s),\xi)|^2|l(\psi^{\varepsilon}(s))-l(\psi(s)|^2)\lambda(d\xi)ds\\
&+\left[\epsilon\mathcal{K}_1+(\varepsilon^2+\varepsilon)C\mathcal{K}_1^2\right]\mathbb{E}\int_0^t(1+|\nabla X^{\varepsilon}_{\zeta^{\varepsilon}}(s))|^2ds. \qquad (5.13)
\end{aligned}
$$

Using Gronwall's inequality, we get

$$
\begin{aligned}
&\mathbb{E}\sup_{0\le t\le T}|\Upsilon^{\varepsilon}(t)|^2+\Lambda_1\mathbb{E}\int_0^t|\nabla\Upsilon^{\varepsilon}(s)|^2ds\le\exp(\varrho_1T)\\
&\times\left\{\mathbb{E}\int_0^t|\sigma(X_{\zeta}(s))|^2|\phi^{\varepsilon}(s)-\phi(s)|^2ds+C(\varepsilon)\mathbb{E}\int_0^t(1+|\nabla X^{\varepsilon}_{\zeta^{\varepsilon}}(s))|^2ds\right.\\
&\left.+\mathbb{E}\int_0^t\int_{\mathbb{Z}}|g(X_{\zeta}(s),\xi)|^2|l(\psi^{\varepsilon}(s))-l(\psi(s)|^2)\lambda(d\xi)ds\right\}, \qquad (5.14)
\end{aligned}
$$

where $C(\varepsilon)=\epsilon\mathcal{K}_1+(\varepsilon^2+\varepsilon)C\mathcal{K}_1^2$. Since $C(\varepsilon)\to 0$ when letting $\varepsilon\to 0$, we get$\Upsilon^{\varepsilon}\to 0$. Hence, $X^{\varepsilon}_{\zeta^{\varepsilon}}$ converges to X_{ζ}.

Theorem 5.5 If there exists a measurable map $\mathcal{G}^0:\mathcal{X}_0\times\mathbb{V}\to\mathbb{X}$ such that

(i) Let $\mathfrak{M}<\infty$ and $\{\zeta^{\varepsilon}=(\phi^{\varepsilon},\psi^{\varepsilon})\in\mathcal{U}:\zeta^{\varepsilon}(\omega)\in S_{\mathfrak{M}}$ for $a.e.\omega\}\subset\mathcal{U}^{\mathfrak{M}}$, if $(\phi^{\varepsilon},\psi^{\varepsilon})\to(\phi,\psi)$ in distribution in $S_{\mathfrak{M}}$ as $\varepsilon\to 0$, then

$$\mathcal{G}^{\varepsilon}\left(\sqrt{\varepsilon}B(\cdot)+\int_0^{\cdot}\phi^{\varepsilon}ds,\varepsilon N^{\varepsilon^{-1}\psi^{\varepsilon}}\right)\to\mathcal{G}^0\left(\int_0^{\cdot}\phi^{\varepsilon}ds,\lambda_T^{\psi}\right).$$

(ii) For every finite $\mathfrak{M}$, $K_{\mathfrak{M}}=\{\mathcal{G}^0\left(\int_0^{\cdot}\phi^{\varepsilon}ds,\lambda_T^{\psi}\right):(\phi,\psi)\in\mathcal{U}^{\mathfrak{M}}\}$ is a compact subset in $\mathbb{X}$.

Then the family of solutions $\{X^{\varepsilon},\varepsilon>0\}$ of the stochastic reaction-diffusion coral reef model with Lévy noise equation (1.2) satisfies the Laplace principle with the rate function I given by

$$I(g)=\inf_{(\phi,\psi)\in S_g}\left\{\frac{1}{2}\int_0^T\int_{\mathbb{Z}}|l(\psi(s,\xi))|^2\lambda(d\xi)ds+\frac{1}{2}\int_0^T|\phi|_0^2ds\right\},$$

where $S_g=\left\{(\phi,\psi)\in\bigcup_{\mathfrak{M}>1}S_{\mathfrak{M}}:g=\{\mathcal{G}^0\left(\int_0^{\cdot}\phi^{\varepsilon}ds,\lambda_T^{\psi}\right)\right\}$, and where infimum over the empty set is taken as ∞.

Proof. By a weak convergence approach based on a variational representation of functionals of infinite-dimensional Brownian motion, combining Lemmas 5.3 and 5.4, it now guarantees that the solution of stochastic reaction-diffusion coral reef model with Lévy noise equation (1.2) satisfies the Laplace principle thereby satisfying the large deviation principle with the same rate function as well.

6. Numerical simulation results

According to our analytical results, the stochastic reaction-diffusion coral reef model with Lévy noise equation (1.2) has a unique weak $\mathcal{D}$-pullback mean random attractor under conditions specified in Theorem 3.3. We now try and support our analytical results by simulations. Interestingly, this shows the effect of Lévy noise which can stabilize or destabilize systems and is significantly different from the classical Brownian motion process. The typical values [11–13]: $d_1 = 0.001, d_2 = 0.002, d_3 = 0.003, d_4 = 0.004, r_1 = 3.23, r_2 = 1.59, r_3 = 1.2, \rho_1 = 1/100, \rho_2 = 1/300, \rho_3 = 1/250, \theta = 50, \chi = 0.0572, \alpha_1 = 2.6, \alpha_2 = 2.5, \varrho_1 = 0.75, \varrho_2 = 0.34, \varrho_3 = 1.67, \varrho_4 = 5.36, \beta_1 = 8.5, \beta_2 = 1.4, \delta_1 = 0.95, \delta_2 = 0.9, \sigma_1(x,t) = \sigma_1 x, \sigma_2(y,t) = \sigma_2 y, \sigma_3(z,t) = \sigma_3 z, \sigma_4(w,t) = \sigma_4 w, g_1(x,\xi) = xg_1(\xi), g_2(y,\xi) = yg_2(\xi), g_3(z,\xi) = zg_3(\xi), g_4(w,\xi) = wg_4(\xi)$.

The spatiotemporal dynamics plot of the stochastic reaction-diffusion coral reef model with Lévy noise equation (1.2) with the initial conditions $x_0 = 0.015 + 0.01\cos(4x), y_0 = 0.025 + 0.02\cos(4x)$, $z_0 = 0.25 + 0.03\cos(4x)$ and $w_0 = 0.45 + 0.04\cos(4x), \sigma_1 = 1.5, \sigma_2 = 1.4, \sigma_3 = 1.2, \sigma_3 = 1.1$. We show the simulation of susceptible corals density(x, u, t) in Figure 1. On the left side of Figure 1 represents the simulation of susceptible corals density(x, u, t) in the stochastic reaction-diffusion coral reef model with Lévy. On the right side of Figure 1 represents the simulation of susceptible corals density(x, u, t) in the deterministic reaction-diffusion coral reef model with Lévy.

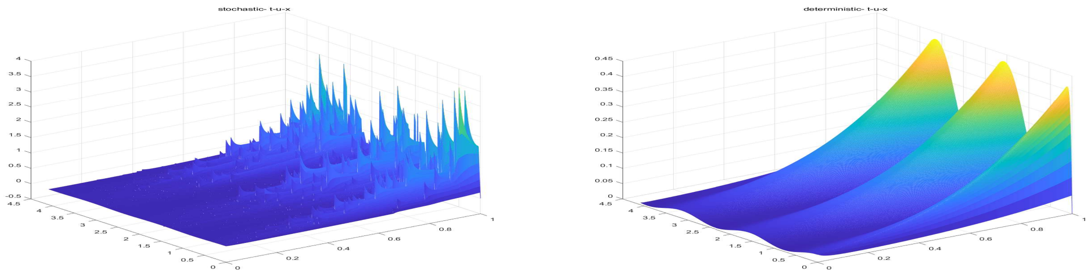

Figure 1. Simulated phase portraits of susceptible corals density(x, u, t) in deterministic and stochastic reaction-diffusion coral reef model with Lévy.

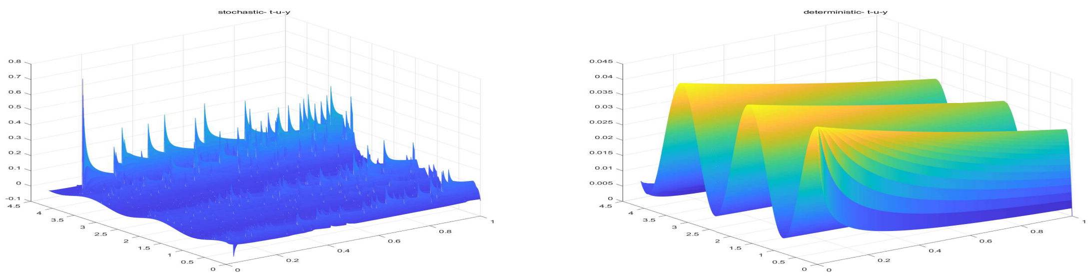

Figure 2. Simulated phase portraits of infected corals density (y, u, t) in deterministic and stochastic reaction-diffusion coral reef model with Lévy.

The spatiotemporal dynamics plot of the stochastic reaction-diffusion coral reef model with Lévy noise equation (1.2) with the initial conditions $x_0 = 0.015 + 0.01\cos(4x), y_0 = 0.025 + 0.02\cos(4x)$, $z_0 = 0.25 + 0.03\cos(4x)$ and $w_0 = 0.45 + 0.04\cos(4x), \sigma_1 = 1.5, \sigma_2 = 1.4, \sigma_3 = 1.2, \sigma_3 = 1.1$. We show the simulation of infected corals density (y, u, t) in Figure 2. On the left side of Figure 2 represents the simulation of infected corals density (y, u, t) in the stochastic reaction-diffusion coral reef model with Lévy. On the right side of Figure 2 represents the simulation of infected corals density (y, u, t) in the deterministic reaction-diffusion coral reef model with Lévy.

The spatiotemporal dynamics plot of the stochastic reaction-diffusion coral reef model with Lévy noise equation (1.2) with the initial conditions $x_0 = 0.015 + 0.01\cos(4x), y_0 = 0.025 + 0.02\cos(4x)$, $z_0 = 0.25 + 0.03\cos(4x)$ and $w_0 = 0.45 + 0.04\cos(4x)$, $\sigma_1 = 1.5, \sigma_2 = 1.4, \sigma_3 = 1.2, \sigma_3 = 1.1$. We show the simulation of Crown-of-thorns starfish (Acanthaster planci) density (z, u, t) in Figure 3. On the left side of Figure 3 represents the simulation of Crown-of-thorns starfish (Acanthaster planci) density (z, u, t) in the stochastic reaction-diffusion coral reef model with Lévy. On the right side of Figure 3 represents the simulation of Crown-of-thorns starfish (Acanthaster planci) density (z, u, t) in the deterministic reaction-diffusion coral reef model with Lévy.

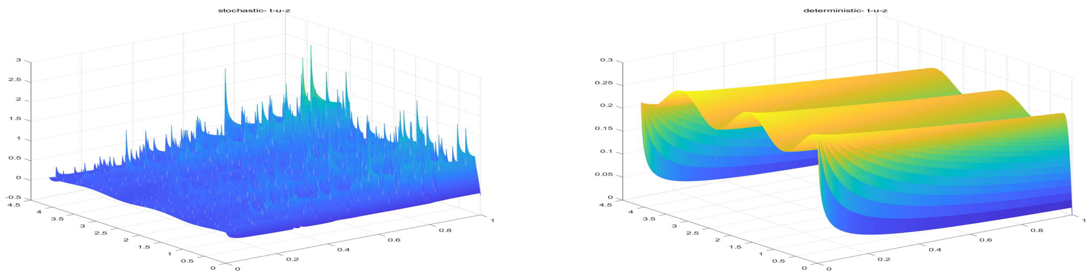

Figure 3. Simulated phase portraits of Crown-of-thorns starfish (Acanthaster planci) density (z, u, t) in deterministic and stochastic reaction-diffusion coral reef model with Lévy.

The spatiotemporal dynamics plot of the stochastic reaction-diffusion coral reef model with Lévy noise equation (1.2) with the initial conditions $x_0 = 0.015 + 0.01\cos(4x), y_0 = 0.025 + 0.02\cos(4x)$, $z_0 = 0.25 + 0.03\cos(4x)$ and $w_0 = 0.45 + 0.04\cos(4x)$, $\sigma_1 = 1.5, \sigma_2 = 1.4, \sigma_3 = 1.2, \sigma_3 = 1.1$. We show the simulation of Humphead wrasse (Cheilinus undulatus) density (w, u, t) in Figure 4. On the left side of Figure 4 represents the simulation of Humphead wrasse (Cheilinus undulatus) density (w, u, t) in the stochastic reaction-diffusion coral reef model with Lévy. On the right side of Figure 4 represents the simulation of Humphead wrasse (Cheilinus undulatus) density (w, u, t) in the deterministic reaction-diffusion coral reef model with Lévy.

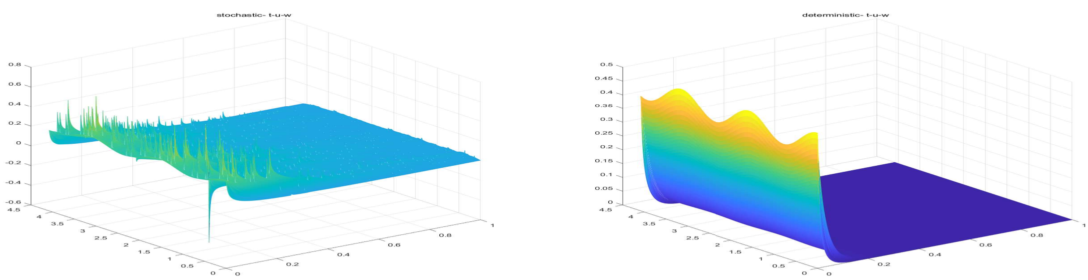

Figure 4. Simulated phase portraits of Humphead wrasse (Cheilinus undulatus) density (w, u, t) in deterministic and stochastic reaction-diffusion coral reef model with Lévy.

Simulated phase portraits of the stochastic reaction-diffusion coral reef model with Lévy noise equation (1.2) with the initial conditions $x_0 = 0.015 + 0.01\cos(4x), y_0 = 0.025 + 0.02\cos(4x), z_0 = 0.25 + 0.03\cos(4x)$ and $w_0 = 0.45 + 0.04\cos(4x)$, $\sigma_1 = 1.5, \sigma_2 = 1.4, \sigma_3 = 1.2, \sigma_3 = 1.1$. Blue represents the simulation of susceptible corals density in the deterministic reaction-diffusion coral reef model with Lévy. Red represents the simulation of susceptible corals density(x-t) in the stochastic reaction-diffusion coral reef model with Lévy. We show the simulation of susceptible corals density(x-t) in Figure 5.

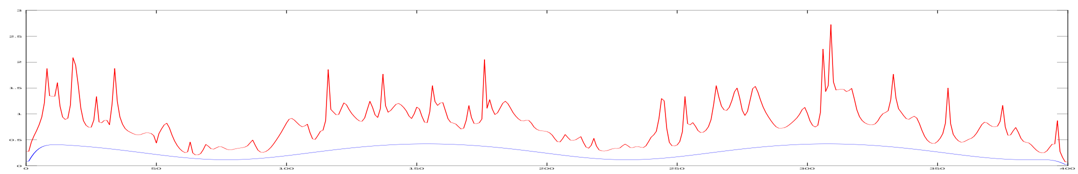

Figure 5. Simulated phase portraits of susceptible corals density(x-t) in deterministic and stochastic reaction-diffusion coral reef model with Lévy.

Simulated phase portraits of the stochastic reaction-diffusion coral reef model with Lévy noise equation (1.2) with the initial conditions $x_0 = 0.015 + 0.01\cos(4x), y_0 = 0.025 + 0.02\cos(4x), z_0 = 0.25 + 0.03\cos(4x)$ and $w_0 = 0.45 + 0.04\cos(4x)$, $\sigma_1 = 1.5, \sigma_2 = 1.4, \sigma_3 = 1.2, \sigma_3 = 1.1$. Blue represents the simulation of infected corals density(y-t) in the deterministic reaction-diffusion coral reef model with Lévy. Red represents the simulation of infected corals density(y-t) in the stochastic reaction-diffusion coral reef model with Lévy. We show the simulation of infected corals density(y-t) in Figure 6.

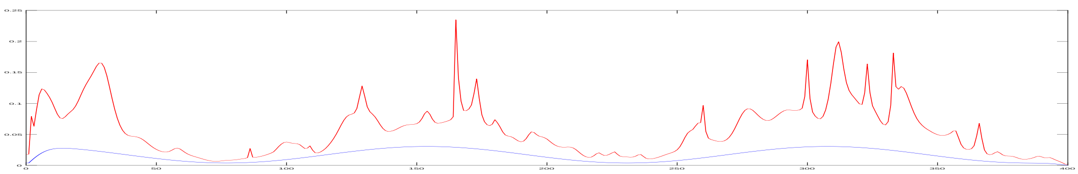

Figure 6. Simulated phase portraits of infected corals density(y-t) in deterministic and stochastic reaction-diffusion coral reef model with Lévy.

Simulated phase portraits of the stochastic reaction-diffusion coral reef model with Lévy noise equation (1.2) with the initial conditions $x_0 = 0.015 + 0.01\cos(4x)$, $y_0 = 0.025 + 0.02\cos(4x)$, $z_0 = 0.25 + 0.03\cos(4x)$ and $w_0 = 0.45 + 0.04\cos(4x)$, $\sigma_1 = 1.5, \sigma_2 = 1.4, \sigma_3 = 1.2, \sigma_3 = 1.1$. Blue represents the simulation of Crown-of-thorns starfish (Acanthaster planci) density(z-t) in the deterministic reaction-diffusion coral reef model with Lévy. Red represents the simulation of Crown-of-thorns starfish (Acanthaster planci) density(z-t) in the stochastic reaction-diffusion coral reef model with Lévy. We show the simulation of Crown-of-thorns starfish (Acanthaster planci) density(z-t) in Figure 7.

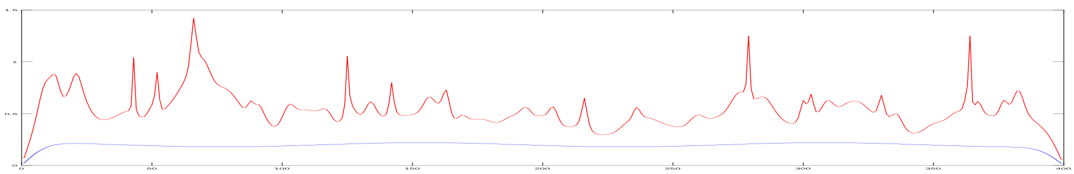

Figure 7. Simulated phase portraits of Crown-of-thorns starfish (Acanthaster planci) density(z-t) in deterministic and stochastic reaction-diffusion coral reef model with Lévy.

Simulated phase portraits of the stochastic reaction-diffusion coral reef model with Lévy noise equation (1.2) with the initial conditions $x_0 = 0.015 + 0.01\cos(4x)$, $y_0 = 0.025 + 0.02\cos(4x)$, $z_0 = 0.25 + 0.03\cos(4x)$ and $w_0 = 0.45 + 0.04\cos(4x)$, $\sigma_1 = 1.5, \sigma_2 = 1.4, \sigma_3 = 1.2, \sigma_3 = 1.1$. Blue represent the simulation of Humphead wrasse (Cheilinus undulatus) density density(w-t) in deterministic reaction-diffusion coral reef model with Lévy. Red represent the simulation of Humphead wrasse (Cheilinus undulatus) density(w-t) in stochastic reaction-diffusion coral reef model with Lévy. We show the simulation of Humphead wrasse (Cheilinus undulatus) density(w-t) in Figure 8.

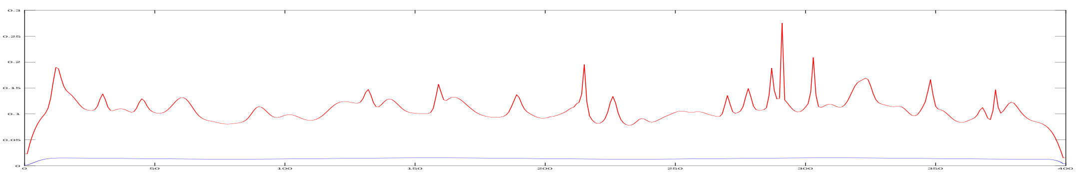

Figure 8. Simulated phase portraits of Humphead wrasse (Cheilinus undulatus) density density(w-t) in deterministic and stochastic reaction-diffusion coral reef model with Lévy.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

We declare that we have no financial and personal relationships with other people or organizations that can inappropriately influence our work, there is no professional or other personal interest of any nature or kind in any product, service and/or company that could be construed as influencing the position presented in, or the review of, the manuscript entitled.

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