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Research article

The Stochastic Gause Predator-Prey model: Noise-induced extinctions and invariance

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Abstract: We consider the Gause predator-prey with general bounded or sub-linear functional responses, – which includes those of Holling types I–IV. – and multiplicative Gaussian noise. In contrast to previous studies, the prey in our model follows logistic dynamics while the predator's population is solely regulated by consumption of the prey. To ensure well-posedeness, we derive explicit Lyapunov-type criteria ensuring global positivity and moment boundedness of solutions. We find conditions for noise-induced extinctions, proving that stochasticity can drive either population to collapse even when the deterministic analogue predicts stable coexistence. In the case when the predator becomes extinct, we establish a limiting distribution for the predator's population. Last, for functional responses of Holling type I, we provide sufficient conditions on the intensity of the noise for the existence and uniqueness of a stationary distribution.

Keywords: Stochastic differential equations; Gause model; stochastic population; noise-induced extinction; invariant distribution

1. Introduction

Over the past fifty years, there has been a growing interest in incorporating stochasticity into mathematical models in theoretical biology. This interest is driven by the need to capture environmental unpredictability, aleatoric errors in model parameters, and variability within populations. For a foundational discussion, see [1], and for modern and comprehensive treatments, see [2, 3]. Stochastic differential equations (SDEs) have emerged as powerful tools for modeling population dynamics in uncertain environments. In fact, numerous textbooks on stochastic calculus now feature population ecology models as key examples [4–6]. Also, studies on population stability in the presence of environmental noise have significantly advanced our understanding of how stochastic fluctuations can either drive species extinction or promote persistence [7, 8]. Among many other insights, the inclusion of Brownian-type noise in these models has revealed that stochastic effects can either stabilize or destabilize populations, depending on initial conditions and environmental factors [9, 10].

In this paper we study a stochastic version of the Gause model with logistic prey dynamics, with a specific focus elucidating the effects that multiplicative noise has on the long-term behavior of the population, and the possibility of noise-induced extinctions. The Gause model is a family of two-species, predator-prey, ordinary differential equation models that generalize the classical Lotka-Volterra model to the case where the intensity of predation depends on the availability of prey. See [11] for a historical review. Specifically, we consider the dynamics in time $t \geq 0$ of the populations of prey $x(t)$ and predator $y(t)$ via the following system of stochastic differential equations

$$\begin{aligned} \mathrm{d}x &= \left(rx\left(1 - \frac{x}{K}\right) - f(x)y\right) \mathrm{d}t + xs_x \,\mathrm{d}B_x, \\ \mathrm{d}y &= (bf(x)y - my)\, \mathrm{d}t + ys_y \,\mathrm{d}B_y, \end{aligned} \tag{1.1}$$

where f is a non-negative function called the functional response of the predator to the prey. It describes the dependence between the predator's behavior and the availability of prey [11–13], and plays a crucial role in the dynamics of the system, both in the deterministic and stochastic models. Our analyses cover very general functional responses, including those of Holling types I to IV, which are summarized in Table 1.

The model parameters in Eq. (1.1) are all assumed positive and interpreted as follows: r is the intrinsic growth rate of the prey, K is its carrying capacity, b is the interaction coefficient of the predator, and m is its mortality rate. See [3, 14, 15].

The diffusion terms in Eq. (1.1) are driven by standard independent Wiener processes B_x, B_y and have diffusion coefficients that depend linearly on x and y with intensities $s_x, s_y > 0$. These multiplicative-noise terms model the situation in which, for example, there is demographic or environmental uncertainty in the parameters r and m that can be modeled as a Gaussian process [16]. Specifically, at any time t, the actual reproductive rates for prey and predator are random variables $r + s_x\dot{B}_x$ and $-m + s_y\dot{B}_y$, respectively. Here, r and $-m$ are mean values, and $\dot{B}_x, \dot{B}_y$ denote independent white noise processes. The coefficients s_x and s_y are the intensities of the noise and model the amount of variability or uncertainty around the mean values of the reproductive rates. The system of SDEs is considered in the Itô sense. See [1, 17]

Our analysis focuses on the probabilistic structural properties of the solution to Eq. (1.1) as a function of the intensity of the noise. We provide sufficient conditions for the existence and boundedness of solutions to Eq. (1.1), noise-induced extinctions of prey and/or predator, and the existence of a unique stationary distribution. Except for our result on invariant distributions, which holds only for type I Holling functional responses, our results do not assume a specific form of f. Rather, we always assume that f in Eq. (1.1) is nonnegative, satisfies $f(0) = 0$, and is either bounded or grows sublinearly.

Recent studies have extensively investigated the long-term behavior of predator-prey systems under stochastic influences. Here, we focus on the literature most relevant to our findings. Various approaches have been employed to determine the conditions under which populations can persist over time. In particular, Markov process theory has been applied in this context, as explored by [18]. Foster-Lyapunov criteria, in particular, have proven useful for assessing the stability of continuous processes and evaluating extinction probabilities in stochastic ecological systems [19]. Also, in [20] the authors studied more complex systems incorporating modifications to the Leslie-Gower and Holling type IV frameworks. These studies emphasize that more sophisticated ecological behaviors, such as group defense mechanisms and nonlinear prey responses, can substantially alter system dynamics and persistence, see [21].

In [22], the authors present a very general framework that shares most of the same goals as this manuscript. The important distinction is that in their study, they assume an underlying logistic growth for both the predator and the prey, whereas in Eq. (1.1) we assign logistic dynamics only to the prey. Namely, the predator in our model cannot grow in the absence of prey and it experiences population regulation primarily from prey availability rather from intraspecific competition for resources. Our techniques for analysis also differ from those of [22]. While they directly compute Lyapunov exponents, we base our derivations on a judicious application of basic comparison theorems and the Lyapunov method for stability of SDEs (see [5] for an introduction). Lastly, in [9] the authors also use the Lyapunov method to establish the existence of a unique stationary distribution but only for the case of a functional response of Holling type II and assuming intraspecific competition among the predators. In contrast with these two studies, our model focuses on an ecological context where the predator's population growth rate is simultaneously highly vulnerable to the scarcity of prey and potentially unlimited as the prey becomes readily available.

The organization of this paper is as follows. In Section 2.1 we review and summarize the structural properties of the deterministic Gause model and introduce the stochastic model in non-dimensional variables. Subsequently, Section 2.1 contains preliminary results that ensure the well-posedness of the stochastic problem, which in this context includes existence, uniqueness, positivity, and boundedness of the solutions. The analysis then explores noise-induced extinction phenomena, demonstrating that certain noise intensities can lead to the eventual extinction of one or both species, even in situations where the deterministic model predicts coexistence. Finally, we provide conditions for the existence of an invariant distribution in the case of a Type I functional response, ensuring the persistence of populations under certain parameter configurations of the model. For each of the results, we provide graphical examples that, as a whole, are not meant to constitute a comprehensive numerical exploration of all possible behaviors but instead provide illustrative examples of typical model behavior.

The main contributions of this work are threefold. First, to the best of our knowledge, this is the first stochastic version of the classical Gause-type predator–prey model with logistic prey growth and general functional response. Second, we extend previous approaches by allowing a broad class of functional responses, including bounded or sublinear ones, thus encompassing ecologically relevant interactions such as Holling types I to IV. This generality enhances the applicability of our results. Third, we provide new sufficient conditions for boundedness, noise-induced extinction, and the existence of stationary distributions, showing how multiplicative noise affects long-term dynamics. The results obtained underscore the significant impact that uncertainty, modeled as multiplicative noise within an SDE, can have on the dynamics of ecological systems, altering behaviors predicted by deterministic models and leading to extinction or coexistence scenarios that depend on the level of noise.

2. Deterministic and stochastic Gause models

We begin with a dimensionless formulation of the deterministic Gause model and a review of its relevant structural properties. See [11] for an in-depth treatment. We consider the following system of ordinary differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = rx\left(1 - \frac{x}{K}\right) - f(x)y, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = bf(x)y - my, \tag{2.1}$$

where all model parameters are assumed positive. We distinguish between two different classes of the functional response f in Eq. (2.1): linear response (also known as Holling type I); and the class of bounded, non-negative continuous functional responses such that $f(0) = 0$, which includes Holling's types II, III, and IV. See Table 1. For both classes, we arrive at the same non-dimensional model but use slightly different normalizations.

For a linear functional response $f(x) = cx$ for some $c > 0$, we change to dimensionless variables by

$$\frac{x}{K} \to x, \quad \frac{c}{r}y \to y, \quad rt \to t, \tag{2.2}$$

and define the following dimensionless functional response and parameters

$$\varphi(x) = x, \quad \beta = \frac{bcK}{r}, \quad \delta = \frac{m}{r}. \tag{2.3}$$

For a bounded functional response $f(x)$ with $F := \sup_{x \geq 0} f(x) < \infty$, we change variables by

$$\frac{x}{K} \to x, \quad \frac{F}{Kr}y \to y, \quad rt \to t, \tag{2.4}$$

and define

$$\varphi(x) = \frac{f(Kx)}{F}, \quad \beta = \frac{Fb}{r}, \quad \delta = \frac{m}{r}. \tag{2.5}$$

Under the corresponding change of variables and notation, the Gause model Eq. (2.1) takes the adimensional form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x(1 - x) - \varphi(x)y, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = \beta\varphi(x)y - \delta y. \tag{2.6}$$

Since $\varphi(0) = 0$, Eq. (2.6) always has two equilibrium points corresponding to extinction events: the global extinction state $(0, 0)$ and predator extinction $(1, 0)$; furthermore,

$$\begin{aligned} &(0,0) \text{ is always unstable,} \\ &(1,0) \text{ is asymptotically stable if and only if } \beta < \beta^* := \tfrac{\delta}{\varphi(1)}. \end{aligned} \tag{2.7}$$

A bifurcation occurs at $\beta = \beta^*$, giving rise to various types of stable manifolds that guarantee persistence of both species, including stable solutions and limit cycles. These have been characterized for some functional response types and parameter ranges. See [11, 12, 23].

In order to write the noise terms without units, note that the term $s_x\,\mathrm{d}B_x$ in Eq. (1.1) is dimensionless but $B_x(t)$ has units of $\sqrt{t}$. Making the change $t \to t/r$ in the Brownian motion process gives $B_x(t/r)$ which has equal distribution to $\frac{1}{\sqrt{r}}B_x(t)$, now in terms of a dimensionless $B_x(t)$. Similarly for the noise term $s_y\,\mathrm{d}B_y$. We thus normalize the noise intensities as

$$\sigma_x = \frac{s_x}{\sqrt{r}}, \quad \sigma_y = \frac{s_y}{\sqrt{r}}. \tag{2.8}$$

which, along the change of variables in Eq. (2.2) or Eq. (2.4), yield the system of SDEs of interest,

$$\mathrm{d}x = (x(1-x) - \varphi(x)y)\,\mathrm{d}t + x\sigma_x\,\mathrm{d}B_x,\quad \mathrm{d}y = (\beta\varphi(x)y - \delta y)\,\mathrm{d}t + y\sigma_y\,\mathrm{d}B_y, \tag{2.9}$$

where the parameters β, δ are as in Eqs. (2.3) and (2.5); B_x and B_y are independent Brownian processes of the dimensionless time variable t; and φ is a continuous, positive function such that $\varphi(0) = 0$. Note that under this hypothesis, the coefficients of Eq. (2.9) are locally Lipschitz, so standard SDE theory guarantees the existence of strong solutions up to a possibly finite explosion time.

Table 1. Functional responses, their adimensionalization, and parameters. To obtain the expressions for type IV, note that c is adimensional, i has units of x^2 and a of x, the specified value of F is the global maximum under the usual condition $4a > i$, and the value of $\varphi(x)$ is obtained by simplifying $f(Kx)/F$ in terms of α and γ.

Type	$f(x)$	F	$\varphi(x)$	Parameters
I	cx	∞	x	
II	$\frac{cx}{x+a}$	c	$\frac{x}{x+\alpha}$	$\alpha = \frac{a}{K}$
III	$\frac{cx^2}{x^2+a}$	c	$\frac{x^2}{x^2+\alpha}$	$\alpha = \frac{a}{K^2}$
IV	$\frac{cx}{\frac{x^2}{i}+x+a}$	$\frac{ci}{1+2\sqrt{ai}}$	$\frac{(\alpha+2\sqrt{\gamma})x}{\gamma+\alpha x+x^2}$	$\alpha = \frac{i}{K}$, $\gamma = \frac{ai}{K^2}$

2.1. Regularity and boundedness

In this section, we present fundamental results concerning the model in Eq. (2.9). The first result guarantees that the solution to Eq. (2.9) for functional responses of all types in Table 1 is well-posed and regular. Namely, that solutions 'remain' within the positive quadrant $\mathbb{R}^2_+ := \{(x,y) \in \mathbb{R}^2 : x > 0, y > 0\}$ and are bounded in probability for all $t \geq 0$.

Theorem 1. *Suppose that all the parameters in* (2.9) *are positive and that the functional response is continuous, non-negative, and satisfies*

$$\sup_{x>0} \frac{\varphi(x)}{x} \leq 1.$$

Then, for any initial value $(x(0), y(0)) \in \mathbb{R}^2_+$, *Eq.* (2.9) *has, with probability one, a unique solution* $(x(t), y(t)) \in \mathbb{R}^2_+$ *for all* $t \geq 0$.

Proof. The main tool for analysis is the Lyapunov operator associated to Eq. (2.9)

$$L = [x(1-x) - \varphi(x)y]\frac{\partial}{\partial x} + [\beta\varphi(x)y - \delta y]\frac{\partial}{\partial y} + \frac{1}{2}\sigma_x^2 x^2 \frac{\partial^2}{\partial x^2} + \frac{1}{2}\sigma_y^2 y^2 \frac{\partial^2}{\partial y^2}. \tag{2.10}$$

Following standard techniques, it is enough to show that there is a function $V : \mathbb{R}^2_+ \to \mathbb{R}_+$ such that $L[V](x,y) \leq M$ for some positive constant M and for all (x,y) in $\mathbb{R}^2_+$. See [5,6].

Since the drift and diffusion coefficients in Eq. (2.9) are locally Lipschitz functions, then for any initial value $(x(0), y(0)) \in \mathbb{R}^2_+$ there is a unique solution $(x(t), y(t))$ with $t \in [0, \tau_e)$, where τ_e is an explosion time. We must show that $\mathbb{P}(\tau_e = \infty) = 1$. Indeed, let n_0 be large enough such that $1/n_0 \leq x(0), y(0) \leq n_0$. For each $n \geq n_0$ define the stopping times

$$\tau_n = \inf_{t\in[0,\tau_e]} \{x(t) \notin (1/n, n) \text{ or } y(t) \notin (1/n, n)\},$$

with $\inf \emptyset := \infty$. Clearly, τ_n is an increasing random time and we can define $\tau_\infty = \lim_{n\to+\infty} \tau_n \leq \tau_e$ almost surely. Note that for $0 \leq t \leq \tau_\infty$, the solution $(x(t), y(t))$ remains in $\mathbb{R}^2_+$.

We proceed by contradiction. If $\mathbb{P}(\tau_\infty < \infty) > 0$, then there are constants $T > 0$ and $\epsilon \in (0, 1)$ such that $\mathbb{P}(\tau_\infty \leq T) > \epsilon$. Therefore, there is $n_1 \geq n_0$ such that $\mathbb{P}(\tau_n \leq T) \geq \epsilon$ for all $n \geq n_1$. Define $V : \mathbb{R}^2_+ \to \mathbb{R}_+$ by

$$V(x, y) = ax - 1 - \ln(ax) + by - 1 - \ln(by),$$

where a and b are positive constants to be determined. Applying the operator in Eq. (2.10) to V gives

$$L[V] = ax - ax^2 - a\varphi(x)y - 1 + x + \frac{\varphi(x)}{x}y + b\beta\varphi(x)y - b\delta y - \beta\varphi(x) + \delta + \frac{\sigma_x^2}{2} + \frac{\sigma_y^2}{2}. \tag{2.11}$$

Itô's formula and the fact that $\varphi(x) \leq x$ for all x, allows us to write

$$\begin{aligned} \mathrm{d}V &= L[V]\,\mathrm{d}t + \sigma_x\,(ax - 1)\,x\,\mathrm{d}B_x(t) + \sigma_y\,(ay - 1)\,y\,\mathrm{d}B_y(t) \\ &\leq \left((a + 1)x - ax^2 + (b\beta - a)\varphi(x)y + (1 - \delta b)y + \delta + \frac{\sigma_x^2}{2} + \frac{\sigma_y^2}{2}\right)\mathrm{d}t \\ &\quad + \sigma_x\,(ax - 1)\,x\,\mathrm{d}B_x(t) + \sigma_y\,(ay - 1)\,y\,\mathrm{d}B_y(t). \end{aligned}$$

The function $p(x) = (a + 1)x - ax^2 + \delta + \sigma_x^2/2 + \sigma_y^2/2$ is bounded above; that is, there exists a constant M such that $p(x) < M$. Therefore, if we choose constants a and b such that $b > 1/\delta$ and $\dfrac{a}{b} > \beta$, we observe that $b\beta - a < 0$ and $1 - b\delta < 0$ and this guarantees that

$$\mathrm{d}V \leq M\,\mathrm{d}t + \sigma_x\,(ax - 1)\,y\,\mathrm{d}B_x(t) + \sigma_y\,(ay - 1)\,y\,\mathrm{d}B_y(t). \tag{2.12}$$

Integrating both sides of Eq. (2.12) between 0 and $\tau_n \wedge T$, and taking expected value, we obtain

$$\mathbb{E}\,V(x(\tau_n \wedge T), y(\tau_n \wedge T)) \leq V(x_0, y_0) + MT. \tag{2.13}$$

Finally, since V is a non-negative function, we can bound

$$\begin{aligned} \mathbb{E}\,V(x(\tau_n \wedge T), y(\tau_n \wedge T)) &\geq \mathbb{E}\,V(x(\tau_n \wedge T), y(\tau_n \wedge T)I(\tau_n \leq T)) \\ &= \mathbb{E}\,V(x(\tau_n), y(\tau_n)) \\ &\geq \min\left\{an - 1 - \log(an),\ a/n - 1 - \log(a/n)\right\} \end{aligned} \tag{2.14}$$

which contradicts Eq. (2.13). Therefore, $\mathbb{P}(\tau_\infty < \infty) = 1$ and the solution exists and remains in $\mathbb{R}^2_+$ for all $t \geq 0$. □

Another important feature of the stochastic Gause model is that it has solutions that are bounded in probability. Borrowing the techniques in [24], and comparing our model to a mutualistic system, we establish boundedness in the following general way.

Theorem 2. *Let $\theta = (\theta_1, \theta_2)$ be a vector of positive numbers such that $\theta_1 + \theta_2 < \frac{1}{2}$. Then, under the hypotheses of Theorem 1, the solution $(x(t), y(t))$ to the stochastic Gause model in Eq. (2.9) satisfies*

$$\log\left(\mathbb{E}[x(t)^{\theta_1} y(t)^{\theta_2}]\right) \leq e^{-c_1 t}(\theta_1 \log x(0) + \theta_2 \log y(0)) + \frac{c_2}{c_1}\,(1 - e^{-c_1 t}) \tag{2.15}$$

for all $t \geq 0$*, where the positive constants* c_1, c_2 *are given by*

$$c_1 = \frac{1}{4}(1 - \theta_1 - \theta_2)\min\{\theta_1\sigma_x^2, \theta_2\sigma_y^2\},$$
$$c_2 = |\theta|\sqrt{1+\delta^2} + \frac{|\theta|^2(2+\beta^2+\sqrt{4+\beta^4})}{8c_1}.$$

Proof. Consider the stochastic system

$$\mathrm{d}\tilde{x} = \tilde{x}(1-\tilde{x})\,\mathrm{d}t + \sigma_x\tilde{x}\,\mathrm{d}B_x, \quad \mathrm{d}\tilde{y} = (\beta\tilde{x}\tilde{y} - \delta\tilde{y})\,\mathrm{d}t + \sigma_y\tilde{y}\,\mathrm{d}B_y.$$

The model in Eq. (2.16) represents a mutualistic relationship where species $\tilde{y}$ benefits from the presence of $\tilde{x}$ without affecting $\tilde{x}$. This model is studied in [17], and from theorem 3.1 of that work, Eq. (2.15) follows for the process $(\tilde{x}, \tilde{y})$.

By the comparison theorem, if $x(0) \leq \tilde{x}(0)$ and $y(0) \leq \tilde{y}(0)$, then $x(t) \leq \tilde{x}(t)$ and $y(t) \leq \tilde{y}(t)$ almost surely for all $t \geq 0$. Since $\theta_1, \theta_2 > 0$ and $\theta_1 + \theta_2 < \frac{1}{2}$, then $x(t)^{\theta_1}y(t)^{\theta_2} \leq \tilde{x}(t)^{\theta_1}\tilde{y}(t)^{\theta_2}$ and Eq. (2.15) follows for the process (x, y).

□

Note that by letting $t \to \infty$ in Eq. (2.15), we obtain the asymptotic mean bound

$$\limsup_{t\to\infty} \mathbb{E}(x(t)^{\theta_1}y(t)^{\theta_2}) \leq e^{c_2/c_1}. \tag{2.16}$$

Moreover, due to the Markov inequality for all $\epsilon > 0$, there exists $M > 0$ such that

$$\limsup_{t\to\infty} \mathbb{P}\left(x(t)^{\theta_1}y(t)^{\theta_2} > M\right) \leq \epsilon. \tag{2.17}$$

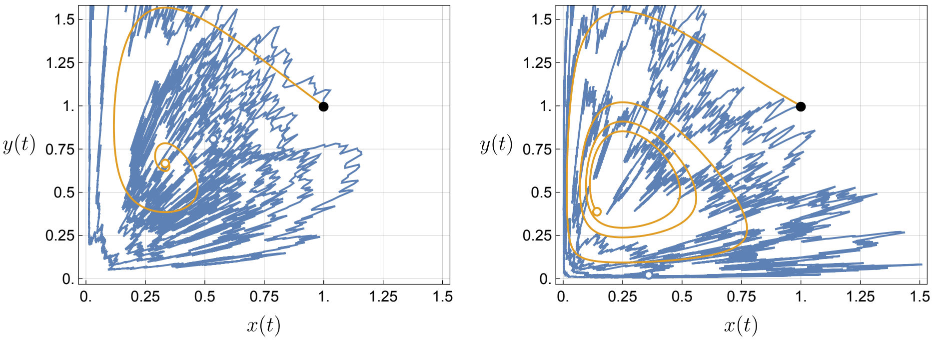
In Figure 1 we present examples of solution paths to Eq. (2.9) under various functional responses and compare them with the solutions to the deterministic system.

3. Noise-induced extinctions

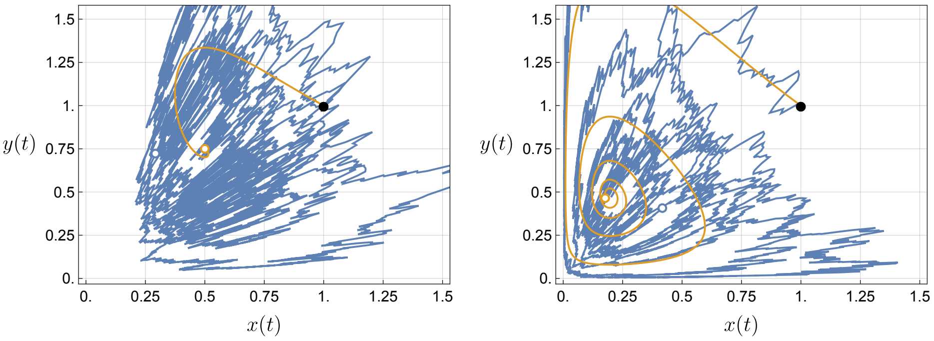
We now turn our attention to extinction events, namely, conditions under which the population of one or both species approaches zero. Note from the conditions in Eq. (2.7) that if $(x(0), y(0)) \in \mathbb{R}_+^2$, no extinction can occur in finite time in the deterministic model Eq. (2.6). It follows from Theorem 1 that finite-time extinction is also precluded in the stochastic Gause model. We thus focus on the events of eventual extinction, namely when the population of one or both species approaches zero as $t \to \infty$. We denote these eventual extinction events as

$$\mathcal{E}_x = \left[\lim_{t\to\infty} x(t) = 0\right], \quad \mathcal{E}_y = \left[\lim_{t\to\infty} y(t) = 0\right]. \tag{3.1}$$

It follows from Eq. (2.7) that in the deterministic model, the only possibility for eventual extinction corresponds to the case where $(1, 0)$ is asymptotically stable, and the predator $y(t) \to 0$ as $t \to \infty$. This occurs when the mortality rate δ is higher than $\beta/\varphi(1)$. The prey x in the deterministic model always persists. In contrast, we will show that the presence of noise in the stochastic model can drive either species to eventual extinction. First, we show that as is natural to expect, if the prey becomes extinct eventually, so will the predators.



(a) Type I. **(b)** Type II, $\alpha = 0.5$.



(c) Type III, $\alpha = 0.5$. **(d)** Type IV, $\alpha = 0.25$, $\gamma = 1.5$.

Figure 1. Examples comparing the solution to the deterministic Gause model in Eq. (2.6) with a realization of the solution to the stochastic counterpart in Eq. (2.9) for each functional response in Table 1. Black markers show the initial conditions and blank markers show the value of the paths at the final time. Every path has $t \in [0, 100]$, $(x(0), y(0)) = (1, 1)$, $\beta = 1.5$, $\delta = 0.5$, $\sigma_x^2 = \sigma_y^2 = 0.1$. In every case, $\beta/\delta = 3 > \frac{1}{\varphi(1)}$ guaranteeing coexistence in the deterministic model. This value of β/δ corresponds to a situation in which the maximum predator effective growth rate Fb from consumption of the prey, is three times that of its intrinsic mortality rate m.

Proposition 1. *Let* $(x(t), y(t))$*,* $t \geq 0$ *be the solution to model Eq.* (2.9) *under the hypotheses of Theorem 1. Then,* $\mathbb{P}(\mathcal{E}_y | \mathcal{E}_x) = 1$.

Proof. Let $\epsilon \in (0, \frac{\delta}{\beta})$ be fixed. There exists a strictly positive random variable T, finite on $\mathcal{E}_x$, such that $0 < x(t) < \epsilon$ for all $t \geq T$ on $\mathcal{E}_x$. Since $\varphi(x) \leq x$ for all $x \geq 0$, then the equation for $\mathrm{d}y$ in Eq. (2.9) can be bounded on $\mathcal{E}_x$ as

$$\mathrm{d}y \leq (\beta\epsilon - \delta) y \, \mathrm{d}t + \sigma_y y \, \mathrm{d}B_y(t) \tag{3.2}$$

for all $t \geq T$. Further, it follows from the positivity of $y(t)$ that

$$\ln(y(t)) - \ln(y(T)) \leq (\beta\epsilon - \delta)t + \sigma_y [B_y(t) - B_y(T)] \tag{3.3}$$

with probability one conditionally on $\mathcal{E}_x$. Finally, as a consequence of the law of the iterated logarithm and the fact $(\beta\epsilon - \delta) < 0$, we obtain that $\limsup_{t \to \infty} \frac{1}{t} \log(y(t)) < 0$ and therefore $\mathcal{E}_y$ has probability one conditionally on $\mathcal{E}_x$. □

Eventual extinctions can be caused by a sufficiently large noise, even if the values of the model parameters allow for coexistence under deterministic dynamics. We consider first the effect that large noise has on the prey x. The next result shows that a sufficiently large value of the variability σ_x on the prey dynamics causes its eventual extinction, regardless of the value or the parameters governing the predator. This phenomenon is shared by the one-dimensional logistic model equation and extends to this case because the predator's role in the dynamics of the x is to increase mortality. See [4, 17]. The proof of Theorem 3 makes this connection explicit.

Theorem 3. *Let* $(x(t), y(t))$*,* $t \geq 0$ *be the solution to model Eq.* (2.9) *under the hypotheses of Theorem 1. If* $\sigma_x^2 > 2$*, then* $\mathbb{P}(\mathcal{E}_x) = 1$.

Proof. Let $\hat{x}$ be the solution to $\mathrm{d}\hat{x} = \hat{x} \, \mathrm{d}t + \sigma_x \hat{x} \, \mathrm{d}B_x$. Namely,

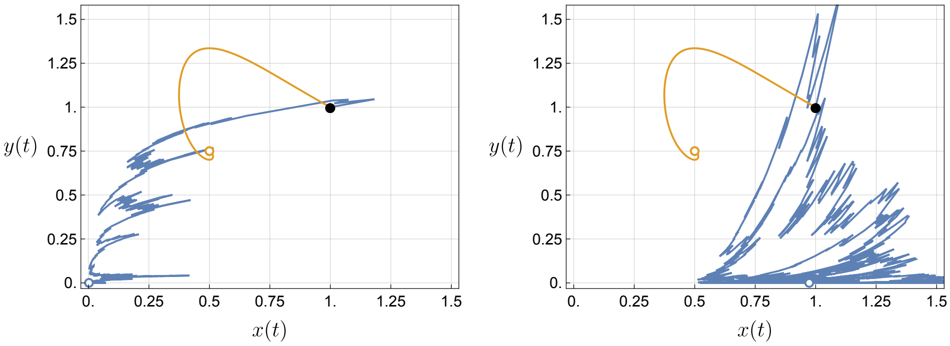
$$\hat{x}(t) = \hat{x}(0) \exp\left(\left(1 - \frac{\sigma_x^2}{2}\right)t + \sigma_x B_x(t)\right), \quad t \geq 0.$$

If $\sigma_x^2 > 2$, the logarithm iterated theorem implies $\hat{x}(t) \to 0$ as $t \to \infty$ almost surely. Since $\varphi(x) \leq x$, the comparison theorem (see [24]) implies that $x(t) \leq \hat{x}(t)$ for all $t \geq 0$ with probability one. Therefore $x(t) \to 0$ as $t \to \infty$ almost surely. □

It follows from Proposition 1 and Theorem 3 that if $\sigma_x^2 > 2$, both the predator and prey are driven to eventual extinction with probability one. Namely, the solution $(0, 0)$ to Eq. (2.9) is stochastically asymptotically stable. See [6]. Sufficiently large noise on the prey can therefore make certain the scenario of complete eventual extinction, which is always an impossibility in the deterministic model.

For functional responses of types II-IV where φ is bounded, the next result shows that noise alone can cause the extinction of the predator.

Theorem 4. *Suppose all parameters in Eq.* (2.9) *are positive and that the functional response satisfies* $\varphi(x) \leq 1$ *for all* $x \geq 0$*. Let* $(x(t), y(t))$*,* $t \geq 0$ *be the solution to model Eq.* (2.9) *for* $(x(0), y(0)) \in \mathbb{R}_+^2$*. If* $\sigma_y^2 > 2(\beta - \delta)$*, then* $\mathbb{P}(\mathcal{E}_y) = 1$.



(a) $\sigma_x^2 = 2.1, \sigma_y^2 = 0.1$.

(b) $\sigma_x^2 = 0.1, \sigma_y^2 = 2.1$.

Figure 2. Examples illustrating Proposition 1, Theorem 3 and Theorem 4 for solutions with functional type II. Every path has $t \in [0, 100]$, $(x(0), y(0)) = (1, 1)$, $\beta = 1.5$, $\delta = 0.5$, $\alpha = 0.5$. Black markers show the initial conditions and blank markers show the value of the paths at the final time. In (a) the conditions of Theorem 3 for $\mathbb{P}(\mathcal{E}_x) = 1$ are satisfied, and hence by Proposition 1, y also becomes extinct. In (b) only the conditions for the eventual extinction of the predator y are satisfied.

The proof of Theorem 4 follows exactly the same arguments as that of Theorem 3, using the comparison theorem with respect to $\mathrm{d}\hat{y} = (\beta - \delta)\hat{y}\,\mathrm{d}t + \sigma_y \hat{y}\,\mathrm{d}B_y(t)$. Note that, since σ_y is assumed positive, Theorem 4 implies that if $\beta < \delta$, any amount of noise will drive the predator y to eventual extinction when the functional response in bounded. This is not unexpected because under those conditions, $\beta < \delta$ implies $\beta < \beta^*$ in Eq. (2.7), which ensures predator extinction in the deterministic model.

As illustrated in Figure 2b, the stochastic Gause model allows for situations where the predator y eventually becomes extinct but the prey x persists. We now turn our attention to the long-term behavior of the prey in those cases. We prove, in fact, that as $t \to \infty$ and $y(t) \to 0$, the probability distribution of the prey $x(t)$ can converge to a distribution supported on $(0, \infty)$. Moreover, this limiting distribution equals precisely the invariant distribution of the one-dimensional stochastic logistic model.

Theorem 5. *Let $(x(t), y(t))$, $t \geq 0$ be the solution to the model Eq. (2.9) under the hypotheses of Theorem 1 and assume $\sigma_x^2 < 2$. Then, for any $y(0) \in (0, \infty)$, and conditional on $\mathcal{E}_y$, the prey process $x(t)$ converges in distribution to a random variable $\tilde{X}$ with a Gamma distribution $\Gamma(\frac{2}{\sigma_x^2} - 1, \frac{\sigma_x^2}{2})$.*

Proof. Let $\epsilon \in (0, 1 - \frac{1}{2}\sigma_x^2)$ and consider the following systems of stochastic differential equations:

$$\begin{aligned} \mathrm{d}\tilde{x} &= (\tilde{x}(1 - \tilde{x}))\,\mathrm{d}t + \sigma_x \tilde{x}\ \mathrm{d}B_x(t), \\ \mathrm{d}\tilde{y} &= (\beta\tilde{x}\tilde{y} - \delta\tilde{y})\,\mathrm{d}t + \sigma_y\,\tilde{y}\,\mathrm{d}B_y(t), \end{aligned} \tag{3.4}$$

$$\begin{aligned} \mathrm{d}\hat{x} &= (\hat{x}(1 - \hat{x}) - \varphi(\hat{x})\epsilon)\,\mathrm{d}t + \sigma_x \hat{x}\ \mathrm{d}B_x(t), \\ \mathrm{d}\hat{y} &= (\beta\hat{x}\hat{y} - \delta\hat{y})\,\mathrm{d}t + \sigma_y\,\hat{y}\,\mathrm{d}B_y(t), \end{aligned} \tag{3.5}$$

where B_x and B_y are the same Brownian motions in Eq. (2.9). Note that the equations for $\tilde{x}$ and $\hat{x}$ do not depend on their corresponding of $\tilde{y}$ and $\hat{y}$. In fact, the equation for $\tilde{x}$ is that of the one-dimensional stochastic logistic model.

Let $\{(\tilde{x}(t), \tilde{y}(t)), t \geq 0\}$ be the solution to Eq. (3.4) with $(\tilde{x}(0), \tilde{y}(0)) = (x(0), y(0))$. Since $\sigma_x^2 < 2$, it is known that $\tilde{x}(t) \to \tilde{X}$ with probability one as $t \to \infty$, where $\tilde{X}$ has the gamma distribution claimed in the theorem (see [25] page 171). Moreover, since $\varphi(x) \geq 0$ for all x, the comparison theorem of [24] implies that $x(t) \leq \tilde{x}(t)$ for all $t \geq 0$. It follows that $\limsup_{t\to\infty} x(t) \leq \tilde{X}$ with probability one.

To establish the behavior of $\liminf x(t)$, fix $T > 0$ and define the event $A_{\epsilon,T} = [\sup_{t\geq T} y(t) \leq \epsilon]$. Note the dependence of Eq. (3.5) on ϵ and let $\{(\hat{x}_{\epsilon,T}(t), \hat{y}_{\epsilon,T}(t)), t \geq T\}$ be its solution starting at $(\tilde{x}_{\epsilon,T}(T), \tilde{y}_{\epsilon,T}(T)) = (x(T), y(T))$. Applying the strong Markov property and the comparison theorem on $A_{\epsilon,T}$, gives that $\mathbb{P}(\hat{x}_{\epsilon,T}(t) \leq x(t) \text{ for all } t \geq T | A_{\epsilon,T}) = 1$. Similarly, since $0 < \sigma_x^2 < 2(1-\epsilon)$ the process $\hat{x}_{\epsilon,T}(t)$ converges almost surely as $t \to \infty$ to a random variable $\hat{X}_\epsilon$ supported on $(0,\infty)$. Furthermore,

$$\mathbb{P}\left(\hat{X}_\epsilon \leq \liminf_{t\to\infty} x(t) \leq \limsup_{t\to\infty} x(t) \leq \tilde{X} \middle| A_{\epsilon,T}\right) = 1, \tag{3.6}$$

Letting $T \to \infty$ in Eq. (3.6) we obtain

$$\mathbb{P}\left(\hat{X}_\epsilon \leq \liminf_{t\to\infty} x(t) \leq \limsup_{t\to\infty} x(t) \leq \tilde{X} \middle| \limsup_{t\to\infty} y(t) \leq \epsilon\right) = 1, \tag{3.7}$$

As $\epsilon \to 0$, $\hat{X}_\epsilon$ converges in distribution to $\tilde{X}$ and $[\limsup_{t\to\infty} y(t) \leq \epsilon] \downarrow \mathcal{E}_y$. This implies that conditional on $\mathcal{E}_y$, $\lim_{t\to\infty} x(t)$ exists with probability one and has the same distribution as $\tilde{X}$. □

Theorem 5 indicates that under the extinction of the predator, if the prey's noise is small enough to allow for its persistence, the prey will behave precisely like in the situation modeled by the one-dimensional stochastic logistic equation. In particular, for large t, $x(t)$ exhibits a noise-induced reduction of its carrying capacity. See [25]. Namely, the asymptotic behavior of the prey mean abundance goes from $x(t) \to 1$ in the deterministic case to

$$\lim_{t\to\infty} \mathbb{E}(x(t)|\mathcal{E}_y) = 1 - \frac{\sigma_x^2}{2}, \tag{3.8}$$

in the stochastic case.

The following corollary gives a reformulation of the results of this section in terms of the dimensional parameters of the original model Eq. (1.1).

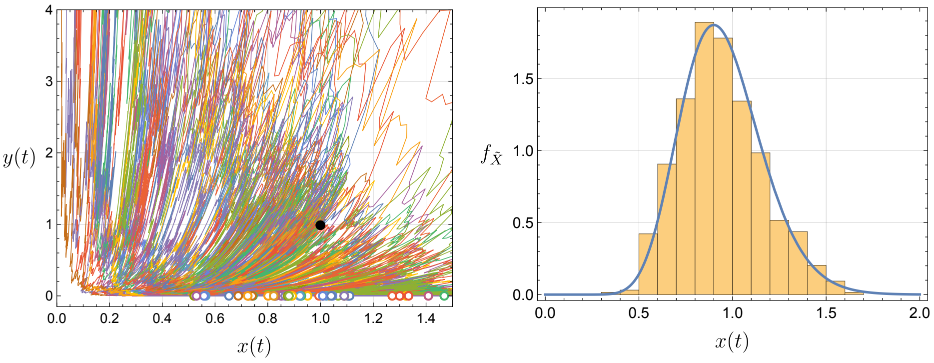
Corollary 1. *Assume the hypotheses of Theorem 1. If $s_x^2 > 2r$, then the prey and predator populations will eventually become extinct. Furthermore, if the functional response f is bounded and $s_y^2 > 2(Fb - m)$, then the predator population will eventually become extinct. In the case of $s_x^2 < 2r$, eventual extinction of the predator implies that the prey population converges to a gamma random variable with mean $K(1 - \frac{s_x^2}{r})$ and variance $\frac{K^2 s_x^2}{4r^2}(s_x^2 - 2r)$.*

4. Invariant distribution for type I functional response

We now specialize in the case of type I functional response, namely $\varphi(x) = x$ in Eq. (2.9). Note from Eq. (2.7) that for the corresponding deterministic case, the point

$$x_* = \frac{\delta}{\beta}, \quad y_* = 1 - \frac{\delta}{\beta}, \tag{4.1}$$

is a coexistence equilibrium solution that is positive and asymptotically stable if $\delta < \beta$. We will provide conditions on model parameters for the stochastic model Eq. (2.9) to have a unique invariant distribution supported on $\mathbb{R}_+^2$. This corresponds to a situation of stochastic coexistence of prey and predator, and precludes all the eventual extinction scenarios covered in Section 3.



(a) **(b)**

Figure 3. Illustration of Theorem 5 for the same setting as Figure 2b: $\sigma_x^2 < 1$ and $\sigma_y^2 > 2(\beta - \delta)$ so that persistence of prey is possible but extinction of prey is sure by Theorem 4. (a) 50 realizations of the solution for $t \in [0, 1000]$. The final values of $x((t), y(t))$ are shown with open markers. (b) Comparison between the sample histogram of the final value of $x(t)$ for 640 realizations, with the density $f_{\tilde{X}}$ of the $\Gamma(\frac{2}{\sigma_x^2} - 1, \frac{\sigma_x^2}{2})$ distribution.

Theorem 6. *Consider the Gause model* Eq. (2.9) *with* $\varphi(x) = x$, $(x(0), y(0)) \in \mathbb{R}_+^2$ *and parameters* β, δ *such that*

$$0 < \frac{\beta}{\beta + 1} < \delta < \min\left\{\beta, \frac{\beta}{|\beta - 1|}\right\}. \tag{4.2}$$

If σ_x^2, σ_y^2 *satisfy that*

$$\frac{\beta^2(\beta + 4\delta)}{\delta((\beta - 1)\delta + \beta)}\sigma_x^2 + \frac{(\beta(\beta + 2) - 2\delta)(\beta - \delta)}{\delta^2((\beta - 1)\delta + \beta)}\sigma_y^2 < 1, \tag{4.3}$$

$$\frac{\beta^2\delta(\beta + 2\delta)}{(\beta - \delta)^2(\beta(1 - \delta) + \delta)}\sigma_x^2 + \frac{(\beta(\beta + 4) - 4\delta)}{(\beta - \delta)(\beta(1 - \delta) + \delta)}\sigma_y^2 < 1, \tag{4.4}$$

then the model has a unique stationary distribution supported on $\mathbb{R}_+^2$.

Remark 1. *In the particular case* $\beta = 1$, *the condition* (4.2) *reduces to*

$$\frac{1}{2} < \delta < 1.$$

Proof. We establish the existence of an invariant method by the method of Lyapunov functions. See [5] for a good introduction. Let $V(x, y) = V_1(x, y) + V_2(x, y)$ with

$$V_1(x, y) = x - x_* - x_* \log\left(\frac{x}{x_*}\right) + \frac{1}{\beta}\left[y_* - y_* - \frac{1}{\beta}\log\left(\frac{y}{y_*}\right)\right],$$

$$V_2(x, y) = \frac{1}{2}\left[(x - x_*) + \frac{1}{\beta}(y - y_*)\right]^2.$$

Let L be the Lyapunov operator in Eq. (2.11). Then,

$$L[V_1](x, y) = -(x - x_*)^2 + \frac{1}{2}\sigma_x^2 x_* + \frac{1}{2\beta}\sigma_y^2 y_*,$$

$$\begin{aligned}L[V_2](x,y) &= (\sigma_x^2 + y_*)(x-x_*)^2 - \left(x_* - \frac{y_*}{\beta}\right)(x-x_*)(y-y_*) \\ &\quad - \left(\frac{x_*}{\beta} - \frac{\sigma_y^2}{\beta^2}\right)(y-y_*)^2 + \sigma_x^2 x_*^2 + \frac{\sigma_y^2}{\beta^2} y_*^2.\end{aligned}$$

The conditions for β, δ, σ_x, and σ_y in Eqs. (4.2) to (4.4) can be equivalently written in terms of conditions on x_* and y_* as

$$x_* - \frac{y_*}{\beta} > 0, \quad \ell < \min\{\ell_1 x_*^2, \ell_2 y_*^2\}, \tag{4.5}$$

where

$$\ell = \sigma_x^2 x_* \left(x_* + \frac{1}{2}\right) + \frac{\sigma_y^2 y_*}{\beta^2}\left(y_* + \frac{\beta}{2}\right), \tag{4.6}$$

$$\ell_1 = 1 - \sigma_x^2 - y_* - \frac{1}{2}\left(x_* - \frac{y_*}{\beta}\right), \tag{4.7}$$

$$\ell_2 = \frac{x_*}{\beta} - \frac{\sigma_y^2}{\beta^2} - \frac{1}{2}\left(x_* - \frac{y_*}{\beta}\right). \tag{4.8}$$

We first bound $L[V_2]$. Note that for any $s \geq 0$, $w, c \in \mathbb{R}$,

$$wx^2 - sxy - cy^2 < \left(w + \frac{s}{2}\right)x^2 - \left(c - \frac{s}{2}\right)y^2. \tag{4.9}$$

Making $w = \sigma_x^2 + y_*$, $c = \dfrac{x_*}{\beta} - \dfrac{\sigma_y^2}{\beta^2}$, and $s = x_* - \dfrac{y_*}{\beta}$, which is positive by Eq. (4.5), we obtain

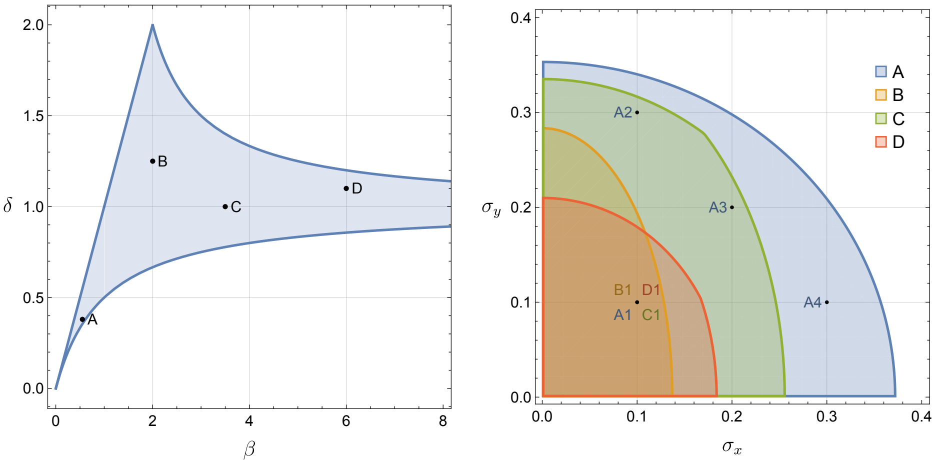
$$\begin{aligned}L[V_2] &\leq \left(\sigma_x^2 + y_* - \frac{y_*}{2\beta} + \frac{x_*}{2}\right)(x-x_*)^2 - \left(\frac{x_*}{\beta} - \frac{\sigma_y^2}{\beta^2} + \frac{y_*}{2\beta} - \frac{x_*}{2}\right)(y-y_*)^2 \\ &\quad + \sigma_x^2 x_*^2 + \frac{\sigma_y^2}{\beta^2} y_*^2.\end{aligned}$$

Adding $L[V_1]$ to this bound for $L[V_2]$ gives

$$L[V](x,y) \leq -\ell_1 (x-x_*)^2 - \ell_2 (y-y_*)^2 + \ell.$$

Note that Eq. (4.5) implies that the ellipsoid $\ell_1(x-x_*)^2 + \ell_2(y-y_*)^2 \leq \ell$ is entirely contained in $\mathbb{R}_+^2$. We can then choose a neighborhood U of such ellipsoid, and $\epsilon > 0$, such that $\bar{U} \in \mathbb{R}_+^2$, and $L[V](x,y) \leq -\epsilon$ for all $(x,y) \in \mathbb{R}_+^2 \setminus U$. Furthermore, the smallest eigenvalue of the diffusion matrix is $\min\{\sigma_x^2 x^2, \sigma_y^2 y^2\}$, which is bounded away from zero in U. The existence of the invariant distribution thus follows from Theorem 4.1 in [5]. □

Note that the condition in Eq. (4.2) for β and δ ensures that the coefficients of σ_x^2, σ_y^2 in Eqs. (4.3) and (4.4) are all positive. Therefore, for each admissible pair (β, δ), the set of values (σ_x, σ_y) for which we can guarantee the existence of an invariant distribution is given by the intersection of two ellipses. This is illustrated in Figure 4.



(a) **(b)**

Figure 4. Illustration of the conditions of Theorem 6 for existence of an invariant distribution. (a) Region in the β-δ plane determined by Eq. (4.2). The labeled points are: A $= (0.55, 0.38)$, B $= (2, 1.25)$, C $= (3.5, 1)$ and D $= (6, 1.1)$, which correspond to the cases illustrated in Figure 6 (b) Regions in the σ_x-σ_y plane determined by the ergodicity conditions Eqs. (4.3) and (4.4) for each of the four points labeled in panel (a).

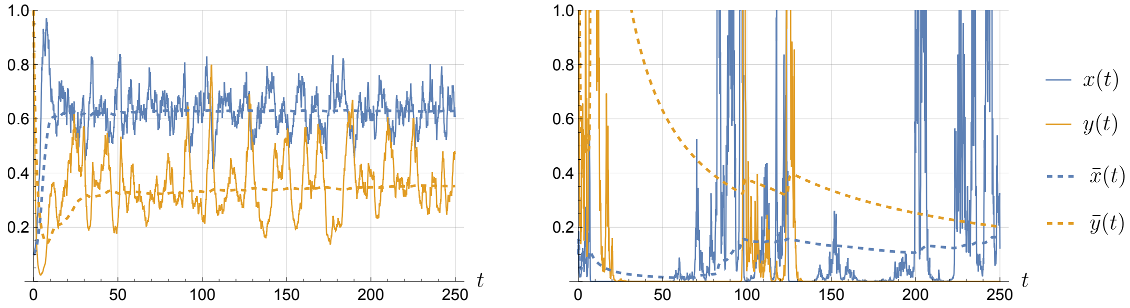
Under the conditions of Theorem 6, the solution $(x(t), y(t))$ to Eq. (2.9) is an ergodic stochastic process. Both the predator and the prey persist with probability one and their populations oscillate around their invariant mean. See Figure 5. Note that in Figure 5b the conditions for eventual extinction are not met either and both species seem to persist while having highly intermittent paths. Theorem 6 does not preclude the existence of an invariant distribution in this case, but our numerical computations did not provide evidence of convergence.

The dependence of the invariant distribution on the values of the model parameters is illustrated next. Figure 6 shows the dependence of the histogram of $(x(t), y(t))$ for large t on the parameters β, δ for fixed but small values of the noise intensities σ_x, σ_y. Note that the invariant distribution approximately concentrates around the equilibrium solution (x_*, y_*) of the deterministic model, which is stable because $\delta < \beta$. It is apparent that the covariance matrix of the invariant distribution is anisotropic and depends strongly on the values of β and δ. Figure 7 shows the sensibility of the invariant distribution to the values of σ_x, σ_y for a fixed pair of (β, δ). Consistently with the observation in Eq. (3.8), the difference between the invariant mean and the equilibrium (x_*, y_*) becomes significant for larger values of the noise intensity.

The following corollary states the conditions for the existence of an invariant distribution in terms of the dimensional parameters of the original model Eq. (1.1).

Corollary 2. *Suppose that the functional response is linear, $f(x) = cx$ and that the hypotheses of Theorem 1 hold. The corresponding deterministic Gause in Eq.* (2.1) *has a stable equilibrium solution* $(x_*, y_*) = (\frac{m}{bck}, rc(1 - \frac{m}{bck}))$. *If the following condition holds on the population parameters,*

$$\frac{bcK}{bcK + r} < \frac{m}{r} < \frac{bcK}{r\left|\frac{bcK}{r} - 1\right|}, \tag{4.10}$$



(a) $\beta = 2$, $\delta = 1.25$, $\sigma_x = \sigma_y = 0.1$. **(b)** $\beta = 4$, $\delta = 0.1$, $\sigma_x = \sigma_y = 1.2$.

Figure 5. Examples of solutions to Eq. (2.9) that (a) satisfy the conditions in Theorem 6 for existence of an invariant distribution, and (b) that do not. Dashed curves denote the Cesàro means, namely $\bar{x}(t) := \frac{1}{t}\int_0^t x(s)\,\mathrm{d}s$ and similarly for $\bar{y}$, which converge almost surely as $t \to \infty$ whenever a unique invariant distribution exists. Note the apparent lack of convergence in panel (b). In both cases the functional response is of Type I, the initial condition is $(x(0), y(0)) = (0.1, 1)$ and the solutions are computed for $t \in [0, 250]$.

and the noise intensities s_x, s_y satisfy

$$\frac{r(bcK-m)\left(b^2c^2K^2+2bcKr-2mr\right)s_y^2+b^2c^2K^2m(bcK+4m)s_x^2}{m^2r(bcK(m+r)-mr)} < 1, \tag{4.11}$$

$$\frac{r(bcK-m)\left(b^2c^2K^2+4bcKr-4mr\right)s_y^2+b^2c^2K^2m(bcK+2m)s_x^2}{r(m-bcK)^2(mr-bcK(m-r))} < 1 \tag{4.12}$$

then the model Eq. (1.1) *has a unique invariant distribution.*

5. Conclusions and discussion

We have conducted a comprehensive analysis of a stochastic version of the Gause population model with multiplicative noise. The model considers a prey population x subject to logistic growth and predation by a predator y, whose functional response can be very general. The specific form of the diffusion term used can be interpreted as modeling the effect of uncertain population parameters with Gaussian errors. In Section 2.1, we show that the resulting model in Eq. (2.9) is 'natural' in the sense that, with probability one, the populations remain bounded and non-negative for all time.

Our results highlight the effect that uncertainty can have on the qualitative properties of predator-prey systems. Perhaps the most intriguing effect is the possibility of noise-induced extinctions, as discussed in Section 3. We demonstrate in Theorem 3 and Proposition 1 that sufficiently large noise in the prey's dynamics x can drive both species to eventual extinction, a scenario impossible in the deterministic model. The sufficient threshold $\sigma_x^2 > 2$ for this total collapse was obtained by an elementary comparison with a one-dimensional stochastic Malthus model, but could have been equally obtained by comparison with the stochastic logistic model in Eq. (3.4).

A more nuanced issue is the possibility of eventual extinctions induced by the predator's noise intensity σ_y. Our Theorem 4 indicates that if $\beta > \delta$, there will always be a level of noise that drives

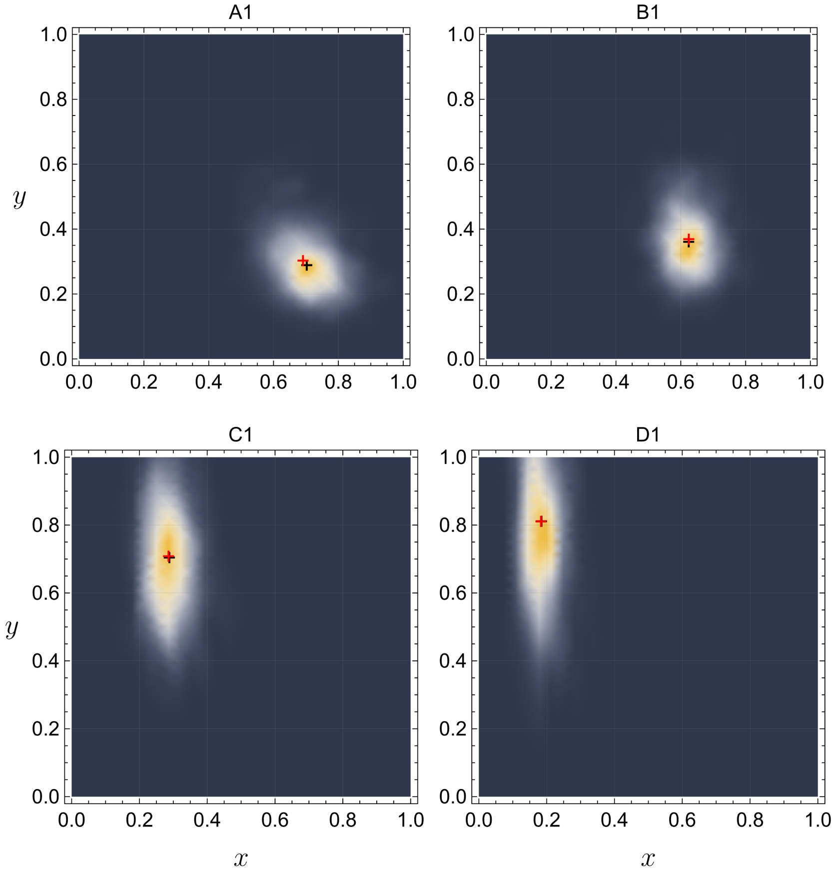


Figure 6. Smooth density histogram estimated from 1000 sample realizations of $(x(t), y(t))$ with $t = 2000$. The initial condition was $(x(0), y(0)) = (1, 1)$, and the parameters where $\sigma_x^2 = \sigma_y^2 = 0.1$ with four cases of (β, δ) corresponding to the labels in Figure 4b. The red markers indicate the equilibrium (x_*, y_*) of the deterministic system, and the black shows the estimated mean of $(x(t), y(t))$ at the final time.

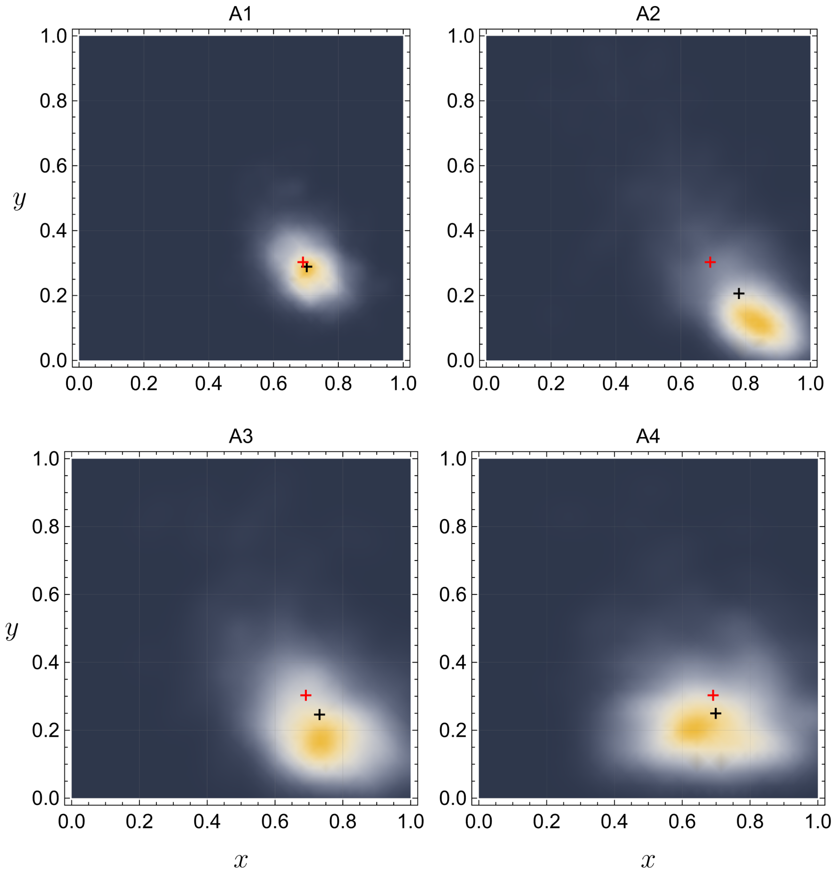


Figure 7. Smooth density histogram estimated from 1000 sample realizations of $(x(t), y(t))$ with $t = 2000$. The initial condition was $(x(0), y(0)) = (1, 1)$. The parameters where $(\beta, \delta) =$ A in Figure 4a with the four different values of (σ_x, σ_y) as labeled in Figure 4b. The red markers indicate the equilibrium (x_*, y_*) of the deterministic system, and the black shows the estimated mean of $(x(t), y(t))$ at the final time.

x to eventual extinction, regardless of the type of coexistence (periodic or otherwise) exhibited in the deterministic case. Moreover, for type II and III responses, if $\beta > \beta^* = (1+\alpha)\delta$ and α is very small, even very small values of σ_y^2 are sufficient to drive the predator to extinction. The exclusion of the unbounded functional response of type I from Theorem 4 stems from a technical issue when using the comparison theorem with respect to a univariate process. Our numerical simulations lead us to conjecture that the result holds in this case as well, although different proof techniques would be necessary.

Our results of noise-induced extinctions for the Gause model echo those obtained for other ecological models using various analytical techniques (see for example the seminal works of [16, 26] or more recent reviews in [27, 28]). As in most previous studies, our conditions on the noise intensity for extinction are only sufficient and most likely not tight enough to be directly applicable to real ecosystems. However, they show that caution should be exercised when assigning diffusion coefficients in stochastic predator-prey models. Moreover, if Holling type functional responses are applicable, we can deduce that demographic or environmental variability can drive entire ecosystems to collapse as has been observed in various real cases, as reviewed in [29]. From an ecological management perspective, these results underscore the importance of buffering populations from excessive environmental variability by preserving or restoring habitat, for example.

Theorem 5 provides a conditional result in the case of an eventual extinction of the predator, whether this extinction is predicted by Theorem 4 or not. It shows that, in the limit, the prey population behaves as in a single-species system, retaining no information from its interaction with the now-extinct predator. Furthermore, according to Eq. (3.8), the limiting mean prey population is always lower than the carrying capacity $y = 1$ in the deterministic system, with uncertainty consistently having a detrimental effect. This noise-induced reduction in the limiting mean appears to extend to the case of coexistence, as illustrated in Figures 6 and 7.

Our results offer an interesting contrasting picture with the results obtained in [22] where the predator dynamics include logistic growth. Namely, the equation for y is instead $\mathrm{d}y = (\delta y(1-y)+\beta\varphi(x)y)\,\mathrm{d}t + y\sigma_y\,\mathrm{d}B_y$ for $\delta, \beta > 0$. In this case, the predator does not require predation to survive, which implies that its persistence is possible even if the prey becomes extinct, an impossibility in our case as per Proposition 1. Further, its population is regulated not only by predation but also by its own intraspecific competition. It follows, in particular, that boundedness of $\varphi(x(t))/x(t)$ is automatically satisfied for all t with probability one.

The conditions provided in Section 3 for noise-induced extinctions are only sufficient, meaning we cannot offer safe ranges for noise intensities that would guarantee coexistence. An exception lies in the cases explored in Section 4, where we present conditions for the existence of an invariant distribution, ensuring coexistence for all time with probability one. This result is however limited to the case of a functional response of Holling type I. Moreover, we do not claim that our conditions for the existence of an invariant distribution are sharp, and note that the critical values for σ_x and σ_y for extinction according to Theorem 3 and Theorem 4 are well separated from the region defined by Eqs. (4.3) and (4.4), where ergodicity is guaranteed. Our numerical simulations suggest that extinctions occur at noise intensities lower than those identified here, and that an ergodic solution is achieved under much less restrictive conditions than those presented in Theorem 6. This state of affairs stems from the difficulty of finding the right Lyapunov functions that ensure the application of available ergodic theory and leave plenty of room for future research.

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Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there is no conflict of interest.

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