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Research article

Studies on invariant measures of fractional stochastic delay Ginzburg-Landau equations on $\mathbb{R}^n$

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Abstract: This paper is concerned with invariant measures of fractional stochastic delay Ginzburg-Landau equations on the entire space $\mathbb{R}^n$. We first derive the uniform estimates and the mean-square uniform smallness of the tails of solutions in corresponding space. Then we deduce the weak compactness of a set of probability distributions of the solutions applying the Ascoli-Arzelà. We finally prove the existence of invariant measures by applying Krylov-Bogolyubov's method.

Keywords: stochastic delay equation; unbounded domain; fractional Ginzburg-Landau equation; invariant measure

1. Introduction

The nonlinear Ginzburg-Landau equation plays an important role in the studies of physics, which describes many interesting phenomena and has been studied extensively (see [1] for a more detailed description). The fractional Ginzburg-Landau equation [2–4] is employed to describe processes in media with fractional dispersion or long-range interaction. It becomes very popular because the fractional derivative and fractional integral have broad applications in different fields of science [5–10].

Our work focuses on the existence of invariant measures of the autonomous fractional stochastic delay Ginzburg-Landau equations on $\mathbb{R}^n$:

$$du(t) + (1 + i\nu)(-\Delta)^\alpha u(t)dt + (1 + i\mu)|u(t)|^{2\beta} u(t)dt + \lambda u(t)dt = G(x, u(t - \rho))dt,$$

$$+ \sum_{k=1}^{\infty} (\sigma_{1,k}(x) + \kappa(x)\sigma_{2,k}(u(t)))\, dW_k(t),\ t > 0, \tag{1.1}$$

with initial condition

$$u(s) = \varphi(s), \quad s \in [-\rho, 0], \tag{1.2}$$

where $u(x,t)$ is a complex-valued function on $\mathbb{R}^n \times [0, +\infty)$. In (1.1), i is the imaginary unit, α, β, μ, ν and λ are real constants with $\beta > 0, \lambda > 0$ and $\rho > 0$. $(-\Delta)^\alpha$ with $0 < \alpha < 1$ is the fractional Laplace operator, $\sigma_{1,k}(x) \in L^2(\mathbb{R}^n)$ and $\sigma_{2,k}(u) : \mathbb{C} \to \mathbb{R}$ are nonlinear functions, $\kappa(x) \in L^2(\mathbb{R}^n) \bigcap L^\infty(\mathbb{R}^n)$ and $\{W_k\}_{k=1}^\infty$ is a sequence of independent standard real-valued Wiener process on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, P)$, where $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$ is an increasing right continuous family of sub-σ-algebras of $\mathcal{F}$ that contains all P-null sets.

The Ginzburg-Landau equation with fractional derivative was first introduced in [2]. There is a large amount of literature which was used for investigating fractional deterministic Ginzburg-Landau equations such as [1] and stochastic equations such as [11–17]. These papers had respectively researched the long-time deterministic as well as random dynamical systems of fractional equations with autonomous forms and non-autonomous forms. However, in spite of quite a lot of contribution of the works, no result is provided for the existence of pathwise pullback random attractors and invariant measures for the delay stochastic Ginzburg-Landau equations.

The delay differential equations [18] was described the dynamical systems that rely on current and past historical states. For the past few years, researchers had made great progress in the study of linear and nonlinear delay differential equations, see [19–21]. Delay differential equations are widely used in many fields, so investigating the solutions of equations has profound significance. Therefore, it's necessary that we establish the dynamics of delay stochastic Ginzburg-Landau equations.

The goal of this paper is to prove the existence of invariant measures of the stochastic Eqs (1.1) and (1.2) in $L^2(\Omega; C([-\rho, 0], L^2(\mathbb{R}^n)))$ by applying Krylov-Bogolyubov's method. The main difficulty of this paper is that deducing the uniform estimates of solutions (because of the nonlinear term $(1 + i\mu)|u(t)|^{2\beta}u(t)$ and complex-valued solutions), proving the weak compactness of a set distribution laws of the segments of solutions in $L^2(\Omega; C([-\rho, 0], L^2(\mathbb{R}^n)))$ (because the standard Sobolev embeddings are not compact on unbounded domains $\mathbb{R}^n$), and establishing the equicontinuity of solutions in $L^2(\Omega; C([-\rho, 0], L^2(\mathbb{R}^n)))$ (because the uniform estimates in $L^2(\Omega; C([-\rho, 0], L^2(\mathbb{R}^n)))$ are not sufficient, and the uniform estimates in $L^2(\Omega; C([-\rho, 0], H^1(\mathbb{R}^n)))$ are needed).

For the estimates of the nonlinear term $(1 + i\mu)|u(t)|^{2\beta}u(t)$, we apply integrating by parts and nonnegative definite quadratic form. There are Several methods to handle the noncompact on unbounded domain, including weighted spaces [22–24], weak Feller approach [25, 26] and uniform tail-estimates [23, 27]. We first obtain the uniform estimates of the tail of the solution as well as the technique of dyadic division, then establish the weak compactness of a set of probability distribution of solutions in $C([-\rho, 0], L^2(\mathbb{R}^n))$ applying the Ascoli-Arzel$\grave{a}$ theorem.

Let $\mathcal{S}$ be the Schwartz space of rapidly decaying C^∞ functions on $\mathbb{R}^n$. The fractional Laplace operator $(-\Delta)^\alpha$ for $0 < \alpha < 1$ is defined by, for $u \in \mathcal{S}$,

$$(-\Delta)^\alpha u(x) = -\frac{1}{2}C(n, \alpha) \int_{\mathbb{R}^n} \frac{u(x + y) + u(x - y) - 2u(x)}{|y|^{n+2\alpha}} dy, \quad x \in \mathbb{R}^n,$$

where $C(n, \alpha)$ is a positive constant given by

$$C(n, \alpha) = \frac{\alpha 4^\alpha \Gamma(\frac{n+2\alpha}{2})}{\pi^{\frac{n}{2}} \Gamma(1 - \alpha)}.$$

By [28], the inner product $\left((-\Delta)^{\frac{\alpha}{2}}u, (-\Delta)^{\frac{\alpha}{2}}v\right)$ in the complex field is defined by

$$\left((-\Delta)^{\frac{\alpha}{2}}u, (-\Delta)^{\frac{\alpha}{2}}v\right) = \frac{C(n,\alpha)}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(u(x)-u(y))(\bar{v}(x)-\bar{v}(y))}{|x-y|^{n+2\alpha}}dxdy,$$

for $u \in H^\alpha(\mathbb{R}^n)$. The fractional Sobolev space $H^\alpha(\mathbb{R}^n)$ is endowed with the norm

$$\|u\|^2_{H^\alpha(\mathbb{R}^n)} = \|u\|^2_{L^2(\mathbb{R}^n)} + \frac{2}{C(n,\alpha)}\|(-\Delta)^{\frac{\alpha}{2}}u\|^2_{L^2(\mathbb{R}^n)}.$$

About the fractional derivative of fractional Ginzburg-Landau equations, there is another statement in [29].

We organize the article as follows. In Section 2, we establish the well-posedness of (1.1) and (1.2) in $L^2(\Omega; C([-\rho, 0], H))$. In Sections 3 and 4, we derive the uniform estimates of solutions in $L^2(\Omega; C([-\rho, 0], H))$ and $L^2(\Omega; C([-\rho, 0], V))$, respectively. In Section 5, the existence of invariant measures is obtained.

2. Preliminaries

In this section, we show the nonlinear drift term and the diffusion term in (1.1) which are needed for the well-posedness of the stochastic delay Ginzburg-Landau Eqs (1.1) and (1.2) defined on $\mathbb{R}^n$.

We assume that $G : \mathbb{R}^n \times \mathbb{C} \to \mathbb{C}$ is continuous and satisfies

$$|G(x,u)| \leq |h(x)| + a|u|, \ \forall x \in \mathbb{R}^n, \ u \in \mathbb{C} \tag{2.1}$$

and

$$|\nabla G(x,u)| \leq |\hat{h}(x)| + \hat{a}|\nabla u|, \ \forall x \in \mathbb{R}^n, \ u \in \mathbb{C}, \tag{2.2}$$

where a and $\hat{a} > 0$ are constants and $h(x), \hat{h}(x) \in L^2(\mathbb{R}^n)$. Moreover, $G(x,u)$ is Lipschitz continuous in $u \in \mathbb{C}$ uniformly with respect to $x \in \mathbb{R}^n$. More precisely, there exists a constant $C_G > 0$ such that

$$|G(x,u_1) - G(x,u_2)| \leq C_G|u_1 - u_2|, \ \forall x \in \mathbb{R}^n, \ u_1, u_2 \in \mathbb{C}. \tag{2.3}$$

For the diffusion coefficients of noise, we suppose that for each $k \in \mathbb{N}^+$

$$\sum_{k=1}^{\infty}\|\sigma_{1,k}\|^2 < \infty, \tag{2.4}$$

and that $\sigma_{2,k}(u) : \mathbb{C} \to \mathbb{R}$ is globally Lipschitz continuous; namely, for every $k \in \mathbb{N}^+$, there exists a positive number α_k such that for all $s_1, s_2 \in \mathbb{C}$,

$$|\sigma_{2,k}(s_1) - \sigma_{2,k}(s_2)| \leq \alpha_k|s_1 - s_2|. \tag{2.5}$$

We further assume that for each $k \in \mathbb{N}^+$, there exist positive numbers β_k, $\hat{\beta}_k$, γ_k and $\hat{\gamma}_k$ such that

$$|\sigma_{2,k}(s)| \leq \beta_k + \gamma_k|s|, \ \forall s \in \mathbb{C}, \tag{2.6}$$

and

$$|\nabla \sigma_{2,k}(s)| \le \hat{\beta}_k + \hat{\gamma}_k |\nabla s|, \ \forall s \in \mathbb{C}, \tag{2.7}$$

where $\sum_{k=1}^{\infty}(\alpha_k^2 + \beta_k^2 + \gamma_k^2 + \hat{\beta}_k^2 + \hat{\gamma}_k^2) < +\infty$. In this paper, we deal with the stochastic Eqs (1.1) and (1.2) in the space $C([-\rho, 0], L^2(\mathbb{R}^n))$. In the following discussion, we denote by $H = L^2(\mathbb{R}^n)$, $V = H^1(\mathbb{R}^n)$.

A solution of problems (1.1) and (1.2) will be understood in the following sense.

Definition 2.1. *We suppose that $\varphi(s) \in L^2(\Omega, C([-\rho, 0], H))$ is $\mathcal{F}_0$-measurable. Then, a continuous H-valued $\mathcal{F}_t$-adapted stochastic process $u(x,t)$ is named a solution of problems* (1.1) *and* (1.2)*, if*

1) *u is pathwise continuous on $[0, +\infty)$, and $\mathcal{F}_t$-adapted for all $t \ge 0$,*

$$u \in L^2(\Omega, C([0,T], H)) \bigcap L^2(\Omega, L^2([0,T], V))$$

for all $T > 0$,

2) *$u(s) = \varphi(s)$ for $-\rho \le s \le 0$,*
3) *For all $t \ge 0$ and $\xi \in V$,*

$$\begin{aligned}
&(u(t), \xi) + (1 + i\nu) \int_0^t \left((-\Delta)^{\frac{\alpha}{2}} u(s), (-\Delta)^{\frac{\alpha}{2}} \xi\right) ds + \int_0^t \int_{\mathbb{R}^n} (1 + \mathrm{i}\mu)|u(s)|^{2\beta} u(s) \xi(x) dx ds \\
&+ \lambda \int_0^t (u(s), \xi) ds \\
&= (\varphi(0), \xi) + \int_0^t (G(s, u(s-\rho)), \xi) ds + \sum_{k=1}^{\infty} \int_0^t (\sigma_{1,k}(x) + \kappa(x)\sigma_{2,k}(u(s)), \xi)\, dW_k(s),
\end{aligned} \tag{2.8}$$

for almost all $\omega \in \Omega$.

By the Galerkin method and the argument of Theorem 3.1 in [30], one can verify that if (2.1)–(2.7) hold true, then, for every $\mathcal{F}_0$-measurable function $\varphi(s) \in L^2(\Omega, C([-\rho, 0], H))$, the problems (1.1) and (1.2) has a unique solution $u(x,t)$ in the sense of Definition 2.1.

Now, we establish the Lipschitz continuity of the solutions of the problems (1.1) and (1.2) with respect to the initial data in $L^2(\Omega, C([-\rho, 0], H))$.

Theorem 2.2. *Suppose* (2.1)–(2.6) *hold, and $\mathcal{F}_0$-measurable function $\varphi_1, \varphi_2 \in L^2(\Omega, C([-\rho, 0], H))$. If $u_1 = u(t, \varphi_1)$ and $u_2 = u(t, \varphi_2)$ are the solutions of the problems* (1.1) *and* (1.2) *with initial data φ_1 and φ_2, respectively, then, for any $t \ge 0$,*

$$\mathbb{E}\left[\sup_{-\rho \le s \le t} \|u(s, \varphi_1) - u(s, \varphi_2)\|^2\right] + \mathbb{E}\left[\int_0^t \|u(s, \varphi_1) - u(s, \varphi_2)\|_V^2 ds\right]$$

$$\le C_1 e^{\tilde{C}_1 t} \mathbb{E}\left[\sup_{-\rho \le s \le 0} \|\varphi_1(s) - \varphi_2(s)]\|^2\right],$$

where C_1 and $\tilde{C}_1$ are positive constants independent of φ_1 and φ_2.

Proof. Since both u_1 and u_2 are the solutions of the problems (1.1) and (1.2), we have, for all $t \geq 0$,

$$\begin{aligned} & u_1 - u_2 + (1+\mathrm{i}\nu)\int_0^t (-\Delta)^\alpha (u_1-u_2)ds + (1+\mathrm{i}\mu)\int_0^t \left(|u_1|^{2\beta}u_1 - |u_2|^{2\beta}u_2\right)ds \\ & \quad + \lambda\int_0^t (u_1-u_2)ds \\ & = \varphi_1(0) - \varphi_2(0) + \int_0^t (G(x,u_1(s-\rho)) - G(x,u_2(s-\rho)))ds \\ & \quad + \sum_{k=1}^\infty \int_0^t \kappa(x)(\sigma_{2,k}(u_1) - \sigma_{2,k}(u_2))dW_k. \end{aligned} \tag{2.9}$$

By (2.9), the integration by parts of Ito's formula and taking the real parts, we get, for all $t \geq 0$,

$$\begin{aligned} & \|u_1-u_2\|^2 + 2\int_0^t \|(-\Delta)^{\frac{\alpha}{2}}(u_1-u_2)\|^2 ds + 2\mathrm{Re}\int_0^t\int_{\mathbb{R}^n}(\bar{u}_1-\bar{u}_2)[|u_1|^{2\beta}u_1 - |u_2|^{2\beta}u_2]dxds \\ & \quad + 2\lambda\int_0^t \|u_1-u_2\|^2 ds \\ & = \|\varphi_1(0)-\varphi_2(0)\|^2 + 2\mathrm{Re}\int_0^t (u_1-u_2, G(x,u_1(s-\rho)) - G(x,u_2(s-\rho)))ds \\ & \quad + \sum_{k=1}^\infty\int_0^t \|\kappa(x)(\sigma_{2,k}(u_1)-\sigma_{2,k}(u_2))\|^2 ds \\ & \quad + 2\mathrm{Re}\int_0^t \left(u_1-u_2, \sum_{k=1}^\infty \kappa(x)(\sigma_{2,k}(u_1)-\sigma_{2,k}(u_2))\right)dW_k(s). \end{aligned} \tag{2.10}$$

For the third term in the first row of (2.10), one has

$$\begin{aligned} & 2\mathrm{Re}\int_0^t\int_{\mathbb{R}^n}(\bar{u}_1-\bar{u}_2)[|u_1|^{2\beta}u_1 - |u_2|^{2\beta}u_2]dxds \\ & = \int_0^t\int_{\mathbb{R}^n} 2|u_1|^{2\beta+2} + 2|u_2|^{2\beta+2} - 2\mathrm{Re}(u_1\bar{u}_2)(|u_1|^{2\beta}+|u_2|^{2\beta})dxds \\ & \geq \int_0^t\int_{\mathbb{R}^n} 2|u_1|^{2\beta+2} + 2|u_2|^{2\beta+2} - 2|u_1||u_2|(|u_1|^{2\beta}+|u_2|^{2\beta})dxds \\ & \geq \int_0^t\int_{\mathbb{R}^n} 2|u_1|^{2\beta+2} + 2|u_2|^{2\beta+2} - (|u_1|^2+|u_2|^2)(|u_1|^{2\beta}+|u_2|^{2\beta})dxds \\ & = \int_0^t\int_{\mathbb{R}^n} |u_1|^{2\beta+2} + |u_2|^{2\beta+2} - |u_1|^{2\beta}|u_2|^2 - |u_2|^{2\beta}|u_1|^2 dxds \\ & = \int_0^t\int_{\mathbb{R}^n} (|u_1|^{2\beta}-|u_2|^{2\beta})(|u_1|^2-|u_2|^2)dxds \geq 0. \end{aligned}$$

By (2.10), we deduce that for $t \geq 0$,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\leq r\leq t}\|u_1(r)-u_2(r)\|^2\right]\\
&\leq \mathbb{E}\left[\sup_{-\rho\leq s\leq 0}\|\varphi_1(s)-\varphi_2(s)\|^2\right]+2\mathbb{E}\left[\int_0^t \|u_1-u_2\|\cdot\|G(x,u_1(s-\rho))-G(x,u_2(s-\rho))\|ds\right]\\
&+\sum_{k=1}^{\infty}\mathbb{E}\left[\int_0^t \|\kappa(\sigma_{2,k}(u_1)-\sigma_{2,k}(u_2))\|^2 ds\right]\\
&+2\mathbb{E}\left[\sup_{0\leq r\leq t}\left|\sum_{k=1}^{\infty}\int_0^r (u_1-u_2,\kappa(x)(\sigma_{2,k}(u_1)-\sigma_{2,k}(u_2))dW_k(s))\right|\right].
\end{aligned} \tag{2.11}$$

For the second term on the right-hand side of (2.11), by (2.3), one has

$$\begin{aligned}
&2\mathbb{E}\left[\int_0^t \|u_1-u_2\|\cdot\|G(x,u_1(s-\rho))-G(x,u_2(s-\rho))\|ds\right]\\
&\leq \mathbb{E}\left[\int_0^t \|u_1-u_2\|^2 ds\right]+\mathbb{E}\left[\int_0^t \|G(x,u_1(s-\rho))-G(x,u_2(s-\rho))\|^2 ds\right]\\
&\leq \mathbb{E}\left[\int_0^t \|u_1-u_2\|^2 ds\right]+C_G^2\mathbb{E}\left[\int_0^t \|u_1(s-\rho)-u_2(s-\rho)\|^2 ds\right]\\
&= \mathbb{E}\left[\int_0^t \|u_1-u_2\|^2 ds\right]+C_G^2\mathbb{E}\left[\int_{-\rho}^{t-\rho} \|u_1-u_2\|^2 ds\right]\\
&\leq (1+C_G^2)\mathbb{E}\left[\int_0^t \|u_1-u_2\|^2 ds\right]+C_G^2\mathbb{E}\left[\int_{-\rho}^0 \|\varphi_1(s)-\varphi_2(s)\|^2 ds\right]\\
&\leq (1+C_G^2)\int_0^t \mathbb{E}\left[\sup_{0\leq r\leq s}\|u_1-u_2\|^2\right]ds+\rho C_G^2\mathbb{E}\left[\sup_{-\rho\leq s\leq 0}\|\varphi_1(s)-\varphi_2(s)\|^2\right].
\end{aligned}$$

For the third term on the right-hand side of (2.11), by (2.5), we have

$$\begin{aligned}
&\sum_{k=1}^{\infty}\mathbb{E}\left[\int_0^t \|\kappa(x)(\sigma_{2,k}(u_1)-\sigma_{2,k}(u_2))\|^2 ds\right]\\
&\leq \|\kappa(x)\|_{L^\infty}^2\sum_{k=1}^{\infty}\alpha_k^2\mathbb{E}\left[\int_0^t \|u_1-u_2\|^2 ds\right]\leq \|\kappa(x)\|_{L^\infty}^2\sum_{k=1}^{\infty}\alpha_k^2\int_0^t \mathbb{E}\left[\sup_{0\leq r\leq s}\|u_1-u_2\|^2\right]ds.
\end{aligned} \tag{2.12}$$

For the forth term on the right-hand side of (2.11), by Burkholder-Davis-Gundy's inequality, one has

$$\begin{aligned}
&2\mathbb{E}\left[\sup_{0\le r\le t}\left|\sum_{k=1}^{\infty}\int_0^r (u_1-u_2,\kappa(x)(\sigma_{2,k}(u_1)-\sigma_{2,k}(u_2))dW_k(s))\right|\right]\\
&\le B_1\mathbb{E}\left[\left(\int_0^t\sum_{k=1}^{\infty}|(u_1-u_2,\kappa(x)(\sigma_{2,k}(u_1)-\sigma_{2,k}(u_2)))|^2ds\right)^{\frac{1}{2}}\right]\\
&\le B_1\mathbb{E}\left[\left(\int_0^t\sum_{k=1}^{\infty}\|u_1-u_2\|^2\cdot\|\kappa\|_{L^\infty}^2\cdot\|\sigma_{2,k}(u_1)-\sigma_{2,k}(u_2)\|^2ds\right)^{\frac{1}{2}}\right]\\
&\le B_1\mathbb{E}\left[\sup_{0\le s\le t}\|u_1-u_2\|\cdot\|\kappa\|_{L^\infty}\cdot\left(\sum_{k=1}^{\infty}\alpha_k^2\right)^{\frac{1}{2}}\left(\int_0^t\|u_1-u_2\|^2ds\right)^{\frac{1}{2}}\right]\\
&\le \frac{1}{2}\mathbb{E}\left[\sup_{0\le s\le t}\|u_1-u_2\|^2\right]+\frac{1}{2}B_1^2\|\kappa\|_{L^\infty}^2\sum_{k=1}^{\infty}\alpha_k^2\mathbb{E}\left[\int_0^t\sup_{0\le r\le s}\|u_1-u_2\|^2ds\right],
\end{aligned} \tag{2.13}$$

where B_1 is a constant produced by Burkholder-Davis-Gundy's inequality.

It follows from (2.11)–(2.13) that for all $t\ge 0$,

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\le r\le t}\|u_1(r)-u_2(r)\|^2\right]&\le 2(1+\rho C_G^2)\mathbb{E}[\sup_{-\rho\le s\le 0}\|\varphi_1(s)-\varphi_2(s)\|^2]\\
&+2\left[1+C_G^2+\left(1+\frac{1}{2}B_1^2\right)\|\kappa\|_{L^\infty}^2\sum_{k=1}^{\infty}\alpha_k^2\right]\int_0^t\mathbb{E}\left[\sup_{0\le r\le s}\|u_1(r)-u_2(r)\|^2\right]ds.
\end{aligned} \tag{2.14}$$

Applying Gronwall inequality to (2.14), we obtain that for all $t\ge 0$,

$$\mathbb{E}\left[\sup_{0\le r\le t}\|u_1(r)-u_2(r)\|^2\right]\le 2(1+\rho C_G^2)e^{c_1t}\mathbb{E}[\sup_{-\rho\le s\le 0}\|\varphi_1(s)-\varphi_2(s)\|^2], \tag{2.15}$$

where $c_1=2\left[1+C_G^2+\left(1+\frac{1}{2}B_1^2\right)\|\kappa\|_{L^\infty}^2\sum_{k=1}^{\infty}\alpha_k^2\right]$. By (2.10), there exists c_2 such that for all $t\ge 0$,

$$\mathbb{E}\left[\int_0^t\|u_1-u_2\|_V^2ds\right]\le \tilde{c}_2e^{c_2t}\mathbb{E}[\sup_{-\rho\le s\le 0}\|\varphi_1(s)-\varphi_2(s)\|^2].$$

3. Uniform estimates of solutions with initial data in $C([-\rho,0],H)$

We assume that a, α_k and γ_k are small enough in the sense, there exists a constant $p\ge 2$ such that

$$2^{1-\frac{1}{2p}}(2p-1)^{\frac{2p-1}{2p}}a+2p(2p-1)\|\kappa\|_{L^\infty}^2\sum_{k=1}^{\infty}(\alpha_k^2+\gamma_k^2)<p\lambda. \tag{3.1}$$

By (3.1), one has

$$2\|\kappa\|_{L^\infty}^2\sum_{k=1}^{\infty}\gamma_k^2<\lambda, \tag{3.2}$$

and

$$\sqrt{2}a + 2\|\kappa\|_{L^\infty}^2 \sum_{k=1}^{\infty} \gamma_k^2 < \lambda. \tag{3.3}$$

The inequalities (3.1)–(3.3) are used to establish the uniform tail-estimate of the solution of (1.1) and (1.2).

Lemma 3.1. *Suppose* (2.1)–(2.6) *and* (3.2) *hold. If* $\varphi(s) \in L^2(\Omega; C([-\rho, 0], H))$*, then, for all* $t \geq 0$*, there exists a positive constant* μ_1 *such that the solution* u *of* (1.1) *and* (1.2) *satisfies*

$$\mathbb{E}[\|u(t)\|^2] + \int_0^t e^{\mu_1(s-t)}\mathbb{E}(\|u(s)\|_V^2)ds + \int_0^t e^{\mu_1(s-t)}\mathbb{E}(\|u(s)\|_{L^{2\beta+2}}^{2\beta+2})ds$$

$$\leq M_1\mathbb{E}\left[\sup_{-\rho\leq s\leq 0} \|\varphi(s)\|^2\right] + \tilde{M}_1, \tag{3.4}$$

and

$$\int_0^{t+\rho} \mathbb{E}[\|u(s)\|_V^2]ds \leq \left(M_1(t+\rho) + \frac{1+\sqrt{2}a\rho}{C(n,\alpha)}\right)\mathbb{E}[\sup_{-\rho\leq s\leq 0} \|\varphi(s)\|^2] + \frac{\sqrt{2}(t+\rho)}{aC(n,\alpha)}\|h(x)\|^2$$
$$+\frac{2(t+\rho)}{C(n,\alpha)}\sum_{k=1}^{\infty}(\|\sigma_{1,k}\|^2 + 2\beta_k^2\|\kappa(x)\|^2) + \tilde{M}_1(t+\rho),$$

where $\tilde{M}_1$ *is a positive constant independent of* φ*.*

Proof. By (1.1) and the integration by parts of Ito's formula, we have for all $t \geq 0$,

$$\begin{aligned}
&\|u(t)\|^2 + 2\int_0^t \|(-\Delta)^{\frac{\alpha}{2}}u(s)\|^2ds + 2\int_0^t \|u(s)\|_{L^{2\beta+2}}^{2\beta+2}ds + 2\lambda\int_0^t \|u(s)\|^2ds\\
&= 2\mathrm{Re}\int_0^t (u(s), G(x, u(s-\rho)))ds + \|\varphi(0)\|^2 + \sum_{k=1}^{\infty}\int_0^t \|\sigma_{1,k}(x) + \kappa(x)\sigma_{2,k}(u(s))\|^2ds\\
&\quad + 2\mathrm{Re}\int_0^t (u(s), \sum_{k=1}^{\infty}\sigma_{1,k}(x) + \kappa(x)\sigma_{2,k}(u(s)))dW_k(s).
\end{aligned} \tag{3.5}$$

The system (3.5) can be rewritten as

$$\begin{aligned}
&d(\|u(t)\|^2) + 2\|(-\Delta)^{\frac{\alpha}{2}}u(t)\|^2dt + 2\|u(t)\|_{L^{2\beta+2}}^{2\beta+2}dt + 2\lambda\|u(t)\|^2dt\\
&= 2\mathrm{Re}(u(t), G(x, u(t-\rho)))dt + \sum_{k=1}^{\infty}\|\sigma_{1,k}(x) + \kappa(x)\sigma_{2,k}(u(t))\|^2dt\\
&\quad + 2\mathrm{Re}(u(t), \sum_{k=1}^{\infty}\sigma_{1,k}(x) + \kappa(x)\sigma_{2,k}(u(t)))dW_k(t).
\end{aligned} \tag{3.6}$$

Assume that μ_1 is a positive constant, one has

$$
\begin{aligned}
&e^{\mu_1 t}\|u(t)\|^2 + 2\int_0^t e^{\mu_1 s}\|(-\Delta)^{\frac{\alpha}{2}}u(s)\|^2 ds + 2\int_0^t e^{\mu_1 s}\|u(s)\|_{L^{2\beta+2}}^{2\beta+2} ds \\
&= (\mu_1 - 2\lambda)\int_0^t e^{\mu_1 s}\|u(s)\|^2 ds + \|\varphi(0)\|^2 + 2\mathrm{Re}\int_0^t e^{\mu_1 s}(u(s), G(x, u(s-\rho)))ds \\
&+ \sum_{k=1}^{\infty}\int_0^t e^{\mu_1 s}\|\sigma_{1,k} + \kappa\sigma_{2,k}(u(s))\|^2 ds + 2\mathrm{Re}\int_0^t e^{\mu_1 s}(u(s), \sum_{k=1}^{\infty}\sigma_{1,k}(x) + \kappa(x)\sigma_{2,k}(u(s)))dW_k(s).
\end{aligned}
$$

Taking the expectation, we have for all $t \geq 0$,

$$
\begin{aligned}
&e^{\mu_1 t}\mathbb{E}(\|u(t)\|^2) + 2\mathbb{E}\left[\int_0^t e^{\mu_1 s}\|(-\Delta)^{\frac{\alpha}{2}}u(s)\|^2 ds\right] + 2\mathbb{E}\left[\int_0^t e^{\mu_1 s}\|u(s)\|_{L^p}^p ds\right] \\
&= \mathbb{E}(\|\varphi(0)\|^2) + (\mu_1 - 2\lambda)\mathbb{E}\left[\int_0^t e^{\mu_1 s}\|u(s)\|^2 ds\right] + 2\mathbb{E}\left[\int_0^t e^{\mu_1 s}\mathrm{Re}(u(s), G(x, u(s-\rho)))ds\right] \\
&\quad + \sum_{k=1}^{\infty}\mathbb{E}\left[\int_0^t e^{\mu_1 s}\|\sigma_{1,k}(x) + \kappa(x)\sigma_{2,k}(u(s))\|^2 ds\right].
\end{aligned} \tag{3.7}
$$

For the third term on the right-hand side (3.7), by (2.1), we have

$$
\begin{aligned}
&2\mathbb{E}\left[\int_0^t e^{\mu_1 s}\mathrm{Re}(u(s), G(x, u(s-\rho)))ds\right] \leq 2\int_0^t e^{\mu_1 s}\mathbb{E}\left[\|u(s)\|\|G(x, u(s-\rho))\|\right] ds \\
&\leq \sqrt{2}a\int_0^t e^{\mu_1 s}\mathbb{E}(\|u(s)\|^2)ds + \frac{\sqrt{2}}{2a}\int_0^t e^{\mu_1 s}\mathbb{E}\left[\|G(x, u(s-\rho))\|^2\right] ds \\
&\leq \sqrt{2}a\int_0^t e^{\mu_1 s}\mathbb{E}(\|u(s)\|^2)ds + \frac{\sqrt{2}}{a}\int_0^t e^{\mu_1 s}\|h(x)\|^2 ds + \sqrt{2}a\int_0^t e^{\mu_1 s}\mathbb{E}\left[\|u(s-\rho)\|^2\right] ds \\
&\leq \sqrt{2}a(1 + e^{\mu_1\rho})\int_0^t e^{\mu_1 s}\mathbb{E}\left[\|u(s)\|^2\right] ds + \frac{\sqrt{2}}{a}\|h(x)\|^2\int_0^t e^{\mu_1 s} ds \\
&\quad + \sqrt{2}a e^{\mu_1\rho}\int_{-\rho}^0 e^{\mu_1 s}\mathbb{E}[\|\varphi(s)\|^2]ds \\
&\leq \sqrt{2}a(1 + e^{\mu_1\rho})\int_0^t e^{\mu_1 s}\mathbb{E}\left[\|u(s)\|^2\right] ds + \frac{\sqrt{2}e^{\mu_1 t}}{a\mu_1}\|h(x)\|^2 + \sqrt{2}a\rho e^{\mu_1\rho}\mathbb{E}[\sup_{-\rho\leq s\leq 0}\|\varphi(s)\|^2].
\end{aligned} \tag{3.8}
$$

For the forth term on the right-hand side (3.7), by (2.6), we have

$$
\begin{aligned}
&\sum_{k=1}^{\infty}\mathbb{E}\left[\int_0^t e^{\mu_1 s}\|\sigma_{1,k} + \kappa\sigma_{2,k}(u(s))\|^2 ds\right] \\
&\leq \sum_{k=1}^{\infty}\mathbb{E}\left[\int_0^t e^{\mu_1 s}(2\|\sigma_{1,k}\|^2 + 2\|\kappa\sigma_{2,k}(u(s))\|^2)ds\right] \\
&\leq \frac{2}{\mu_1}\sum_{k=1}^{\infty}\|\sigma_{1,k}\|^2 e^{\mu_1 t} + 4\sum_{k=1}^{\infty}\int_0^t e^{\mu_1 s}\mathbb{E}[\beta_k^2\|\kappa\|^2 + \gamma_k^2\|\kappa\|_{L^\infty}^2\|u(s)\|^2]ds \\
&\leq \frac{2}{\mu_1}\sum_{k=1}^{\infty}\left(\|\sigma_{1,k}\|^2 + 2\beta_k^2\|\kappa(x)\|^2\right)e^{\mu_1 t} + 4\sum_{k=1}^{\infty}\gamma_k^2\|\kappa(x)\|_{L^\infty}^2\int_0^t e^{\mu_1 s}\mathbb{E}(\|u(s)\|^2)ds.
\end{aligned} \tag{3.9}
$$

By (3.7)–(3.9), we obtain for all $t \geq 0$,

$$
\begin{aligned}
&e^{\mu_1 t}\mathbb{E}(\|u(t)\|^2) + 2\mathbb{E}\left[\int_0^t e^{\mu_1 s}\|(-\Delta)^{\frac{\alpha}{2}}u(s)\|^2 ds\right] + 2\mathbb{E}\left[\int_0^t e^{\mu_1 s}\|u(s)\|_{L^{2\beta+2}}^{2\beta+2} ds\right]\\
&\leq (1+\sqrt{2}a\rho e^{\mu_1\rho})\mathbb{E}\left[\sup_{-\rho\leq s\leq 0}\|\varphi(s)\|^2\right]\\
&\quad+\left[\mu_1 - 2\lambda + \sqrt{2}a(1+e^{\mu_1\rho}) + 4\sum_{k=1}^{\infty}\gamma_k^2\|\kappa\|_{L^\infty}^2\right]\int_0^t e^{\mu_1 s}\mathbb{E}\left[\|u\|^2\right]ds\\
&\quad+\frac{\sqrt{2}}{a\mu_1}e^{\mu_1 t}\|h(x)\|^2 + \frac{2}{\mu_1}\sum_{k=1}^{\infty}\left(\|\sigma_{1,k}\|^2 + 2\beta_k^2\|\kappa(x)\|^2\right)e^{\mu_1 t}.
\end{aligned}
\tag{3.10}
$$

By (3.2), there exists a positive constant μ_1 sufficiently small such that

$$
2\mu_1 + \sqrt{2}a + \sqrt{2}ae^{\mu_1\rho} + 4\sum_{k=1}^{\infty}\gamma_k^2\|\kappa(x)\|_{L^\infty}^2 \leq 2\lambda.
$$

Then, we have, for all $t \geq 0$,

$$
\begin{aligned}
&\mathbb{E}(\|u(t)\|^2) + 2\int_0^t e^{\mu_1(s-t)}\mathbb{E}(\|(-\Delta)^{\frac{\alpha}{2}}u(s)\|^2)ds\\
&\quad+\mu_1\int_0^t e^{\mu_1(s-t)}\mathbb{E}(\|u(s)\|^2)ds + 2\int_0^t e^{\mu_1(s-t)}\mathbb{E}(\|u(s)\|_{L^{2\beta+2}}^{2\beta+2})ds\\
&\leq (1+\sqrt{2}a\rho e^{\mu_1\rho})\mathbb{E}\Big(\sup_{-\rho\leq s\leq 0}\|\varphi(s)\|^2\Big) + \frac{1}{\mu_1}\left(\frac{\sqrt{2}}{a}\|h(x)\|^2 + 2\sum_{k=1}^{\infty}\left(\|\sigma_{1,k}\|^2 + 2\beta_k^2\|\kappa(x)\|^2\right)\right),
\end{aligned}
$$

which completes the proof of (3.4).

Integrating (3.6) on $[0, t+\rho]$ and taking the expectation, one has

$$
\begin{aligned}
&\mathbb{E}[\|u(t+\rho)\|^2] + 2\mathbb{E}\left[\int_0^{t+\rho}\|(-\Delta)^{\frac{\alpha}{2}}u(s)\|^2 ds\right] + 2\mathbb{E}\left[\int_0^{t+\rho}\|u(s)\|_{L^{2\beta+2}}^{2\beta+2}ds\right] + 2\lambda\mathbb{E}\left[\int_0^{t+\rho}\|u(s)\|^2 ds\right]\\
&= \mathbb{E}[\|\varphi(0)\|^2] + 2\mathbb{E}\left[\int_0^{t+\rho}\mathrm{Re}(u(s), G(x, u(s-\rho)))ds\right] + \sum_{k=1}^{\infty}\mathbb{E}\left[\int_0^{t+\rho}\|\sigma_{1,k} + \kappa(x)\sigma_{2,k}(u(s))ds\right].
\end{aligned}
\tag{3.11}
$$

For the second term on the right-hand side of (3.11), by (2.1), we have

$$
\begin{aligned}
&2\mathbb{E}\left[\int_0^{t+\rho}\mathrm{Re}(u, G(x, u(s-\rho)))ds\right]\\
&\leq 2\sqrt{2}a\mathbb{E}\left[\int_0^{t+\rho}\|u\|^2 ds\right] + \sqrt{2}a\rho\mathbb{E}\left[\sup_{-\rho\leq s\leq 0}\|\varphi(s)\|^2\right] + \frac{\sqrt{2}(t+\rho)}{a}\|h\|^2.
\end{aligned}
\tag{3.12}
$$

For the third term on the right-hand side of (3.11), one has

$$
\begin{aligned}
&\sum_{k=1}^{\infty}\mathbb{E}\left[\int_0^{t+\rho}\|\sigma_{1,k} + \kappa\sigma_{2,k}(u(s))ds\right]\\
&\leq 2(t+\rho)\sum_{k=1}^{\infty}(\|\sigma_{1,k}\|^2 + 2\beta_k^2\|\kappa\|^2) + 4\sum_{k=1}^{\infty}\gamma_k^2\|\kappa(x)\|_{L^\infty}^2\mathbb{E}\left[\int_0^{t+\rho}\|u\|^2 ds\right].
\end{aligned}
\tag{3.13}
$$

Then, by (3.2) and (3.11)–(3.13), for all $t \geq 0$, we obtain,

$$2\mathbb{E}\left[\int_0^{t+\rho} \|(-\Delta)^{\frac{\alpha}{2}} u(s)\|^2 ds\right] \leq (1+\sqrt{2}a\rho)\mathbb{E}\left[\sup_{-\rho\leq s\leq 0} \|\varphi(s)\|^2\right]$$
$$+2(t+\rho)\sum_{k=1}^{\infty}(\|\sigma_{1,k}\|^2 + 2\beta_k^2\|\kappa(x)\|^2) + \frac{\sqrt{2}(t+\rho)}{a}\|h(x)\|^2.$$

The result then follows from (3.4).

The next lemma is used to obtain the uniform estimates of the segments of solutions in $C([-\rho,0],H)$.

Lemma 3.2. *Suppose* (2.1)–(2.6) *and* (3.2) *hold. Then, for any* $\varphi(s) \in L^2(\Omega,\mathcal{F}_0;C([-\rho,0],H))$*, the solution of* (1.1) *satisfies that, for all* $t \geq \rho$*,*

$$\mathbb{E}\left(\sup_{t-\rho\leq r\leq t} \|u(r)\|^2\right) \leq M_2\mathbb{E}[\sup_{-\rho\leq s\leq 0} \|\varphi(s)\|^2] + \tilde{M}_2,$$

where M_2 *and* $\tilde{M}_2$ *are positive constants independent of* φ*.*

Proof. By (1.1) and integration by parts of Ito's formula and taking the real part, we get for all $t \geq \rho$ and $t-\rho \leq r \leq t$,

$$\begin{aligned} &\|u(r)\|^2 + 2\int_{t-\rho}^{r} \|(-\Delta)^{\frac{\alpha}{2}} u(s)\|^2 ds + 2\int_{t-\rho}^{r} \|u(s)\|_{L^{2\beta+2}}^{2\beta+2} ds + 2\lambda\int_{t-\rho}^{r} \|u(s)\|^2 ds \\ &= \|u(t-\rho)\|^2 + 2\mathrm{Re}\int_{t-\rho}^{r}(u(s),G(x,u(s-\rho)))ds + \sum_{k=1}^{\infty}\int_{t-\rho}^{r} \|\sigma_{1,k}(x)+\kappa(x)\sigma_{2,k}(u(s)))\|^2 ds \\ &\quad + 2\mathrm{Re}\sum_{k=1}^{\infty}\int_{t-\rho}^{r}(u(s),(\sigma_{1,k}(x)+\kappa(x)\sigma_{2,k}(u(s))dW_k(s)). \end{aligned} \quad (3.14)$$

For the second term on the right-hand side of (3.14), by (2.1) we have, for all $t \geq \rho$ and $t-\rho \leq r \leq t$,

$$\begin{aligned} &2\mathrm{Re}\int_{t-\rho}^{r}(u(s),G(x,u(s-\rho)))ds \\ &\leq 2\int_{t-\rho}^{r} \|u(s)\|\cdot\|G(x,u(s-\rho))\|ds \\ &\leq \int_{t-\rho}^{r} \|u(s)\|^2 ds + \int_{t-\rho}^{r} \|G(x,u(s-\rho))\|^2 ds \\ &\leq \int_{t-\rho}^{r} \|u(s)\|^2 ds + 2\int_{t-\rho}^{r} \|h\|^2 ds + 2a^2\int_{t-\rho}^{r} \|u(s-\rho)\|^2 ds \\ &\leq \int_{t-\rho}^{r} \|u(s)\|^2 ds + 2\rho\|h\|^2 + 2a^2\int_{t-2\rho}^{t-\rho} \|u(s)\|^2 ds. \end{aligned} \quad (3.15)$$

For the third term on the right-hand side of of (3.14), for all $t \geq \rho$ and $t-\rho \leq r \leq t$, by (2.6), we have

$$\begin{aligned} &\sum_{k=1}^{\infty}\int_{t-\rho}^{r} \|\sigma_{1,k}(x)+\kappa(x)\sigma_{2,k}(u(s))\|^2 ds \\ &\leq 2\rho\sum_{k=1}^{\infty}\|\sigma_{1,k}\|^2 + 4\rho\|\kappa\|^2\sum_{k=1}^{\infty}\beta_k^2 + 4\|\kappa\|_{L^\infty}^2\sum_{k=1}^{\infty}\gamma_k^2\int_{t-\rho}^{r}\|u(s)\|^2 ds. \end{aligned} \quad (3.16)$$

By (3.14)–(3.16), we obtain for all $t \geq \rho$ and $t-\rho \leq r \leq t$,

$$\|u(r)\|^2 \leq c_3 + \|u(t-\rho)\|^2 + c_4 \int_{t-2\rho}^{r} \|u(s)\|^2 ds$$

$$+2\mathrm{Re} \sum_{k=1}^{\infty} \int_{t-\rho}^{r} (u(s), (\sigma_{1,k}(x) + \kappa(x)\sigma_{2,k}(u(s))dW_k(s)), \tag{3.17}$$

where $c_3 = 2\rho\|h\|^2 + 2\rho \sum_{k=1}^{\infty} \|\sigma_{1,k}\|^2 + 4\rho\|\kappa\|^2 \sum_{k=1}^{\infty} \beta_k^2$ and $c_4 = 1 + 2a^2 + 4\|\kappa\|^2_{L^\infty} \sum_{k=1}^{\infty} \gamma_k^2$. By (3.17), we find that for all $t \geq \rho$,

$$\begin{aligned}\mathbb{E}\left[\sup_{t-\rho\leq r\leq t} \|u(r)\|^2\right] &\leq c_3 + \mathbb{E}\left[\|u(t-\rho)\|^2\right] + c_4 \int_{t-2\rho}^{t} \mathbb{E}\left[\|u(s)\|^2\right] ds \\ &+ 2\mathbb{E}\left[\sup_{t-\rho\leq r\leq t} \left|\sum_{k=1}^{\infty} \int_{t-\rho}^{r} (u(s), (\sigma_{1,k}(x) + \kappa(x)\sigma_{2,k}(u(s))dW_k(s))\right|\right].\end{aligned} \tag{3.18}$$

For the second term and the third term on the right-hand side of (3.18), by Lemma 3.1, we deduce for all $t \geq \rho$,

$$\mathbb{E}\left[\|u(t-\rho)\|^2\right] \leq \sup_{s\geq 0} \mathbb{E}[\|u(s)\|^2] \leq M_1 \mathbb{E}[\sup_{-\rho\leq s\leq 0} \|\varphi\|^2] + \tilde{M}_1 \tag{3.19}$$

and

$$c_4 \int_{t-2\rho}^{t} \mathbb{E}\left[\|u(s)\|^2\right] ds \leq 2\rho c_4 \sup_{s\geq -\rho} \mathbb{E}[\|u(s)\|^2] \leq c_5 \mathbb{E}[\sup_{-\rho\leq s\leq 0} \|\varphi\|^2] + c_5. \tag{3.20}$$

For the last term on the right-hand side of (3.18), by Burkholder-Davis-Gundy's inequality and Lemma 3.1, we obtain for all $t \geq \rho$,

$$\begin{aligned}&2\mathbb{E}\left[\sup_{t-\rho\leq r\leq t} \left|\sum_{k=1}^{\infty} \int_{t-\rho}^{r} (u(s), \sigma_{1,k}(x) + \kappa(x)\sigma_{2,k}(u(s))dW_k(s))\right|\right] \\ &\leq 2B_2 \mathbb{E}\left[\left(\sum_{k=1}^{\infty} \int_{t-\rho}^{t} |(u(s), \sigma_{1,k} + \kappa\sigma_{2,k}(u(s)))|^2 ds\right)^{\frac{1}{2}}\right] \\ &\leq \frac{1}{2}\mathbb{E}\left[\sup_{t-\rho\leq s\leq t} \|u(s)\|^2\right] + 2B_2^2 \mathbb{E}\left[\sum_{k=1}^{\infty} \int_{t-\rho}^{t} \|\sigma_{1,k} + \kappa\sigma_{2,k}(u(s))\|^2 ds\right] \\ &\leq \frac{1}{2}\mathbb{E}\left[\sup_{t-\rho\leq s\leq t} \|u(s)\|^2\right] + 2B_2^2 (2\rho \sum_{k=1}^{\infty} \|\sigma_{1,k}\|^2 + 4\rho\|\kappa\|^2 \sum_{k=1}^{\infty} \beta_k^2) \\ &\quad + 8B_2^2 \rho \|\kappa\|^2_{L^\infty} \sum_{k=1}^{\infty} \gamma_k^2 \sup_{s\geq 0} \mathbb{E}[\|u(s)\|^2].\end{aligned} \tag{3.21}$$

By Lemma 3.1 and (3.18)–(3.21), we deduce that for all $t \geq \rho$,

$$\mathbb{E}\left[\sup_{t-\rho\leq r\leq t} \|u(r)\|^2\right] \leq M_2 \mathbb{E}[\sup_{-\rho\leq s\leq 0} \|\varphi(s)\|^2] + \tilde{M}_2.$$

This completes the proof.

To establish the tightness of a family of distributions of solutions, we now derive uniform estimates on the tails of solutions to the problems (1.1) and (1.2).

Lemma 3.3. *Suppose* (2.1)–(2.6) *and* (3.2) *hold. If* $\varphi(s) \in L^2(\Omega, C([-\rho, 0], H))$*. Then, for all* $t \geq 0$*, the solution* u *of* (1.1) *and* (1.2) *satisfies*

$$\limsup_{m\to\infty} \sup_{t\geq-\rho} \int_{|x|\geq m} \mathbb{E}[|u(t,x)|^2]dx = 0.$$

Proof. We suppose that $\theta(x) : \mathbb{R}^n \to \mathbb{R}$ is a smooth function with $0 \leq \theta(x) \leq 1$, for all $x \in \mathbb{R}^n$ defined by

$$\theta(x) = \begin{cases} 0 & if\ |x| \leq 1, \\ 1 & if\ |x| \geq 2. \end{cases}$$

For fixed $m \in \mathbb{N}$, we denote that $\theta_m(x) = \theta(\frac{x}{m})$. By (1.1), we have

$$d(\theta_m u) + (1 + \mathrm{i}\nu)\theta_m(-\Delta)^\alpha u dt + (1 + \mathrm{i}\mu)\theta_m|u|^{2\beta}u dt + \lambda\theta_m u dt = \theta_m G(x, u(t - \rho))dt$$

$$+ \sum_{k=1}^{\infty} \theta_m(\sigma_{1,k} + \kappa\sigma_{2,k})dW_k(t). \tag{3.22}$$

By (3.2), We can find μ_2 sufficiently small such that

$$\mu_2 + 2\sqrt{2}a + 4\|\kappa\|^2_{L^\infty} \sum_{k=1}^{\infty} \gamma_k^2 - 2\lambda < 0. \tag{3.23}$$

By (3.22) and integration by parts of Ito's formula and taking the expectation, we obtain

$$\begin{aligned} &\mathbb{E}[\|\theta_m u\|^2] + 2\int_0^t e^{\mu_2(s-t)}\mathbb{E}\left[\int_{\mathbb{R}^n} \theta_m^2|u|^{2\beta+2}dx\right]ds \\ &= e^{-\mu_2 t}\mathbb{E}[\|\theta_m\varphi(0)\|^2] - 2\int_0^t e^{\mu_2(s-t)}\mathbb{E}\left[\mathrm{Re}(1 + \mathrm{i}\nu)((-\Delta)^{\frac{\alpha}{2}}u, (-\Delta)^{\frac{\alpha}{2}}(\theta_m^2 u))\right]ds \\ &\quad + (\mu_2 - 2\lambda)\int_0^t e^{\mu_2(s-t)}\mathbb{E}\left[\|\theta_m u\|^2\right]ds + 2\int_0^t e^{\mu_2(s-t)}\mathbb{E}\left[\mathrm{Re}\left(\theta_m u, \theta_m G(x, u(s - \rho))\right)\right]ds \\ &\quad + \sum_{k=1}^{\infty}\int_0^t e^{\mu_2(s-t)}\mathbb{E}\left[\|\theta_m(\sigma_{1,k} + \kappa(x)\sigma_{2,k}(u(s)))\|^2\right]ds. \end{aligned} \tag{3.24}$$

For the first term in the second row of (3.24), since $\varphi(s) \in L^2(\Omega, C([-\rho, 0], H))$, we have for all $s \in [-\rho, 0]$, $\mathbb{E}[\|\varphi(0)\|^2] < \infty$. It follows that for any $\varepsilon > 0$, there exists a positive $N_1 = N_1(\varepsilon, \varphi) \geq 1$, for all $m \geq N_1$, one has $\int_{|x|\geq m} \mathbb{E}[\varphi^2(0, x)]dx < \varepsilon$. Consequently,

$$\begin{aligned} \mathbb{E}\left[\|\theta_m\varphi(0)\|^2\right] &= \mathbb{E}\left[\int_{\mathbb{R}^n} |\theta(\frac{x}{m})\varphi(0, x)|^2dx\right] \\ &= \mathbb{E}\left[\int_{|x|\geq m} |\theta(\frac{x}{m})\varphi(0, x)|^2dx\right] \leq \int_{|x|\geq m} \mathbb{E}[|\varphi(0, x)|^2]dx < \varepsilon, \quad \forall m \geq N_1. \end{aligned} \tag{3.25}$$

Now we consider the second term on the right-hand side of (3.24). We first have

$$
\begin{aligned}
&-2\mathbb{E}\Big[\mathrm{Re}(1+\mathrm{i}\nu)((-\Delta)^{\frac{\alpha}{2}}u(s),(-\Delta)^{\frac{\alpha}{2}}(\theta_m^2u(s)))\Big]\\
&=-C(n,\alpha)\mathbb{E}\left[\mathrm{Re}(1+\mathrm{i}\nu)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{[u(x)-u(y)][\theta_m^2(x)\bar{u}(x)-\theta_m^2(y)\bar{u}(y)]}{|x-y|^{n+2\alpha}}\right]dxdy\\
&=-C(n,\alpha)\mathbb{E}\left[\mathrm{Re}(1+\mathrm{i}\nu)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{[u(x)-u(y)][\theta_m^2(x)(\bar{u}(x)-\bar{u}(y))+\bar{u}(y)(\theta_m^2(x)-\theta_m^2(y))]}{|x-y|^{n+2\alpha}}\right]dxdy\\
&=-C(n,\alpha)\mathbb{E}\left[\mathrm{Re}(1+\mathrm{i}\nu)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\theta_m^2(x)|u(x)-u(y)|^2}{|x-y|^{n+2\alpha}}dxdy\right]\\
&-C(n,\alpha)\mathbb{E}\left[\mathrm{Re}(1+\mathrm{i}\nu)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(u(x)-u(y))(\theta_m^2(x)-\theta_m^2(y))\bar{u}(y)}{|x-y|^{n+2\alpha}}dxdy\right]\\
&\leq-C(n,\alpha)\mathbb{E}\left[\mathrm{Re}(1+\mathrm{i}\nu)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(u(x)-u(y))(\theta_m^2(x)-\theta_m^2(y))\bar{u}(y)}{|x-y|^{n+2\alpha}}dxdy\right]\\
&\leq C(n,\alpha)\sqrt{1+\nu^2}\mathbb{E}\left[\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(u(x)-u(y))(\theta_m^2(x)-\theta_m^2(y))\bar{u}(y)}{|x-y|^{n+2\alpha}}dxdy\right|\right]\\
&\leq 2C(n,\alpha)\sqrt{1+\nu^2}\mathbb{E}\left[\int_{\mathbb{R}^n}|\bar{u}(y)|\left(\int_{\mathbb{R}^n}\frac{|(u(x)-u(y))(\theta_m(x)-\theta_m(y))|}{|x-y|^{n+2\alpha}}dx\right)dy\right]\\
&\leq 2C(n,\alpha)\sqrt{1+\nu^2}\mathbb{E}\left[\|u(s)\|\left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\frac{|(u(x)-u(y))(\theta_m(x)-\theta_m(y))|}{|x-y|^{n+2\alpha}}dx\right)^2dy\right)^{\frac{1}{2}}\right]\\
&\leq 2C(n,\alpha)\sqrt{1+\nu^2}\mathbb{E}\left[\|u(s)\|\left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2\alpha}}dx\int_{\mathbb{R}^n}\frac{|(\theta_m(x)-\theta_m(y))|^2}{|x-y|^{n+2\alpha}}dx\right)dy\right)^{\frac{1}{2}}\right].
\end{aligned}
\tag{3.26}
$$

We now prove the following inequality:

$$
\int_{\mathbb{R}^n}\frac{|(\theta_m(x)-\theta_m(y))|^2}{|x-y|^{n+2\alpha}}dx\leq\frac{c_6}{m^{2\alpha}}. \tag{3.27}
$$

Let $x-y=h$ and $\frac{h}{m}=z$, then, we obtain,

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\frac{|(\theta_m(x)-\theta_m(y))|^2}{|x-y|^{n+2\alpha}}dx=\int_{\mathbb{R}^n}\frac{|\theta(\frac{y+h}{m})-\theta(\frac{y}{m})|^2}{|h|^{n+2\alpha}}dh=\int_{\mathbb{R}^n}\frac{|\theta(\frac{y}{m}+z)-\theta(\frac{y}{m})|^2}{m^{n+2\alpha}|z|^{n+2\alpha}}m^ndz\\
&=\frac{1}{m^{2\alpha}}\int_{\mathbb{R}^n}\frac{|\theta(\frac{y}{m}+z)-\theta(\frac{y}{m})|^2}{|z|^{n+2\alpha}}dz\\
&=\frac{1}{m^{2\alpha}}\int_{|z|\leq1}\frac{|\theta(\frac{y}{m}+z)-\theta(\frac{y}{m})|^2}{|z|^{n+2\alpha}}dz+\frac{1}{m^{2\alpha}}\int_{|z|>1}\frac{|\theta(\frac{y}{m}+z)-\theta(\frac{y}{m})|^2}{|z|^{n+2\alpha}}dz\\
&\leq\frac{c_6^*}{m^{2\alpha}}\int_{|z|\leq1}\frac{|z|^2}{|z|^{n+2\alpha}}dz+\frac{4}{m^{2\alpha}}\int_{|z|>1}\frac{1}{|z|^{n+2\alpha}}dz\\
&\leq\frac{c_6^*}{m^{2\alpha}}\int_{|z|\leq1}\frac{1}{|z|^{n+2\alpha-2}}dz+\frac{4}{m^{2\alpha}}\int_{|z|>1}\frac{1}{|z|^{n+2\alpha}}dz\\
&\leq\frac{c_6^*\bar{c}_6}{m^{2\alpha}}+\frac{4\tilde{c}_6}{m^{2\alpha}}=\frac{c_6^*\bar{c}_6+4\tilde{c}_6}{m^{2\alpha}}.
\end{aligned}
\tag{3.28}
$$

This proves (3.27) with $c_6 := c_6^*\bar{c}_6 + 4\tilde{c}_6$. By (3.26) and (3.27), we obtain,

$$\begin{aligned}
&-2\mathbb{E}\left[\mathrm{Re}(1+\mathrm{i}\nu)((-\Delta)^{\frac{\alpha}{2}}u(s),(-\Delta)^{\frac{\alpha}{2}}\theta_m^2 u(s))\right]\\
&\le 2\sqrt{c_6(1+\nu^2)}C(n,\alpha)m^{-\alpha}\mathbb{E}\left[\|u(s)\|\sqrt{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2\alpha}}dxdy}\right]\\
&\le \sqrt{c_6(1+\nu^2)}C(n,\alpha)m^{-\alpha}\left(\mathbb{E}(\|u(s)\|^2)+\mathbb{E}\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2\alpha}}dxdy\right)\right)\\
&\le \sqrt{c_6(1+\nu^2)}C(n,\alpha)m^{-\alpha}\mathbb{E}(\|u(s)\|^2)+2\sqrt{c_6(1+\nu^2)}m^{-\alpha}\mathbb{E}(\|(-\Delta)^{\frac{\alpha}{2}}u(s)\|^2).
\end{aligned} \tag{3.29}$$

By (3.29), for the second term on the right-hand side of (3.24), we get

$$\begin{aligned}
&-2\int_0^t e^{\mu_2 s}\mathbb{E}\left[\mathrm{Re}(1+\mathrm{i}\nu)((-\Delta)^{\frac{\alpha}{2}}u(s),(-\Delta)^{\frac{\alpha}{2}}\theta_m^2 u(s))\right]ds\\
&\le \sqrt{c_6(1+\nu^2)}C(n,\alpha)m^{-\alpha}\int_0^t e^{\mu_2 s}\mathbb{E}[\|u(s)\|^2]ds\\
&\quad+2\sqrt{c_6(1+\nu^2)}m^{-\alpha}\int_0^t e^{\mu_2 s}\mathbb{E}[\|(-\Delta)^{\frac{\alpha}{2}}u(s)\|^2]ds.
\end{aligned} \tag{3.30}$$

By Lemma 3.1, we have

$$\begin{aligned}
&\sqrt{c_6(1+\nu^2)}C(n,\alpha)m^{-\alpha}\int_0^t e^{\mu_2(s-t)}\mathbb{E}[\|u(s)\|^2]ds\\
&\le \sqrt{c_6(1+\nu^2)}C(n,\alpha)m^{-\alpha}\left[M_1\mathbb{E}[\sup_{-\rho\le s\le 0}\|\varphi(s)\|^2]+\tilde{M}_1\right]\int_0^t e^{\mu_2(s-t)ds}\\
&\le \sqrt{c_6(1+\nu^2)}C(n,\alpha)m^{-\alpha}\frac{1}{\mu_2}\left[M_1\mathbb{E}[\sup_{-\rho\le s\le 0}\|\varphi(s)\|^2]+\tilde{M}_1\right].
\end{aligned} \tag{3.31}$$

By (3.31), we deduce that there exists $N_2(\varepsilon,\varphi)\ge N_1$, for all $t\ge 0$ and $m\ge N_2$,

$$\sqrt{c_6(1+\nu^2)}C(n,\alpha)m^{-\alpha}\int_0^t e^{\mu_2(s-t)}\mathbb{E}[\|u(s)\|^2]ds<\varepsilon.$$

By Lemma 3.1, there exists $N_3(\varepsilon,\varphi)\ge N_2$ such that for all $t\ge 0$ and $m\ge N_3$,

$$\begin{aligned}
&2\sqrt{c_6(1+\nu^2)}m^{-\alpha}\int_0^t e^{\mu_2(s-t)}\mathbb{E}[\|(-\Delta)^{\frac{\alpha}{2}}u\|^2]ds\\
&\le 2\sqrt{c_6(1+\nu^2)}m^{-\alpha}\left[M_1\mathbb{E}[\sup_{-\rho\le s\le 0}\|\varphi(s)\|^2]+\tilde{M}_1\right]<\varepsilon.
\end{aligned}$$

For the forth term on the right-hand side of (3.24), we obtain that there exists $N_4(\varepsilon,\varphi)\ge N_3$, for all $t\ge 0$ and $m\ge N_4$,

$$
\begin{aligned}
&2\int_0^t e^{\mu_2(s-t)}\mathbb{E}\left[\mathrm{Re}(\theta_m u, \theta_m G(x,u(s-\rho)))\right]ds\\
&\le \sqrt{2}a\int_0^t e^{\mu_2(s-t)}\mathbb{E}[\|\theta_m u(s)\|^2]ds + \frac{1}{\sqrt{2}a}\int_0^t e^{\mu_2(s-t)}\mathbb{E}[\|\theta_m G(x,u(s-\rho))\|^2]ds\\
&\le \frac{\sqrt{2}}{a\mu_2}\int_{|x|\ge m} h^2(x)dx + \sqrt{2}a\int_{-\rho}^0 e^{\mu_2(s-t)}\mathbb{E}[\|\theta_m\varphi(s)\|^2]ds + 2\sqrt{2}a\int_0^t e^{\mu_2(s-t)}\mathbb{E}[\|\theta_m u(s)\|^2]ds\\
&\le \frac{\sqrt{2}}{a\mu_2}\varepsilon + \sqrt{2}a\int_{-\rho}^0 e^{\mu_2(s-t)}\mathbb{E}[\|\theta_m\varphi(s)\|^2]ds + 2\sqrt{2}a\int_0^t e^{\mu_2(s-t)}\mathbb{E}[\|\theta_m u(s)\|^2]ds.
\end{aligned}
$$

Since $\{\varphi(s)\in L^2(\Omega,H)|s\in[-\rho,0]\}$ is compact, it has a open cover of balls with radius $\frac{\sqrt{\varepsilon}}{2}$ which denoted by $\{B(\varphi^i,\frac{\sqrt{\varepsilon}}{2})\}_{i=1}^l$. Since $\varphi^i=\varphi(s_i)\in L^2(\Omega;C([-\rho,0],H))$ for $i=1,2,\cdots,l$, we obtain that for given $\varepsilon>0$,

$$\{\varphi(s)\in L^2(\Omega;C([-\rho,0],H))\}\subseteq\cup_{i=1}^l\left\{X\in L^2(\Omega,H)|\|X-\varphi^i\|_{L^2(\Omega,H)}<\frac{\sqrt{\varepsilon}}{2}\right\}.$$

Since $\varphi^i\in L^2(\Omega,H)$, there exists a positive constant $N_5=N_5(\varepsilon,\varphi)\ge N_4$, for $m\ge N_5$, we have

$$\sup_{i=1,2,\cdots,l}\int_{|x|\ge m}\mathbb{E}[|\varphi(s_i,x)|^2]dx<\frac{\varepsilon}{4}.$$

Then,

$$\sup_{s\in[-\rho,0]}\int_{|x|\ge m}\mathbb{E}[|\varphi(s,x)|^2]dx<\frac{\varepsilon}{2},\forall m\ge N_5.$$

Consequently, one has

$$
\begin{aligned}
&2\int_0^t e^{\mu_2(s-t)}\mathbb{E}\left[\mathrm{Re}(\theta_m u, \theta_m G(x,u(s-\rho)))\right]ds\\
&\le \frac{\sqrt{2}}{a\mu_2}\varepsilon + \sqrt{2}a\rho\frac{\varepsilon}{2} + 2\sqrt{2}a\int_0^t e^{\mu_2(s-t)}\mathbb{E}[\|\theta_m u(s)\|^2]ds.
\end{aligned}
\tag{3.32}
$$

For the fifth term on the right-hand side of (3.24), by (2.6), we obtain

$$
\begin{aligned}
&\sum_{k=1}^\infty\int_0^t e^{\mu_2(s-t)}\mathbb{E}\left[\|\theta_m(\sigma_{1,k}+\kappa(x)\sigma_{2,k}(u(s)))\|^2\right]ds\\
&\le 2\sum_{k=1}^\infty\int_0^t e^{\mu_2(s-t)}\|\theta_m\sigma_{1,k}\|^2ds + 2\sum_{k=1}^\infty\int_0^t e^{\mu_2(s-t)}\mathbb{E}\left[\|\theta_m\kappa(x)\sigma_{2,k}(u(s))\|^2\right]ds\\
&\le \frac{2}{\mu_2}\sum_{k=1}^\infty\int_{|x|\ge m}|\sigma_{1,k}(x)|^2dx + \frac{4}{\mu_2}\sum_{k=1}^\infty\beta_k^2\int_{|x|\ge m}\kappa^2(x)dx\\
&\quad + 4\|\kappa(x)\|_{L^\infty}^2\sum_{k=1}^\infty\gamma_k^2\int_0^t e^{\mu_2(s-t)}\mathbb{E}[\|\theta_m u(s)\|^2]ds.
\end{aligned}
$$

Since $\sum_{k=1}^{\infty}\|\sigma_{1,k}\|^2<\infty$ and $\kappa(x)\in L^2(\mathbb{R}^n)\bigcap L^\infty(\mathbb{R}^n)$, there exists $N_6=N_6(\varepsilon,\varphi)\geq N_5$, for all $t\geq 0$ and $m\geq N_6$, we have

$$\sum_{k=1}^{\infty}\int_{|x|\geq m}|\sigma_{1,k}(x)|^2dx+\int_{|x|\geq m}\kappa^2(x)dx<\varepsilon.$$

Consequently, for the fifth term on the right-hand side of (3.24), we get for all $t\geq 0$ and $m\geq N_6$,

$$\sum_{k=1}^{\infty}\int_0^t e^{\mu_2(s-t)}\mathbb{E}\left[\|\theta_m\left(\sigma_{1,k}+\kappa\sigma_{2,k}\right)\|^2\right]ds\leq\frac{2}{\mu_2}(1+2\sum_{k=1}^{\infty}\beta_k^2)\varepsilon$$

$$+4\|\kappa\|_{L^\infty}^2\sum_{k=1}^{\infty}\gamma_k^2\int_0^t e^{\mu_2(s-t)}\mathbb{E}[\|\theta_m u(s)\|^2]ds.$$

Therefore, for all $t\geq 0$ and $m\geq N_6$,

$$\mathbb{E}[\|\theta_m u(t)\|^2]\leq\left[2+e^{-\mu_2 t}+\frac{\sqrt{2}}{a\mu_2}+\frac{\sqrt{2}}{2}a\rho+\frac{2}{\mu_2}(1+2\sum_{k=1}^{\infty}\beta_k^2)\right]\varepsilon$$

$$+\left(\mu_2-2\lambda+2\sqrt{2}a+4\|\kappa\|_{L^\infty}^2\sum_{k=1}^{\infty}\gamma_k^2\right)\int_0^t e^{\mu_2(s-t)}\mathbb{E}[\|\theta_m u(s)\|^2]ds.$$

Taking the limit in the above equation and by (3.23), we have

$$\limsup_{m\to\infty}\sup_{t\geq-\rho}\int_{|x|\geq m}\mathbb{E}[|u(t,x)|^2]dx=0,$$

which completes the proof.

Lemma 3.4. *Suppose* (2.1)–(2.6) *and* (3.2) *hold. If* $\varphi(s)\in L^2(\Omega,C([-\rho,0],H))$*, then the solution* u *of* (1.1) *and* (1.2) *satisfies*

$$\limsup_{m\to\infty}\sup_{t\geq 0}\mathbb{E}\left[\sup_{r\in[t-\rho,t]}\int_{|x|\geq m}|u(r,x)|^2dx\right]=0.$$

Proof. By (3.22) and integration by parts of Ito's formula and taking the real part, for all $t\geq\rho$ and $r\in[t-\rho,t]$, we have

$$\begin{aligned}
&e^{\mu_2 r}\|\theta_m u(r)\|^2+2\int_{t-\rho}^r e^{\mu_2 s}\int_{\mathbb{R}^n}\theta_m^2|u|^{2\beta+2}dxds\\
&=e^{\mu_2(t-\rho)}\|\theta_m u(t-\rho)\|^2-2\int_{t-\rho}^r e^{\mu_2 s}\mathrm{Re}(1+\mathrm{i}\nu)\left((-\Delta)^{\frac{\alpha}{2}}u(s),(-\Delta)^{\frac{\alpha}{2}}\theta_m^2u(s)\right)ds\\
&\quad+(\mu_2-2\lambda)\int_{t-\rho}^r e^{\mu_2 s}\|\theta_m u(s)\|^2ds+2\mathrm{Re}\int_{t-\rho}^r e^{\mu_2 s}(\theta_m u(s),\theta_m G(x,u(s-\rho)))ds\\
&\quad+\sum_{k=1}^{\infty}\int_{t-\rho}^r e^{\mu_2 s}\|\theta_m(\sigma_{1,k}+\kappa(x)\sigma_{2,k}(u(s)))\|^2ds\\
&\quad+2\mathrm{Re}\sum_{k=1}^{\infty}\int_{t-\rho}^r e^{\mu_2 s}(\theta_m u(s),\theta_m(\sigma_{1,k}+\kappa\sigma_{2,k}(u(s))))dW_k(s).
\end{aligned}\tag{3.33}$$

By (3.33), we deduce,

$$
\begin{aligned}
&\mathbb{E}[\sup_{t-\rho\le r\le t}\|\theta_m u(r)\|^2] \\
&\le \mathbb{E}[\|\theta_m u(t-\rho)\|^2] - 2\mathbb{E}\left[\sup_{t-\rho\le r\le t}\int_{t-\rho}^{r} e^{\mu_2(s-r)}\mathrm{Re}(1+\mathrm{i}\nu)\left((-\Delta)^{\frac{\alpha}{2}}u, (-\Delta)^{\frac{\alpha}{2}}\theta_m^2 u\right)ds\right] \\
&\quad + |\mu_2 - 2\lambda|\mathbb{E}\left[\sup_{t-\rho\le r\le t}\int_{t-\rho}^{r}\|\theta_m u\|^2 e^{\mu_2(s-r)}ds\right] \\
&\quad + 2\mathbb{E}\left[\sup_{t-\rho\le r\le t}\int_{t-\rho}^{r} e^{\mu_2(s-r)}\|\theta_m u\|\cdot\|\theta_m G(x,u(s-\rho))\|ds\right] \\
&\quad + \sum_{k=1}^{\infty}\mathbb{E}\left[\sup_{t-\rho\le r\le t}\int_{t-\rho}^{r} e^{\mu_2(s-r)}\|\theta_m(\sigma_{1,k}+\kappa\sigma_{2,k}(u(s)))\|^2 ds\right] \\
&\quad + 2\mathbb{E}\left[\sup_{t-\rho\le r\le t}\left|\sum_{k=1}^{\infty}\int_{t-\rho}^{r} e^{\mu_2(s-r)}(\theta_m u(s), \theta_m(\sigma_{1,k}+\kappa\sigma_{2,k}(u(s))))dW_k(s)\right|\right].
\end{aligned}
\tag{3.34}
$$

For the first term on the right-hand side of (3.34), by Lemma 3.3, one has for any $\varepsilon > 0$, there exists $\tilde{N}_1(\varepsilon,\varphi) \ge 1$ such that for all $m \ge \tilde{N}_1$ and $t \ge \rho$,

$$
\mathbb{E}[\|\theta_m u(t-\rho)\|^2] \le \int_{|x|\ge m}\mathbb{E}[|u(t-\rho,x)|^2]dx < \varepsilon. \tag{3.35}
$$

For the second term on the right-hand side of (3.34), by (3.29), we have

$$
\begin{aligned}
&-2\mathbb{E}\left[\sup_{t-\rho\le r\le t}\int_{t-\rho}^{r} e^{\mu_2(s-r)}\mathrm{Re}(1+\mathrm{i}\nu)\left((-\Delta)^{\frac{\alpha}{2}}u(s), (-\Delta)^{\frac{\alpha}{2}}\theta_m^2 u(s)\right)ds\right] \\
&\le 2\sqrt{c_6(1+\nu^2)}C(n,\alpha)m^{-\alpha}\mathbb{E}\left[\sup_{t-\rho\le r\le t}\left(\int_{t-\rho}^{r} e^{\mu_2(s-r)}\|u(s)\|\|(-\Delta)^{\frac{\alpha}{2}}u(s)\|ds\right)\right] \\
&\le 2\sqrt{c_6(1+\nu^2)}C(n,\alpha)m^{-\alpha}e^{\mu_2\rho}\mathbb{E}\left[\left(\int_{t-\rho}^{t} e^{\mu_2(s-t)}\|u(s)\|\|(-\Delta)^{\frac{\alpha}{2}}u(s)\|ds\right)\right] \\
&\le \sqrt{c_6(1+\nu^2)}C(n,\alpha)m^{-\alpha}e^{\mu_2\rho}\left\{\int_{t-\rho}^{t} e^{\mu_2(s-t)}\mathbb{E}[\|u\|^2]ds + \mathbb{E}\left[\int_{t-\rho}^{t} e^{\mu_2(s-t)}\|(-\Delta)^{\frac{\alpha}{2}}u\|^2 ds\right]\right\} \\
&\le \sqrt{c_6(1+\nu^2)}C(n,\alpha)m^{-\alpha}e^{\mu_2\rho}\left\{\rho\sup_{s\in[t-\rho,t]}\mathbb{E}[\|u(s)\|^2] + \mathbb{E}\left[\int_{t-\rho}^{t} e^{\mu_2(s-t)}\|(-\Delta)^{\frac{\alpha}{2}}u\|^2 ds\right]\right\}.
\end{aligned}
\tag{3.36}
$$

By Lemma 3.1 and (3.36), we deduce that there exists $\tilde{N}_2(\varepsilon,\varphi) \ge \tilde{N}_1$ such that for all $m \ge \tilde{N}_2$ and $t \ge \rho$,

$$
-2\mathbb{E}\left[\sup_{t-\rho\le r\le t}\int_{t-\rho}^{r} e^{\mu_2(s-r)}\mathrm{Re}(1+\mathrm{i}\nu)\left((-\Delta)^{\frac{\alpha}{2}}u(s), (-\Delta)^{\frac{\alpha}{2}}\theta_m^2 u(s)\right)ds\right] < \varepsilon. \tag{3.37}
$$

For the third term on the right-hand side of (3.34), by Lemma 3.3, we obtain that for all $m \ge \tilde{N}_2$ and $t \ge \rho$,

$$|\mu_2-2\lambda|\mathbb{E}\left[\sup_{t-\rho\le r\le t}\int_{t-\rho}^{r}\|\theta_m u(s)\|^2e^{\mu_2(s-r)}ds\right]\le|\mu_2-2\lambda|\mathbb{E}\left[\int_{t-\rho}^{t}\|\theta_m u(s)\|^2ds\right]$$

$$\le|\mu_2-2\lambda|\rho\sup_{t-\rho\le s\le t}\mathbb{E}[\|\theta_m u(s)\|^2]<|\mu_2-2\lambda|\rho\varepsilon. \quad (3.38)$$

For the forth term on the right-hand side of (3.34), by (2.1), we obtain

$$2\mathbb{E}\left[\sup_{t-\rho\le r\le t}\int_{t-\rho}^{r}e^{\mu_2(s-r)}\|\theta_m u(s)\|\cdot\|\theta_m G(x,u(s-\rho))\|ds\right]$$
$$\le\int_{t-\rho}^{t}\mathbb{E}[\|\theta_m u(s)\|^2]ds+2\rho\|\theta_m h\|^2+2a^2\int_{t-2\rho}^{t-\rho}\mathbb{E}[\|\theta_m u(s)\|^2]ds$$
$$\le\rho\sup_{t-\rho\le s\le t}\mathbb{E}[\|\theta_m u(s)\|^2]+2\rho\|\theta_m h\|^2+2a^2\rho\sup_{t-2\rho\le s\le t-\rho}\mathbb{E}[\|\theta_m u(s)\|^2],$$

which along with Lemma 3.3, we deduce that there exists $\tilde{N}_3(\varepsilon,\varphi)\ge\tilde{N}_2$ such that for all $m\ge\tilde{N}_3$ and $t\ge\rho$,

$$2\mathbb{E}\left[\sup_{t-\rho\le r\le t}\int_{t-\rho}^{r}e^{\mu_2(s-r)}\|\theta_m u(s)\|\cdot\|\theta_m G(x,u(s-\rho))\|ds\right]<(3+2a^2)\rho\varepsilon. \quad (3.39)$$

For the fifth term on the right-hand side of (3.34), by (2.6), we have

$$\sum_{k=1}^{\infty}\mathbb{E}\left[\sup_{t-\rho\le r\le t}\int_{t-\rho}^{r}e^{\mu_2(s-r)}\|\theta_m(\sigma_{1,k}+\kappa\sigma_{2,k}(u(s)))\|^2ds\right]$$
$$\le2\rho\sum_{k=1}^{\infty}\|\theta_m\sigma_{1,k}\|^2+2\rho\sum_{k=1}^{\infty}\sup_{t-\rho\le s\le t}\mathbb{E}[\|\theta_m\kappa(x)\sigma_{2,k}(u(s))\|^2]$$
$$\le2\rho\sum_{k=1}^{\infty}\int_{|x|\ge m}|\sigma_{1,k}(x)|^2dx+4\rho\sum_{k=1}^{\infty}\beta_k^2\int_{|x|\ge m}|\kappa(x)|^2dx+4\rho\|\kappa(x)\|_{L^\infty}^2\sum_{k=1}^{\infty}\gamma_k^2\sup_{t-\rho\le s\le t}\mathbb{E}[\|\theta_m u(s)\|^2].$$

By the condition $\kappa(x)\in L^2(\mathbb{R}^n)\bigcap L^\infty(\mathbb{R}^n)$, (2.4) and Lemma 3.3, we deduce that there exists $\tilde{N}_4(\varepsilon,\varphi)\ge\tilde{N}_3$ such that for all $m\ge\tilde{N}_4$ and $t\ge\rho$,

$$\sum_{k=1}^{\infty}\mathbb{E}\left[\sup_{t-\rho\le r\le t}\int_{t-\rho}^{r}e^{\mu_2(s-r)}\|\theta_m(\sigma_{1,k}+\kappa\sigma_{2,k}(u(s)))\|^2ds\right]<2\rho(1+\lambda+2\sum_{k=1}^{\infty}\beta_k^2)\varepsilon. \quad (3.40)$$

For the sixth term on the right-hand side of (3.34), by (2.6), (3.40) and Burkholder-Davis-Gundy's inequality, we have,

$$2\mathbb{E}\left[\sup_{t-\rho\le r\le t}\left|\sum_{k=1}^{\infty}\int_{t-\rho}^{r}e^{\mu_2(s-r)}(\theta_m u(s),\theta_m(\sigma_{1,k}+\kappa\sigma_{2,k}(u(s))))dW_k(s)\right|\right]$$
$$\le2e^{-\mu_2(t-\rho)}\mathbb{E}\left[\sup_{t-\rho\le r\le t}\left|\sum_{k=1}^{\infty}\int_{t-\rho}^{r}e^{\mu_2 s}(\theta_m u(s),\theta_m\sigma_{1,k}+\theta_m\kappa(x)\sigma_{2,k}(u(s)))dW_k(s)\right|\right]$$

$$\le 2\tilde{B}_2 e^{-\mu_2(t-\rho)}\mathbb{E}\left[\left(\int_{t-\rho}^{t} e^{2\mu_2 s}\sum_{k=1}^{\infty}|(\theta_m u(s),\theta_m\sigma_{1,k}+\theta_m\kappa(x)\sigma_{2,k}(u(s)))|^2 ds\right)^{\frac{1}{2}}\right]$$
$$\le 2\tilde{B}_2 e^{-\mu_2(t-\rho)}\mathbb{E}\left[\sup_{t-\rho\le s\le t}\|\theta_m u(s)\|\left(\int_{t-\rho}^{t} e^{2\mu_2 s}\sum_{k=1}^{\infty}\|\theta_m\sigma_{1,k}+\theta_m\kappa\sigma_{2,k}(u(s))\|^2 ds\right)^{\frac{1}{2}}\right]$$
$$\le \frac{1}{2}\mathbb{E}\left[\sup_{t-\rho\le s\le t}\|\theta_m u(s)\|^2\right]+2\tilde{B}_2^2\mathbb{E}\left[e^{2\mu_2\rho}\int_{t-\rho}^{t} e^{2\mu_2(s-t)}\sum_{k=1}^{\infty}\|\theta_m\sigma_{1,k}+\theta_m\kappa(x)\sigma_{2,k}(u(s))\|^2 ds\right]$$
$$\le \frac{1}{2}\mathbb{E}\left[\sup_{t-\rho\le s\le t}\|\theta_m u(s)\|^2\right]+2\tilde{B}_2^2 e^{2\mu_2\rho}\sum_{k=1}^{\infty}\mathbb{E}\left[\sup_{t-\rho\le r\le t}\int_{t-\rho}^{r} e^{\mu_2(s-r)}\|\theta_m\sigma_{1,k}+\theta_m\kappa(x)\sigma_{2,k}(u(s))\|^2 ds\right]$$
$$\le \frac{1}{2}\mathbb{E}\left[\sup_{t-\rho\le s\le t}\|\theta_m u(s)\|^2\right]+4\rho(1+\lambda+2\sum_{k=1}^{\infty}\beta_k^2)\tilde{B}_2^2 e^{2\mu_2\rho}\varepsilon.$$

Above all, for all $m \ge \tilde{N}_4$ and $t \ge \rho$, we obtain,

$$\mathbb{E}\left[\sup_{t-\rho\le r\le t}\|\theta_m u(r)\|^2\right]\le\left[4+2|\mu_2-2\lambda|\rho+(6+4a^2)\rho+4\rho(1+2\tilde{B}_2^2 e^{2\mu_2\rho})(1+\lambda+2\sum_{k=1}^{\infty}\beta_k^2)\right]\varepsilon.$$

Therefore, we conclude

$$\lim_{m\to\infty}\sup\sup_{t\ge0}\mathbb{E}\left[\sup_{t-\rho\le r\le t}\int_{|x|\ge m}|u(r,x)|^2 dx\right]=0.$$

Lemma 3.5. *Suppose* (2.1)–(2.6) *and* (3.1) *hold. If* $\varphi(s)\in L^2(\Omega, C([-\rho,0],H))$*, then there exists a positive constant* μ_3 *such that the solution* u *of* (1.1) *and* (1.2) *satisfies*

$$\begin{aligned}&\sup_{t\ge-\rho}\mathbb{E}[\|u(t)\|^{2p}]+\sup_{t\ge0}\mathbb{E}\left[\int_0^t e^{\mu_3(s-t)}\|u(s)\|^{2p-2}\|(-\Delta)^{\frac{\alpha}{2}}u(s)\|^2 ds\right]\\&\quad\le\left(1+a\rho e^{\frac{\mu\rho}{2p}}(4p-2)^{\frac{2p-1}{2p}}\right)\mathbb{E}\left[\|\varphi\|_{C_H}^{2p}\right]+M_3,\end{aligned}\tag{3.41}$$

where M_3 *is a positive constant independent of* φ*.*

Proof. By (3.1), there exist positive constants μ and ϵ_1 such that

$$\begin{aligned}&\mu+ae^{\frac{\mu\rho}{2p}}2^{1-\frac{1}{2p}}(2p-1)^{\frac{2p-1}{2p}}+4(p-1)(2p-1)\epsilon_1^{\frac{2p}{2p-2}}\sum_{k=1}^{\infty}(\|\sigma_{1,k}\|^2+\|\kappa\|^2\beta_k^2)\\&\quad+4\theta(2p-1)\|\kappa\|_{L^\infty}^2\sum_{k=1}^{\infty}\gamma_k^2\le2p\lambda.\end{aligned}\tag{3.42}$$

Given $n\in\mathbb{N}$, let τ_n be a stopping time as defined by

$$\tau_n=\inf\{t\ge0:\|u(t)\|>n\},$$

and as usual, we set $\tau_n = +\infty$ if $\{t \geq 0 : \|u(t)\| > n\} = \emptyset$. By the continuity of solutions, we have

$$\lim_{n\to\infty} \tau_n = +\infty.$$

Applying Ito's formula, we obtain

$$\begin{aligned} &d(\|u(t)\|^{2p}) = d((\|u(t)\|^2)^p) \\ &= p\|u(t)\|^{2(p-1)} d(\|u(t)\|^2) + 2p(p-1)\|u(t)\|^{2(p-2)} \\ &\quad \times \sum_{k=1}^{\infty} \left|(u(t), \sigma_{1,k} + \kappa\sigma_{2,k}(u(t)))\right|^2 dt. \end{aligned} \tag{3.43}$$

Substituting (3.6) into (3.43), we infer

$$\begin{aligned} d(\|u(t)\|^{2p}) = &-2p\|u(t)\|^{2(p-1)}\|(-\Delta)^{\frac{\alpha}{2}} u(t)\|^2 dt - 2p\|u(t)\|^{2(p-1)}\|u(t)\|^{2\beta+2}_{L^{2\beta+2}} dt - 2p\lambda\|u(t)\|^{2p} dt \\ &+ 2p\|u(t)\|^{2(p-1)}\mathrm{Re}(u(t), G(x, u(t-\rho)))dt \\ &+ p\|u(t)\|^{2(p-1)} \sum_{k=1}^{\infty} \|\sigma_{1,k}(x) + \kappa(x)\sigma_{2,k}(u(t))\|^2 dt \\ &+ 2p\|u(t)\|^{2(p-1)}\mathrm{Re}(u(t), \sum_{k=1}^{\infty} \sigma_{1,k}(x) + \kappa(x)\sigma_{2,k}(u(t)))dW_k(t) \\ &+ 2p(p-1)\|u(t)\|^{2(p-2)} \sum_{k=1}^{\infty} \left|(u(t), \sigma_{1,k} + \kappa\sigma_{2,k}(u(t)))\right|^2 dt. \end{aligned} \tag{3.44}$$

We also get the formula

$$d(e^{\mu t}\|u(t)\|^{2p}) = \mu e^{\mu t}\|u(t)\|^{2p} dt + e^{\mu t} d(\|u(t)\|^{2p}). \tag{3.45}$$

Substituting (3.44) into (3.45) and integrating on $(0, t \wedge \tau_n)$ with $t \geq 0$, we deduce

$$\begin{aligned} &e^{\mu(t\wedge\tau_n)}\|u(t \wedge \tau_n)\|^{2p} + 2p \int_0^{t\wedge\tau_n} e^{\mu s}\|u(s)\|^{2(p-1)}\|(-\Delta)^{\frac{\alpha}{2}} u(s)\|^2 ds \\ &= -2p \int_0^{t\wedge\tau_n} e^{\mu s}\|u(s)\|^{2(p-1)}\|u(s)\|^{2\beta+2}_{L^{2\beta+2}} ds + \|\varphi(0)\|^{2p} + (\mu - 2p\lambda) \int_0^{t\wedge\tau_n} e^{\mu s}\|u(s)\|^{2p} ds \\ &\quad + 2p \int_0^{t\wedge\tau_n} e^{\mu s}\|u(s)\|^{2(p-1)}\mathrm{Re}(u(s), G(x, u(s-\rho)))ds \\ &\quad + p \sum_{k=1}^{\infty} \int_0^{t\wedge\tau_n} e^{\mu s}\|u(s)\|^{2(p-1)}\|\sigma_{1,k} + \kappa\sigma_{2,k}(u(s))\|^2 ds \\ &\quad + 2p \sum_{k=1}^{\infty} \int_0^{t\wedge\tau_n} e^{\mu s}\|u(t)\|^{2(p-1)}\mathrm{Re}(u(s), \sigma_{1,k} + \kappa\sigma_{2,k}(u(s)))dW_k(s) \\ &\quad + 2p(p-1) \sum_{k=1}^{\infty} \int_0^{t\wedge\tau_n} e^{\mu s}\|u(s)\|^{2(p-2)} \left|(u(s), \sigma_{1,k} + \kappa\sigma_{2,k}(u(s)))\right|^2 ds. \end{aligned} \tag{3.46}$$

Taking the expectation, we obtain for $t \geq 0$,

$$
\begin{aligned}
&\mathbb{E}\left[e^{\mu(t\wedge\tau_n)}\|u(t\wedge\tau_n)\|^{2p}\right] + 2p\mathbb{E}\left[\int_0^{t\wedge\tau_n} e^{\mu s}\|u(s)\|^{2(p-1)}\|(-\Delta)^{\frac{\alpha}{2}}u(s)\|^2 ds\right]\\
&= -2p\mathbb{E}\left[\int_0^{t\wedge\tau_n} e^{\mu s}\|u(s)\|^{2(p-1)}\|u(s)\|^{2\beta+2}_{L^{2\beta+2}}ds\right]+\mathbb{E}\left[\|\varphi(0)\|^{2p}\right]+(\mu-2p\lambda)\mathbb{E}\left[\int_0^{t\wedge\tau_n} e^{\mu s}\|u(s)\|^{2p}ds\right]\\
&\quad + 2p\mathbb{E}\left[\int_0^{t\wedge\tau_n} e^{\mu s}\|u(s)\|^{2(p-1)}\mathrm{Re}(u(s), G(x, u(s-\rho)))ds\right]\\
&\quad + p\sum_{k=1}^{\infty}\mathbb{E}\left[\int_0^{t\wedge\tau_n} e^{\mu s}\|u(s)\|^{2(p-1)}\|\sigma_{1,k}+\kappa\sigma_{2,k}(u(s))\|^2 ds\right]\\
&\quad + 2p(p-1)\sum_{k=1}^{\infty}\mathbb{E}\left[\int_0^{t\wedge\tau_n} e^{\mu s}\|u(s)\|^{2(p-2)}\left|(u(s), \sigma_{1,k}+\kappa\sigma_{2,k}(u(s)))\right|^2 ds\right]\\
&\leq \mathbb{E}\left[\|\varphi(0)\|^{2p}\right] + (\mu-2p\lambda)\mathbb{E}\left[\int_0^{t\wedge\tau_n} e^{\mu s}\|u(s)\|^{2p}ds\right]\\
&\quad + 2p\mathbb{E}\left[\int_0^{t\wedge\tau_n} e^{\mu s}\|u(s)\|^{2(p-1)}\mathrm{Re}(u(s), G(x, u(s-\rho)))ds\right]\\
&\quad + p\sum_{k=1}^{\infty}\mathbb{E}\left[\int_0^{t\wedge\tau_n} e^{\mu s}\|u(s)\|^{2(p-1)}\|\sigma_{1,k}+\kappa\sigma_{2,k}(u(s))\|^2 ds\right]\\
&\quad + 2p(p-1)\sum_{k=1}^{\infty}\mathbb{E}\left[\int_0^{t\wedge\tau_n} e^{\mu s}\|u(s)\|^{2(p-2)}\left|(u(s), \sigma_{1,k}+\kappa\sigma_{2,k}(u(s)))\right|^2 ds\right].
\end{aligned}
\tag{3.47}
$$

Next, we estimate the terms on the right-hand side of (3.47).
For the third term on the right-hand side of (3.47), by Young's inequality and (2.1), we infer

$$
\begin{aligned}
&2\theta\mathbb{E}\left[\int_0^{t\wedge\tau_n} e^{\mu s}\|u(s)\|^{2(p-1)}\mathrm{Re}(u(s), G(x, u(s-\rho)))ds\right]\\
&\leq 2\theta\mathbb{E}\left[\int_0^{t\wedge\tau_n} e^{\mu s}\|u(s)\|^{2p-1}\|G(x, u(s-\rho))\|^2 ds\right]\\
&\leq ae^{\frac{\mu\rho}{2p}}2^{1-\frac{1}{2p}}(2p-1)^{\frac{2p-1}{2p}}\mathbb{E}\left[\int_0^{t\wedge\tau_n} e^{\mu s}\|u(s)\|^{2p}ds\right]\\
&\quad + \left(\frac{2p-1}{2^{2p-1}a^{2p}e^{\mu\rho}}\right)^{\frac{2p-1}{2p}}\mathbb{E}\left[\int_0^{t\wedge\tau_n} e^{\mu s}\|G(x, u(s-\rho))\|^2 ds\right]\\
&\leq ae^{\frac{\mu\rho}{2p}}2^{1-\frac{1}{2p}}(2p-1)^{\frac{2p-1}{2p}}\mathbb{E}\left[\int_0^{t\wedge\tau_n} e^{\mu s}\|u(s)\|^{2p}ds\right]\\
&\quad + 2^{2p-1}\left(\frac{2p-1}{2^{2p-1}a^{2p}e^{\mu\rho}}\right)^{\frac{2p-1}{2p}}\mathbb{E}\left[\int_0^{t\wedge\tau_n} e^{\mu s}\left(\|h\|^{2p}+a^{2p}\|u(s-\rho)\|^{2p}\right)ds\right]\\
&\leq ae^{\frac{\mu\rho}{2p}}2^{1-\frac{1}{2p}}(2p-1)^{\frac{2p-1}{2p}}\mathbb{E}\left[\int_0^{t\wedge\tau_n} e^{\mu s}\|u(s)\|^{2p}ds\right]\\
&\quad + \frac{1}{\mu}\left(\frac{4p-2}{a^{2p}e^{\mu\rho}}\right)^{\frac{2p-1}{2p}}\|h\|^{2p}e^{\mu t} + a\rho e^{\frac{\mu\rho}{2p}}(4p-2)^{\frac{2p-1}{2p}}\mathbb{E}\left[\|\varphi\|^{2p}_{C_H}\right].
\end{aligned}
\tag{3.48}
$$

For the forth term on the right-hand side of (3.47), we infer

$$
\begin{aligned}
& p\sum_{k=1}^{\infty}\mathbb{E}\left[\int_0^{t\wedge\tau_n} e^{\mu s}\|u(s)\|^{2(p-1)}\|\sigma_{1,k}+\kappa\sigma_{2,k}(u(s))\|^2 ds\right] \\
\leq\ & 2p\sum_{k=1}^{\infty}\mathbb{E}\left[\int_0^{t\wedge\tau_n} e^{\mu s}\|u(s)\|^{2(p-1)}\|\sigma_{1,k}\|^2 ds\right] \\
& + 2p\sum_{k=1}^{\infty}\mathbb{E}\left[\int_0^{t\wedge\tau_n} e^{\mu s}\|u(s)\|^{2(p-1)}\|\kappa\sigma_{2,k}(u(s))\|^2 ds\right].
\end{aligned} \tag{3.49}
$$

For the first term on the right-hand side of (3.49), we have

$$
\begin{aligned}
& 2p\sum_{k=1}^{\infty}\mathbb{E}\left[\int_0^{t\wedge\tau_n} e^{\mu s}\|u(s)\|^{2(p-1)}\|\sigma_{1,k}\|^2 ds\right] \\
\leq\ & 2(p-1)\epsilon_1^{\frac{2p}{2p-2}}\sum_{k=1}^{\infty}\|\sigma_{1,k}\|^2\mathbb{E}\left[\int_0^{t\wedge\tau_n} e^{\mu s}\|u(s)\|^{2p} ds\right] + \frac{2}{\mu\epsilon_1^p}\sum_{k=1}^{\infty}\|\sigma_{1,k}\|^2 e^{\mu t}.
\end{aligned} \tag{3.50}
$$

For the second term on the right-hand side of (3.49), we have

$$
\begin{aligned}
& 2p\sum_{k=1}^{\infty}\mathbb{E}\left[\int_0^{t\wedge\tau_n} e^{\mu s}\|u(s)\|^{2(p-1)}\|\kappa\sigma_{2,k}(u(s))\|^2 ds\right] \\
\leq\ & 4p\|\kappa\|^2\sum_{k=1}^{\infty}\beta_k^2\mathbb{E}\left[\int_0^{t\wedge\tau_n} e^{\mu s}\|u(s)\|^{2(p-1)} ds\right] + 4p\|\kappa\|_{L^\infty}^2\sum_{k=1}^{\infty}\gamma_k^2\mathbb{E}\left[\int_0^{t\wedge\tau_n} e^{\mu s}\|u(s)\|^{2p} ds\right] \\
\leq\ & 4(p-1)\epsilon_1^{\frac{2p}{2p-2}}\|\kappa\|^2\sum_{k=1}^{\infty}\beta_k^2\mathbb{E}\left[\int_0^{t\wedge\tau_n} e^{\mu s}\|u(s)\|^{2p} ds\right] \\
& + \frac{4}{\mu\epsilon_1^p}\|\kappa\|^2\sum_{k=1}^{\infty}\beta_k^2 e^{\mu t} + 4p\|\kappa\|_{L^\infty}^2\sum_{k=1}^{\infty}\gamma_k^2\mathbb{E}\left[\int_0^{t\wedge\tau_n} e^{\mu s}\|u(s)\|^{2p} ds\right].
\end{aligned} \tag{3.51}
$$

By (3.49)–(3.51), we obtain

$$
\begin{aligned}
& p\sum_{k=1}^{\infty}\mathbb{E}\left[\int_0^{t\wedge\tau_n} e^{\mu s}\|u(s)\|^{2(p-1)}\|\sigma_{1,k}+\kappa\sigma_{2,k}(u(s))\|^2 ds\right] \\
\leq\ & \left[4(p-1)\epsilon_1^{\frac{2p}{2p-2}}\sum_{k=1}^{\infty}(\|\sigma_{1,k}\|^2+\|\kappa\|^2\beta_k^2) + 4p\|\kappa\|_{L^\infty}^2\sum_{k=1}^{\infty}\gamma_k^2\right]\mathbb{E}\left[\int_0^{t\wedge\tau_n} e^{\mu s}\|u(s)\|^{2p} ds\right] \\
& + \frac{2}{\mu\epsilon_1^p}\sum_{k=1}^{\infty}(\|\sigma_{1,k}\|^2 + 2\|\kappa\|^2\beta_k^2))e^{\mu t}.
\end{aligned} \tag{3.52}
$$

For the fifth term on the right-hand side of (3.47), applying (3.52), we have

$$
\begin{aligned}
&2p(p-1)\sum_{k=1}^{\infty}\mathbb{E}\left[\int_0^{t\wedge\tau_n} e^{\mu s}\|u(s)\|^{2(p-2)}\left|(u(s),\sigma_{1,k}+\kappa\sigma_{2,k}(u(s)))\right|^2 ds\right]\\
&\le 2p(p-1)\sum_{k=1}^{\infty}\mathbb{E}\left[\int_0^{t\wedge\tau_n} e^{\mu s}\|u(s)\|^{2p-2}\|\sigma_{1,k}+\kappa\sigma_{2,k}(u(s))\|^2 ds\right]\\
&\le\left[8(p-1)^2\epsilon_1^{\frac{2p}{2p-2}}\sum_{k=1}^{\infty}(\|\sigma_{1,k}\|^2+\|\kappa\|^2\beta_k^2)+8p(p-1)\|\kappa\|_{L^\infty}^2\sum_{k=1}^{\infty}\gamma_k^2\right]\mathbb{E}\left[\int_0^{t\wedge\tau_n} e^{\mu s}\|u(s)\|^{2pa}ds\right]\\
&\quad+\frac{4(p-1)}{\mu\epsilon_1^p}\sum_{k=1}^{\infty}(\|\sigma_{1,k}\|^2+2\|\kappa\|^2\beta_k^2))e^{\mu t}.
\end{aligned}
\tag{3.53}
$$

From (3.47), (3.48), (3.52) and (3.53), we obtain that for $t\ge 0$,

$$
\begin{aligned}
&\mathbb{E}\left[e^{\mu(t\wedge\tau_n)}\|u(t\wedge\tau_n)\|^{2p}\right]+2p\mathbb{E}\left[\int_0^{t\wedge\tau_n} e^{\mu s}\|u(s)\|^{2(p-1)}\|(-\Delta)^{\frac{\alpha}{2}}u(s)\|^2 ds\right]\\
&\le\left(1+a\rho e^{\frac{\mu\rho}{2p}}(4p-2)^{\frac{2p-1}{2p}}\right)\mathbb{E}\left[\|\varphi\|_{C_H}^{2p}\right]\\
&\quad+\left(\mu-2p\lambda+ae^{\frac{\mu\rho}{2p}}2^{1-\frac{1}{2p}}(2p-1)^{\frac{2p-1}{2p}}+4(p-1)(2p-1)\epsilon_1^{\frac{2p}{2p-2}}\right.\\
&\quad\left.\times\sum_{k=1}^{\infty}(\|\sigma_{1,k}\|^2+\|\kappa\|^2\beta_k^2)+4p(2p-1)\|\kappa\|_{L^\infty}^2\sum_{k=1}^{\infty}\gamma_k^2\right)\mathbb{E}\left[\int_0^{t\wedge\tau_n} e^{\mu s}\|u(s)\|^{2p}ds\right]\\
&\quad+\frac{1}{\mu}\left(\frac{4p-2}{a^{2p}e^{\mu\rho}}\right)^{\frac{2p-1}{2p}}\|h\|^{2p}e^{\mu t}+\frac{4(p-1)}{\mu\epsilon_1^p}\sum_{k=1}^{\infty}(\|\sigma_{1,k}\|^2+2\|\kappa\|^2\beta_k^2))e^{\mu t}.
\end{aligned}
\tag{3.54}
$$

Then by (3.42) and (3.54), we obtain that for $t\ge 0$,

$$
\begin{aligned}
&\mathbb{E}\left[e^{\mu(t\wedge\tau_n)}\|u(t\wedge\tau_n)\|^{2p}\right]+2p\mathbb{E}\left[\int_0^{t\wedge\tau_n} e^{\mu s}\|u(s)\|^{2(p-1)}\|(-\Delta)^{\frac{\alpha}{2}}u(s)\|^2 ds\right]\\
&\le\left(1+a\rho e^{\frac{\mu\rho}{2p}}(4p-2)^{\frac{2p-1}{2p}}\right)\mathbb{E}\left[\|\varphi\|_{C_H}^{2p}\right]+\frac{1}{\mu}\left(\frac{4p-2}{a^{2pa}e^{\mu\rho}}\right)^{\frac{2p-1}{2p}}\|h\|^{2p}e^{\mu t}\\
&\quad+\frac{4(p-1)}{\mu\epsilon_1^p}\sum_{k=1}^{\infty}(\|\sigma_{1,k}\|^2+2\|\kappa\|^2\beta_k^2))e^{\mu t}.
\end{aligned}
\tag{3.55}
$$

Letting $n\to\infty$, by Fatou's Lemma, we deduce that for $t\ge 0$,

$$
\begin{aligned}
&\mathbb{E}\left[e^{\mu t}\|u(t)\|^{2p}\right]+2p\mathbb{E}\left[\int_0^{t} e^{\mu s}\|u(s)\|^{2(p-1)}\|(-\Delta)^{\frac{\alpha}{2}}u(s)\|^2 ds\right]\\
&\le\left(1+a\rho e^{\frac{\mu\rho}{2p}}(4\theta-2)^{\frac{2\theta-1}{2p}}\right)\mathbb{E}\left[\|\varphi\|_{C_H}^{2p}\right]+\frac{1}{\mu}\left(\frac{4p-2}{a^{2p}e^{\mu\rho}}\right)^{\frac{2p-1}{2p}}\|h\|^{2p}e^{\mu t}\\
&\quad+\frac{4(p-1)}{\mu\epsilon_1^p}\sum_{k=1}^{\infty}(\|\sigma_{1,k}\|^2+2\|\kappa\|^2\beta_k^2))e^{\mu t}.
\end{aligned}
$$

Hence, we have for $t \geq 0$,

$$\mathbb{E}\left[\|u(t)\|^{2p}\right] + 2p\mathbb{E}\left[\int_0^t e^{\mu(s-t)}\|u(s)\|^{2(p-1)}\|(-\Delta)^{\frac{\alpha}{2}}u(s)\|^2 ds\right]$$
$$\leq \left(1 + a\rho e^{\frac{\mu\rho}{2p}}(4p-2)^{\frac{2p-1}{2p}}\right)\mathbb{E}\left[\|\varphi\|_{C_H}^{2p}\right] + \frac{1}{\mu}\left(\frac{4p-2}{a^{2p}e^{\mu\rho}}\right)^{\frac{2p-1}{2p}} \|h\|^{2p}$$
$$+ \frac{4(p-1)}{\mu\epsilon_1^p}\sum_{k=1}^{\infty}(\|\sigma_{1,k}\|^2 + 2\|\kappa\|^2\beta_k^2)).$$

This implies the desired estimate.

4. Uniform estimates of solutions with initial data in $C([-\rho, 0], V)$

In this section, we establish the uniform estimates of solutions of problems (1.1) and (1.2) with initial data in $C([-\rho, 0], V)$. To the end, we assume that for each $k \in \mathbb{N}$, the function $\sigma_{1,k} \in V$ and

$$\sum_{k=1}^{\infty}\|\sigma_{1,k}\|_V^2 < \infty. \tag{4.1}$$

Furthermore, we assume that the function $\kappa \in V$ and there exists a constant $C > 0$ such that

$$|\nabla\kappa(x)| \leq C. \tag{4.2}$$

In the sequel, we further assume that the constant a, $\hat{\gamma}_k$ in (2.7) are sufficiently small in the sense that there exists a constant $p \geq 2$ such that

$$\hat{a}2^{1-\frac{1}{2p}}(2p-1)^{\frac{2p-1}{2p}} + 2p(2p-1)\|\kappa\|_{L^\infty}^2\sum_{k=1}^{\infty}(\beta_k^2 + \hat{\beta}_k^2 + \gamma_k^2 + \hat{\gamma}_k^2) < p\frac{\lambda}{2}. \tag{4.3}$$

By (4.3), we can find

$$\sqrt{2}\hat{a} + 2\|\kappa\|_{L^\infty}^2\sum_{k=1}^{\infty}\hat{\gamma}_k^2 < \frac{\lambda}{2}. \tag{4.4}$$

Lemma 4.1. *Suppose* (2.1)–(2.7) *and* (4.4) *hold. If* $\varphi(s) \in L^2(\Omega; C([-\rho, 0], V))$*, then, for all* $t \geq 0$*, there exists a positive constant* μ_4 *such that the solution* u *of* (1.1) *and* (1.2) *satisfies*

$$\sup_{s\geq-\rho}\mathbb{E}[\|\nabla u(t)\|^2] + \sup_{s\geq0}\mathbb{E}\left[\int_0^t e^{\mu_4(s-t)}\|(-\Delta)^{\frac{\alpha+1}{2}}u(s)\|^2 ds\right] \leq M_4\left(\mathbb{E}[\|\varphi\|_{C_V}^2] + 1\right), \tag{4.5}$$

where M_4 *is a positive constant independent of* φ*.*

Proof. By (4.4), there exists a positive constant μ_1 such that

$$\mu_1 - 2\lambda + 8\|\kappa\|_{L^\infty}^2\sum_{k=1}^{\infty}\hat{\gamma}_k^2 < 0. \tag{4.6}$$

By (1.1) and applying Ito's formula to $e^{\mu_1 t}\|\nabla u(t)\|^2$, we have for $t \geq 0$,

$$
\begin{aligned}
&e^{\mu_1 t}\|\nabla u(t)\|^2 + 2\int_0^t e^{\mu_1 s}\|(-\Delta)^{\frac{\alpha+1}{2}}u(s)\|^2 ds + 2\int_0^t e^{\mu_1 s}\mathrm{Re}\left((1+\mathrm{i}\mu)|u(s)|^{2\beta}u(s), -\Delta u(s)\right) ds \\
&= (\mu_1 - 2\lambda)\int_0^t e^{\mu_1 s}\|\nabla u(s)\|^2 ds + \|\nabla\varphi(0)\|^2 + 2\mathrm{Re}\int_0^t e^{\mu_1 s}(G(x, u(s-\rho)), -\Delta u(s))ds \\
&\quad + \sum_{k=1}^{\infty}\int_0^t e^{\mu_1 s}\|\nabla(\sigma_{1,k} + \kappa\sigma_{2,k}(u(s)))\|^2 ds \\
&\quad + 2\sum_{k=1}^{\infty}\mathrm{Re}\int_0^t e^{\mu_1 s}\left(\sigma_{1,k}(x) + \kappa(x)\sigma_{2,k}(u(s)), -\Delta u(s)\right) dW_k(s).
\end{aligned}
$$

Taking the expectation, we have for all $t \geq 0$,

$$
\begin{aligned}
&e^{\mu_1 t}\mathbb{E}[\|\nabla u(t)\|^2] + 2\mathbb{E}\left[\int_0^t e^{\mu_1 s}\|(-\Delta)^{\frac{\alpha+1}{2}}u(s)\|^2 ds\right] + 2\mathbb{E}\left[\int_0^t e^{\mu_1 s}\mathrm{Re}\left((1+\mathrm{i}\mu)|u(s)|^{2\beta}u(s), -\Delta u(s)\right) ds\right] \\
&= (\mu_1 - 2\lambda)\mathbb{E}\left[\int_0^t e^{\mu_1 s}\|\nabla u(s)\|^2 ds\right] + \mathbb{E}\left[\|\nabla\varphi(0)\|^2\right] + 2\mathbb{E}\left[\mathrm{Re}\int_0^t e^{\mu_1 s}(G(x, u(s-\rho)), -\Delta u(s))ds\right] \\
&\quad + \sum_{k=1}^{\infty}\mathbb{E}\left[\int_0^t e^{\mu_1 s}\|\nabla(\sigma_{1,k} + \kappa\sigma_{2,k}(u(s)))\|^2 ds\right].
\end{aligned}
\tag{4.7}
$$

First, we estimate the third term on the left-hand side of (4.7). Applying integrating by parts, we have

$$
\begin{aligned}
&\mathrm{Re}\left((1+\mathrm{i}\mu)|u|^{2\beta}u, \Delta u\right) \\
&= -\mathrm{Re}(1+\mathrm{i}\mu)\int_{\mathbb{R}^n}\left((\beta+1)|u|^{2\beta}|\nabla u|^2 + \beta|u|^{2(\beta-1)}(u\nabla\overline{u})^2\right) dx \\
&= \int_{\mathbb{R}^n}|u|^{2(\beta-1)}\left(-(\beta+1)|u|^2|\nabla u|^2 + \frac{\beta(1+\mathrm{i}\mu)}{2}(u\nabla\overline{u})^2 + \frac{\beta(1-\mathrm{i}\mu)}{2}(\overline{u}\nabla u)^2\right) dx \\
&= \int_{\mathbb{R}^n}|u|^{2(\beta-1)}trace(YMY^H),
\end{aligned}
\tag{4.8}
$$

where

$$
Y = \begin{pmatrix} \overline{u}\nabla u \\ u\nabla\overline{u} \end{pmatrix}^H, M = \begin{pmatrix} -\frac{\beta+1}{2} & \frac{\beta(1+\mathrm{i}\mu)}{2} \\ \frac{\beta(1-\mathrm{i}\mu)}{2} & -\frac{\beta+1}{2} \end{pmatrix},
$$

and Y^H is the conjugate transpose of the matrix Y. We observe that the condition $\beta \leq \frac{1}{\sqrt{1+\mu^2}-1}$ implies that the matrix M is nonpositive definite. Hence, we obtain

$$
2\mathbb{E}\left[\int_0^t e^{\mu_1 s}\mathrm{Re}\left((1+\mathrm{i}\mu)|u(s)|^{2\beta}u(s), \Delta u(s)\right) ds\right] \leq 0. \tag{4.9}
$$

Next, we estimate the terms on the right-hand side of (4.7). For the third term on the right-hand side of (4.7), applying (2.2) and Gagliardo-Nirenberg inequality, we have

$$
\begin{aligned}
&2\mathbb{E}\left[\mathrm{Re}\int_0^t e^{\mu_1 s}(G(x,u(s-\rho)),-\Delta u(s))ds\right] \le 2\mathbb{E}\left[\int_0^t e^{\mu_1 s}\|\nabla u(s)\|\|\nabla G(x,u(s-\rho))\|ds\right]\\
&\le \mathbb{E}\left[\int_0^t e^{\mu_1 s}\|\nabla u(s)\|^2 ds\right] + \mathbb{E}\left[\int_0^t e^{\mu_1 s}\|\nabla G(x,u(s-\rho))\|^2 ds\right]\\
&\le \mathbb{E}\left[\int_0^t e^{\mu_1 s}\|\nabla u(s)\|^2 ds\right] + 2\mathbb{E}\left[\int_0^t e^{\mu_1 s}\|\hat{h}(x)\|^2 ds\right] + 2\hat{a}^2\mathbb{E}\left[\int_0^t e^{\mu_1 s}\|\nabla u(s-\rho)\|^2 ds\right] \qquad (4.10)\\
&\le \mathbb{E}\left[\int_0^t e^{\mu_1 s}\|(-\Delta)^{\frac{\alpha+1}{2}}u(s)\|^2 ds\right] + \frac{2}{\mu_1}\|\hat{h}(x)\|^2 e^{\mu_1 t}\\
&\quad + \frac{c}{\mu_1}\sup_{s\ge 0}\mathbb{E}[\|u(s)\|^2]e^{\mu_1 t} + \frac{2\hat{a}^2}{\mu_1}\sup_{-\rho\le s\le 0}\mathbb{E}[\|\nabla\varphi(s)\|^2]e^{\mu_1 t},
\end{aligned}
$$

where c is a positive constant from Gagliardo-Nirenberg inequality. For the forth term on the right-hand side of (4.7), applying (2.6) and (2.7), we have

$$
\begin{aligned}
&\sum_{k=1}^{\infty}\mathbb{E}\left[\int_0^t e^{\mu_1 s}\|\nabla(\sigma_{1,k}+\kappa\sigma_{2,k}(u(s)))\|^2 ds\right] \le 2\sum_{k=1}^{\infty}\mathbb{E}\left[\int_0^t e^{\mu_1 s}(\|\nabla\sigma_{1,k}\|^2 + \|\nabla(\kappa\sigma_{2,k}(u(s)))\|^2)ds\right]\\
&\le \frac{2}{\mu_1}\sum_{k=1}^{\infty}\|\nabla\sigma_{1,k}\|^2 e^{\mu_1 t} + 8\sum_{k=1}^{\infty}\mathbb{E}\left[\int_0^t e^{\mu_1 s}\left(\beta_k^2\|\nabla\kappa\|^2 + \hat{\beta}_k^2\|\kappa\|^2 + \gamma_k^2 C^2\|u(s)\|^2 + \hat{\gamma}_k^2\|\kappa\|_{L^\infty}^2\|\nabla u(s)\|^2\right)ds\right]\\
&\le \frac{2}{\mu_1}\sum_{k=1}^{\infty}\left(\|\nabla\sigma_{1,k}\|^2 + 4\beta_k^2\|\nabla\kappa\|^2 + 4\hat{\beta}_k^2\|\kappa\|^2 + 4C^2\gamma_k^2\sup_{s\ge 0}\mathbb{E}[\|u(s)\|^2]\right)e^{\mu_1 t} \qquad (4.11)\\
&\quad + 8\sum_{k=1}^{\infty}\hat{\gamma}_k^2\|\kappa(x)\|_{L^\infty}^2\mathbb{E}\left[\int_0^t e^{\mu_1 s}\|\nabla u(s)\|^2 ds\right].
\end{aligned}
$$

By (4.7), (4.10) and (4.11), we obtain

$$
\begin{aligned}
&\mathbb{E}[\|\nabla u(t)\|^2] + \mathbb{E}\left[\int_0^t e^{\mu_1(s-t)}\|(-\Delta)^{\frac{\alpha+1}{2}}u(s)\|^2 ds\right]\\
&\le \mathbb{E}\left[\|\nabla\varphi(0)\|^2\right]e^{-\mu_1 t} + \frac{2}{\mu_1}\|\hat{h}(x)\|^2 + \left(\mu_1 - 2\lambda + 8\|\kappa\|_{L^\infty}^2\sum_{k=1}^{\infty}\hat{\gamma}_k^2\right)\mathbb{E}\left[\int_0^t e^{\mu_1 s}\|\nabla u(s)\|^2 ds\right]\\
&\quad + \frac{2}{\mu_1}\left(\frac{c}{2} + 4\left(C^2\sum_{k=1}^{\infty}\gamma_k^2 + c\|\kappa\|_{L^\infty}^2\sum_{k=1}^{\infty}\hat{\gamma}_k^2\right)\right)\sup_{s\ge-\rho}\mathbb{E}[\|u(s)\|^2] \qquad (4.12)\\
&\quad + \frac{2}{\mu_1}\sum_{k=1}^{\infty}\left(\|\nabla\sigma_{1,k}\|^2 + 4(\beta_k^2+\hat{\beta}_k^2)\|\kappa\|_V^2\right) + \frac{2\hat{a}^2}{\mu_1}\sup_{-\rho\le s\le 0}\mathbb{E}[\|\nabla\varphi(s)\|^2].
\end{aligned}
$$

Then by (4.6) and (4.12), we obtain that for all $t \geq 0$,

$$\begin{aligned} &\mathbb{E}[\|\nabla u(t)\|^2] + \mathbb{E}\left[\int_0^t e^{\mu_1(s-t)}\|(-\Delta)^{\frac{\alpha+1}{2}}u(s)\|^2 ds\right] \\ &\leq \mathbb{E}\left[\|\nabla\varphi(0)\|^2\right]e^{-\mu_1 t} + \frac{2}{\mu_1}\|\hat{h}(x)\|^2 + \frac{2}{\mu_1}\left(\frac{c}{2} + 4(C^2\sum_{k=1}^{\infty}\gamma_k^2 + c\|\kappa\|_{L^\infty}^2\sum_{k=1}^{\infty}\hat{\gamma}_k^2)\right)\sup_{s\geq-\rho}\mathbb{E}[\|u(s)\|^2] \\ &\quad + \frac{2}{\mu_1}\sum_{k=1}^{\infty}\left(\|\nabla\sigma_{1,k}\|^2 + 4(\beta_k^2 + \hat{\beta}_k^2)\|\kappa\|_V^2\right) + \frac{2\hat{a}^2}{\mu_1}\sup_{-\rho\leq s\leq 0}\mathbb{E}[\|\nabla\varphi(s)\|^2]. \end{aligned} \tag{4.13}$$

Then by (4.13) and Lemma 3.1, we obtain the estimates (4.5).

Lemma 4.2. *Suppose* (2.1)–(2.7) *and* (4.4) *hold. If* $\varphi(s) \in L^2(\Omega; C([-\rho, 0], V))$*, then the solution* u *of* (1.1) *and* (1.2) *satisfies*

$$\sup_{t\geq\rho}\left\{\mathbb{E}\left[\sup_{t-\rho\leq r\leq t}\|\nabla u(r)\|^2\right]\right\} \leq M_5\left(\mathbb{E}[\|\varphi\|_{C_V}^2] + 1\right), \tag{4.14}$$

where M_5 *is a positive constant independent of* φ*.*

Proof. By (1.1) and Ito's formula, we get for all $t \geq \rho$ and $t - \rho \leq r \leq t$,

$$\begin{aligned} &\|\nabla u(r)\|^2 + 2\int_{t-\rho}^r \|(-\Delta)^{\frac{\alpha+1}{2}}u(s)\|^2 ds \\ &\quad + 2\int_{t-\rho}^r \mathrm{Re}\left((1+\mathrm{i}\mu)|u(s)|^{2\beta}u(s), -\Delta u(s)\right)ds + 2\lambda\int_{t-\rho}^r \|\nabla u(s)\|^2 ds \\ &= \|\nabla u(t-\rho)\|^2 + 2\mathrm{Re}\int_{t-\rho}^r (G(x, u(s-\rho)), -\Delta u(s))ds \\ &\quad + \sum_{k=1}^{\infty}\int_{t-\rho}^r \|\nabla(\sigma_{1,k} + \kappa\sigma_{2,k}(u(s)))\|^2 ds \\ &\quad + 2\sum_{k=1}^{\infty}\mathrm{Re}\int_{t-\rho}^r (\sigma_{1,k}(x) + \kappa(x)\sigma_{2,k}(u(s)), -\Delta u(s))\, dW_k(s). \end{aligned} \tag{4.15}$$

For the third term on the left-hand side of (4.15), applying (4.8), we have

$$-2\int_{t-\rho}^r \mathrm{Re}\left((1+\mathrm{i}\mu)|u(s)|^{2\beta}u(s), -\Delta u(s)\right)ds \leq 0. \tag{4.16}$$

For the second term on the right-hand side of (4.15), applying (2.2) and Gagliardo-Nirenberg inequality, we have

$$
\begin{aligned}
&2\mathrm{Re}\int_{t-\rho}^{r}(G(x,u(s-\rho)),-\Delta u(s))ds \le 2\int_{t-\rho}^{r}\|\nabla u(s)\|\|\nabla G(x,u(s-\rho))\|ds\\
&\le \int_{t-\rho}^{r}\|\nabla u(s)\|^2ds+\int_{t-\rho}^{r}\|\nabla G(x,u(s-\rho))\|^2ds\\
&\le \int_{t-\rho}^{r}\|\nabla u(s)\|^2ds+2\int_{t-\rho}^{r}\|\hat{h}(x)\|^2ds+2\hat{a}^2\int_{t-\rho}^{r}\|\nabla u(s-\rho)\|^2ds\\
&\le \int_{t-\rho}^{r}\|(-\Delta)^{\frac{\alpha+1}{2}}u(s)\|^2ds+2\rho\|\hat{h}(x)\|^2+2\hat{a}^2\int_{t-2\rho}^{r-\rho}\|\nabla u(s)\|^2ds+c\int_{t-\rho}^{r}\|u(s)\|^2ds.
\end{aligned} \tag{4.17}
$$

For the third term on the right-hand side of (4.15), applying (2.6) and (2.7), we have

$$
\begin{aligned}
&\sum_{k=1}^{\infty}\int_{t-\rho}^{r}\|\nabla(\sigma_{1,k}+\kappa\sigma_{2,k}(u(s)))\|^2ds \le 2\sum_{k=1}^{\infty}\int_{t-\rho}^{r}(\|\nabla\sigma_{1,k}\|^2+\|\nabla(\kappa\sigma_{2,k}(u(s)))\|^2)ds\\
&\le 2\rho\sum_{k=1}^{\infty}\|\nabla\sigma_{1,k}\|^2+8\rho\left(\|\nabla\kappa\|^2\sum_{k=1}^{\infty}\beta_k^2+\|\kappa\|^2\sum_{k=1}^{\infty}\hat{\beta}_k^2\right)\\
&\quad+8C^2\sum_{k=1}^{\infty}\gamma_k^2\int_{t-\rho}^{r}\|u(s)\|^2ds+8\|\kappa\|_{L^\infty}^2\sum_{k=1}^{\infty}\hat{\gamma}_k^2\int_{t-\rho}^{r}\|\nabla u(s)\|^2ds.
\end{aligned} \tag{4.18}
$$

By (4.4) and (4.15)–(4.18), we infer that for all $t \ge \rho$ and $t-\rho \le r \le t$,

$$
\begin{aligned}
\|\nabla u(r)\|^2 &\le c_1+\|\nabla u(t-\rho)\|^2+c_2\int_{t-2\rho}^{r}\|u(s)\|^2ds+2\hat{a}^2\int_{t-2\rho}^{r}\|\nabla u(s)\|^2ds\\
&\quad+2\sum_{k=1}^{\infty}\mathrm{Re}\int_{t-\rho}^{r}(\sigma_{1,k}(x)+\kappa(x)\sigma_{2,k}(u(s)),-\Delta u(s))\,dW_k(s),
\end{aligned} \tag{4.19}
$$

where c_1 and c_2 are positive constants. By (4.19), we deduce that for all $t \ge \rho$,

$$
\begin{aligned}
\mathbb{E}\left[\sup_{t-\rho\le r\le t}\|\nabla u(r)\|^2\right] &\le c_1+\mathbb{E}\left[\|\nabla u(t-\rho)\|^2\right]+c_2\int_{t-2\rho}^{r}\mathbb{E}\left[\|u(s)\|^2+\|\nabla u(s)\|^2\right]ds\\
&\quad+2\mathbb{E}\left[\sup_{t-\rho\le r\le t}\left|\sum_{k=1}^{\infty}\int_{t-\rho}^{r}(\sigma_{1,k}(x)+\kappa(x)\sigma_{2,k}(u(s)),-\Delta u(s))\,dW_k(s)\right|\right].
\end{aligned} \tag{4.20}
$$

For the second term on the right-hand side of (4.20), by Lemma 4.1 we infer that for all $t \ge \rho$,

$$
\mathbb{E}\left[\|\nabla u(t-\rho)\|^2\right] \le \sup_{s\ge-\rho}\mathbb{E}\left[\|\nabla u(s)\|^2\right] \le c_3\mathbb{E}\left[\|\varphi\|_{C_V}^2\right]+c_3. \tag{4.21}
$$

For the third term on the right-hand side of (4.20), by Lemmas 3.1 and 4.1 we infer that for all $t \ge \rho$,

$$
c_2\int_{t-2\rho}^{r}\mathbb{E}\left[\|u(s)\|^2+\|\nabla u(s)\|^2\right]ds \le 2\rho c_2\sup_{s\ge-\rho}\mathbb{E}\left[\|u(s)\|^2+\|\nabla u(s)\|^2\right] \le c_4\mathbb{E}\left[\|\varphi\|_{C_V}^2\right]+c_4. \tag{4.22}
$$

For the last term on the right-hand side of (4.20), by BDG inequality, (4.18), Lemmas 3.1 and 4.1, we deduce that for all $t \geq \rho$,

$$\begin{aligned}
&2\mathbb{E}\left[\sup_{t-\rho\leq r\leq t}\left|\sum_{k=1}^{\infty}\int_{t-\rho}^{r}\left(\sigma_{1,k}(x)+\kappa(x)\sigma_{2,k}(u(s)),-\Delta u(s)\right)dW_k(s)\right|\right]\\
&\leq 2c_5\mathbb{E}\left[\left(\sum_{k=1}^{\infty}\int_{t-\rho}^{t}\left|\left(\sigma_{1,k}(x)+\kappa(x)\sigma_{2,k}(u(s)),-\Delta u(s)\right)\right|^2ds\right)^{\frac{1}{2}}\right]\\
&\leq 2c_5\mathbb{E}\left[\left(\sum_{k=1}^{\infty}\int_{t-\rho}^{t}\|\nabla u(s)\|^2\|\nabla\left(\sigma_{1,k}(x)+\kappa(x)\sigma_{2,k}(u(s))\right)\|^2ds\right)^{\frac{1}{2}}\right]\\
&\leq 2c_5\mathbb{E}\left[\sup_{t-\rho\leq s\leq t}\|\nabla u(s)\|\left(\sum_{k=1}^{\infty}\int_{t-\rho}^{t}\|\nabla\left(\sigma_{1,k}(x)+\kappa(x)\sigma_{2,k}(u(s))\right)\|^2ds\right)^{\frac{1}{2}}\right]\\
&\leq \frac{1}{2}\mathbb{E}\left[\sup_{t-\rho\leq s\leq t}\|\nabla u(s)\|^2\right]+2c_5^2\mathbb{E}\left[\sum_{k=1}^{\infty}\int_{t-\rho}^{t}\|\nabla\left(\sigma_{1,k}(x)+\kappa(x)\sigma_{2,k}(u(s))\right)\|^2ds\right]\\
&\leq \frac{1}{2}\mathbb{E}\left[\sup_{t-\rho\leq s\leq t}\|\nabla u(s)\|^2\right]+c_6+c_6\int_{t-\rho}^{t}\mathbb{E}\left[\|u(s)\|^2+\|\nabla u(s)\|^2\right]ds\\
&\leq \frac{1}{2}\mathbb{E}\left[\sup_{t-\rho\leq s\leq t}\|\nabla u(s)\|^2\right]+c_6+\rho c_6\left(\sup_{s\geq 0}\mathbb{E}\|u(s)\|^2+\sup_{s\geq 0}\|\nabla u(s)\|^2\right)\\
&\leq \frac{1}{2}\mathbb{E}\left[\sup_{t-\rho\leq s\leq t}\|\nabla u(s)\|^2\right]+c_7\mathbb{E}\left[\|\varphi\|_{C_V}^2\right]+c_7.
\end{aligned} \tag{4.23}$$

By (4.20)–(4.23), we obtain that for all $t \geq \rho$,

$$\mathbb{E}\left[\sup_{t-\rho\leq r\leq t}\|\nabla u(r)\|^2\right]\leq c_8\mathbb{E}\left[\|\varphi\|_{C_V}^2\right]+c_9,$$

which completes the proof.

Lemma 4.3. *Suppose* (2.1)–(2.7) *and* (3.1) *hold. If* $\varphi(s) \in L^{2p}(\Omega, C([-\rho, 0], V))$*, then there exists a positive constant* μ_5 *such that the solution* u *of* (1.1) *and* (1.2) *satisfies*

$$\begin{aligned}
&\sup_{t\geq-\rho}\mathbb{E}[\|\nabla u(t)\|^{2p}]+\sup_{t\geq 0}\mathbb{E}\left[\int_0^t e^{\mu_5(s-t)}\|\nabla u(s)\|^{2(p-1)}\|(-\Delta)^{\frac{\alpha+1}{2}}u(s)\|^2ds\right]\\
&\qquad\leq M_5\left(\mathbb{E}\left[\|\varphi\|_{C_V}^{2p}\right]+1\right),
\end{aligned} \tag{4.24}$$

where M_5 *is a positive constant independent of* φ.

Proof. By (3.1), there exist positive constants μ and ϵ_1 such that

$$\begin{aligned}
&\mu+4(p-1)\epsilon_1^{\frac{p}{p-1}}+\frac{2p\hat{a}^2}{\mu\epsilon_1^p}+8C^2(p-1)(2p-1)\epsilon_1^{\frac{p}{p-1}}\sum_{k=1}^{\infty}\gamma_k^2+8p(2p-1)\|\kappa\|_{L^\infty}^2\sum_{k=1}^{\infty}\hat{\gamma}_k^2\\
&\quad+2(p-1)(2p-1)\epsilon_1^{\frac{p}{p-1}}\sum_{k=1}^{\infty}\left(\|\nabla\sigma_{1,k}\|^2+4\beta_k^2\|\nabla\kappa\|^2+4\hat{\beta}_k^2\|\kappa\|^2\right)\leq 2p\lambda.
\end{aligned} \tag{4.25}$$

By (1.1) and applying Ito's formula to $e^{\mu t}\|\nabla u(t)\|^{2p}$, we get for $t \geq 0$,

$$
\begin{aligned}
& e^{\mu t}\|\nabla u(t)\|^{2p} + 2p \int_0^t e^{\mu s}\|\nabla u(s)\|^{2(p-1)}\|(-\Delta)^{\frac{\alpha+1}{2}} u(s)\|^2 ds \\
& \quad + 2p \int_0^t e^{\mu s}\|\nabla u(s)\|^{2(p-1)} \mathrm{Re}\left((1+\mathrm{i}\mu)|u(s)|^{2\beta}u(s), -\Delta u(s)\right) ds \\
&= \|\nabla\varphi(0)\|^{2p} + (\mu - 2p\lambda)\int_0^t e^{\mu s}\|\nabla u(s)\|^{2p} ds \\
& \quad + 2p\mathrm{Re}\int_0^t e^{\mu s}\|\nabla u(s)\|^{2(p-1)}(G(x, u(s-\rho)), -\Delta u(s)) ds \\
& \quad + p\sum_{k=1}^{\infty}\int_0^t e^{\mu s}\|\nabla u(s)\|^{2(p-1)}\|\nabla(\sigma_{1,k} + \kappa\sigma_{2,k}(u(s)))\|^2 ds \\
& \quad + 2p\sum_{k=1}^{\infty}\mathrm{Re}\int_0^t e^{\mu_1 s}\|\nabla u(s)\|^{2(p-1)}\left(\sigma_{1,k} + \kappa\sigma_{2,k}(u(s)), -\Delta u(s)\right) dW_k(s) \\
& \quad + 2p(p-1)\sum_{k=1}^{\infty}\mathrm{Re}\int_0^t e^{\mu_1 s}\|\nabla u(s)\|^{2(p-2)}\left|(\sigma_{1,k} + \kappa\sigma_{2,k}(u(s)), -\Delta u(s))\right|^2 ds.
\end{aligned}
$$

Taking the expectation, we have for $t \geq 0$,

$$
\begin{aligned}
& e^{\mu t}\mathbb{E}\left[\|\nabla u(t)\|^{2p}\right] + 2p\mathbb{E}\left[\int_0^t e^{\mu s}\|\nabla u(s)\|^{2(p-1)}\|(-\Delta)^{\frac{\alpha+1}{2}} u(s)\|^2 ds\right] \\
& \quad + 2p\mathbb{E}\left[\int_0^t e^{\mu s}\|\nabla u(s)\|^{2(p-1)}\mathrm{Re}\left((1+\mathrm{i}\mu)|u(s)|^{2\beta}u(s), -\Delta u(s)\right) ds\right] \\
&= \mathbb{E}\left[\|\nabla\varphi(0)\|^{2p}\right] + (\mu - 2p\lambda)\mathbb{E}\left[\int_0^t e^{\mu s}\|\nabla u(s)\|^{2p} ds\right] \\
& \quad + 2p\mathbb{E}\left[\mathrm{Re}\int_0^t e^{\mu s}\|\nabla u(s)\|^{2(p-1)}(G(x, u(s-\rho)), -\Delta u(s)) ds\right] \\
& \quad + p\sum_{k=1}^{\infty}\mathbb{E}\left[\int_0^t e^{\mu s}\|\nabla u(s)\|^{2(p-1)}\|\nabla(\sigma_{1,k} + \kappa\sigma_{2,k}(u(s)))\|^2 ds\right] \\
& \quad + 2p(p-1)\sum_{k=1}^{\infty}\mathbb{E}\left[\mathrm{Re}\int_0^t e^{\mu_1 s}\|\nabla u(s)\|^{2(p-2)}\left|(\sigma_{1,k} + \kappa\sigma_{2,k}(u(s)), -\Delta u(s))\right|^2 ds\right].
\end{aligned}
\tag{4.26}
$$

By (4.8), we get the third term on the left-hand side of (4.26) is nonnegative. Next, we estimate each term on the right-hand side of (4.26). For the third term on the right-hand side of (4.26), applying (2.2), Gagliardo-Nirenberg inequality and Young's inequality, we deduce

$$
\begin{aligned}
&2p\mathbb{E}\left[\mathrm{Re}\int_0^t e^{\mu s}\|\nabla u(s)\|^{2(p-1)}(G(x,u(s-\rho)),-\Delta u(s))ds\right]\\
&\le 2p\mathbb{E}\left[\int_0^t e^{\mu s}\|\nabla u(s)\|^{2(p-1)}\|\nabla u(s)\|\|\nabla G(x,u(s-\rho))\|ds\right]\\
&\le p\mathbb{E}\left[\int_0^t e^{\mu s}\|\nabla u(s)\|^{2(p-1)}\|\nabla u(s)\|^2ds\right]+p\mathbb{E}\left[\int_0^t e^{\mu s}\|\nabla u(s)\|^{2(p-1)}\|\nabla G(x,u(s-\rho))\|^2ds\right]\\
&\le p\mathbb{E}\left[\int_0^t e^{\mu s}\|\nabla u(s)\|^{2(p-1)}\|\nabla u(s)\|^2ds\right]\\
&\quad+2p\mathbb{E}\left[\int_0^t e^{\mu s}\|\nabla u(s)\|^{2(p-1)}\|\hat{h}\|^2ds\right]+2p\hat{a}^2\mathbb{E}\left[\int_0^t e^{\mu s}\|\nabla u(s)\|^{2(p-1)}\|\nabla u(s-\rho)\|^2ds\right]\\
&\le p\mathbb{E}\left[\int_0^t e^{\mu s}\|\nabla u(s)\|^{2(p-1)}(\|(-\Delta)^{\frac{\alpha+1}{2}}u(s)\|^2+c\|u(s)\|^2)ds\right]+\frac{2}{\epsilon_1^p}\|\hat{h}(x)\|^{2p}\int_0^t e^{\mu s}ds\\
&\quad+4(p-1)\epsilon_1^{\frac{p}{p-1}}\mathbb{E}\left[\int_0^t e^{\mu s}\|\nabla u(s)\|^{2p}ds\right]+\frac{2p\hat{a}^2}{\epsilon_1^p}\mathbb{E}\left[\int_0^t e^{\mu s}\|\nabla u(s)\|^{2p}ds\right]\\
&\le p\mathbb{E}\left[\int_0^t e^{\mu s}\|\nabla u(s)\|^{2(p-1)}\|(-\Delta)^{\frac{\alpha+1}{2}}u(s)\|^2ds\right]+\left(4(p-1)\epsilon_1^{\frac{p}{p-1}}+\frac{2p\hat{a}^2}{\mu\epsilon_1^p}\right)\mathbb{E}\left[\int_0^t e^{\mu s}\|\nabla u(s)\|^{2p}ds\right]\\
&\quad+\frac{1}{\mu\epsilon_1^p}\|\hat{h}(x)\|^{2p}e^{\mu t}+\frac{2p\hat{a}^2}{\mu\epsilon_1^p}\sup_{-\rho\le s\le 0}\mathbb{E}\left[\|\nabla\varphi(s)\|^{2p}\right]e^{\mu t}+c\sup_{s\ge 0}\mathbb{E}\left[\|u(s)\|^{2p}\right]e^{\mu t}.
\end{aligned}
\tag{4.27}
$$

For the forth term on the right-hand side of (4.26), applying (2.7), we infer

$$
\begin{aligned}
&p\sum_{k=1}^{\infty}\mathbb{E}\left[\int_0^t e^{\mu s}\|\nabla u(s)\|^{2(p-1)}\|\nabla(\sigma_{1,k}+\kappa\sigma_{2,k}(u(s)))\|^2ds\right]\\
&\le 2p\sum_{k=1}^{\infty}\mathbb{E}\left[\int_0^t e^{\mu s}\|\nabla u(s)\|^{2(p-1)}\|\nabla\sigma_{1,k}\|^2ds\right]+2p\sum_{k=1}^{\infty}\mathbb{E}\left[\int_0^t e^{\mu s}\|\nabla u(s)\|^{2(p-1)}\|\nabla(\kappa\sigma_{2,k}(u(s)))\|^2ds\right]\\
&\le 2p\sum_{k=1}^{\infty}\mathbb{E}\left[\int_0^t e^{\mu s}\|\nabla u(s)\|^{2(p-1)}\|\nabla\sigma_{1,k}\|^2ds\right]\\
&\quad+8p\sum_{k=1}^{\infty}\mathbb{E}\left[\int_0^t e^{\mu s}\|\nabla u(s)\|^{2(p-1)}\left(\beta_k^2\|\nabla\kappa\|^2+\hat{\beta}_k^2\|\kappa\|^2+\gamma_k^2C^2\|u(s)\|^2+\hat{\gamma}_k^2\|\kappa\|_{L^\infty}^2\|\nabla u(s)\|^2\right)ds\right]\\
&\le 2p\sum_{k=1}^{\infty}\mathbb{E}\left[\int_0^t e^{\mu s}\|\nabla u(s)\|^{2(p-1)}\left(\|\nabla\sigma_{1,k}\|^2+4\beta_k^2\|\nabla\kappa\|^2+4\hat{\beta}_k^2\|\kappa\|^2\right)ds\right]\\
&\quad+8C^2p\sum_{k=1}^{\infty}\gamma_k^2\mathbb{E}\left[\int_0^t e^{\mu s}\|\nabla u(s)\|^{2(p-1)}\|u(s)\|^2ds\right]+8p\|\kappa\|_{L^\infty}^2\sum_{k=1}^{\infty}\hat{\gamma}_k^2\mathbb{E}\left[\int_0^t e^{\mu s}\|\nabla u(s)\|^{2p}ds\right].
\end{aligned}
\tag{4.28}
$$

Then applying Young's inequality, (4.28) can be estimated by

$$
\begin{aligned}
& p\sum_{k=1}^{\infty}\mathbb{E}\left[\int_0^t e^{\mu s}\|\nabla u(s)\|^{2(p-1)}\|\nabla(\sigma_{1,k}+\kappa\sigma_{2,k}(u(s)))\|^2 ds\right] \\
\le & \sum_{k=1}^{\infty}\left(\|\nabla\sigma_{1,k}\|^2+4\beta_k^2\|\nabla\kappa\|^2+4\hat{\beta}_k^2\|\kappa\|^2\right)\times\mathbb{E}\left[\int_0^t e^{\mu s}\left((2p-2)\epsilon_1^{\frac{p}{p-1}}\|\nabla u(s)\|^{2p}+\frac{2}{\epsilon_1^p}\right)ds\right] \\
& +2C^2\sum_{k=1}^{\infty}\gamma_k^2\mathbb{E}\left[\int_0^t e^{\mu s}\left((4p-4)\epsilon_1^{\frac{p}{p-1}}\|\nabla u(s)\|^{2p}+\frac{4}{\epsilon_1^p}\|u(s)\|^{2p}\right)ds\right] \\
& +8p\|\kappa\|_{L^\infty}^2\sum_{k=1}^{\infty}\hat{\gamma}_k^2\mathbb{E}\left[\int_0^t e^{\mu s}\|\nabla u(s)\|^{2p}ds\right] \qquad (4.29) \\
= & \Bigg[(2p-2)\epsilon_1^{\frac{p}{p-1}}\sum_{k=1}^{\infty}\left(\|\nabla\sigma_{1,k}\|^2+4\beta_k^2\|\nabla\kappa\|^2+4\hat{\beta}_k^2\|\kappa\|^2\right) \\
& +2C^2(4p-4)\epsilon_1^{\frac{p}{p-1}}\sum_{k=1}^{\infty}\gamma_k^2+8p\|\kappa\|_{L^\infty}^2\sum_{k=1}^{\infty}\hat{\gamma}_k^2\Bigg]\mathbb{E}\left[\int_0^t e^{\mu s}\|\nabla u(s)\|^{2p}ds\right] \\
& +\sum_{k=1}^{\infty}\left(\|\nabla\sigma_{1,k}\|^2+4\beta_k^2\|\nabla\kappa\|^2+4\hat{\beta}_k^2\|\kappa\|^2\right)\frac{2}{\epsilon_1^p}e^{\mu t}+\frac{8C^2}{\mu\epsilon_1^p}\sum_{k=1}^{\infty}\gamma_k^2\sup_{s\ge0}\mathbb{E}\left[\|u(s)\|^{2p}\right]e^{\mu t}.
\end{aligned}
$$

For the fifth term on the right-hand side of (4.26), applying integrating by parts and (4.29), we get

$$
\begin{aligned}
& 2p(p-1)\sum_{k=1}^{\infty}\mathbb{E}\left[\mathrm{Re}\int_0^t e^{\mu_1 s}\|\nabla u(s)\|^{2(p-2)}\left|(\sigma_{1,k}+\kappa\sigma_{2,k}(u(s)),-\Delta u(s))\right|^2 ds\right] \\
\le & 2p(p-1)\sum_{k=1}^{\infty}\mathbb{E}\left[\int_0^t e^{\mu_1 s}\|\nabla u(s)\|^{2p-2}\|\nabla(\sigma_{1,k}+\kappa\sigma_{2,k}(u(s)))\|^2 ds\right] \\
\le & 2(p-1)\Bigg[(2p-2)\epsilon_1^{\frac{p}{p-1}}\sum_{k=1}^{\infty}\left(\|\nabla\sigma_{1,k}\|^2+4\beta_k^2\|\nabla\kappa\|^2+4\hat{\beta}_k^2\|\kappa\|^2\right) \qquad (4.30) \\
& +2C^2(4p-4)\epsilon_1^{\frac{p}{p-1}}\sum_{k=1}^{\infty}\gamma_k^2+8p\|\kappa\|_{L^\infty}^2\sum_{k=1}^{\infty}\hat{\gamma}_k^2\Bigg]\mathbb{E}\left[\int_0^t e^{\mu s}\|\nabla u(s)\|^{2p}ds\right] \\
& +\sum_{k=1}^{\infty}\left(\|\nabla\sigma_{1,k}\|^2+4\beta_k^2\|\nabla\kappa\|^2+4\hat{\beta}_k^2\|\kappa\|^2\right)\frac{4(p-1)}{\epsilon_1^p}e^{\mu t}+\frac{16C^2(p-1)}{\mu\epsilon_1^p}\sum_{k=1}^{\infty}\gamma_k^2\sup_{s\ge0}\mathbb{E}\left[\|u(s)\|^{2p}\right]e^{\mu t}.
\end{aligned}
$$

By (4.26), (4.27), (4.29) and (4.30), we obtain

$$\begin{aligned}
&\mathbb{E}\left[\|\nabla u(t)\|^{2p}\right] + p\mathbb{E}\left[\int_0^t e^{\mu(s-t)}\|\nabla u(s)\|^{2(p-1)}\|(-\Delta)^{\frac{\alpha+1}{2}}u(s)\|^2 ds\right] \\
&\leq \mathbb{E}\left[\|\nabla\varphi(0)\|^{2p}\right]e^{-\mu t} + \left[\mu - 2p\lambda + 4(\varrho-1)\epsilon_1^{\frac{p}{p-1}} + \frac{2p\hat{a}^2}{\mu\epsilon_1^p} + 8C^2(p-1)(2p-1)\epsilon_1^{\frac{p}{p-1}}\sum_{k=1}^{\infty}\gamma_k^2\right. \\
&\left. +8p(2p-1)\|\kappa\|^2_{L^\infty}\sum_{k=1}^{\infty}\hat{\gamma}_k^2 + 2(p-1)(2p-1)\epsilon_1^{\frac{p}{p-1}}\sum_{k=1}^{\infty}\left(\|\nabla\sigma_{1,k}\|^2 + 4\beta_k^2\|\nabla\kappa\|^2 + 4\hat{\beta}_k^2\|\kappa\|^2\right)\right] \\
&\times\mathbb{E}\left[\int_0^t e^{\mu(s-t)}\|\nabla u(s)\|^{2p}ds\right] + \frac{1}{\mu\epsilon_1^p}\|\hat{h}(x)\|^{2p} + \frac{2p\hat{a}^2}{\mu\epsilon_1^p}\sup_{-\rho\leq s\leq 0}\mathbb{E}\left[\|\nabla\varphi(s)\|^{2p}\right] \\
&+ c\sup_{s\geq 0}\mathbb{E}\left[\|u(s)\|^{2p}\right] + \sum_{k=1}^{\infty}\left(\|\nabla\sigma_{1,k}\|^2 + 4\beta_k^2\|\nabla\kappa\|^2 + 4\hat{\beta}_k^2\|\kappa\|^2\right)\frac{4p-2}{\epsilon_1^p} \\
&+ \frac{8C^2(2p-1)}{\mu\epsilon_1^p}\sum_{k=1}^{\infty}\gamma_k^2\sup_{s\geq 0}\mathbb{E}\left[\|u(s)\|^{2p}\right].
\end{aligned} \tag{4.31}$$

Then by (4.25) and (4.31), we deduce that for all $t \geq 0$,

$$\begin{aligned}
&\mathbb{E}\left[\|\nabla u(t)\|^{2p}\right] + p\mathbb{E}\left[\int_0^t e^{\mu(s-t)}\|\nabla u(s)\|^{2(p-1)}\|(-\Delta)^{\frac{\alpha+1}{2}}u(s)\|^2 ds\right] \\
&\leq \mathbb{E}\left[\|\nabla\varphi(0)\|^{2p}\right]e^{-\mu t} + \frac{1}{\mu\epsilon_1^p}\|\hat{h}(x)\|^{2p} + \frac{2p\hat{a}^2}{\mu\epsilon_1^p}\sup_{-\rho\leq s\leq 0}\mathbb{E}\left[\|\nabla\varphi(s)\|^{2p}\right] \\
&+ c\sup_{s\geq 0}\mathbb{E}\left[\|u(s)\|^{2p}\right] + \sum_{k=1}^{\infty}\left(\|\nabla\sigma_{1,k}\|^2 + 4\beta_k^2\|\nabla\kappa\|^2 + 4\hat{\beta}_k^2\|\kappa\|^2\right)\frac{4p-2}{\epsilon_1^p} \\
&+ \frac{8C^2(2p-1)}{\mu\epsilon_1^p}\sum_{k=1}^{\infty}\gamma_k^2\sup_{s\geq 0}\mathbb{E}\left[\|u(s)\|^{2p}\right].
\end{aligned} \tag{4.32}$$

Therefore, by (4.33) and Lemma 3.5, there exists a constant M_5 independent of φ such that

$$\begin{aligned}
&\sup_{t\geq-\rho}\mathbb{E}\left[\|\nabla u(t)\|^{2p}\right] + \sup_{t\geq 0}\mathbb{E}\left[\int_0^t e^{\mu(s-t)}\|\nabla u(s)\|^{2(p-1)}\|(-\Delta)^{\frac{\alpha+1}{2}}u(s)\|^2 ds\right] \\
&\leq M_5(\mathbb{E}\left[\|\varphi\|^{2p}_{C_V}\right] + 1).
\end{aligned} \tag{4.33}$$

For convenience, we write $A = (1 + \mathrm{i}\nu)(-\triangle)^\alpha + \lambda I$. Then, similar to Theorem 6.5 in [31], the solution of (1.1) and (1.2) can be expressed as

$$\begin{aligned}
u(t) &= e^{-At}u(0) - \int_0^t e^{-A(t-s)}(1+\mathrm{i}\mu)|u(s)|^{2\beta}u(s)ds \\
&+ \int_0^t e^{-A(t-s)}G(\cdot, u(s-\rho))ds + \sum_{k=1}^{\infty}\int_0^t e^{-A(t-s)}(\sigma_{1,k} + \kappa\sigma_{2,k}(u(s)))dW_k(s).
\end{aligned} \tag{4.34}$$

The next lemma is concerned with the Hölder continuity ofsolutions in time which is needed to prove the tightness of distributions of solutions.

Lemma 4.4. *Suppose* (2.1)–(2.7) *and* (3.1) *hold. If* $\varphi(s) \in L^{2p}(\Omega, C([-\rho, 0], V))$*, then the solution* u *of* (1.1) *and* (1.2) *satisfies, for any* $t > r \geq 0$,

$$\mathbb{E}[\|u(t) - u(r)\|^{2p}] \leq M_6(|t - r|^p + |t - r|^{2p}), \tag{4.35}$$

where M_6 *is a positive constant depending on* φ*, but independent of* t *and* r.

Proof. By (4.34), we get for $t > r \geq 0$,

$$\begin{aligned} u(t) = e^{-A(t-r)}u(r) - \int_r^t e^{-A(t-s)}(1 + \mathrm{i}\mu)|u(s)|^{2\beta}u(s)ds \\ + \int_r^t e^{-A(t-s)}G(\cdot, u(s - \rho))ds + \sum_{k=1}^{\infty}\int_r^t e^{-A(t-s)}(\sigma_{1,k} + \kappa\sigma_{2,k}(u(s)))dW_k(s). \end{aligned} \tag{4.36}$$

Then we infer

$$\begin{aligned} \|u(t) - u(r)\|^{2p} \leq \frac{5^{2p}}{4}\Bigg[\|(e^{-A(t-r)} - I)u(r)\|^{2p} + \|\int_r^t e^{-A(t-s)}(1 + \mathrm{i}\mu)|u(s)|^{2\beta}u(s)ds\|^{2p} \\ +\|\int_r^t e^{-A(t-s)}G(\cdot, u(s - \rho))ds\|^{2p} + \|\sum_{k=1}^{\infty}\int_r^t e^{-A(t-s)}(\sigma_{1,k} + \kappa\sigma_{2,k}(u(s)))dW_k(s)\|^{2p}\Bigg]. \end{aligned} \tag{4.37}$$

Taking the expectation of (4.37), we have for all $t > r \geq 0$,

$$\begin{aligned} \mathbb{E}[\|u(t) - u(r)\|^{2p}] \leq \frac{5^{2p}}{4}\mathbb{E}[\|(e^{-A(t-r)} - I)u(r)\|^{2p}] + \frac{5^{2p}}{4}\mathbb{E}\left[\|\int_r^t e^{-A(t-s)}(1 + \mathrm{i}\mu)|u(s)|^{2\beta}u(s)ds\|^{2p}\right] \\ + \frac{5^{2p}}{4}\mathbb{E}\left[\|\int_r^t e^{-A(t-s)}G(\cdot, u(s-\rho))ds\|^{2p}\right] + \frac{5^{2p}}{4}\mathbb{E}\left[\|\sum_{k=1}^{\infty}\int_r^t e^{-A(t-s)}(\sigma_{1,k} + \kappa\sigma_{2,k}(u(s)))dW_k(s)\|^{2p}\right]. \end{aligned} \tag{4.38}$$

For the first term on the right-hand side of (4.38), by Theorem 1.4.3 in [32] , we find that there exists a positive number C_0 depending on ϱ such that for all $t > r \geq 0$,

$$\frac{5^{2p}}{4}\mathbb{E}[\|(e^{-A(t-r)} - I)u(r)\|^{2p}] \leq C_0(t - r)^p\mathbb{E}[\|u(r)\|_{C_V}^{2p}].$$

Applying Lemmas 3.5 and 4.3, we obtain for all $t > r \geq 0$,

$$\frac{5^{2p}}{4}\mathbb{E}[\|(e^{-A(t-r)} - I)u(r)\|^{2p}] \leq C_1(t - r)^p. \tag{4.39}$$

For the second term on the right-hand side of (4.38), by the contraction property of e^{-At}, we infer that for all $t > r \geq 0$,

$$
\begin{aligned}
&\mathbb{E}[\|\int_r^t e^{-A(t-s)}(1+\mathrm{i}\mu)|u(s)|^{2\beta}u(s)ds\|^{2p}] \le (1+\mu^2)^p\mathbb{E}\left[\left(\int_r^t \||u(s)|^{2\beta+1}\|ds\right)^{2p}\right]\\
&\le (1+\mu^2)^p\mathbb{E}\left[\left(\int_r^t \||u(s)|^{2\beta+1}\|^{2p}ds\right)\right](t-r)^{2p-1}\\
&\le (1+\mu^2)^p \sup_{s\ge 0}\mathbb{E}\left[\left(\|u(s)\|^{2(2\beta+1)}_{L^{2(2\beta+1)}}\right)^p\right](t-r)^{2p}.
\end{aligned}
$$

We deduce the estimate $\sup_{s\ge 0}\mathbb{E}\left[\left(\|u(s)\|^{2(2\beta+1)}_{L^{2(2\beta+1)}}\right)^p\right] \le M'\left(\mathbb{E}[\|\varphi\|^2_{C_V}]+1\right)$ similarly to Lemma 3.5 together with Lemma 3.3 in [1]. Hence, the second term on the right-hand side of (4.38) can be estimated by

$$
\mathbb{E}\left[\|\int_r^t e^{-A(t-s)}(1+\mathrm{i}\mu)|u(s)|^{2\beta}u(s)ds\|^{2p}\right] \le C_2(t-r)^{2p}. \tag{4.40}
$$

For the third term on the right-hand side of (4.38), by the contraction property of e^{-At} and (2.1) and Lemma 3.5, we deduce that for all $t > r \ge 0$,

$$
\begin{aligned}
&\frac{5^{2p}}{4}\mathbb{E}\left[\|\int_r^t e^{-A(t-s)}G(\cdot,u(s-\rho))ds\|^{2p}\right] \le \frac{5^{2p}}{4}\mathbb{E}\left[\left(\int_r^t \|G(\cdot,u(s-\rho))\|ds\right)^{2p}\right]\\
&\le \frac{5^{2p}}{4}\mathbb{E}\left[\left(\int_r^t (\|h\|+a\|u(s-\rho)\|)ds\right)^{2p}\right]\\
&\le \frac{5^{2p}}{4}\mathbb{E}\left[\left(\int_r^t (\|h\|+a\|u(s-\rho)\|)^{2p}ds\right)\right](t-r)^{2p-1}\\
&\le \frac{10^{2p}}{8}(t-r)^{2p-1}\int_r^t\left(\|h\|^{2p}+a^{2p}\mathbb{E}\left[\|u(s-\rho)\|^{2p}\right]\right)ds\\
&\le \frac{10^{2p}}{8}\left(\|h\|^{2p}+a^{2p}\sup_{t\ge-\rho}\mathbb{E}\left[\|u(s)\|^{2p}\right]\right)(t-r)^{2p} \le C_3(t-r)^{2p}.
\end{aligned} \tag{4.41}
$$

For the forth term the right-hand side of (4.38), from the BDG inequality, the contraction property of e^{-At}, (2.6) Hölder's inequality and Lemma 3.5, we deduce

$$
\begin{aligned}
&\frac{5^{2p}}{4}\mathbb{E}\left[\|\sum_{k=1}^\infty\int_r^t e^{-A(t-s)}(\sigma_{1,k}+\kappa\sigma_{2,k}(u(s)))dW_k(s)\|^{2p}\right]\\
&\le \frac{5^{2p}}{4}C_4\mathbb{E}\left[\left(\int_r^t\sum_{k=1}^\infty \|e^{-A(t-s)}(\sigma_{1,k}+\kappa\sigma_{2,k}(u(s)))\|^2ds\right)^p\right]\\
&\le \frac{5^{2p}}{4}C_4\mathbb{E}\left[\left(\int_r^t\sum_{k=1}^\infty 2(\|\sigma_{1,k}\|^2+\|\kappa\sigma_{2,k}(u(s))\|^2)ds\right)^p\right]\\
&\le \frac{5^{2p}}{4}C_4\mathbb{E}\left[\left(\int_r^t\sum_{k=1}^\infty 2(\|\sigma_{1,k}\|^2+2\|\kappa\|^2\beta_k^2+2\|\kappa\|^2_{L^\infty}\gamma_k^2\|(u(s))\|^2)ds\right)^p\right]
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{5^{2p}}{2}C_4\mathbb{E}\left[\left(2\sum_{k=1}^{\infty}(\|\sigma_{1,k}\|^2+2\|\kappa\|^2\beta_k^2)(t-r)+4\sum_{k=1}^{\infty}\|\kappa\|_{L^\infty}^2\gamma_k^2\int_r^t\|u(s)\|^2ds\right)^p\right]\\
&\le \frac{10^{2p}}{8}C_4\left(\sum_{k=1}^{\infty}(\|\sigma_{1,k}\|^2+2\|\kappa\|^2\beta_k^2\right)^p(t-r)^p+\frac{10^{2p}}{8}C_4\left(2\sum_{k=1}^{\infty}\|\kappa\|_{L^\infty}^2\gamma_k^2\right)^p\mathbb{E}\left[\left(\int_r^t\|u(s)\|^2ds\right)^p\right]\\
&\le \frac{10^{2p}}{8}C_4\left(\sum_{k=1}^{\infty}(\|\sigma_{1,k}\|^2+2\|\kappa\|^2\beta_k^2\right)^p(t-r)^p\\
&\quad+\frac{10^{2p}}{8}C_4\left(2\sum_{k=1}^{\infty}\|\kappa\|_{L^\infty}^2\gamma_k^2\right)^p(t-r)^{p-1}\int_r^t\mathbb{E}\left[\|u(s)\|^{2p}\right]ds\\
&\le \frac{10^{2p}}{8}C_4\left(\sum_{k=1}^{\infty}(\|\sigma_{1,k}\|^2+2\|\kappa\|^2\beta_k^2\right)^p(t-r)^p\\
&\quad+\frac{10^{2p}}{8}C_4\left(2\sum_{k=1}^{\infty}\|\kappa\|_{L^\infty}^2\gamma_k^2\right)^p(t-r)^p\sup_{s\ge 0}\mathbb{E}\left[\|u(s)\|^{2p}\right]\\
&\le C_5(t-r)^p.
\end{aligned}
\tag{4.42}
$$

Therefore, from (4.38)–(4.42), we obtain there exists $C_6 > 0$ independent of t and r, such that for all $t > r \ge 0$,

$$\mathbb{E}[\|u(t)-u(r)\|^{2p}] \le C_6(|t-r|^p + |t-r|^{2p}).$$

The proof is complete.

5. Existence of invariant measures

In this section, we first recall the definition of invariant measure and transition operator. Then we construct a compact subset of $C([-\rho,0];H)$ in order to prove the tightness of the sequence of invariant measure m_k on $C([-\rho,0];H)$.

Recall that for any initial time t_0 and every $\mathcal{F}_{t_0}$-measurable function $\varphi(s) \in L^2(\Omega, C([-\rho,0],H))$, problems (1.1) and (1.2) has a unique solution $u(t;t_0,\varphi)$ for $t \in [t_0-\rho,\infty)$. For convenience, given $t \ge t_0$ and $\mathcal{F}_{t_0}$-measurable function $\varphi(s) \in L^2(\Omega, C([-\rho,0],H))$, the segment of $u(t;t_0,\varphi)$ on $[t-\rho,t]$ is written as

$$u_t(t_0,\varphi)(s) = u(t+s;t_0,\varphi)\ for\ every\ s \in [-\rho,0].$$

Then $u_t(t_0,\varphi) \in L^2(\Omega, C([-\rho,0],H))$ for all $t \ge t_0$. We introduce the transition operator for (1.1). If $\phi(s) : C([-\rho,0],H) \to \mathbb{R}$ is a bounded Borel function, then for initial time r with $0 \le r \le t$ and $\varphi(s) \in C([-\rho,0],H)$, we write

$$(p_{r,t}\phi)(\varphi) = \mathbb{E}[\phi(u_t(r,\varphi))].$$

Particularly, for $\Gamma \in \mathcal{B}(C([-\rho,0],H))$, $0 \le r \le t$ and $\varphi \in C([-\rho,0],H)$, we have

$$p(r,\varphi;t,\Gamma) = (p_{r,t}1_\Gamma)(\varphi) = P\{\omega \in \Omega | u_t(r,\varphi) \in \Gamma\},$$

where 1_Γ is the characteristic function of Γ. Then $p(r,\varphi;t,\cdot)$ is the distribution of $u_t(0,\varphi)$ in $C([-\rho,0],H)$. In the following context, we will write $p_{0,t}$ as p_t.

Recall that a probability measure $\mathscr{M}$ on $C([-\rho, 0], H)$ is called an invariant measure, if for all $t \geq 0$ and every bounded and continuous function $\phi : C([-\rho, 0]; H) \to \mathbb{R}$,

$$\int_{C([-\rho,0];H)} (p_t\phi)(\varphi)d\mathscr{M}(\varphi) = \int_{C([-\rho,0];H)} \phi(\varphi)d\mathscr{M}(\varphi), \quad for\ all\ t \geq 0.$$

According to [33], we infer that the transition operator $\{p_{r,t}\}_{0\leq r\leq t}$ has the following properties.

Lemma 5.1. *Suppose* (2.1)–(2.7) *and* (4.1)–(4.3) *hold. One has*

(a) The family $\{p_{r,t}\}_{0\leq r\leq t}$ is Feller; that is, if $\phi : C([-\rho, 0], H) \to \mathbb{R}$ is bounded and continuous, then for any $0 \leq r \leq t$, the function $p_{r,t}\phi : C([-\rho, 0], H) \to \mathbb{R}$ is also bounded and continuous.

(b) The family $\{p_{r,t}\}_{0\leq r\leq t}$ is homogeneous (in time); that is, for any $0 \leq r \leq t$,

$$p(r, \varphi; t, \cdot) = p(0, \varphi; t - r, \cdot), \forall \varphi \in C([-\rho, 0], H).$$

(c) Given $r \geq 0$ and $\varphi \in C([-\rho, 0], H)$, the process $\{u_t(r, \varphi)\}_{t\geq r}$ is a $C([-\rho, 0], H)$-valued Markov process. Consequently, if $\phi : C([-\rho, 0], H) \to \mathbb{R}$ is a bounded Borel function, then for any $0 \leq s \leq r \leq t$, P-almost surely,

$$(p_{s,t}\phi)(\varphi) = (p_{s,r}(p_{r,t}\phi))(\varphi), \forall \varphi \in C([-\rho, 0], H),$$

and the Chapman-Kolmogorov equation is valid:

$$p(s, \varphi; t, \Gamma) = \int_{C([-\rho,0],H)} p(s, \varphi; r, dy)p(r, y; t, \Gamma),$$

for any $\varphi \in C([-\rho, 0], H)$ and $\Gamma \in \mathcal{B}(C([-\rho, 0], H))$.

Now, we establish the existence of invariant measures of problems (1.1) and (1.2).

Theorem 5.2. *Suppose* (2.1)–(2.7) *and* (4.1)–(4.3) *hold. Then* (1.1) *and* (1.2) *processes an invariant measure on $C([-\rho, 0], H)$.*

Proof. We employ Krylov-Bogolyubov's method to the solution $u(t, 0, 0)$ of problems (1.1) and (1.2), where the initial condition $\varphi \equiv 0$ at the initial time 0. Because of this particular $\varphi \in C([-\rho, 0], V) \subseteq C([-\rho, 0], H)$, we know that all results obtained in the previous Sections 3 and 4 are valid. For simplicity, the solution $u(t, 0, 0)$ is written as $u(t)$ and the segment $u_t(0, 0)$ as u_t. For $k \in \mathbb{N}^+$, we set

$$\mathscr{M}_k = \frac{1}{k}\int_{\rho}^{k+\rho} p(0, 0; t, \cdot)dt. \tag{5.1}$$

Step 1. We prove the tightness of $\{\mathscr{M}_k\}_{k=1}^{\infty}$ in $C([-\rho, 0], H)$. Applying Lemmas 3.2 and 4.2, we get that there exists $C_1 > 0$ such that for all $t \geq \rho$,

$$\mathbb{E}\left[\sup_{-\rho\leq s\leq 0} \|u_t(s)\|_V^2\right] \leq C_1. \tag{5.2}$$

By (5.2) and Chebyshev's inequality, we have that for all $t \geq \rho$,

$$P\left(\left\{\sup_{-\rho\leq s\leq 0} \|u_t(s)\|_V \geq R\right\}\right) \leq \frac{1}{R^2}\mathbb{E}\left[\sup_{-\rho\leq s\leq 0} \|u_t(s)\|_V^2\right] \leq \frac{C_1}{R^2} \to 0 \ \ as \ \ R \to \infty,$$

and hence for every $\varepsilon > 0$, there exists $R_1 = R_1(\varepsilon) > 0$ such that for all $t \geq \rho$,

$$P\left(\left\{\sup_{-\rho\leq s\leq 0}\|u_t(s)\|_V \geq R_1\right\}\right) \leq \frac{1}{3}\varepsilon. \tag{5.3}$$

By Lemma 4.4, we get that there exists $C_2 > 0$ such that for all $t \geq \rho$ and $r, s \in [-\rho, 0]$,

$$\mathbb{E}[\|u_t(r) - u_t(s)\|^{2p}] \leq C_2(1 + |r - s|^p)|r - s|^p,$$

and hence for all $t \geq \rho$ and $r, s \in [-\rho, 0]$,

$$\mathbb{E}[\|u_t(r) - u_t(s)\|^{2p}] \leq C_2(1 + \rho^p)|r - s|^p. \tag{5.4}$$

Since $p \geq 2$, applying (5.4) and the usual technique of dyadic division, we obtain that there exists $R_2 = R_2(\varepsilon) > 0$ such that for all $t \geq \rho$,

$$P\left(\left\{\sup_{-\rho\leq s\leq r\leq 0}\frac{\|u_t(r) - u_t(s)\|}{|r - s|^{\frac{p-1}{4p}}} \leq R_2\right\}\right) \geq 1 - \frac{1}{3}\varepsilon. \tag{5.5}$$

By Lemma 3.4, we get that for given $\varepsilon > 0$ and $m \in \mathbb{N}$, there exists an integer $n_m = n_m(\varepsilon, m) \geq 1$ such that for all $t \geq \rho$,

$$\mathbb{E}\left[\sup_{-\rho\leq s\leq 0}\int_{|x|\geq n_m}|u(t + s, x)|^2dx\right] \leq \frac{\varepsilon}{2^{2m+2}},$$

which implies that for all $t \geq \rho$ and $m \in \mathbb{N}$,

$$P\left(\left\{\sup_{-\rho\leq s\leq 0}\int_{|x|\geq n_m}|u(t + s, x)|^2dx \geq \frac{1}{2^m}\right\}\right) \leq 2^m\mathbb{E}\left[\sup_{-\rho\leq s\leq 0}\int_{|x|\geq n_m}|u(t + s, x)|^2dx\right] \leq \frac{\varepsilon}{2^{m+2}}. \tag{5.6}$$

By (5.6), we infer that for all $t \geq \rho$,

$$P\left(\bigcup_{m=1}^{\infty}\left\{\sup_{-\rho\leq s\leq 0}\int_{|x|\geq n_m}|u(t + s, x)|^2dx \geq \frac{1}{2^m}\right\}\right) \leq \sum_{k=1}^{\infty}\frac{\varepsilon}{2^{m+2}} \leq \frac{1}{4}\varepsilon,$$

and hence for all $t \geq \rho$,

$$P\left(\left\{\sup_{-\rho\leq s\leq 0}\int_{|x|\geq n_m}|u(t + s, x)|^2dx \leq \frac{1}{2^m}\ for\ all\ m \in \mathbb{N}\right\}\right) \geq 1 - \frac{1}{4}\varepsilon. \tag{5.7}$$

Let

$$\mathcal{M}_{1,\varepsilon} = \left\{\zeta : [-\rho, 0] \to V,\ \sup_{-\rho\leq s\leq 0}\|\zeta(s)\|_V \leq R_1(\varepsilon)\right\}, \tag{5.8}$$

$$\mathcal{M}_{2,\varepsilon} = \left\{\zeta \in C([-\rho, 0], H) : \sup_{-\rho\leq s\leq r\leq 0}\frac{\|\zeta(r) - \zeta(s)\|}{|r - s|^{\frac{\varrho-1}{4\varrho}}} \leq R_2(\varepsilon)\right\}, \tag{5.9}$$

$$\mathcal{M}_{3,\varepsilon} = \left\{ \zeta \in C([-\rho, 0], H) : \sup_{-\rho \le s \le 0} \int_{|x| \ge n_m} |\zeta(s, x)|^2 dx \le \frac{1}{2^m} \; for\; all\; m \in \mathbb{N} \right\}, \tag{5.10}$$

and

$$\mathcal{M}_{\varepsilon} = \mathcal{M}_{1,\varepsilon} \bigcap \mathcal{M}_{2,\varepsilon} \bigcap \mathcal{M}_{3,\varepsilon}. \tag{5.11}$$

From (5.3), (5.5) and (5.7)–(5.11), we obtain that for all $t \ge \rho$,

$$P\left(u_t \in \mathcal{M}_{\varepsilon}\right) > 1 - \varepsilon. \tag{5.12}$$

By (5.1) and (5.12), we deduce that for all $k \in \mathbb{N}$,

$$\mathscr{M}_k\left(\mathcal{M}_{\varepsilon}\right) > 1 - \varepsilon. \tag{5.13}$$

Next, we prove the set $\mathcal{M}_{\varepsilon}$ is precompact in $C([-\rho, 0], H)$. First, we prove for every $s \in [-\rho, 0]$ the set $\{\zeta(s) : \zeta \in \mathcal{M}_{\varepsilon}\}$ is a precompact subset of H. By (5.8) and (5.11), we obtain that for every $s \in [-\rho, 0]$, the set $\{\zeta(s) : \zeta \in \mathcal{M}_{\varepsilon}\}$ is bounded in V. Let $Q_{m_0} = \{x \in \mathbb{R}^n : |x| < n_{m_0}\}$. Then we get that the set $\{\zeta(s)|_{Q_{m_0}} : \zeta \in \mathcal{M}_{\varepsilon}\}$ is bounded in $H^1(Q_{m_0})$ and hence precompact in $L^2(Q_{m_0})$ due to compactness of the embedding $H^1(Q_{m_0}) \hookrightarrow L^2(Q_{m_0})$. This implies that the set $\{\zeta(s)|_{Q_{m_0}} : \zeta \in \mathcal{M}_{\varepsilon}\}$ has a finite open cover of balls with radius $\frac{1}{2}\delta$ in $L^2(Q_{m_0})$. Note that for every $\delta > 0$, there exists $m_0 = m_0(\delta) \in \mathbb{N}$ such that for all $\zeta \in \mathcal{M}_{\varepsilon}$,

$$\int_{|x| \ge n_{m_0}} |\zeta(s, x)|^2 dx \le \frac{1}{2^{m_0}} < \frac{\delta^2}{8}. \tag{5.14}$$

Hence, by (5.14), the set $\{\zeta(s) : \zeta \in \mathcal{M}_{\varepsilon}\}$ has a finite open cover of balls with radius $\frac{1}{2}\delta$ in $L^2(\mathbb{R}^n)$. Since $\delta > 0$ is arbitrary, we obtain that the set $\{\zeta(s) : \zeta \in \mathcal{M}_{\varepsilon}\}$ is percompact in H. Then from (5.9) and (5.11), we obtain that $\mathcal{M}_{\varepsilon}$ is equicontinuous in $C([-\rho, 0], H)$. Therefore, by the Ascoli-Arzel*à* theorem we deduce that $\mathcal{M}_{\varepsilon}$ is precompact in $C([-\rho, 0], H)$, which along with (5.13) shows that $\{m_k\}_{k=1}^{\infty}$ is tight on $C([-\rho, 0], H)$.

Step 2. We prove the existence of invariant measures of problems (1.1) and (1.2). Since the sequence $\{\mathscr{M}_k\}_{k=1}^{\infty}$ is tight on $C([-\rho, 0]; H)$, there exists a probability measure m on $C([-\rho, 0]; H)$, we take a subsequence of $\{\mathscr{M}_k\}$(not rebel) such that $\mathscr{M}_k \to m, \;\; as \;\; k \to \infty$. In the following, we prove $\mathscr{M}$ is an invariant measure of (1.1) and (1.2). Applying (5.1) and the Chapman-Kolmogorov equation, we obtain that for every $t \ge 0$ and every $\phi : C([-\rho, 0]; H) \to \mathbb{R}$,

$$\begin{aligned}
&\int_{C([-\rho,0];H)} \phi(v) d\mathscr{M}(v) = \lim_{k\to\infty} \frac{1}{k} \int_{\rho}^{k+\rho} \left(\int_{C([-\rho,0];H)} \phi(v) p(0, 0; s, dv) \right) ds \\
&= \lim_{k\to\infty} \frac{1}{k} \int_{\rho-t}^{k+\rho-t} \left(\int_{C([-\rho,0];H)} \phi(v) p(0, 0; s + t, dv) \right) ds \\
&= \lim_{k\to\infty} \frac{1}{k} \int_{\rho}^{k+\rho} \left(\int_{C([-\rho,0];H)} \phi(v) p(0, 0; s + t, dv) \right) ds \\
&= \lim_{k\to\infty} \frac{1}{k} \int_{\rho}^{k+\rho} \left(\int_{C([-\rho,0];H)} \left(\int_{C([-\rho,0];H)} \phi(v) p(s, \varphi; s + t, dv) \right) p(0, 0; s, d\varphi) \right) ds
\end{aligned}$$

$$= \lim_{k\to\infty} \frac{1}{k} \int_{\rho}^{k+\rho} \left(\int_{C([-\rho,0];H)} \left(\int_{C([-\rho,0];H)} \phi(v) p(0,\varphi;t,dv) \right) p(0,0;s,d\varphi) \right) ds$$
$$= \int_{C([-\rho,0];H)} \left(\int_{C([-\rho,0];H)} \phi(v) p(0,\varphi;t,dv) \right) d\mathscr{M}(\varphi)$$
$$= \int_{C([-\rho,0];H)} (p_{0,t}\phi)(\varphi) d\mathscr{M}(\varphi),$$

which completes the proof.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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