



Research article

Analysis of a heterogeneous SEIRS patch model with asymmetric mobility kernel

Shuangshuang Yin¹, Jianhong Wu² and Pengfei Song^{1,2,*}

¹ School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, Shaanxi, China

² Laboratory for Industrial and Applied Mathematics, Department of Mathematics and Statistics
York University, Ontario, Toronto, CA

* **Correspondence:** Email: song921012@xjtu.edu.cn.

Abstract: In this paper, we establish a spatial heterogeneous SEIRS patch model with asymmetric mobility kernel. The basic reproduction ratio \mathcal{R}_0 is defined, and threshold-type results on global dynamics are investigated in terms of \mathcal{R}_0 . In certain cases, the monotonicity of \mathcal{R}_0 with respect to the heterogeneous diffusion coefficients is established, but this is not true in all cases. Finally, when the diffusion rate of susceptible individuals approaches zero, the long-term behavior of the endemic equilibrium is explored. In contrast to most prior studies, which focused primarily on the mobility of susceptible and symptomatic infected individuals, our findings indicate the significance of the mobility of exposed and recovered persons in disease dynamics.

Keywords: SEIRS epidemic model; asymmetric mobility kernel; basic reproduction number; spatial heterogeneity

1. Introduction

There is mounting evidence in epidemiology that spatial-temporal heterogeneity and human mobility have a substantial impact on the spread of infectious diseases [1–4]. Different population, social contact rates, individual response, control measures, and medical resources (sickbeds and physicians per thousand people) at different regions all contribute to spatial heterogeneity. Many reaction diffusion or patch models have been presented in recent years to examine the role of diffusion and spatial heterogeneity in disease transmission [3, 5–15]. Among these works, Allen et al. [5] developed a susceptible-infected-susceptible (SIS) patch system that collaborated with spatial heterogeneity in the

following way:

$$\begin{cases} \frac{d\bar{S}_j}{dt} = d_S \sum_{k \in \Omega} (L_{jk}\bar{S}_k - L_{kj}\bar{S}_j) - \frac{\beta_j \bar{S}_j \bar{I}_j}{\bar{S}_j + \bar{I}_j} + \gamma_j \bar{I}_j, & j \in \Omega, \\ \frac{d\bar{I}_j}{dt} = d_I \sum_{k \in \Omega} (L_{jk}\bar{I}_k - L_{kj}\bar{I}_j) + \frac{\beta_j \bar{S}_j \bar{I}_j}{\bar{S}_j + \bar{I}_j} - \gamma_j \bar{I}_j, & j \in \Omega, \end{cases} \quad (1.1)$$

where $\Omega = \{1, 2, 3, \dots, n\}$ with $n \geq 2$ denoting the patch numbers, $\bar{S}_j(t)$ and $\bar{I}_j(t)$ represent the number of susceptible and infected individuals, respectively, in patch j at any given time t . The transmission rate of the disease in patch j is denoted by β_j , while the recovery rate is denoted by γ_j . Furthermore, d_S and d_I are the diffusion rates of the susceptible and infected populations, respectively. Matrix $(L_{jk})_{n \times n}$ is symmetric with L_{jk} representing the degree of movement from patch k into patch j .

In [5], a spatial SIS reaction-diffusion model is studied, with the focus on the existence, uniqueness and particularly the asymptotic profile of the steady states. More works related to system (1.1) can be seen in [6] (e.g., asymptotic behaviors of the endemic equilibrium (EE) as the diffusion rate of the infected individuals (d_I) approaches to zero) and [16, 17] (e.g., asymmetric matrix $(L_{jk})_{n \times n}$).

However, these models neglected the mobility of exposed persons and did not incorporate the class of exposed individuals. Exposed individuals, without exhibiting symptoms, can be seen in various epidemic diseases, such as malaria, HIV/AIDS, and SARS-COV-2. The disease can spread spatial if exposed individuals with no symptoms travel, making it more difficult to control [18, 19]. As a result, it appears critical to incorporate the exposed individuals subclass and examine the effects of exposed individuals' mobility on infection disease spread, which is mathematically linked to the basic reproduction number's dependence on the diffusion rates of exposed individuals. Earlier studies have investigated discrete-space multi-patch models concerning this issue [20, 21]. The exposed and recovery classes will be extended to the SIS patch model, and the associated SEIRS patch model will be analyzed in this study.

1.1. SEIRS patch model

We consider a system of differential equations:

$$\begin{cases} \frac{dS}{dt} = d_S \mathbf{L}S - \text{diag} \left\{ \frac{\beta_i S_i}{S_i + E_i + I_i + R_i} \right\} I + \alpha R, & i \in \Omega, t > 0, \\ \frac{dE}{dt} = d_E \mathbf{L}E + \text{diag} \left\{ \frac{\beta_i S_i}{S_i + E_i + I_i + R_i} \right\} I - \sigma E, & i \in \Omega, t > 0, \\ \frac{dI}{dt} = d_I \mathbf{L}I + \sigma E - \text{diag}\{\gamma_i\}I, & i \in \Omega, t > 0, \\ \frac{dR}{dt} = d_R \mathbf{L}R + \text{diag}\{\gamma_i\}I - \alpha R, & i \in \Omega, t > 0, \end{cases} \quad (1.2)$$

where S , E , I , and R represent the number of individuals in the susceptible, exposed, infectious, and recovered compartments, respectively. The parameters β , σ , γ , and α denote the rates of transmission, latent period, recovery, and loss of immunity, respectively.

Here $\Omega = \{1, 2, 3, \dots, n\}$ with $n \geq 2$ denoting the patch numbers, $S_i(t)$, $E_i(t)$, $I_i(t)$ and $R_i(t)$ denote the population of susceptible, exposed, infected and recovered individuals at time t and patch i , respectively. Furthermore, d_S , d_E , d_I and d_R represent the diffusion coefficients associated with susceptible,

exposed, infected and recovered individuals, respectively. We assume that $\beta := (\beta_i)_{n \times 1}$ and $\gamma := (\gamma_i)_{n \times 1}$ depends on the environment, could be spatial heterogeneous and are assumed to be positive throughout this paper. Given that the latent period $1/\sigma$ which represents the time takes for an infection to become detectable after exposure and the rate of loss of immunity α which describes the proportion of individuals who lose their immunity to a disease over time, are inherent properties of the individual and not influenced by the external environment, it is reasonable to consider σ and α as constants in this study.

$\mathbf{L} = (\ell_{ij})$ is an $n \times n$ cooperative, irreducible matrix with $\ell_{ii} = -\sum_{j \neq i} \ell_{ij}$ referred as Laplacian matrix, where ℓ_{ij} , denoting the degree of movement from patch k into patch j , is independent of t . Here an $n \times n$ matrix is said to be cooperative if all of its off-diagonal entries are non-negative, and to be irreducible if its index set $\{1, 2, \dots, n\}$ cannot be split into two complementary sets (without common indices) $\{p_1, p_2, \dots, p_r\}$ and $\{q_1, q_2, \dots, q_s\}$ ($r + s = n$) such that $a_{p_i q_j} = 0$ for all $1 \leq i \leq r, 1 \leq j \leq s$. An $n \times n$ matrix is called quasi-positive (also called Metzler matrix) if $\ell_{ij} \geq 0$ for all $i \neq j$. Here we mention that \mathbf{L} can be asymmetric. Moreover, to match the real human mobility patterns and transportation phenomena such as gravity [22, 23] or radiation mobility model [24], we assume that \mathbf{L} can be asymmetric. We also assume that the following initial conditions are satisfied throughout the study.

(A1) $S_i(0), E_i(0), I_i(0), R_i(0) \geq 0$ for $i \in \Omega$ and $\sum_{i=1}^n I_i(0) > 0$.

It is easy to verify that $\frac{\beta_i S_i I_i}{S_i + I_i + E_i + R_i}$ is a Lipschitz continuous function of S_i and I_i , therefore we define the quantity to be zero in cases where $S_i = 0$ or $I_i = 0$. According to [34, Chapter I], it can be stated that the system (1.2) has a unique classical solution $S, E, I, R \in C^1((0, \infty); \mathbb{R}^n)$, and $S_i(t), E_i(t), I_i(t), R_i(t) > 0$ for $i \in \Omega$ and $t > 0$.

The total population at time t is defined as

$$N(t) = \sum_{i=1}^n (S_i(t) + E_i(t) + I_i(t) + R_i(t)),$$

and let the total number of individuals is a fixed positive constant at the initial time $t = 0$, denoted by N_0 . By the system (1.2), we conclude that the total population size is constant, i.e.,

$$N(t) = N_0 \text{ for any } t \geq 0. \quad (1.3)$$

1.2. Equilibrium problems

Let $u, v \in \mathbb{R}^n$. We write $u \geq v$ if $u_i \geq v_i$ for any $i \in \Omega$; $u > v$ if $u_i \geq v_i$ for any $i \in \Omega$, and there exists j such that $u_j > v_j$; $u \gg v$ if $u_i > v_i$ for any $i \in \Omega$. We say u is non-negative, positive and strongly positive if $u \geq 0, u > 0, u \gg 0$, respectively.

Our primary focus will be on equilibrium solutions of the system (1.2), i.e., non-negative solutions

of the following system:

$$\begin{cases} d_S \mathbf{L}\tilde{S} - \text{diag} \left\{ \frac{\beta_i \tilde{S}_i}{\tilde{S}_i + \tilde{I}_i + \tilde{E}_i + \tilde{R}_i} \right\} \tilde{I} + \alpha \tilde{R} = 0, \\ d_E \mathbf{L}\tilde{E} + \text{diag} \left\{ \frac{\beta_i \tilde{S}_i}{\tilde{S}_i + \tilde{I}_i + \tilde{E}_i + \tilde{R}_i} \right\} \tilde{I} - \sigma \tilde{E} = 0, \\ d_I \mathbf{L}\tilde{I} + \sigma \tilde{E} - \text{diag} \{ \gamma_i \} \tilde{I} = 0, \\ d_R \mathbf{L}\tilde{R} + \text{diag} \{ \gamma_i \} \tilde{I} - \alpha \tilde{R} = 0, \\ \sum_{i=1}^n (\tilde{S}_i + \tilde{E}_i + \tilde{I}_i + \tilde{R}_i) = N_0, \end{cases} \quad (1.4)$$

where the population of susceptible, exposed, infected and recovered individuals at equilibrium are represented by the variables \tilde{S} , \tilde{E} , \tilde{I} , and \tilde{R} , respectively. A solution of (1.4) in which $\tilde{I}_i = 0$ for every $i \in \Omega$ is a *disease-free equilibrium* (DFE), while a solution in which $\tilde{I}_i > 0$ for some $i \in \Omega$ is referred to as an *endemic equilibrium* (EE). It is straightforward to observe that DFE is unique, given by $E_0 = (\tilde{N}, 0, 0, 0)$, where \tilde{N} is the unique positive eigenvector corresponds to the eigenvalue of 0 in the following eigenvalue problem:

$$\mathbf{L}\phi = 0, \quad \sum_{i=1}^n \phi = N_0. \quad (1.5)$$

Moreover, for any EE, $\tilde{S}, \tilde{E}, \tilde{I}, \tilde{R} \gg 0$.

1.3. Main results

The purpose of this research is to examine how changes in human mobility and environment heterogeneity affect the persistence or eradication of infectious diseases. Our research will investigate the threshold-type dynamics of system (1.2), the characteristics of the basic production ratio, and when the diffusion rate of the susceptible individuals approaches zero, the asymptotic behaviors of the endemic equilibria of (1.4).

The basic reproduction number is a fundamental concept in the study of infectious disease transmission. It is defined as the expected number of secondary infections generated by a single infected individual in a completely susceptible population [25, 26]. More importantly, it is frequently used to determine the behaviour of various epidemic models' thresholds. Specifically, when the basic reproduction number is less than unity, the disease is expected to fade out, while a basic reproduction number greater than unity indicates the potential for the disease to establish itself within the population. The next-generation operator method has been widely utilized for the computation of fundamental reproduction number, see [25, 27, 28] and for related investigations, see [3, 28, 29].

By [28, 29], the basic reproduction number, denoted as \mathcal{R}_0 , can be expressed as $\mathcal{R}_0 = \frac{1}{\mu_0}$ (see Lemma 2.2). Here, μ_0 represents the unique positive eigenvalue associated with a positive eigenvector of the following problem:

$$\begin{cases} -d_E \mathbf{L}\varphi_E + \sigma \varphi_E = \mu \text{diag} \{ \beta_i \} \varphi_I, & i \in \Omega, \\ -d_I \mathbf{L}\varphi_I + \text{diag} \{ \gamma_i \} \varphi_I - \sigma \varphi_E = 0, & i \in \Omega. \end{cases} \quad (1.6)$$

In terms of \mathcal{R}_0 , we state the dynamics for system (1.2) as follows:

Theorem 1.1. (i) If $\mathcal{R}_0 \leq 1$, then DFE is globally asymptotically stable, that is $E(t), I(t), R(t) \rightarrow 0$ as $t \rightarrow \infty$, and $S \rightarrow \bar{N}$ as $t \rightarrow \infty$;

(ii) If $\mathcal{R}_0 > 1$, then there exists a positive constant ϵ_0 such that any positive solution of (1.2) satisfies

$$\liminf_{t \rightarrow \infty} \left\| (S(t), E(t), I(t), R(t)) - (\bar{N}, 0, 0, 0) \right\| > \epsilon_0. \quad (1.7)$$

Furthermore, system described by (1.2) has at least one endemic equilibrium.

It is not hard to see that the value of the basic reproduction number gives an indication on the persistence or extinction of infectious diseases. Therefore, we focus on the dependence of \mathcal{R}_0 on d_E, d_I, d_S and d_R . It is easy to know that \mathcal{R}_0 is independent of d_S and d_R . The dependence of \mathcal{R}_0 on d_E, d_I can have important implications for disease control because it explores how changes in human mobility affect the persistence or extinction of infectious diseases. [16] demonstrates that \mathcal{R}_0 for model (1.1) is decreasing in d_I . However, for the *SEIRS* system (1.2), the monotonicity of \mathcal{R}_0 with respect to d_I is more subtle due to the presence of mobility of exposed individuals.

To start with, we give the asymptotic properties of \mathcal{R}_0 when d_E, d_I tend to 0 or infinity in Theorem 1.2. Define $\beta = (\beta_i)_{n \times 1}, \gamma = (\gamma_i)_{n \times 1}$ and set

$$B = \text{diag} \{ \beta_i \}, Y = \text{diag} \{ \gamma_i \}, Z = \text{diag} \{ \beta_i / \gamma_i \}.$$

Moreover, ξ is defined as the positive eigenvector of \mathbf{L} that is unique, satisfying $\sum_{i \in \Omega} \xi_i = 1$.

Theorem 1.2. (i) Fix $d_I > 0$. Then $\mathcal{R}_0 \rightarrow \rho((-d_I \mathbf{L} + Y)^{-1} B)$ as $d_E \rightarrow 0$, and $\mathcal{R}_0 \rightarrow \frac{\|B(-d_I \mathbf{L} + Y)^{-1} \xi\|_1}{\|\xi\|_1}$ as $d_E \rightarrow \infty$.

(ii) Fix $d_E > 0$. Then $\mathcal{R}_0 \rightarrow \rho(\sigma(-d_E \mathbf{L} + \sigma I_{n \times n})^{-1} Z)$ as $d_I \rightarrow 0$ and $\mathcal{R}_0 \rightarrow \frac{\sum_{i=1}^n \xi_i \beta_i}{\sum_{i=1}^n \xi_i \gamma_i}$ as $d_I \rightarrow \infty$.

(iii) As $d_E, d_I \rightarrow 0$, then $\mathcal{R}_0 \rightarrow \max \left\{ \frac{\beta_i}{\gamma_i}, i \in \Omega \right\}$;

(iv) As $d_E \rightarrow \infty$ and $d_I \rightarrow 0$, then $\mathcal{R}_0 \rightarrow \sum_{i=1}^n \frac{\xi_i \beta_i}{\gamma_i}$.

The following result is concerned with the monotonicity of \mathcal{R}_0 with respect to d_E, d_I :

Theorem 1.3. If \mathbf{L} is symmetric and either β or γ is a multiple of vector $\mathbf{1}$, then \mathcal{R}_0 exhibits a monotone decreasing behaviour with respect to the parameters d_E and d_I . Furthermore, strict monotonicity is observed if and only if one of them is not multiple of vector $\mathbf{1}$.

If both β and γ are not multiples of vector $\mathbf{1}$, Theorem 1.3 may be not valid. Refer to Theorem 3.3.

Theorem 1.4. Let $\ell_{ij} = 1, i \neq j, \ell_{ii} = -(n-1)$ and

$$\text{sign}\{\beta_i - \beta_j\} = \text{sign}\{\gamma_j - \gamma_i\} \quad (1.8)$$

holds for any $i, j \in \Omega$. Then \mathcal{R}_0 is nonincreasing in d_E, d_I .

Under the conditions in Theorems 1.3 and 1.4, the nonincreasing behaviour of \mathcal{R}_0 with respect to d_E and d_I seems counter-intuitive. If no traffic control or transportation restrictions policies are adopted during the disease spread, large diffusion rate also means accessing more health resources (β is a multiple of vector $\mathbf{1}$; condition (1.8)) or escaping from high risk areas (γ is a multiple of vector $\mathbf{1}$; condition (1.8)).

Finally, to better understand the influence of the suspected individuals movement on the spread of disease, we demonstrate that the asymptotic behavior of the endemic equilibrium as the diffusion rate of susceptible individuals approaches zero. By Theorem 1.1, when $\mathcal{R}_0 > 1$, there exist at least one endemic equilibrium. To this end, let's examine this linear eigenvalue problem

$$-d_R \mathbf{L} \phi + \alpha \text{diag} \left\{ 1 - \frac{\gamma_i}{\beta_i} \right\} \phi = \lambda \phi. \quad (1.9)$$

And here represent the minimum eigenvalue of (1.9) as $\Lambda_1 \left(-d_R \mathbf{L} + \alpha \text{diag} \left\{ 1 - \frac{\gamma_i}{\beta_i} \right\} \right)$ ([33, Theorems 1 and 2]).

Theorem 1.5. *Assume $\mathcal{R}_0 > 1$ and $\Lambda_1 \left(-d_R \mathbf{L} + \alpha \text{diag} \left\{ 1 - \frac{\gamma_i}{\beta_i} \right\} \right) < 0$. Then*

(i) *There exist positive constants C_1, C_2 , independent of d_S , such that for sufficiently small d_S ,*

$$C_1 \leq \frac{\widetilde{E}_i}{d_S}, \frac{\widetilde{I}_i}{d_S}, \frac{\widetilde{R}_i}{d_S} \leq C_2, \quad \forall i \in \Omega;$$

(ii) *As d_S approaches zero, with respect to a given sequence,*

$$\widetilde{S} \rightarrow \widetilde{S}^* = \frac{N_0(\xi - M^*)}{\sum_{i=1}^n (\xi - M^*)},$$

where $M^* \in \mathbb{R}^n$ satisfying $0 \leq M^* \leq \xi$ and $\{i \in \Omega : M_i^* = \xi_i\} \in (0, n)$.

In particular, Theorem 1.5 demonstrates that as d_S approaches zero, the variables \widetilde{E} , \widetilde{I} , and \widetilde{R} converge uniformly to zero in Ω . Additionally, \widetilde{S}^* is positive and there exists a non-empty subset $\{i \in \Omega : \widetilde{S}_i^* = 0\}$. From a biological standpoint, limiting the diffusion rate of susceptible individuals can effectively reduce the number of exposed and infected individuals in a population. We point out that if $\Lambda_1 \left(d_R \mathbf{L} - \alpha \text{diag} \left\{ 1 - \frac{\gamma_i}{\beta_i} \right\} \right) > 0$, Theorem 1.5 may be not valid.

Most of the results of SEIRS patch model (1.2) match the contents of SEIRS reaction diffusion model in [19], which deals with the continuous-space case to the discrete case. Compared with [19], the main improvement of this paper is as follows: investigation of threshold dynamics and asymptotic behaviours of \mathcal{R}_0 with respect to d_E and d_I are extended to asymmetric mobility kernel such as gravity [22,23] or radiation mobility model [24]. In [19, Theorem 1.2], under the condition that γ is a constant function, the proof of the monotonicity of \mathcal{R}_0 with respect to d_E, d_I is complicated and difficult for readers to access the core ideas. For SEIRS patch model in this paper, the proof is shortened (see Theorem 3.2) and can give more intuitiveness to understand the proof of [19, Theorem 1.2]. Moreover, though the results in Theorem 1.4 is similar in spirit to [19, Theorem 1.3], the proof for patch model is novel and totally different from continuous-space case.

This paper is organized as follows. In Section 2, we establish the well-posedness of the system (1.2), introduce the basic reproduction number \mathcal{R}_0 , and analyze the system's dynamics in terms of \mathcal{R}_0 . Section 3 is dedicated to exploring the asymptotic stability and directional monotonicity of \mathcal{R}_0 in relation to d_E and d_I . Finally, Section 4 focuses on the behavior of the endemic equilibrium as d_S approaches zero.

2. The stability of disease free equilibrium

To start with, we have the following uniform bound result, which can be directly derived from (1.3).

Lemma 2.1. *For some positive constants C_1 , which are independent of initial values, and $T > 0$, the solution $(S, E, I, R) \in C^1((0, \infty); \mathbb{R}^{4n})$ of the system (1.2) satisfies*

$$|S_i(t)| + |E_i(t)| + |I_i(t)| + |R_i(t)| \leq C_1, \quad \forall i \in \Omega, t > T. \quad (2.1)$$

We now make use of the theory developed in [28] to derive \mathcal{R}_0 of system (1.2).

Lemma 2.2. (i). $\mathcal{R}_0 = \rho(FV^{-1}) = \rho(\sigma B(-d_E \mathbf{L} + \sigma I_{n \times n})^{-1}(-d_I \mathbf{L} + Y)^{-1})$.

(ii). *The eigenvalue problem (1.6) has a positive eigenvalue that is unique, denoted by μ_0 , along with a corresponding positive eigenvector. Additionally, the basic reproduction number of system (1.2), denoted by \mathcal{R}_0 , satisfies*

$$\mathcal{R}_0 = \frac{1}{\mu_0}. \quad (2.2)$$

Proof. (i) The linearization of system (1.2) at E_0 is given by

$$\begin{cases} \frac{d\bar{S}}{dt} = d_S \mathbf{L} \bar{S} - B \bar{I} + \alpha \bar{R}, t > 0, \\ \frac{d\bar{E}}{dt} = d_E \mathbf{L} \bar{E} + B \bar{I} - \sigma \bar{E}, t > 0, \\ \frac{d\bar{I}}{dt} = d_I \mathbf{L} \bar{I} + \sigma \bar{E} - Y \bar{I}, t > 0, \\ \frac{d\bar{R}}{dt} = d_R \mathbf{L} \bar{R} + Y \bar{I} - \alpha \bar{R}, t > 0. \end{cases} \quad (2.3)$$

Note that E and I represent the infected population in (1.2). Furthermore, here F and V in [28] is denoted as

$$F = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}_{2n \times 2n}, \quad V = L + \begin{pmatrix} \sigma I_{n \times n} & 0 \\ -\sigma I_{n \times n} & Y \end{pmatrix}, \quad (2.4)$$

where $L = \text{diag}\{-d_E \mathbf{L}, -d_I \mathbf{L}\}$. Part (i) follow immediately from [28]. We only need to prove part (ii). By Perron-Frobenius theorem [33, Theorems 1 and 2], it is possible to establish the existence and uniqueness of a positive eigenvalue μ_0 with respect to a positive eigenvector φ within the system described in Eq (1.6), i.e.,

$$V\varphi = \mu_0 F\varphi.$$

Note that there exists a positive eigenvector ϕ such that

$$FV^{-1}\phi = \mathcal{R}_0\phi,$$

which implies

$$V\phi = \frac{1}{\mathcal{R}_0} F\phi.$$

By the uniqueness of μ_0 , we obtain $\mathcal{R}_0 = \frac{1}{\mu_0}$. This completes the proof.

We now demonstrate that the stability of E_0 can be determined by the basic reproduction number \mathcal{R}_0 .

Lemma 2.3. *The disease-free equilibrium E_0 is locally asymptotically stable if $\mathcal{R}_0 < 1$, unstable if $\mathcal{R}_0 > 1$.*

Proof. The proof of Lemma 2.3 can follow from [28]. For readers' convenience, we also give some details here. To prove the locally asymptotically stability if $\mathcal{R}_0 < 1$, we only need to prove $\inf\{\operatorname{Re}\lambda, \lambda \in \Lambda\} > 0$, where Λ is the spectrum of the following eigenvalue problem

$$\begin{cases} d_S \mathbf{L}\phi_S - B\phi_I + \alpha\phi_R + \lambda\phi_S = 0, \\ d_E \mathbf{L}\phi_E + B\phi_I - \sigma\phi_E + \lambda\phi_E = 0, \\ d_I \mathbf{L}\phi_I + \sigma\phi_E - Y\phi_I + \lambda\phi_I = 0, \\ d_R \mathbf{L}\phi_R + Y\phi_I - \alpha\phi_R + \lambda\phi_R = 0. \end{cases} \quad (2.5)$$

Note that $\Lambda = \Lambda_{\{(\phi_E, \phi_I) \neq 0\}} \cup \Lambda_{\{(\phi_E, \phi_I) = 0\}}$. Obviously, $\inf\{\operatorname{Re}\lambda, \lambda \in \Lambda_{\{(\phi_E, \phi_I) = 0\}}\} > 0$. Moreover, it can be shown that $\Lambda_{\{(\phi_E, \phi_I) \neq 0\}}$ is a subset of the set of $\sigma(L - F + V)$, where L, F, V are defined in (2.4) and $\sigma(L - F + V)$ denotes the eigenvalues of the matrix $L - F + V$. The infimum of the real parts of the eigenvalues of $L - F + V$ is positive, here denoted by λ_1 , i.e., $\inf\{\operatorname{Re}\lambda, \lambda \in \sigma(L - F + V)\} = \lambda_1 > 0$. Moreover, λ_1 represents the principal eigenvalue of

$$\begin{cases} -d_E \mathbf{L}\phi_E - B\phi_I + \sigma\phi_E = \lambda\phi_E, \\ -d_I \mathbf{L}\phi_I + Y\phi_I - \sigma\phi_E = \lambda\phi_I. \end{cases} \quad (2.6)$$

Consider the left eigenvector corresponding to principal eigenvalue of (2.6), i.e.,

$$\begin{cases} -d_E \mathbf{L}^T \phi_E^* + \sigma\phi_E^* - \sigma\phi_I^* = \lambda_1 \phi_E^*, \\ -d_I \mathbf{L}^T \phi_I^* - B\phi_E^* + Y\phi_I^* = \lambda_1 \phi_I^*. \end{cases} \quad (2.7)$$

Multiply (1.6)'s first equation by $(\phi_E^*)^T$ and (2.7)'s first equation by φ_E^T . Subtracting the resulting equations yields

$$\lambda_1 \varphi_E^T \phi_E^* = \frac{1}{\mathcal{R}_0} \varphi_I^T B \phi_E^* - \sigma (\phi_I^*)^T \varphi_E. \quad (2.8)$$

Moreover, multiply (1.6)'s second equation by $(\phi_I^*)^T$ and (2.7)'s second equation by φ_I^T . Subtracting the resulting equations yields

$$\lambda_1 \varphi_I^T \phi_I^* = -\varphi_I^T B \phi_E^* - \sigma (\phi_I^*)^T \varphi_E. \quad (2.9)$$

Adding two Eqs (2.8) and (2.9) yields

$$\lambda_1 (\varphi_E^T \phi_E^* + \varphi_I^T \phi_I^*) = \frac{1 - \mathcal{R}_0}{\mathcal{R}_0} \varphi_I^T B \phi_E^*.$$

Then we have $\operatorname{sign}(1 - \mathcal{R}_0) = \operatorname{sign}(\lambda_1)$ for the positive of $\varphi_E, \phi_E^*, \varphi_I, \phi_I^*, \beta$. We obtain $\inf\{\operatorname{Re}\lambda, \lambda \in \Lambda_{\{(\phi_E, \phi_I) \neq 0\}}\} > 0$. Therefore, if $\mathcal{R}_0 < 1$, E_0 is locally asymptotically stable.

To establish the linear instability of E_0 when $\mathcal{R}_0 > 1$, it is necessary to demonstrate the existence of a non-trivial solution to (2.5) with $\operatorname{Re}\lambda < 0$. To this end, let $\lambda = \lambda_1 < 0$, where λ_1 denotes the principal eigenvalue of (2.6) and select (ϕ_E, ϕ_I) as the eigenvector of (2.6) associated with λ_1 and solving for ϕ_S and ϕ_R in (2.5). Therefore, E_0 is unstable if $\mathcal{R}_0 > 1$.

We now demonstrate that the disease will be extinct if $\mathcal{R}_0 < 1$, i.e., the E_0 is globally asymptotically stable, and we show that when $\mathcal{R}_0 > 1$ at least one EE exists and that both \widetilde{S} and \widetilde{E} are positive.

Proof of Theorem 1.1. (i). In order to analyze the behavior of the system (1.2), we make use of LaSalle's invariance principle (Theorem 1 in [30]) and construct Lyapunov function. We consider the ordered space $X = \mathbb{R}^{4n}$ equipped with the supremum norm, and observe that X has a nonempty interior, which we denote by $\text{int}(P)$, where the cone P is composed of all functions in X that are nonnegative. Set

$$X_0 = \left\{ u = (u_s, u_e, u_i, u_r) \in X \mid \sum_{j \in \Omega} (u_{s,j} + u_{e,j} + u_{i,j} + u_{r,j}) = N_0 \right\}$$

and $U = P \cap X_0$. It can be shown that system (1.2) and (1.3) defines a dynamic system on U . Moreover, for any initial condition $(s_0, e_0, i_0, r_0) \in U$, the unique solution of the system is denoted by $\Phi_t(s_0, e_0, i_0, r_0) = (S, E, I, R)$ for any $t > 0$. It's worth noting that Φ_t is compact, and for each $u_0 \in U$, the orbit of u_0 under the dynamical system generated by (1.2) has compact closure in U .

Define

$$L(u) = u_e^T \phi_E^* + u_i^T \phi_I^*$$

for $u \in U$, where (ϕ_E^*, ϕ_I^*) is the eigenvector corresponding to the principal eigenvalue λ_1 of (2.7), i.e., left eigenvector of (2.6). Next, we show that $L(u)$ serves as a Lyapunov function for (1.2). Considering an arbitrary solution of the system (1.2) coupled with (1.3), we obtain:

$$\begin{aligned} \frac{d}{dt} L(u(t)) &= (\phi_E^*)^T E_t + (\phi_I^*)^T I_t \\ &= (\phi_E^*)^T \left(d_E \mathbf{L} E + \text{diag} \left\{ \frac{\beta_i S_i}{S_i + I_i + E_i + R_i} \right\} I - \sigma E \right) + (\phi_I^*)^T (d_I \mathbf{L} I + \sigma E - \gamma I) \quad (2.10) \\ &= -I^T \text{diag} \left\{ \frac{\beta_i (E_i + I_i + R_i)}{S_i + I_i + E_i + R_i} \right\} \phi_E^* - \lambda_1 (E^T \phi_E^* + I^T \phi_I^*). \end{aligned}$$

Following the same process as in the proof of Lemma 2.3, $\mathcal{R}_0 \leq 1$ yields that $\lambda_1 \geq 0$. Furthermore, $S, E, I, R \geq 0$, and $\beta, \phi_E^*, \phi_I^* > 0$. Consequently, we can assert that $\frac{d}{dt} L(u(t)) \leq 0$, which indicates that $L(u)$ functions as a Lyapunov function for the system (1.2).

Next define

$$\dot{L}(u_0) := \left. \frac{d}{dt} L(u(t)) \right|_{t=0} \quad \text{and} \quad M = \{u_0 \in U \mid \dot{L}(u_0) = 0\},$$

where u is the unique solution of (1.2) with $u_0 = (s_0, e_0, i_0, r_0) \in U$ as initial condition. By (2.10), we have $M = \{u_0 = (s_0, e_0, i_0, r_0) \in U \mid i_0 = 0\}$ if $\lambda_1 = 0$, and $M = \{u_0 = (s_0, e_0, i_0, r_0) \in U \mid e_0 = i_0 = 0\}$ if $\lambda_1 > 0$. Using (1.2), we can infer that when $\lambda_1 \geq 0$, the maximal invariant set in M can be characterized by:

$$\hat{M} := \{u_0 = (s_0, e_0, i_0, r_0) \in U \mid e_0 = i_0 = 0\}.$$

Consequently, we can utilize the LaSalle invariant principle (stated in Theorem 1 in [30]) to obtain:

$$(E(t), I(t)) \rightarrow (0, 0), \quad \text{as } t \rightarrow \infty.$$

By combining the above equation with (1.2), we can infer that $R(t) \rightarrow 0$ as $t \rightarrow \infty$. Consequently, utilizing (1.3), we obtain $\sum_{i=1}^n S(t) \rightarrow \widetilde{N}$ as $t \rightarrow \infty$, where \widetilde{N} is defined in (1.5).

We omit the proof of Theorem 1.1 (ii) because it is standard.

3. The properties of \mathcal{R}_0

In terms of the basic reproduction number \mathcal{R}_0 , we established the threshold dynamics of system (1.2) in the previous section. In this part, we will examine the asymptotic characteristics and monotonicity of the basic reproduction number in relation to the parameters d_E and d_I . The aim is to explore the influence of population movement on the long-term behavior of infectious diseases, particularly persistence or eradication. In this section, we will establish the proofs of Theorems 1.2 and 1.4. Theorem 1.3 is a consequence of Theorems 3.1 and 3.2.

3.1. Preliminary results

For further conveniences, we give some preliminary results here. To start with, we give some definitions. Let $u, v \in \mathbb{R}^n$. We define the relation between u and v as follows: $u \geq v$ if $u_i \geq v_i$ for $\forall i \in \Omega$; $u > v$ if $u_i \geq v_i$ for $\forall i \in \Omega$, and $\exists j$ such that $u_j > v_j$; $u \gg v$ if $u_i > v_i$ for $\forall i \in \Omega$. We say that u is non-negative, positive and strongly positive if $u \geq 0, u > 0, u \gg 0$, respectively. Moreover, we define the vector norm

$$\|u\|_1 = \sum_{i=1}^n |u_i|, \quad \|u\|_\infty = \max_{1 \leq i \leq n} \{|u_i|\}.$$

Consider an $n \times n$ matrix, denoted as A , with the set of eigenvalues $\sigma(A)$. Let $r(A)$ to be the spectral radius of A , i.e.,

$$r(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}.$$

Denote $s(A)$ be the spectral bound of A , i.e.,

$$s(A) = \max\{\operatorname{Re} \lambda : \lambda \in \sigma(A)\}.$$

Recall that a principal eigenvalue of the matrix A means a real and simple eigenvalue with positive eigenvectors.

Now we have the following preliminary results.

Lemma 3.1. $-\mathbf{L}$ admits a unique principal eigenvalue 0 with a unique positive eigenvector ξ satisfying $\sum_{i \in \Omega} \xi_i = 1$. Moreover, for any vector φ , we have $-\varphi^T \mathbf{L} \varphi \geq 0$.

Throughout this paper, we define ξ to be the positive eigenvector that is unique satisfying $\sum_{i \in \Omega} \xi_i = 1$.

Lemma 3.2 ([17]). Consider an $n \times n$ quasi-positive irreducible matrix, denoted as $A = (a_{ij})_{n \times n}$. Let Q be a diagonal matrix, denoted as $Q = \operatorname{diag}\{q_i\}$. Then, we have the subsequent results:

(i). If $s(A) < 0$, then $s(\mu A + Q)$ is strictly decreasing in $\mu \in \mathbb{R}^+$. Furthermore,

$$\lim_{\mu \rightarrow 0} s(\mu A + Q) = \max\{q_i\}$$

and

$$\lim_{\mu \rightarrow \infty} s(\mu A + Q) = -\infty;$$

(ii). If $s(A) = 0$, then $s(\mu A + Q)$ is strictly decreasing provided that Q is not a multiple of $I_{n \times n}$. Furthermore,

$$\lim_{\mu \rightarrow 0} s(\mu A + Q) = \max\{q_i\}$$

and

$$\lim_{\mu \rightarrow \infty} s(\mu A + Q) = \sum_{i \in \Omega} q_i \xi_i,$$

where ξ refers to the positive eigenvector of $s(A)$ that is unique and satisfies the condition $\sum_{i \in \Omega} \xi_i = 1$. (It is worth noting that if every row of matrix A has a sum of zero, then ξ denotes the left positive eigenvector of A .)

3.2. Asymptotic properties of \mathcal{R}_0 associated with d_E, d_I

Lemma 3.3. For any $d_E > 0, d_I > 0$,

$$\min \left\{ \frac{\beta_i}{\gamma_i}, i \in \Omega \right\} \leq \mathcal{R}_0 \leq \max \left\{ \frac{\beta_i}{\gamma_i}, i \in \Omega \right\}. \quad (3.1)$$

Proof. According to Lemma 2.2, the reciprocal of the basic reproduction number, denoted as $\frac{1}{\mathcal{R}_0}$, represents principal eigenvalue that is unique of the eigenvalue problem expressed in Eq (1.6). Therefore, it follows that:

$$\begin{cases} -d_E \mathbf{L} \varphi_E + \sigma \varphi_E = \frac{1}{\mathcal{R}_0} B \varphi_I, \\ -d_I \mathbf{L} \varphi_I + Y \varphi_I - \sigma \varphi_E = 0, \end{cases} \quad (3.2)$$

It follows from the addition of two equations of (3.2) that

$$-d_E \mathbf{L} \varphi_E - d_I \mathbf{L} \varphi_I + Y \varphi_I = \frac{1}{\mathcal{R}_0} B \varphi_I, \quad (3.3)$$

which yields

$$\sum_{i=1}^n \gamma_i \left(\mathcal{R}_0 - \frac{\beta_i}{\gamma_i} \right) \varphi_{I,i} = 0.$$

Since γ_i and φ_I are positive, we obtain (3.1).

Lemma 3.3 showed an estimate of \mathcal{R}_0 and implied that if $\frac{\beta_i}{\gamma_i} \equiv \text{Constant}$, then \mathcal{R}_0 is constant, which is independent of d_E, d_I .

Proof of Theorem 1.2. We only need to prove (i) here because (ii) can be shown through similar arguments. The case $d_E \rightarrow 0$ can be derived by the continuity of eigenvalue with respect to d_E [32, Proposition 2.1]. Now we explore the case $d_E \rightarrow \infty$. By Lemma 3.3, if required, we can pass to a sequence such that $\mathcal{R}_0 \rightarrow \tilde{\mathcal{R}}_0$ as $d_E \rightarrow \infty$. Without loss of generality, we may assume $\|\varphi_E\| + \|\varphi_I\| = 1$. Passing to a sequence if necessary, $\varphi_E \rightarrow \tilde{\varphi}_E, \varphi_I \rightarrow \tilde{\varphi}_I$ in \mathbb{R}_+^n as $d_E \rightarrow \infty$. As shown in Lemma 3.2, $\tilde{\varphi}_E$ is a multiple of ξ and $\tilde{\varphi}_I$ is a solution of a certain equation

$$-d_I \mathbf{L} \tilde{\varphi}_I + Y \tilde{\varphi}_I - \sigma \tilde{\varphi}_E = 0.$$

Summarizing the first equation of (3.2) yields that $\mathcal{R}_0 \rightarrow \frac{\|B(-d_I L + Y)^{-1} \xi\|_1}{\|\xi\|_1}$. This ends the proof of (i).

(iii) The statement can be derived by the continuity of eigenvalue with respect to d_E [32, Proposition 2.1].

(iv) The statement can be directly deduced from Lemma 3.2 and statement (ii).

3.3. Monotonicity of \mathcal{R}_0 with respect to d_E, d_I

Now, we show some cases that \mathcal{R}_0 has monotonicity with respect to d_E, d_I results as follows.

Theorem 3.1. *If β is a multiple of vector $\mathbf{1}$ and \mathbf{L} is symmetric, then \mathcal{R}_0 is monotone decreasing in d_E, d_I . Moreover, the strict monotonicity holds if γ_i is not a multiple of vector $\mathbf{1}$.*

Proof. Firstly, we demonstrate that \mathcal{R}_0 is monotone decreasing with respect to d_E . By [32, Proposition 2.1], we know that \mathcal{R}_0 and the corresponding eigenvectors $(\varphi_E, \varphi_I), (\varphi_E^*, \varphi_I^*)$ are both differentiable functions of d_E, d_I . By direct differentiating both sides of the equations in (3.2) by d_E, d_I , we obtain

$$\begin{cases} -d_E \mathbf{L} \varphi'_E - \mathbf{L} \varphi_E + \sigma \varphi'_E = \frac{1}{\mathcal{R}_0} B \varphi'_I - \frac{\mathcal{R}'_0}{\mathcal{R}_0^2} B \varphi_I, \\ -d_I \mathbf{L} \varphi'_I + Y \varphi'_I - \sigma \varphi'_E = 0, \end{cases} \quad (3.4)$$

and

$$\begin{cases} -d_E \mathbf{L} \varphi'_E + \sigma \varphi'_E = \frac{1}{\mathcal{R}_0} B \varphi'_I - \frac{\mathcal{R}'_0}{\mathcal{R}_0^2} B \varphi_I, \\ -d_I \mathbf{L} \varphi'_I - \mathbf{L} \varphi_I + Y \varphi'_I - \sigma \varphi'_E = 0, \end{cases} \quad (3.5)$$

respectively. Since no confusion occurs in subsequent proofs, here the prime notation is utilized to signify differentiation in relation to either E or I , for the purpose of convenience. The initial equation of (3.2) is multiplied by φ'_E and the initial equation of (3.4) is multiplied by φ_E . The resulting equations are then subtracted to obtain the desired outcome,

$$\frac{\mathcal{R}'_0}{\mathcal{R}_0^2} \varphi_I^T B \varphi_E = \varphi_E^T \mathbf{L} \varphi_E + \frac{1}{\mathcal{R}_0} (\varphi_I'^T B \varphi_E - \varphi_I^T B \varphi'_E).$$

Analogously, the second equation of (3.2) is multiplied by φ'_I and the second equation of (3.4) is multiplied by φ_I . The resulting equations are then subtracted to obtain the desired outcome, (It should be noted that, since σ is constant, it can be treated as a constant throughout the procedure)

$$(\varphi_I'^T \varphi_E - \varphi_I^T \varphi'_E) = 0.$$

If β is a multiple of vector $\mathbf{1}$, we can see that

$$\frac{\beta \mathcal{R}'_0}{\mathcal{R}_0^2} \varphi_I^T \varphi_E = \varphi_E^T \mathbf{L} \varphi_E.$$

By Lemma 3.1, $\varphi_E^T \mathbf{L} \varphi_E \leq 0$. We obtain $\mathcal{R}'_0 \leq 0$ as φ_I, φ_E are positive. In addition, the equality is only possible if φ_E is a multiple of vector $\mathbf{1}$. This fact together with the first equation of (3.2) leads to the conclusion that φ_I must be a multiple of vector $\mathbf{1}$. The obtained expression, along with the second equation of (3.2), implies that the parameter γ must be a multiple of the vector $\mathbf{1}$. Consequently, it follows that the quantity \mathcal{R}_0 exhibits a monotone decreasing behavior with respect to the variable d_E , with strict monotonicity being guaranteed only if the parameter γ is not a multiple of vector $\mathbf{1}$.

Next we show the monotonicity of \mathcal{R}_0 with respect to d_I . The initial equation of (3.2) is multiplied by φ'_E , and initial equation of (3.5) is multiplied by φ_E . The resulting equations are subtracted to obtain the desired outcome,

$$\frac{\mathcal{R}'_0}{\mathcal{R}_0^2} \varphi_I^T B \varphi_E = \varphi_I'^T B \varphi_E - \varphi_I^T B \varphi'_E.$$

Analogously, the second equation of (3.2) is multiplied by φ'_I , and the second equation of (3.5) is multiplied by φ_I . The resulting equations are subtracted to obtain the desired outcome,

$$\varphi_I^T \mathbf{L} \varphi_I + \sigma(\varphi'_I{}^T \varphi_E - \varphi_I^T \varphi'_E) = 0.$$

If β is a multiple of vector $\mathbf{1}$. We obtain

$$\frac{\sigma \mathcal{R}'_0}{\mathcal{R}_0^2} \varphi_I^T \varphi_E = \varphi_I^T \mathbf{L} \varphi_I.$$

Similar to the previously mentioned arguments, it can be stated that the quantity \mathcal{R}_0 is exactly monotonically decreasing in behaviour with respect to the variable d_I if and only if the parameter γ is not a multiple of vector $\mathbf{1}$.

Theorem 3.2. *If γ is a multiple of vector $\mathbf{1}$ and \mathbf{L} is symmetric, then \mathcal{R}_0 is monotone decreasing function of d_E, d_I . Furthermore, it is only when the parameter β is not a multiple of vector $\mathbf{1}$ that the monotonicity of \mathcal{R}_0 becomes strictly decreasing.*

Proof. Note from Lemma 2.2 that $\mathcal{R}_0 = \rho(\sigma B(-d_E \mathbf{L} + \sigma I_{n \times n})^{-1}(-d_I \mathbf{L} + Y)^{-1})$. Let $\gamma = y\mathbf{1}$. Then \mathcal{R}_0 satisfies

$$(-d_E \mathbf{L} + \sigma I_{n \times n})(-d_I \mathbf{L} + y I_{n \times n})\varphi = \frac{1}{\mathcal{R}_0} \sigma B \varphi, \quad (3.6)$$

where φ is a strongly positive eigenvector.

We only prove that \mathcal{R}_0 is monotone decreasing function of d_E and the proof of monotonicity with respect to d_I is similar. Differentiating (3.6) by d_E yields that

$$(-d_E \mathbf{L} + \sigma I_{n \times n})(-d_I \mathbf{L} + y I_{n \times n})\varphi' - \mathbf{L}(-d_I \mathbf{L} + y I_{n \times n})\varphi = \frac{1}{\mathcal{R}_0} \sigma B \varphi' - \frac{\mathcal{R}'_0}{\mathcal{R}_0^2} \sigma B \varphi. \quad (3.7)$$

Multiplying (3.6) by φ^T and (3.7) by φ'^T , and subtracting the resulting equations yield that

$$\frac{\mathcal{R}'_0}{\mathcal{R}_0^2} \sigma \varphi^T B \varphi = -d_I (\mathbf{L} \varphi)^T \mathbf{L} \varphi + \varphi^T \mathbf{L} \varphi \leq 0.$$

Therefore, \mathcal{R}_0 is monotone decreasing function of d_E . Applying analogous arguments to that utilized in the proof of Theorem 3.1, it can be asserted that the quantity \mathcal{R}_0 is exactly monotonically decreasing behaviour with respect to d_E if and only if the parameter β is not a multiple of vector $\mathbf{1}$.

Now we prove Theorem 1.4.

Proof of Theorem 1.4. Note that the sign conditions mean that for any $i, j \in \Omega$, $\beta_i \geq \beta_j$ ($\gamma_i \leq \gamma_j$) or $\beta_i \leq \beta_j$ ($\gamma_i \geq \gamma_j$) holds true. We divide the proof of Theorem 1.4 into two steps.

Step 1. First we show that in (1.6),

$$\text{sign}\{\beta_i - \beta_j\} = \text{sign}\{\varphi_{E,i} - \varphi_{E,j}\} = \text{sign}\{\varphi_{I,i} - \varphi_{I,j}\} \quad (3.8)$$

holds for any $i, j \in \Omega$. Fix i, j , set $\rho_E = \varphi_{E,i} - \varphi_{E,j}$, $\rho_I = \varphi_{I,i} - \varphi_{I,j}$, and without loss generality, let $\beta_j \geq \beta_i$, $\gamma_j \leq \gamma_i$. Note from (1.6) that

$$\begin{cases} (n-1)d_E\varphi_{E,i} - d_E \sum_{k \neq i} \varphi_{E,k} + \sigma\varphi_{E,i} = \mu_0\beta_i\varphi_{I,i}, \\ (n-1)d_E\varphi_{E,j} - d_E \sum_{k \neq j} \varphi_{E,k} + \sigma\varphi_{E,j} = \mu_0\beta_j\varphi_{I,j}, \\ (n-1)d_I\varphi_{I,i} - d_I \sum_{k \neq i} \varphi_{I,k} + \gamma_i\varphi_{I,i} - \sigma\varphi_{E,i} = 0, \\ (n-1)d_I\varphi_{I,j} - d_I \sum_{k \neq j} \varphi_{I,k} + \gamma_j\varphi_{I,j} - \sigma\varphi_{E,j} = 0. \end{cases}$$

Subtracting the last and first two equations yields that

$$\begin{cases} \sigma\rho_E - (nd_I + \gamma_i)\rho_I = (\gamma_i - \gamma_j)\varphi_{I,j}, \\ -(nd_E + \sigma)\rho_E + \mu\beta_i\rho_I = \mu(\beta_j - \beta_i)\varphi_{I,j}. \end{cases} \quad (3.9)$$

Moreover, it follows from (1.6) that

$$\begin{cases} \sigma\varphi_{E,i} - (nd_I + \gamma_i)\varphi_{I,i} = d_I \sum_{j \in \Omega} \varphi_{I,i}, \\ -(nd_E + \sigma)\varphi_{E,i} + \mu\beta_i\varphi_{I,i} = d_E \sum_{j \in \Omega} \varphi_{E,j}. \end{cases} \quad (3.10)$$

Denote $h_0 = [(\gamma_i - \gamma_j)\varphi_{I,j}, \mu(\beta_j - \beta_i)\varphi_{I,j}]^T$, $h_1 = [d_I \sum_{j \in \Omega} \varphi_{I,i}, d_E \sum_{j \in \Omega} \varphi_{E,j}]^T$, $h_2 = [\varphi_{E,i}, \varphi_{I,i}]^T$ and

$$M = \begin{pmatrix} \sigma & -(nd_I + \gamma_i) \\ -(nd_E + \sigma) & \mu\beta_i \end{pmatrix}.$$

Thus, (3.10) can be rewritten as $Mh_2 = h_1 \gg 0$. By [31, Fact 6.11.13 (xii)], M is M-matrix and M^{-1} is a positive matrix. Here a matrix $A = (a_{ij})_{n \times n}$ is called an M-matrix if $a_{ij} \leq 0$ for all $i \neq j$ and $A = sI - B$ with B having all off-diagonal elements negative and $s \geq r(B)$. Therefore, $[\rho_E, \rho_I]^T = M^{-1}h_0 \geq 0$, which implies (3.8).

By similar arguments, we can also obtain that

$$\text{sign}\{\beta_i - \beta_j\} = \text{sign}\{\varphi_{E,i}^* - \varphi_{E,j}^*\} = \text{sign}\{\varphi_{I,i}^* - \varphi_{I,j}^*\} \quad (3.11)$$

Step 2. Moreover, the reciprocal of the basic reproduction number, denoted as $\frac{1}{\mathcal{R}_0}$, represents the principal eigenvalue that is unique of the eigenvalue problem expressed in Eq (1.6). Using the same notations F, V of (2.4), (1.6) can be written as

$$V\varphi = \frac{1}{\mathcal{R}_0}F\varphi,$$

where $\varphi = (\varphi_E, \varphi_I)$. Thus the adjoint problem of (1.6) can be written as

$$V^T\varphi^* = \frac{1}{\mathcal{R}_0}F^T\varphi^*,$$

i.e.,

$$\begin{cases} -d_E \mathbf{L}^T \varphi_E^* + \sigma \varphi_E^* - \sigma \varphi_I^* = 0, \\ -d_I \mathbf{L}^T \varphi_I^* + Y \varphi_I^* = \frac{1}{\mathcal{R}_0} B \varphi_E^*, \end{cases} \quad (3.12)$$

where $\varphi^* = (\varphi_E^*, \varphi_I^*)$ corresponds to a positive eigenvector. We can derive the following equation by multiplying the first equation of (3.4) by the transpose of φ_E^* , denoted by $\varphi_E^{*,T}$, and multiplying the first equation of (3.12) by the transpose of the derivative of φ_E , denoted by $(\varphi'_E)^T$, then subtracting the resulting equations:

$$\frac{\mathcal{R}'_0}{\mathcal{R}_0^2} \varphi_I^T B \varphi_E^* = \varphi_E^{*,T} \mathbf{L} \varphi_E + \frac{1}{\mathcal{R}_0} (\varphi'_I)^T B \varphi_E^* - \sigma \varphi_I^{*,T} \varphi'_E. \quad (3.13)$$

The second equation of (3.4) is multiplied by φ_I^* , and the second equation of (3.12) is multiplied by φ'_I , which yields

$$\frac{1}{\mathcal{R}_0} (\varphi'_I)^T B \varphi_E^* - \sigma \varphi_I^{*,T} \varphi'_E = 0. \quad (3.14)$$

As a consequence of (3.13) and (3.14), we have

$$\begin{aligned} \frac{\mathcal{R}'_0}{\mathcal{R}_0^2} \varphi_I^T B \varphi_E^* &= \varphi_E^{*,T} \mathbf{L} \varphi_E \\ &= -n \sum_{i=1}^n \varphi_{E,i}^* \varphi_{E,i} + \sum_{i,j=1}^n \varphi_{E,j}^* \varphi_{E,i} \\ &= -\frac{1}{2} \left(n \sum_{i=1}^n \varphi_{E,i}^* \varphi_{E,i} + n \sum_{j=1}^n \varphi_{E,j}^* \varphi_{E,j} - \sum_{i,j=1}^n \varphi_{E,i}^* \varphi_{E,j} - \sum_{i,j=1}^n \varphi_{E,j}^* \varphi_{E,i} \right) \\ &= -\frac{1}{2} \sum_{i,j=1}^n (\varphi_{E,i}^* - \varphi_{E,j}^*) (\varphi_{E,i} - \varphi_{E,j}) \leq 0. \end{aligned} \quad (3.15)$$

This combined with (3.15) suggests that $\mathcal{R}'_0 \leq 0$. By similar arguments as the proof of Theorem 3.1, if and only if both the parameters β and γ are constant vectors the equality holds.

We then prove that \mathcal{R}_0 decreases monotonically with respect to d_I . Using similar arguments as before, we demonstrate that,

$$\frac{\mathcal{R}'_0}{\mathcal{R}_0^2} \varphi_I^T B \varphi_E^* = \varphi_I^{*,T} \mathbf{L} \varphi_I = -\frac{1}{2} \sum_{i,j=1}^n (\varphi_{I,i}^* - \varphi_{I,j}^*) (\varphi_{I,i} - \varphi_{I,j}) \leq 0.$$

and the remaining arguments are analogous to the previous ones.

3.4. Non-monotonicity of \mathcal{R}_0 with respect to d_E, d_I

In previous subsections, we show that \mathcal{R}_0 is monotone decreasing in d_E, d_I in some cases. In this part, we will prove that \mathcal{R}_0 isn't always monotone decreasing with respect with d_E, d_I .

Theorem 3.3. *There exist d_E^0 and $d_I^1 < d_I^2$ such that $\mathcal{R}_0(d_E^0, d_I^1) < \mathcal{R}_0(d_E^0, d_I^2)$, if*

$$\frac{\sum_{i=1}^n \xi_i \beta_i}{\sum_{i=1}^n \xi_i \gamma_i} > \sum_{i=1}^n \frac{\xi_i \beta_i}{\gamma_i}.$$

Proof. Based on Theorem 1.2, it can be shown that for any fixed value of $d_E > 0$, $\mathcal{R}_0 \rightarrow \frac{\sum_{i=1}^n \xi_i \beta_i}{\sum_{i=1}^n \xi_i \gamma_i}$ as $d_I \rightarrow \infty$, and $\mathcal{R}_0 \rightarrow \sum_{i=1}^n \frac{\xi_i \beta_i}{\gamma_i}$ as $d_I \rightarrow 0, d_E \rightarrow \infty$. For any arbitrarily small positive value of ϵ , there exists a sufficiently large constant $C_1(\epsilon)$ such that for any $\frac{1}{d_I}, d_E \geq C_1(\epsilon)$, we have

$$\mathcal{R}_0(d_E, d_I^1) \leq (1 + \epsilon) \sum_{i=1}^n \frac{\xi_i \beta_i}{\gamma_i}.$$

Furthermore, there exists $C_2(\epsilon, d_E)$ such that for any $d_I^2 \geq C_2(\epsilon, d_E)$,

$$\mathcal{R}_0(d_E, d_I^2) \geq (1 - \epsilon) \frac{\sum_{i=1}^n \xi_i \beta_i}{\sum_{i=1}^n \xi_i \gamma_i}.$$

Since

$$\frac{\sum_{i=1}^n \xi_i \beta_i}{\sum_{i=1}^n \xi_i \gamma_i} > \sum_{i=1}^n \frac{\xi_i \beta_i}{\gamma_i},$$

we can choose ϵ_0 small enough such that

$$(1 - \epsilon_0) \frac{\sum_{i=1}^n \xi_i \beta_i}{\sum_{i=1}^n \xi_i \gamma_i} > (1 + \epsilon_0) \frac{\xi_i \beta_i}{\gamma_i},$$

and let $d_E^0 = C_1(\epsilon_0), d_I^1 = \frac{1}{C_1(\epsilon_0)}, d_I^2 = C_2(\epsilon_0, d_E^0)$, we have $\mathcal{R}_0(d_E^0, d_I^1) < \mathcal{R}_0(d_E^0, d_I^2)$.

Theorem 3.4. Let $\nu_0 = \frac{\sum_{i=1}^n \xi_i \gamma_i}{\sum_{i=1}^n \xi_i \beta_i}$ and φ_1, ϕ_1 be the unique solutions of

$$-\mathbf{L}\varphi_1 = \nu_0 B\xi - Y\xi$$

and

$$-\mathbf{L}\phi_1 = \nu_0 \left(\sum_{i=1}^n \beta_i \xi_i \right) \xi - Y\xi,$$

respectively. If

$$(\gamma - \nu_0 \beta)^T (\varphi_1 - \phi_1) > 0, \quad (3.16)$$

there exist d_I^0 and $d_E^1 < d_E^2$ such that $\mathcal{R}_0(d_E^1, d_I^0) < \mathcal{R}_0(d_E^2, d_I^0)$.

Proof. Consider the principal eigenvalues of the following two eigenvalue problems, which referred to as μ and ν , respectively,

$$-d_I \mathbf{L}\varphi + Y\varphi = \mu B\varphi, \quad (3.17)$$

and

$$-d_I \mathbf{L}\phi + Y\phi = \nu \left(\sum_{i=1}^n \beta_i \phi_i \right) \xi \quad (3.18)$$

with $\sum_{i=1}^n \varphi_i = \sum_{i=1}^n \phi_i = 1$. Note from Theorem 1.2 that for any fixed $d_I > 0$, $\mathcal{R}_0 \rightarrow \frac{1}{\mu}$ as $d_E \rightarrow 0$ and $\mathcal{R}_0 \rightarrow \frac{1}{\nu}$ as $d_E \rightarrow \infty$. Now we take $\epsilon = \frac{1}{d_I}$ and perform regular expansions on (ϕ, ν) and (φ, μ) to obtain the following expressions.

$$\begin{aligned} \varphi_i &= \varphi_{0,i} + \epsilon \varphi_{1,i} + \epsilon^2 \varphi_{2,i}(\epsilon), \\ \phi_i &= \phi_{0,i} + \epsilon \phi_{1,i} + \epsilon^2 \phi_{2,i}(\epsilon), \\ \mu &= \mu_0 + \epsilon \mu_1 + \epsilon^2 \mu_2(\epsilon), \\ \nu &= \nu_0 + \epsilon \nu_1 + \epsilon^2 \nu_2(\epsilon). \end{aligned} \quad (3.19)$$

Our objective is to establish that the inequality $\mu > \nu$ holds true for small values of ϵ subject to the condition (3.16). Upon performing direct computation, we obtain the following expressions, $\varphi_0 = \phi_0 = \xi$, $\mu_0 = \nu_0 = \frac{\sum_{i=1}^n \xi_i \gamma_i}{\sum_{i=1}^n \xi_i \beta_i}$ and φ_1, ϕ_1 satisfy

$$-\mathbf{L}\varphi_1 = \nu_0 B\xi - Y\xi$$

and

$$-\mathbf{L}\phi_1 = \nu_0 \left(\sum_{i=1}^n \beta_i \xi_i \right) \xi - Y\xi,$$

respectively. Furthermore, we have

$$\sum_{i=1}^n \gamma_i \varphi_{1,i} = \mu_0 \sum_{i=1}^n \beta_i \varphi_{1,i} + \mu_1 \sum_{i=1}^n \beta_i \xi_i \quad (3.20)$$

and

$$\sum_{i=1}^n \gamma_i \phi_{1,i} = \nu_0 \sum_{i=1}^n \beta_i \phi_{1,i} + \nu_1 \sum_{i=1}^n \beta_i \xi_i. \quad (3.21)$$

Therefore, by condition (3.16), (3.20), (3.21) and $\mu_0 = \nu_0$, we obtain

$$(\mu_1 - \nu_1) \sum_{i=1}^n \beta_i \xi_i = \sum_{i=1}^n (\gamma_i - \nu_0 \beta_i) (\varphi_{1,i} - \phi_{1,i}) > 0.$$

Thus $\mu > \nu$ for large d_I . Therefore, we can find d_I^0 large, and d_E^2 large, d_E^1 small such that $\mathcal{R}_0(d_E^1, d_I^0) < \mathcal{R}_0(d_E^2, d_I^0)$.

Remark: We present an example in which condition (3.16) holds. Let $\Omega = \{1, 2\}$, $\beta_1 = 1.01, \beta_2 = 1.424$ and $\gamma_1 = 1.01, \gamma_2 = 2.01$, by direct computation, we can see

$$\sum_{i=1}^n (\gamma_i - \nu_0 \beta_i) (\varphi_{1,i} - \phi_{1,i}) = 0.0043 > 0.$$

4. Asymptotic behaviours of the endemic equilibrium

Theorem 1.5 is a consequence of Lemma 4.1, Theorems 4.1 and 4.2. To analyze the behavior of endemic equilibria as d_S tends towards zero, we shall examine the asymptotic properties of the system. Now we consider the alternative statements of the endemic equilibrium problem (1.4).

Lemma 4.1. $(\tilde{S}, \tilde{E}, \tilde{I}, \tilde{R})$ satisfies (1.4) if and only if (S, E, I, R) satisfies the following system,

$$\begin{cases} d_E \mathbf{L}E + \text{diag} \left\{ \frac{\beta_i S_i}{S_i + I_i + E_i + R_i} \right\} I - \sigma E = 0, \\ d_I \mathbf{L}I + \sigma E - YI = 0, \\ d_R \mathbf{L}R + YI - \alpha R = 0, \\ d_S S + d_E E + d_I I + d_R R = \xi. \end{cases} \quad (4.1)$$

Moreover, $\widetilde{S} = \kappa S, \widetilde{E} = \kappa E, \widetilde{I} = \kappa I, \widetilde{R} = \kappa R$ and

$$\kappa = \frac{N_0}{\sum_{i=1}^n (S_i + E_i + I_i + R_i)}.$$

Proof. Note that

$$\mathbf{L}(d_S \widetilde{S} + d_E \widetilde{E} + d_I \widetilde{I} + d_R \widetilde{R}) = 0,$$

which implies that

$$d_S \widetilde{S} + d_E \widetilde{E} + d_I \widetilde{I} + d_R \widetilde{R} = \kappa \xi, \kappa \in \mathbb{R}^+,$$

where ξ is the unique positive eigenvector of the unique principal eigenvalue 0 of \mathbf{L} satisfying $\sum_{i \in \Omega} \xi_i = 1$. Set $S = \widetilde{S}/\kappa, E = \widetilde{E}/\kappa, I = \widetilde{I}/\kappa, R = \widetilde{R}/\kappa$. (4.1) can be obtained from (1.4).

Now we investigate the asymptotic behavior of the endemic equilibria when $d_S \rightarrow 0$. Recall that $\Lambda_1(-d_R \mathbf{L} + \alpha \text{diag}\{1 - \frac{\gamma_i}{\beta_i}\})$ is the smallest eigenvalue of (1.9).

Theorem 4.1. Suppose that $\mathcal{R}_0 > 1$. Under this assumption, we have

- (i) As $d_S \rightarrow 0, E \rightarrow E^*, I \rightarrow I^*, R \rightarrow R^*$
- (ii) The set $J^+ := \{i | M_i^* = \xi_i, i \in \Omega\}$ is nonempty, where $M_i^* := d_E E_i^* + d_I I_i^* + d_R R_i^*$;
- (iii) If further assume $\Lambda_1(-d_R \mathbf{L} + \alpha \text{diag}\{1 - \frac{\gamma_i}{\beta_i}\}) < 0$, then the set $J^- := \{i | M_i^* < \xi_i, i \in \Omega\}$ is not empty.

Proof. (i). Note that $E_i, I_i, R_i > 0$ for any $i \in \Omega, d_S > 0$. $d_S S + d_E E + d_I I + d_R R = \xi$ yields that $\frac{\beta_i S_i I_i}{S_i + E_i + I_i + R_i}$ exhibit uniform boundedness for any $d_S > 0$. After extracting a subsequence if necessary, $E \rightarrow E^*$ as $d_S \rightarrow 0$ where $E^* \geq 0$. Using analogous analysis, it can be shown that $I \rightarrow I^*, R \rightarrow R^*$ as $d_S \rightarrow 0$ where $I^*, R^* \geq 0$, which satisfy

$$\begin{cases} d_I \mathbf{L} I^* + \sigma E^* - Y I^* = 0, \\ d_R \mathbf{L} R^* + Y I^* - \alpha R^* = 0. \end{cases} \quad (4.2)$$

Now we show that $E^* \neq 0$, i.e., $E^* > 0$. It can be proved by contradiction. Assuming $E = 0$, it follows from Eq (4.2) that $I = R^* = 0$. Consequently, $S \rightarrow \infty$ almost everywhere as $d_S \rightarrow 0$. Thus $\frac{\beta_i S_i}{S_i + E_i + I_i + R_i} \rightarrow \beta_i$ as $d_S \rightarrow 0$. Define

$$K = \|E\|_1 + \|I\|_1 + \|R\|_1, \quad \hat{E} = \frac{E}{K}, \quad \hat{I} = \frac{I}{K}, \quad \hat{R} = \frac{R}{K}.$$

Observe that \hat{E}, \hat{I} and \hat{R} are strictly positive, and $\|\hat{E}\|_1 + \|\hat{I}\|_1 + \|\hat{R}\|_1 = 1$. After extracting a subsequence if necessary, we obtain $\hat{E}, \hat{I}, \hat{R}$ approaches $\hat{E}^*, \hat{I}^*, \hat{R}^*$ respectively as $d_S \rightarrow 0$, where $\hat{E}_i^*, \hat{I}_i^*, \hat{R}_i^* \geq 0$ for $i \in \Omega$ and

$$\|\hat{E}^*\|_1 + \|\hat{I}^*\|_1 + \|\hat{R}^*\|_1 = 1. \quad (4.3)$$

It follows from $\frac{\beta_i S_i}{S_i + E_i + I_i + R_i} \rightarrow \beta_i$ as $d_S \rightarrow 0$ that \hat{E}^* is a solution of

$$d_E \mathbf{L} \hat{E}^* - \sigma \hat{E}^* + B \hat{I}^* = 0,$$

which gives

$$\begin{cases} d_E \mathbf{L} \hat{E}^* - \sigma \hat{E}^* + B \hat{I}^* = 0, \\ d_I \mathbf{L} \hat{I}^* + \sigma \hat{E}^* - Y \hat{I}^* = 0, \\ d_R \mathbf{L} \hat{R}^* + Y \hat{I}^* - \alpha \hat{R}^* = 0. \end{cases} \quad (4.4)$$

It can be deduced from Eq (4.3) that the values of \hat{E}^* , \hat{I}^* , \hat{R}^* are significantly greater than zero, which indicates that $\mathcal{R}_0 = 1$. This lead to a contradiction, implying $E^* > 0$. Therefore, we obtain $I^*, R^* \gg 0$.

To establish the claim that $|J^+| > 0$, we prove by contradiction. Suppose that $|J^+| = 0$, then as $d_S \rightarrow 0$, it follows that $S \rightarrow \infty$, and thus $\frac{\beta_i S i_i}{S_i + E_i + I_i + R_i} \rightarrow \beta_i$ as $d_S \rightarrow 0$. Therefore, E^* is a solution of

$$d_E \mathbf{L} E^* - \sigma E^* + B I^* = 0,$$

which yields

$$\begin{cases} d_E \mathbf{L} E^* - \sigma E^* + B I^* = 0, \\ d_I \mathbf{L} I^* + \sigma E^* - Y I^* = 0, \\ d_R \mathbf{L} R^* + Y I^* - \alpha R^* = 0. \end{cases} \quad (4.5)$$

Based on Eq (4.5) in conjunction with $I \gg 0$ and $R \gg 0$, it can be inferred that $E^* \gg 0$. Consequently, we have $\mathcal{R}_0 = 1$, which leads to a contradiction. Therefore, $|J^+| > 0$.

Part (iii) can be shown through contradiction. Assume now that $|J^-| = 0$. Denote $h_i := \frac{\beta_i S i_i}{S_i + E_i + I_i + R_i} - \alpha R_i$ and choose $\varphi \in \mathbb{R}^n$ subject to the condition that $\varphi \geq 0$. By multiplying the first three equations of (4.1) by φ^T and adding the results together, we obtain

$$\varphi^T \mathbf{L} (d_E E + d_I I + d_R R) + \varphi^T h = 0. \quad (4.6)$$

As $|J^-| = 0$, $M^* = 0$. Thus, we obtain

$$\varphi^T h \rightarrow 0 \text{ as } d_S \rightarrow 0 \quad (4.7)$$

$\varphi \in \mathbb{R}^n$ such that $\varphi \geq 0$.

Let ϕ_0 be a positive left eigenvector of $\Lambda_1 \left(-d_R \mathbf{L} + \alpha \text{diag} \left\{ 1 - \frac{\gamma_i}{\beta_i} \right\} \right)$, i.e.

$$-d_R \mathbf{L}^T \phi_0 + \alpha \text{diag} \left\{ 1 - \frac{\gamma_i}{\beta_i} \right\} \phi_0 = \Lambda_1 \phi_0, \quad (4.8)$$

Since $S, E, I, R \gg 0$ on Ω and $d_R \mathbf{L} R + Y I - \alpha R = 0$, we obtain

$$-d_R \mathbf{L} R + \alpha \text{diag} \left\{ 1 - \frac{\gamma_i}{\beta_i} \right\} R > \text{diag} \left\{ \frac{\gamma_i}{\beta_i} \right\} h, \quad (4.9)$$

Multiplying (4.9) by ϕ_0 and applying (4.8), we obtain

$$\Lambda_1 \phi_0^T R > \sum_{i=1}^n \frac{\gamma_i h_i}{\beta_i} \phi_{0,i}.$$

Let $d_S \rightarrow 0$, it follows from (4.7) that $\Lambda_1 \phi_0^T R \geq 0$. Given that $\phi_0, R^* > 0$ on Ω , it follows that $\Lambda_1 > 0$. This contradiction finishes the proof of (iii).

Theorem 4.2. Assume $\mathcal{R}_0 > 1$ and $\Lambda_1(-d_R \mathbf{L} + \alpha \text{diag}\{1 - \frac{\gamma_i}{\beta_i}\}) < 0$. Then we have

(i) As $d_S \rightarrow 0$,

$$\frac{\xi}{d_S} \rightarrow \frac{N_0}{\sum_{i=1}^n (\xi_i - M_i^*)} \text{ and } \widetilde{S} \rightarrow \widetilde{S}^* = \frac{N_0(\xi - M^*)}{\sum_{i=1}^n (\xi_i - M_i^*)};$$

(ii) There exist positive constants C_1, C_2 , independent of d_S such that for sufficiently small d_S ,

$$C_1 \leq \frac{\widetilde{E}_i}{d_S}, \frac{\widetilde{I}_i}{d_S}, \frac{\widetilde{R}_i}{d_S} \leq C_2, \forall i \in \Omega.$$

Proof. (i). For further purposes, denote $M := d_E E + d_I I + d_R R$. By (4.1), we have

$$\begin{aligned} N_0 &= \sum_{i=1}^n (\widetilde{S}_i + \widetilde{E}_i + \widetilde{I}_i + \widetilde{R}_i) \\ &= \frac{\kappa}{d_S} \left(\sum_{i=1}^n d_S (E_i + I_i + R_i) + \sum_{i=1}^n (\xi_i - M_i) \right). \end{aligned}$$

Given that $S, E, I, R \gg 0$ and $d_S S + d_E E + d_I I + d_R R = \xi$, we can conclude that E, I and R exhibit uniform boundedness with respect to d_S . Therefore,

$$\sum_{i=1}^n d_S (E_i + I_i + R_i) \rightarrow 0 \text{ as } d_S \rightarrow 0.$$

By Theorem 4.1(i),(ii),

$$\sum_{i=1}^n (\xi_i - M_i) \rightarrow \sum_{i=1}^n (\xi_i - M_i^*) > 0 \text{ as } d_S \rightarrow 0.$$

Therefore,

$$\frac{\kappa}{d_S} \rightarrow \frac{N_0}{\sum_{i=1}^n (\xi_i - M_i^*)} \text{ as } d_S \rightarrow 0. \quad (4.10)$$

Furthermore, (4.1) implies $\widetilde{S} = \frac{\kappa}{d_S}(\xi - M)$. Based on (4.10) and Theorem 4.1(i), we obtain

$$\widetilde{S} \rightarrow \widetilde{S}^* = \frac{N_0(\xi - M^*)}{\sum_{i=1}^n (\xi_i - M_i^*)}$$

as $d_S \rightarrow 0$.

Now we proceed to the proof of (ii). By $d_S S + d_E E + d_I I + d_R R = \xi$ and $\widetilde{E} = \frac{\kappa}{d_S} d_S E$, $\widetilde{I} = \frac{\kappa}{d_S} d_S I$, $\widetilde{R} = \frac{\kappa}{d_S} d_S R$, we get

$$0 < \frac{\|\widetilde{E}\|_1}{d_S}, \frac{\|\widetilde{I}\|_1}{d_S}, \frac{\|\widetilde{R}\|_1}{d_S} < \frac{\kappa}{d_S} \max \left\{ \frac{1}{d_E}, \frac{1}{d_I}, \frac{1}{d_R} \right\}.$$

Hence, (i) implies

$$\limsup_{d_S \rightarrow 0} \frac{\|\widetilde{E}\|_\infty}{d_S}, \limsup_{d_S \rightarrow 0} \frac{\|\widetilde{I}\|_\infty}{d_S}, \limsup_{d_S \rightarrow 0} \frac{\|\widetilde{R}\|_\infty}{d_S} \leq \frac{N_0}{\sum_{i=1}^n (\xi_i - M_i^*)} \max \left\{ \frac{1}{d_E}, \frac{1}{d_I}, \frac{1}{d_R} \right\}. \quad (4.11)$$

Now we prove

$$\min\{\widetilde{E}_i, \widetilde{I}_i, \widetilde{R}_i, i \in \Omega\}/d_S \rightarrow 0, \text{ as } d_S \rightarrow 0 \quad (4.12)$$

by contradiction. Assume that $\min\{\widetilde{E}_i, \widetilde{I}_i, \widetilde{R}_i, i \in \Omega\}/d_S = o(d_S)$. By (1.4) and direct calculation, $\sum_{i=1}^n \widetilde{E} dx, \sum_{i=1}^n \widetilde{I}_i, \sum_{i=1}^n \widetilde{R}_i = o(d_S)$, which implies

$$\sum_{i=1}^n \frac{d_E \widetilde{E}_i + d_I \widetilde{I}_i + d_R \widetilde{R}_i}{d_S} \rightarrow 0 \text{ as } d_S \rightarrow 0. \quad (4.13)$$

Note that

$$N_0 = \sum_{i=1}^n \frac{\kappa}{d_S} - \sum_{i=1}^n \frac{d_E \widetilde{E}_i + d_I \widetilde{I}_i + d_R \widetilde{R}_i}{d_S} + \sum_{i=1}^n (\widetilde{E}_i + \widetilde{I}_i + \widetilde{R}_i).$$

Let $d_S \rightarrow 0$, it follows from (i), (4.11) and (4.13) that

$$N_0 = \frac{N_0}{\sum_{i=1}^n (\xi_i - M_i^*)}.$$

As a result, we have $|J^-| = 0$. This contradiction finishes the proof of (ii).

Acknowledgments

The authors of this work receive financial support from various sources. S.Y. is partially funded by the National Natural Science Foundation of China (NSFC) grant number 12101487 and the China Scholarship Council under grant number 202006280442. J.W. is supported by the Canada Research Chair Program with reference number 230720. P.S. receives support from the National Natural Science Foundation of China (NSFC) grants 12101487 (PS) and 12220101001 (PS), the China Postdoctoral Science Foundation under grant number 2020M683445, and the Postdoctoral Fellowship of York University, Toronto, Canada. The authors also express their gratitude to the referees for their valuable assistance in reviewing the manuscript.

Conflict of interest

The authors declare there is no conflict of interest.

References

1. R. S. Cantrell, C. Cosner, *Spatial Ecology via Reaction-diffusion Equations*, John Wiley and Sons, Ltd., Chichester, 2003.
2. J. D. Murray, *Mathematical Biology*, 2nd edition, Springer-Verlag, New York, 2002.
3. X. Q. Zhao, *Dynamical Systems in Population Biology*, 2nd edition, Springer, Cham, 2017.
4. L. Sattenspiel, *The Geographic Spread of Infectious Diseases: Models and Applications*, 2009.
5. L. J. S. Allen, B. M. Bolker, Y. Lou, A. L. Nevai, Asymptotic profiles of the steady states for an SIS epidemic patch model, *SIAM J. Appl. Math.*, **67** (2007), 1283–1309. <https://doi.org/10.1137/060672522>

6. H. Li, R. Peng, Dynamics and asymptotic profiles of endemic equilibrium for sis epidemic patch models, *J. Math. Biol.*, **79** (2019), 1279–1317. <https://doi.org/10.1007/s00285-019-01395-8>
7. L. J. S. Allen, B. M. Bolker, Y. Lou, A. L. Nevai, Asymptotic profiles of the steady states for an SIS epidemic reaction-diffusion model, *Discrete Contin. Dyn. Syst. Ser. A*, **21** (2008), 1–20. <https://doi.org/10.3934/dcds.2008.21.1>
8. W. Wang, X. Q. Zhao, Basic reproduction numbers for reaction-diffusion epidemic models, *SIAM J. Appl. Dyn. Syst.*, **11** (2012), 1652–1673. <https://doi.org/10.1137/120872942>
9. R. Peng, X. Q. Zhao, A reaction-diffusion SIS epidemic model in a time-periodic environment, *Nonlinearity*, **25** (2012), 1451–1471. <https://doi.org/10.1088/0951-7715/25/5/1451>
10. R. Cui, Y. Lou, A spatial SIS model in advective heterogeneous environments, *J. Differ. Equations*, **261** (2016), 3305–3343.
11. R. Cui, K. Y. Lam, Y. Lou, Dynamics and asymptotic profiles of steady states of an epidemic model in advective environments, *J. Differ. Equations*, **263** (2017), 2343–2373. <https://doi.org/10.1016/j.jde.2017.03.045>
12. K. Deng, Y. Wu, Dynamics of a susceptible-infected-susceptible epidemic reaction-diffusion model, *Proc. R. Soc. Edinburgh Sect. A*, **146** (2016), 929–946. <https://doi.org/10.1017/S0308210515000864>
13. Y. Wu, X. Zou, Asymptotic profiles of steady states for a diffusive SIS epidemic model with mass action infection mechanism, *J. Differ. Equations*, **261** (2016), 4424–4447. <https://doi.org/10.1016/j.jde.2016.06.028>
14. H. Li, R. Peng, F. B. Wang, Varying total population enhances disease persistence: qualitative analysis on a diffusive SIS epidemic model, *J. Differ. Equations*, **262** (2017), 885–913. <https://doi.org/10.1016/j.jde.2016.09.044>
15. H. Li, R. Peng, T. Xiang, Dynamics and asymptotic profiles of endemic equilibrium for two frequency-dependent SIS epidemic models with cross-diffusion, *Eur. J. Appl. Math.*, **31** (2020), 26–56. <https://doi.org/10.1017/S0956792518000463>
16. S. Chen, J. Shi, Z. Shuai, Y. Wu, Asymptotic profiles of the steady states for an SIS epidemic patch model with asymmetric connectivity matrix, *J. Math. Biol.*, **80** (2020), 2327–2361. <https://doi.org/10.1007/s00285-020-01497-8>
17. S. Chen, J. Shi, Z. Shuai, Y. Wu, Spectral monotonicity of perturbed quasi-positive matrices with applications in population dynamics, preprint, arXiv:1911.02232.
18. J. Qiu, Covert coronavirus infections could be seeding new outbreaks, *Nature*, 2020. <https://doi.org/10.1038/d41586-020-00822-x>
19. P. Song, Y. Lou, Y. Xiao, A spatial SEIRS reaction-diffusion model in heterogeneous environment, *J. Differ. Equations*, **267** (2019), 5084–5114. <https://doi.org/10.1016/j.jde.2019.05.022>
20. D. Gao, S. Ruan, A multipatch malaria model with logistic growth population, *SIAM J. Appl. Math.*, **72** (2012), 819–841. <https://doi.org/10.1137/110850761>
21. Y. Xiao, X. Zou, Transmission dynamics for vector-borne diseases in a patchy environment, *J. Math. Biol.*, **69** (2014), 113–146. <https://doi.org/10.1007/s00285-013-0695-1>

22. G. K. Zipf, The PJVD hypothesis on the intercity movement of persons, in *American Sociological Review*, 1946.
23. M. Barthelemy, Spatial networks, *Phys. Rep.*, **499** (2010), 1–101. <https://doi.org/10.1016/j.physrep.2010.11.002>
24. F. Simini, M. C. Gonzalez, A. Maritan, A. L. Barabasi, A universal model for mobility and migration patterns, *Nature*, **484** (2012), 96–100. <https://doi.org/10.1038/nature10856>
25. O. Diekmann, J. A. P. Heesterbeek, *Mathematical Epidemiology of Infectious Diseases*, John Wiley and Sons Ltd, Chichester, New York, 2000.
26. R. M. Anderson, R. M. May, *Infectious Diseases of Humans: Dynamics and Control*, Cambridge University Press, 1991.
27. O. Diekmann, J. A. P. Heesterbeek, J. A. J. Metz, On the definition and the computation of the basic reproduction ratio R_0 in models for infectious diseases in heterogeneous populations, *J. Math. Biol.*, **28** (1990), 365–382. <https://doi.org/10.1007/BF00178324>.
28. P. van den Driessche, J. Watmough, Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission, *Math. Biosci.*, **180** (2002), 29–48. [https://doi.org/10.1016/S0025-5564\(02\)00108-6](https://doi.org/10.1016/S0025-5564(02)00108-6)
29. H. R. Thieme, Spectral bound and reproduction number for infinite-dimensional population structure and time heterogeneity, *SIAM J. Appl. Math.*, **70** (2009), 188–211. <https://doi.org/10.1137/080732870>
30. J. K. Hale, Dynamical systems and stability, *J. Math. Anal. Appl.*, **26** (1969), 39–59. [https://doi.org/10.1016/0022-247X\(69\)90175-9](https://doi.org/10.1016/0022-247X(69)90175-9)
31. D. S. Bernstein, Scalar, vector, and matrix mathematics, in *Scalar, Vector, and Matrix Mathematics*, Princeton university press, 2018. <https://doi.org/10.1515/9781400888252>
32. G. Degla, An overview of semi-continuity results on the spectral radius and positivity, *J. Math. Anal. Appl.*, **338** (2008), 101–110. <https://doi.org/10.1016/j.jmaa.2007.05.011>
33. F. R. Gantmakher, *The Theory of Matrices*, American Mathematical Soc., 2000.
34. J. K. Hale, *Ordinary Differential Equations*, Robert E. Krieger Publishing Company, INC., 1980.



AIMS Press

©2023 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (<http://creativecommons.org/licenses/by/4.0>)