Mathematical Biosciences and Engineering

http://www.aimspress.com/journal/mbe

MBE, 20(7): 13222–13249.
DOI: 10.3934/mbe.2023590
Received: 14 January 2023
Revised: 15 March 2023
Accepted: 03 April 2023
Published: 08 June 2023

Research article

Survival analysis and probability density function of switching heroin model

Hui Jiang[1,2], Ling Chen[1], Fengying Wei[1,3]*and Quanxin Zhu[4]

[1] School of Mathematics and Statistics, Fuzhou University, Fuzhou 350116, China
[2] Fujian Key Laboratory of Financial Information Processing, Putian University, Putian 351100, China
[3] Center for Applied Mathematics of Fujian Province, Fuzhou University, Fuzhou 350116, China
[4] School of Mathematics and Statistics, Hunan Normal University, Changsha 410081, China

* **Correspondence:** Email: weifengying@fzu.edu.cn.

Abstract: We study a switching heroin epidemic model in this paper, in which the switching of supply of heroin occurs due to the flowering period and fruiting period of opium poppy plants. Precisely, we give three equations to represent the dynamics of the susceptible, the dynamics of the untreated drug addicts and the dynamics of the drug addicts under treatment, respectively, within a local population, and the coefficients of each equation are functions of Markov chains taking values in a finite state space. The first concern is to prove the existence and uniqueness of a global positive solution to the switching model. Then, the survival dynamics including the extinction and persistence of the untreated drug addicts under some moderate conditions are derived. The corresponding numerical simulations reveal that the densities of sample paths depend on regime switching, and larger intensities of the white noises yield earlier times for extinction of the untreated drug addicts. Especially, when the switching model degenerates to the constant model, we show the existence of the positive equilibrium point under moderate conditions, and we give the expression of the probability density function around the positive equilibrium point.

Keywords: heroin model; stationary distribution; extinction; regime switching; Fokker-Planck equation; probability density function

1. Model establishment

Heroin is a semi-synthetic opioid drug, which is mainly extracted from opium poppy. Heroin was originally developed as a drug to cure morphine addiction, but later it was found to be highly addictive, dependence causing and toxic [1]. Heroin became one of the most popular drugs in the world [2]. White and Comiskey [3] were the first to study the spreading of heroin by using an ordinary differential equation (ODE) compartmental model, and they separated the local population into three compartments

based on the states of drug addicts: the susceptible individuals, the untreated drug addicts and drug addicts under treatment. Based on White's model, many scholars developed different mathematical models to discuss the transmission mechanisms of heroin, such as age structure models [4–6], distributed delay models [7–9] and nonlinear incidence models [10–13] as well. Within the above-mentioned works, the authors found that the consumption of heroin was transmitted from a drug addict to a non-drug addict, which was similar to the mechanism of the spreading of infectious diseases. They further discussed the basic reproduction number R_0 as the threshold, and they determined the stability of the drug-free equilibrium and the endemic equilibrium.

Meanwhile, environmental noises usually affected the dynamics of heroin models in [14–19]. More precisely, Liu et al. [14] proposed a stochastic heroin epidemic model, in which they obtained a threshold for the extinction of the drug addicts. Further, [15] studied a stochastic heroin epidemic model with the bilinear incidence within a varying population. Then, Wei et al. [16] analyzed the long-term dynamics of a perturbed heroin epidemic model under non-degenerate noise. Later, Wei et al. [17] established a heroin population model with the standard incidence rates between distinct patches, and by constructing suitable Lyapunov functions, they established the sufficient criteria for the existence of the addict elimination and the existence of an ergodic stationary distribution. The recent contributions in [20–33] governed the continuous-time Markov chains taking values in a finite-state space to describe the regime switchings, in which Markov-chains were memoryless, and the waiting time from one state to another state usually obeyed the exponential distribution. Therefore, in this paper, we consider the following stochastic heroin model with the bilinear incidence rate under regime switching:

$$\begin{cases} \mathrm{d}S(t) = & \Big[\Lambda(m(t)) - \beta_1(m(t))S(t)U(t) - \mu(m(t))S(t)\Big]\mathrm{d}t \\ & +\sigma_1(m(t))S(t)\mathrm{d}B_1(t), \\ \mathrm{d}U(t) = & \Big[\beta_1(m(t))S(t)U(t) - p(m(t))U(t) + \beta_2(m(t))U(t)T(t) \\ & -(\mu(m(t)) + \delta_1(m(t)))U(t)\Big]\mathrm{d}t + \sigma_2(m(t))U(t)\mathrm{d}B_2(t), \\ \mathrm{d}T(t) = & \Big[p(m(t))U(t) - \beta_2(m(t))U(t)T(t) - (\mu(m(t)) + \delta_2(m(t)))T(t)\Big]\mathrm{d}t \\ & +\sigma_3(m(t))T(t)\mathrm{d}B_3(t), \end{cases} \tag{1.1}$$

where $S(t)$ is the number of the susceptible individuals; $U(t)$ is the number of the untreated drug addicts; and $T(t)$ is the number of the drug addicts under treatment at time t respectively. Moreover, $N(t) = S(t) + U(t) + T(t)$ denotes the total population size at time t; $B_i(t)$ $(i = 1, 2, 3)$ are mutually independent standard Brownian motions defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$, which is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets; and $\sigma_i^2 > 0$ $(i = 1, 2, 3)$ denote the intensities of the white noises. Λ is the population density entering the susceptible per unit of time, μ is the natural death rate of the total population, p is the proportion of drug users who are under treatment, β_1 is the rate that an individual becomes a drug user, β_2 is the rate that drug users under treatment relapsed to the untreated, δ_1 is the drug-related death rate, δ_2 is the successful cure rate. We assume that all parameters of model (1.1) are non-negative.

Let $m(t)$ be a right-continuous Markov chain on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in a finite state space $\mathbb{S} = \{1, 2, \cdots, N\}$ for $t \geq 0$ and $\Delta t > 0$, which is generated by the

transition matrix $\Gamma = (p_{ij})_{N\times N}$, i.e., $\mathbb{P}\{m(t+\Delta t) = j|m(t) = i\} \le p_{ij}\Delta t + o(\Delta t)$ if $i \ne j$; otherwise, $\mathbb{P}\{m(t+\Delta t) = j|m(t) = i\} \le 1 + p_{ii}\Delta t + o(\Delta t)$ if $i = j$, where $p_{ij} \ge 0$ is the transition rate from state i to state j if $i \ne j$ while $\sum_{j=1}^{N} p_{ij} = 1$.

In this paper, we assume that $p_{ij} > 0$ for $i, j = 1, \cdots, N$ with $i \ne j$. In model (1.1), the parameters Λ, p, μ, β_1, β_2, δ_1, δ_2, σ_i $(i = 1, 2, 3)$ are not constants; instead they are generated by a homogeneous continuous-time Markov chain $m(t)$ for $t \ge 0$. That is, for each fixed $k \in \mathbb{S}$, $\Lambda(k)$, $p(k)$, $\mu(k)$, $\beta_1(k)$, $\beta_2(k)$, $\delta_1(k)$, $\delta_2(k)$ and $\sigma_i(k)$ $(i = 1, 2, 3)$ are all positive constants. We assume that the Markov chain $m(t)$ is irreducible, which means that the system can switch from one regime to another regime. It implies that the Markov chain $m(t)$ has a unique stationary distribution $\pi = (\pi_1, \pi_2, \cdots, \pi_N)$ which can be determined by the equation $\pi\Gamma = 0$ subject to $\sum_{k=1}^{N} \pi_k = 1$ and $\pi_k > 0$ for any $k \in \mathbb{S}$. Define $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i > 0, 1 \le i \le n\}$. For any vector $g = (g(1), g(2), \cdots, g(N))$, let $\hat{g} = \min_{k\in\mathbb{S}}\{g(k)\}$ and $\check{g} = \max_{k\in\mathbb{S}}\{g(k)\}$. Next, we will show the existence and uniqueness of a global positive solution. Then, we will discuss the survival dynamics including the extinction and persistence of the untreated drug addicts for the switching model (1.1). Further, we will investigate the probability density function of the degenerated model (2.20) under some sufficient conditions.

2. Main results

In this section, we give the generalized SDEs

$$\mathrm{d}X(t) = f(X(t), m(t))\mathrm{d}t + g(X(t), m(t))\mathrm{d}B(t), \quad t \ge 0, \tag{2.1}$$

with the initial values $X(0) = X_0, m(0) = m$, where $B(\cdot)$ and $m(\cdot)$ are the d-dimensional Brownian motions and the right-continuous Markov chains, respectively. $f(\cdot,\cdot)$ and $g(\cdot,\cdot)$ respectively map $\mathbb{R}^n \times \mathbb{S}$ to $\mathbb{R}^n$ and $\mathbb{R}^{n\times d}$ with $g(X,k)g^T(X,k) = (g_{ij}(X,k))_{n\times n}$. For each $k \in \mathbb{S}$, let $V(\cdot,k)$ be any twice continuously differentiable function, and the operator $\mathcal{L}$ can be defined by

$$\mathcal{L}V(X,k) = \sum_{i=1}^{N} f_i(X,k)\frac{\partial V(X,k)}{\partial X_i} + \frac{1}{2}\sum_{i,j=1}^{N} g_{ij}(X,k)\frac{\partial^2 V(X,k)}{\partial X_i \partial X_j} + \sum_{l\in N} p_{kl}V(X,l).$$

2.1. Existence-and-uniqueness of the solution

We first of all consider the existence and uniqueness of a global positive solution before investigating other long-term properties of model (1.1) in this section.

Theorem 1. *For any initial value $(S(0), U(0), T(0), m(0)) \in \mathbb{R}^3_+ \times \mathbb{S}$, there exists a unique solution $(S(t), U(t), T(t), m(t))$ of model (1.1) on $t \ge 0$, and the solution will remain in $\mathbb{R}^3_+ \times \mathbb{S}$ with probability one.*

Proof. We write down similar lines as we did in [34, 35] and define the stopping time

$$\tau_r = \inf\left\{t \in [0, \tau_e) : \min\{S(t), U(t), T(t)\} \le \frac{1}{r} \text{ or } \max\{S(t), U(t), T(t)\} \ge r\right\}.$$

Therefore, there exists an integer $r_1 \ge r_0$ such that $\mathbb{P}\{\tau_r \le l\} \ge \varepsilon$ for each integer $r \ge r_1$. Define a C^2-function $V : \mathbb{R}^3_+ \to \mathbb{R}_+$ as follows:

$$V(S,U,T) = S - c - \ln\frac{S}{c} + U - 1 - \ln U + T - 1 - \ln T,$$

where c is a positive constant to be determined later. Let $l > 1$ be arbitrary, for any $0 \le t \le \tau_r \wedge l = \min\{\tau_r, l\}$, and applying Itô's formula to V, we get

$$\begin{aligned}\mathcal{L}V =& -\mu(k)S - (\mu(k) + \delta_1(k))U - (\mu(k) + \delta_2(k))T \\ &- \frac{c\Lambda(k)}{S} - \beta_1(k)S - \beta_2(k)T - \frac{p(k)U}{T} + (c\beta_1(k) + \beta_2(k))U + \Lambda(k) \\ &+ (c+2)\mu(k) + p(k) + \delta_1(k) + \delta_2(k) + \frac{1}{2}(c\sigma_1^2(k) + \sigma_2^2(k) + \sigma_3^2(k)) \\ <& \check{\Lambda} + (c+2)\check{\mu} + \check{p} + \check{\delta}_1 + \check{\delta}_2 + \frac{1}{2}(c\check{\sigma}_1^2 + \check{\sigma}_2^2 + \check{\sigma}_3^2) + (c\check{\beta}_1 + \check{\beta}_2 - \hat{\mu} - \hat{\delta}_1)U.\end{aligned}$$

Choosing c such that $c\check{\beta}_1 + \check{\beta}_2 = \hat{\mu} + \hat{\delta}_1$,

$$\mathcal{L}V \le \check{\Lambda} + (c+2)\check{\mu} + \check{p} + \check{\delta}_1 + \check{\delta}_2 + \frac{1}{2}(c\check{\sigma}_1^2 + \check{\sigma}_2^2 + \check{\sigma}_3^2) := K.$$

The rest of the proof is similar to Theorem 1 in [17], so we omit it. The proof is complete.

2.2. Extinction of the untreated drug addicts within local population

For a long time, extinction always refers to the disappearance of infectious diseases in epidemiology. So, the most important concern of the dynamical behaviors for the stochastic heroin model is to control the spreading of heroin and the number of the untreated drug addicts. By the approaches given in [34–39], together with constructing several Lyapunov functions, combining generalized Itô's formula and the strong law of large numbers, we derive the moderate conditions for the extinction of the untreated drug addicts to model (1.1). With these conditions, we find that the spreading of heroin ultimately vanishes in the local population, in other words, the number of the untreated drug addicts declines to zero.

Lemma 1. *Assume that* $\hat{\mu} > \frac{1}{2}(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2)$*, and the solution* $(S(t), U(t), T(t), m(t))$ *of model* (1.1) *satisfies*

$$\lim_{t\mapsto\infty}\frac{1}{t}\int_0^t \check{\sigma}_1 S(s)dB_1(s) = \lim_{t\mapsto\infty}\frac{1}{t}\int_0^t \check{\sigma}_2 U(s)dB_2(s) = 0 \quad a.s.,$$

$$\lim_{t\mapsto\infty}\frac{1}{t}\int_0^t \check{\sigma}_3 T(s)dB_3(s) = 0 \quad a.s..$$

Proof. We write down similar lines by the same approach as in Lemma 2.2 of [28], so the proof is easy to check, and we omit the details.

Lemma 2. *For* $t \ge 0$*, the solution* $(S(t), U(t), T(t), m(t))$ *of model* (1.1) *satisfies*

$$\lim_{t\mapsto\infty}\frac{1}{t}(S(t) + U(t) + T(t)) = 0 \quad a.s..$$

Proof. Define $W(N) = (1+N)^\rho$ with $\rho > 1$, which gives

$$\mathrm{d}W(N) = \mathcal{L}W(N)\mathrm{d}t + \rho(1+N)^{\rho-1}(\sigma_1(k)S\,\mathrm{d}B_1(t) + \sigma_2(k)U\mathrm{d}B_2(t) + \sigma_3(k)T\mathrm{d}B_3(t)),$$

where

$$\begin{aligned}\mathcal{L}W(N) &= \rho(1+N)^{\rho-1}[\Lambda(k)-\mu(k)S-(\mu(k)+\delta_1(k))U-(\mu(k)+\delta_2(k))T]\\ &\quad+\frac{\rho(\rho-1)}{2}(1+N)^{\rho-2}(\sigma_1^2(k)S^2+\sigma_2^2(k)U^2+\sigma_3^2(k)T^2)\\ &= \rho(1+N)^{\rho-2}[(1+N)(\Lambda(k)-\mu(k)S-(\mu(k)+\delta_1(k))U\\ &\quad-(\mu(k)+\delta_2(k))T)+\frac{\rho-1}{2}(\sigma_1^2(k)S^2+\sigma_2^2(k)U^2+\sigma_3^2(k)T^2)]\\ &\le \rho(1+N)^{\rho-2}\Big[(1+N)(\check{\Lambda}-\hat{\mu}N)+\frac{\rho-1}{2}(\check{\sigma}_1^2\vee\check{\sigma}_2^2\vee\check{\sigma}_3^2)N^2\Big]\\ &= \rho(1+N)^{\rho-2}\Big[-\Big(\hat{\mu}-\frac{\rho-1}{2}(\check{\sigma}_1^2\vee\check{\sigma}_2^2\vee\check{\sigma}_3^2)\Big)N^2+(\check{\Lambda}-\hat{\mu})N+\check{\Lambda}\Big].\end{aligned}$$

The remaining proof is referred as Lemma 2.3 of [28]. We omit the details.

Theorem 2. *If the conditions*

$$\hat{\mu}>\frac{1}{2}(\check{\sigma}_1^2\vee\check{\sigma}_2^2\vee\check{\sigma}_3^2)$$

and

$$\frac{\max\{\check{\beta}_1,\check{\beta}_2\}}{\hat{\mu}}\check{\Lambda}-\Big(\hat{p}+\hat{\mu}+\hat{\delta}_1+\frac{1}{2}\hat{\sigma}_2^2\Big)<0$$

hold, then

$$\lim_{t\mapsto\infty}\frac{\ln U(t)}{t}<0.$$

That is, the density of the untreated drug addicts will decline to zero with an exponential rate.

Proof. Model (1.1) gives

$$\begin{aligned}\mathrm{d}(S+U+T)<&\Big[\check{\Lambda}-\hat{\mu}(S+T)-(\hat{\mu}+\hat{\delta}_1)U\Big]\mathrm{d}t\\ &+\check{\sigma}_1S\,\mathrm{d}B_1(t)+\check{\sigma}_2U\mathrm{d}B_2(t)+\check{\sigma}_3T\mathrm{d}B_3(t).\end{aligned}\tag{2.2}$$

Integrating (2.2) from 0 to t, we get

$$\frac{A(t)}{t}\le\frac{1}{t}\int_0^t\Big[\check{\Lambda}-\hat{\mu}(S(s)+T(s))-(\hat{\mu}+\hat{\delta}_1)U(s)\Big]\mathrm{d}s+\frac{M(t)}{t}.\tag{2.3}$$

Together with

$$M(t)=\int_0^t\check{\sigma}_1S(s)\mathrm{d}B_1(s)+\int_0^t\check{\sigma}_2U(s)\mathrm{d}B_2(s)+\int_0^t\check{\sigma}_3T(s)\mathrm{d}B_3(s)$$

and

$$A(t)=S(t)+U(t)+T(t)-S(0)-U(0)-T(0),$$

the expression (2.3) further gives

$$\frac{1}{t}\int_0^t (S(s)+T(s))\mathrm{d}s \le \frac{1}{\hat{\mu}}\Big\{\frac{1}{t}\int_0^t \Big[\check{\Lambda}-(\hat{\mu}+\hat{\delta}_1)U(s)\Big]\mathrm{d}s+\frac{M(t)}{t}-\frac{A(t)}{t}\Big\}. \tag{2.4}$$

From model (1.1), we can see

$$\begin{aligned}\frac{1}{t}\Big(\ln U(t)-\ln U(0)\Big) \le \max\{\check{\beta}_1,\check{\beta}_2\}\frac{1}{t}\int_0^t (S(s)+T(s))\mathrm{d}s \\ -\Big(\hat{p}+\hat{\mu}+\hat{\delta}_1+\frac{1}{2}\hat{\sigma}_2^2\Big)+\frac{\check{\sigma}_2 B_2(t)}{t}.\end{aligned} \tag{2.5}$$

Combining expressions (2.4) and (2.5), we have

$$\begin{aligned}\frac{\ln U(t)}{t} &\le \frac{\max\{\check{\beta}_1,\check{\beta}_2\}}{\hat{\mu}}\Big\{\frac{1}{t}\int_0^t \Big[\check{\Lambda}-(\hat{\mu}+\hat{\delta}_1)U(s)\Big]\mathrm{d}s+\frac{M(t)}{t}-\frac{A(t)}{t}\Big\}\\ &\quad -\Big(\hat{p}+\hat{\mu}+\hat{\delta}_1+\frac{1}{2}\hat{\sigma}_2^2\Big)+\frac{\check{\sigma}_2 B_2(t)}{t}+\frac{\ln U(0)}{t}\\ &< \frac{\max\{\check{\beta}_1,\check{\beta}_2\}}{\hat{\mu}}\Big(\frac{M(t)}{t}-\frac{A(t)}{t}\Big)+\frac{\check{\sigma}_2 B_2(t)}{t}+\frac{\ln U(0)}{t}\\ &\quad +\frac{\check{\Lambda}\max\{\check{\beta}_1,\check{\beta}_2\}}{\hat{\mu}}-\Big(\hat{p}+\hat{\mu}+\hat{\delta}_1+\frac{1}{2}\hat{\sigma}_2^2\Big).\end{aligned}$$

By the strong law of large numbers for local martingales, together with Lemma 1 and Lemma 2, we obtain

$$\lim_{t\mapsto\infty}\frac{1}{t}\int_0^t \check{\sigma}_2\mathrm{d}B_2(s)=0 \quad \text{a.s.},$$

and

$$\lim_{t\mapsto\infty}\frac{1}{t}\Big\{\frac{\max\{\check{\beta}_1,\check{\beta}_2\}}{\hat{\mu}}(M(t)-A(t))+\ln U(0)\Big\}=0 \quad \text{a.s..}$$

If

$$\frac{\max\{\check{\beta}_1,\check{\beta}_2\}}{\hat{\mu}}\check{\Lambda}-\Big(\hat{p}+\hat{\mu}+\hat{\delta}_1+\frac{1}{2}\hat{\sigma}_2^2\Big)<0,$$

this means that

$$\lim_{t\mapsto\infty}\frac{\ln U(t)}{t}<0.$$

In other words, by Definition 3.2 in [34], the density of the untreated drug addicts declines to extinction exponentially. The proof is complete.

2.3. Persistence of the untreated drug addicts within local population

Next, we investigate the sufficient conditions of the existence of an ergodic stationary distribution for model (1.1). Define

$$R_0^s := \sum_{k\in\mathbb{S}}\pi_k R_{0k}, \tag{2.6}$$

where

$$R_{0k} = c_1(k)\Lambda(k) - p(k) - \mu(k) - \delta_1(k) - \frac{1}{2}\sigma_2^2(k),$$

and $c_1(k)$ is the solution of the linear system (2.7).

Lemma 3. *For each $k \in \mathbb{S}$, the linear system*

$$c_1(k)\mu(k) - \beta_1(k) - \sum_{l\in S} p_{kl}c_1(l) = 0 \tag{2.7}$$

has a unique solution $c_1 = (c_1(1), c_1(2), \cdots, c_1(N))^T \gg 0$; moreover,

$$c_2(k)(\mu(k) + \delta_2(k)) - \beta_2(k) - \sum_{l\in S} p_{kl}c_2(l) = 0 \tag{2.8}$$

has a unique solution $c_2 = (c_2(1), c_2(2), \cdots, c_2(N))^T \gg 0$.

Proof. The linear system (2.7) can be rewritten as the form of $AV = \beta_1$, where $V \in \mathbb{R}^N$, $\beta_1 = (\beta_1(1), \beta_1(2), \cdots, \beta_1(N))^{\mathrm{T}}$, and

$$A = \begin{pmatrix} \mu(1) - p_{11} & -p_{12} & \cdots & -p_{1N} \\ -p_{21} & \mu(2) - p_{22} & \cdots & -p_{2N} \\ \vdots & \vdots & & \vdots \\ -p_{N1} & -p_{N2} & \cdots & \mu(N) - p_{NN} \end{pmatrix}.$$

Obviously, $A \in Z^{N\times N}$, and $Z^{N\times N} = \{B = (b_{ij})_{N\times N} : b_{ij} \leqslant 0, i \neq j\}$. By Lemma 5.3 in [40], we obtain that determinant of (A_k) is positive for $k = 1, 2, \cdots, N$, where

$$A_k = \begin{pmatrix} \mu(1) - p_{11} & -p_{12} & \cdots & -p_{1k} \\ -p_{21} & \mu(2) - p_{22} & \cdots & -p_{2k} \\ \vdots & \vdots & & \vdots \\ -p_{k1} & -p_{k2} & \cdots & \mu(k) - p_{kk} \end{pmatrix}.$$

In other words, the leading principal minors of A are all positive, which means that A is a nonsingular M-matrix. For the vector $\beta_1 \in \mathbb{R}^N$, the linear system (2.7) has a solution $c_1 = (c_1(1), c_1(2), \cdots, c_1(N))^{\mathrm{T}}$. Similarly, we show that the linear system (2.8) has a solution $c_2 = (c_2(1), c_2(2), \cdots, c_2(N))^{\mathrm{T}}$.

Theorem 3. If $R_0^s > 0$, then model (1.1) admits a unique ergodic stationary distribution.

Proof. Let

$$x(t) = \ln S(t),\ y(t) = \ln U(t),\ z(t) = \ln T(t),$$

and then model (1.1) is rewritten as follows:

$$\begin{cases} \mathrm{d}x(t) = \Big[\dfrac{\Lambda(m(t))}{e^x} - \beta_1(m(t))e^y - \Big(\mu(m(t)) + \dfrac{1}{2}\sigma_1^2(m(t))\Big)\Big]\mathrm{d}t \\ \qquad\quad + \sigma_1(m(t))\mathrm{d}B_1(t), \\ \mathrm{d}y(t) = \Big[\beta_1(m(t))e^x - p(m(t)) + \beta_2(m(t))e^z - \Big(\mu(m(t)) + \delta_1(m(t)) \\ \qquad\quad + \dfrac{1}{2}\sigma_2^2(m(t))\Big)\Big]\mathrm{d}t + \sigma_2(m(t))\mathrm{d}B_2(t), \\ \mathrm{d}z(t) = \Big[\dfrac{p(m(t))e^y}{e^z} - \beta_2(m(t))e^y - \Big(\mu(m(t)) + \delta_2(m(t)) + \dfrac{1}{2}\sigma_3^2(m(t))\Big)\Big]\mathrm{d}t \\ \qquad\quad + \sigma_3(m(t))\mathrm{d}B_3(t). \end{cases} \tag{2.9}$$

Equivalently, we study the stationary distribution of model (2.9) by using Lemma 2.1 in [41] (also referred as Lemma 5.1 in [36]).

Step 1. The assumption $p_{ij} > 0$ for $i \neq j$ implies that condition (i) in Lemma 2.1 in [41] is satisfied.

Step 2. The diffusion matrix

$$D(x,k) = \begin{pmatrix} \sigma_1^2(k) & 0 & 0 \\ 0 & \sigma_2^2(k) & 0 \\ 0 & 0 & \sigma_3^2(k) \end{pmatrix}$$

of model (2.9) is positive definite, which implies that condition (ii) in Lemma 2.1 in [41] holds.

Step 3. We define a C^2-function

$$\begin{aligned} W(x,y,z,k) = {} & \frac{1}{\theta+1}(e^x + e^y + e^z)^{(\theta+1)} \\ & - B\big[c_1(k)(e^x + e^y) + c_2(k)(e^y + e^z) + y + \omega_k\big] - x - z, \end{aligned}$$

such that $\theta \in (0,1)$ satisfying

$$\hat{\mu} - 0.5\theta\check{\sigma}_1^2 > 0, \;\; \hat{\mu} + \hat{\delta}_1 - 0.5\theta\check{\sigma}_2^2 > 0, \;\; \hat{\mu} + \hat{\delta}_2 - 0.5\theta\check{\sigma}_3^2 > 0,$$

and such that $B > 0$ satisfying

$$f_1^u + f_3^u - BR_0^s \leq -2,$$

here ω_k will be determined later. Obviously, there exists a point (x_0, y_0, z_0, k) at which the minimum value $W(x_0, y_0, z_0, k)$ is taken. We define a non-negative C^2-Lyapunov function as follows:

$$V(x,y,z,k) = W(x,y,z,k) - W(x_0,y_0,z_0,k). \tag{2.10}$$

Denote

$$\begin{aligned} V_1(x,y,z,k) &= \frac{1}{\theta+1}(e^x + e^y + e^z)^{(\theta+1)}, \\ V_2(x,y,z,k) &= -c_1(k)(e^x + e^y) - c_2(k)(e^y + e^z) - y - \omega_k, \\ V_3(y,k) &= -x, \\ V_4(x,k) &= -z. \end{aligned}$$

By using the generalized Itô's formula, together with the elementary equality

$$(a+b+c)^\theta \le 3^\theta(a^\theta+b^\theta+c^\theta), \quad \text{for } a>0, b>0, c>0,$$

we obtain

$$\begin{aligned}
\mathcal{L}V_1 &= (e^x+e^y+e^z)^\theta[\Lambda(k)-\mu(k)e^x-(\mu(k)+\delta_1(k))e^y-(\mu(k)+\delta_2(k))e^z] \\
&\quad+\frac{\theta}{2}(e^x+e^y+e^z)^{\theta-1}(\sigma_1^2(k)e^{2x}+\sigma_2^2(k)e^{2y}+\sigma_3^2(k)e^{2z}) \\
&\le 3^\theta\Lambda(k)(e^{\theta x}+e^{\theta y}+e^{\theta z})+\frac{\theta}{2}\Big(\sigma_1^2(k)e^{(\theta+1)x}+\sigma_2^2(k)e^{(\theta+1)y}+\sigma_3^2(k)e^{(\theta+1)z}\Big) \\
&\quad-(\mu(k)+\delta_2(k))e^{(\theta+1)z}-\mu(k)e^{(\theta+1)x}-(\mu(k)+\delta_1(k))e^{(\theta+1)y},
\end{aligned} \tag{2.11}$$

and picking the coefficients by terms gives that

$$\begin{aligned}
\mathcal{L}V_1 \le &-\Big(\hat{\mu}-\frac{\theta}{2}\check{\sigma}_1^2\Big)e^{(\theta+1)x}-\Big(\hat{\mu}+\hat{\delta}_1-\frac{\theta}{2}\check{\sigma}_2^2\Big)e^{(\theta+1)y}-\Big(\hat{\mu}+\hat{\delta}_2-\frac{\theta}{2}\check{\sigma}_3^2\Big)e^{(\theta+1)z} \\
&+3^\theta\check{\Lambda}(e^{\theta x}+e^{\theta y}+e^{\theta z}).
\end{aligned} \tag{2.12}$$

According to the similar discussion and Lemma 3, we obtain

$$\begin{aligned}
\mathcal{L}V_2 &= -c_1(k)[\Lambda(k)-\mu(k)e^x-p(k)e^y+\beta_2(k)e^{y+z}-(\mu(k)+\delta_1(k))e^y] \\
&\quad-c_2(k)[\beta_1(k)e^{x+y}-(\mu(k)+\delta_1(k))e^y-(\mu(k)+\delta_2(k))e^z] \\
&\quad-\beta_1(k)e^x-\beta_2(k)e^z+p(k)+\mu(k)+\delta_1(k)+\frac{1}{2}\sigma_2^2(k)-\sum_{l\in S}p_{kl}\omega(l) \\
&\quad-\sum_{l\in S}p_{kl}c_1(l)(e^x+e^y)-\sum_{l\in S}p_{kl}c_2(l)(e^y+e^z) \\
&\le \Big[c_1(k)\mu(k)-\beta_1(k)-\sum_{l\in S}p_{kl}c_1(l)\Big]e^x \\
&\quad+\Big[c_2(k)(\mu(k)+\delta_2(k))-\beta_2(k)-\sum_{l\in S}p_{kl}c_2(l)\Big]e^z \\
&\quad+\Big[c_1(k)(p(k)+\mu(k)+\delta_1(k))+c_2(k)(\mu(k)+\delta_1(k)) \\
&\quad-\sum_{l\in S}p_{kl}c_1(l)-\sum_{l\in S}p_{kl}c_2(l)\Big]e^y \\
&\quad+p(k)+\mu(k)+\delta_1(k)+\frac{1}{2}\sigma_2^2(k)-c_1(k)\Lambda(k)-\sum_{l\in S}p_{kl}\omega(l) \\
&=: -R_{0k}-\sum_{l\in S}p_{kl}\omega(l) \\
&\quad+\Big[c_1(k)(p(k)+\delta_1(k))+c_2(k)(\delta_1(k)-\delta_2(k))+\beta_1(k)+\beta_2(k)\Big]e^y,
\end{aligned} \tag{2.13}$$

with

$$R_{0k} = c_1(k)\Lambda(k) - p(k) - \mu(k) - \delta_1(k) - \frac{1}{2}\sigma_2^2(k).$$

We define a vector $R_0 = (R_{01}, R_{02}, \cdots, R_{0N})^{\mathrm{T}}$, since the generator matrix Γ is irreducible, there exists a solution of the Poisson system $\omega = (\omega_1, \cdots, \omega_N)^{\mathrm{T}}$ such that

$$\Gamma\omega = \Big(\sum_{k=1}^{N} \pi_k R_{0k}\Big)\vec{1} - R_0, \tag{2.14}$$

where $\vec{1}$ is a column vector in which all elements are one, which further implies

$$R_{0k} + \sum_{l\in S} p_{kl}\omega(l) = \sum_{k=1}^{N} \pi_k R_{0k},$$

and together with (2.6), the expression (2.13) turns into

$$\mathcal{L}V_2 \le -R_0^s + [\check{c}_1(\check{p} + \check{\delta}_1) + \check{c}_2\check{\delta}_1 + \check{\beta}_1 + \check{\beta}_2]e^y. \tag{2.15}$$

By the same arguments, we derive

$$\mathcal{L}V_3 = -\frac{\Lambda(k)}{e^x} + \beta_1(k)e^y + \mu(k) + \frac{1}{2}\sigma_1^2(k) \le -\frac{\hat{\Lambda}}{e^x} + \check{\beta}_1 e^y + \check{\mu} + \frac{1}{2}\check{\sigma}_1^2, \tag{2.16}$$

$$\mathcal{L}V_4 \le -\frac{\hat{p}(k)}{e^z} + \check{\beta}_2(k)e^y + \check{\mu} + \check{\delta}_2 + \frac{1}{2}\check{\sigma}_3^2. \tag{2.17}$$

Thus the following result is derived

$$\mathcal{L}V = \mathcal{L}V_1 + B\mathcal{L}V_2 + \mathcal{L}V_3 + \mathcal{L}V_4 < f(x, y, z) = f_1(x) + f_2(y) + f_3(z), \tag{2.18}$$

where

$$\begin{aligned}
f_1(x) &= -\Big(\hat{\mu} - \frac{\theta}{2}\check{\sigma}_1^2\Big)e^{(\theta+1)x} + 3^{\theta}\check{\Lambda}e^{\theta x} - \frac{\hat{\Lambda}}{e^x} + 2\check{\mu} + \check{\delta}_2 + \frac{1}{2}(\check{\sigma}_1^2 + \check{\sigma}_3^2),\\
f_2(y) &= -\Big(\hat{\mu} + \hat{\delta}_1 - \frac{\theta}{2}\check{\sigma}_2^2\Big)e^{(\theta+1)y} + 3^{\theta}\check{\Lambda}e^{\theta y} + (\check{\beta}_1 + \check{\beta}_2)e^y\\
&\quad + B[-R_0^s + (\check{c}_1(\check{p} + \check{\delta}_1) + \check{c}_2\check{\delta}_1 + \check{\beta}_1 + \check{\beta}_2)e^y],\\
f_3(z) &= -\Big(\hat{\mu} + \hat{\delta}_2 - \frac{\theta}{2}\check{\sigma}_3^2\Big)e^{(\theta+1)z} + 3^{\theta}\check{\Lambda}e^{\theta z} - \frac{\hat{p}}{e^z}.
\end{aligned}$$

Furthermore, we have

$$f(x, y, z) \le f_1(x) + f_2^u + f_3^u \to -\infty, \text{ if } x \to +\infty \text{ or } x \to -\infty,$$

and the same arguments give that

$$\begin{aligned}
&f(x, y, z) \le f_1^u + f_2(y) + f_3^u \to -\infty, \text{ if } y \to +\infty,\\
&f(x, y, z) \le f_1^u + f_2(y) + f_3^u \to f_1^u + f_3^u - BR_0^s \le -2, \text{ if } y \to -\infty,\\
&f(x, y, z) \le f_1^u + f_2^u + f_3(z) \to -\infty, \text{ if } z \to +\infty \text{ or } z \to -\infty.
\end{aligned}$$

Therefore, we take $\varepsilon > 0$ sufficiently large, and let

$$U = (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon).$$

Then,

$$\mathcal{L}V(x, y, z, k) \leq -1, \ (x, y, z, k) \in U^c \times \mathbb{S}.$$

Hence condition (iii) of Lemma 2.1 in [41] is verified.

2.4. Probability density function of model (2.20) *within local population*

Lemma 4. [42] *Let* Υ_0 *be a symmetric positive definite matrix, such that the three dimensional algebraic equation*

$$G_0^2 + A_0\Upsilon_0 + \Upsilon_0 A_0^T = 0 \tag{2.19}$$

holds, where $G_0 = diag\{1, 0, 0\}$*, and*

$$A_0 = \begin{pmatrix} -c_1 & -c_2 & -c_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

and also that $c_1 > 0, c_3 > 0$ *and* $c_1c_2 - c_3 > 0$*, then* Υ_0 *follows*

$$\Upsilon_0 = \frac{1}{2(c_1c_2 - c_3)} \begin{pmatrix} c_2 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & \frac{c_1}{c_3} \end{pmatrix}.$$

Lemma 5. [42] *Let* Υ_0 *be a symmetric positive semi-definite matrix, such that the three-dimensional algebraic equation*

$$G_0^2 + \tilde{A}_0\Upsilon_1 + \Upsilon_1\tilde{A}_0^T = 0$$

holds, where $G_0 = diag\{1, 0, 0\}$*, and*

$$\tilde{A}_0 = \begin{pmatrix} -d_1 & -d_2 & -d_3 \\ 1 & 0 & 0 \\ 0 & 0 & d_{33}. \end{pmatrix}.$$

Also, $d_1 > 0, d_2 > 0$*, and thus* Υ_1 *takes the form*

$$\Upsilon_1 = diag\Big\{\frac{1}{2d_1}, \frac{1}{2d_1d_2}, 0\Big\}.$$

Next, as a special case, we consider the degenerated model (2.20) as follows:

$$\begin{cases} \mathrm{d}S(t) = [\Lambda - \beta_1 S(t)U(t) - \mu S(t)]\mathrm{d}t + \sigma_1 S(t)\mathrm{d}B_1(t), \\ \mathrm{d}U(t) = [\beta_1 S(t)U(t) - pU(t) + \beta_2 U(t)T(t) - (\mu + \delta_1)U(t)]\mathrm{d}t \\ \qquad\quad + \sigma_2 U(t)\mathrm{d}B_2(t), \\ \mathrm{d}T(t) = [pU(t) - \beta_2 U(t)T(t) - (\mu + \delta_2)T(t)]\mathrm{d}t + \sigma_3 T(t)\mathrm{d}B_3(t). \end{cases} \tag{2.20}$$

Then, we investigate the existence of the probability density function of model (2.20). First of all, we consider the existence of the positive equilibrium point to model (2.20).

Theorem 4. *If the conditions*

$$g_1 = 0, \quad 1 + \frac{m_1 m_2}{m_1 p - \beta_1 \Lambda} > 0, \tag{2.21}$$

or

$$g_1 < 0, \quad \Delta > 0, \quad \beta_1 \Lambda - m_1 p - m_1 m_2 > 0, \tag{2.22}$$

or

$$g_1 < 0, \quad \Delta = 0, \quad (\beta_2 m_1 - \beta_1 m_3)(m_2 + p) + \beta_2 (m_1 p - \beta_1 \Lambda) > 0, \tag{2.23}$$

hold, then, model (2.20) *admits a positive equilibrium point* P^**, where* g_1, m_1, m_2, m_3 *and* Δ *could be found later.*

Proof. Let $(z_1, z_2, z_3)^{\mathrm{T}} = (\ln S, \ln U, \ln T)^{\mathrm{T}}$, by using Itô's formula, the following is derived from model (2.20) that

$$\begin{cases} \mathrm{d}z_1 = \Big[\dfrac{\Lambda}{e^{z_1}} - \beta_1 e^{z_2} - \Big(\mu + \dfrac{\sigma_1^2}{2}\Big)\Big]\mathrm{d}t + \sigma_1 \mathrm{d}B_1(t), \\ \mathrm{d}z_2 = \Big[\beta_1 e^{z_1} - p + \beta_2 e^{z_3} - \Big(\mu + \delta_1 + \dfrac{\sigma_2^2}{2}\Big)\Big]\mathrm{d}t + \sigma_2 \mathrm{d}B_2(t), \\ \mathrm{d}z_3 = \Big[p e^{z_2 - z_3} - \beta_2 e^{z_2} - \Big(\mu + \delta_2 + \dfrac{\sigma_3^2}{2}\Big)\Big]\mathrm{d}t + \sigma_3 \mathrm{d}B_3(t). \end{cases} \tag{2.24}$$

Then, we determine the unique local equilibrium point

$$P^* = (S^*, U^*, T^*) = (e^{z_1^*}, e^{z_2^*}, e^{z_3^*}),$$

by solving the following equations:

$$\begin{cases} \dfrac{\Lambda}{e^{z_1^*}} - \beta_1 e^{z_2^*} - \Big(\mu + \dfrac{\sigma_1^2}{2}\Big) = 0, \\ \beta_1 e^{z_1^*} - p + \beta_2 e^{z_3^*} - \Big(\mu + \delta_1 + \dfrac{\sigma_2^2}{2}\Big) = 0, \\ p e^{z_2^* - z_3^*} - \beta_2 e^{z_2^*} - \Big(\mu + \delta_2 + \dfrac{\sigma_3^2}{2}\Big) = 0, \end{cases} \tag{2.25}$$

in which

$$S^* = \frac{p - \beta_2 T^* + m_2}{\beta_1} > 0, \ U^* = \frac{m_3 T^*}{p - \beta_2 T^*} > 0,$$

and T^* satisfies the following quadratic equation

$$g_1 T^2 + g_2 T + g_3 = 0, \tag{2.26}$$

with

$$\begin{aligned} g_1 &= \beta_1\beta_2 m_3 - \beta_2^2 m_1, \\ g_2 &= 2\beta_2 m_1 p + \beta_2 m_1 m_2 - \beta_1\beta_2\Lambda - \beta_1 m_2 m_3 - \beta_1 m_3 p, \\ g_3 &= \beta_1 p\Lambda - m_1 m_2 p - m_1 p^2, \end{aligned}$$

and

$$m_1 = \mu + \frac{\sigma_1^2}{2}, m_2 = \mu + \delta_1 + \frac{\sigma_2^2}{2}, m_3 = \mu + \delta_2 + \frac{\sigma_3^2}{2}.$$

Next, we discuss the value of g_1 by three cases.

Case 1. If $g_1 = 0$ (i.e., $\beta_1 m_3 - \beta_2 m_1 = 0$), and

$$1 + \frac{m_1 m_2}{m_1 p - \beta_1\Lambda} > 0, \tag{2.27}$$

together with

$$g_2 = \beta_2(m_1 p - \beta_1\Lambda), \quad g_3 = p(\beta_1\Lambda - m_1 m_2 - m_1 p),$$

as (2.27) is valid, then we get

$$\frac{g_3}{g_2} = \frac{p(\beta_1\Lambda - m_1 m_2 - m_1 p)}{\beta_2(m_1 p - \beta_1\Lambda)} = -\frac{p}{\beta_2}\Big(1 + \frac{m_1 m_2}{m_1 p - \beta_1\Lambda}\Big) < 0,$$

and thus Eq (2.26) has a unique positive root.

Case 2. If $g_1 < 0$, then we get

$$\begin{aligned} \Delta =& g_2^2 - 4g_1 g_3 \\ =& [2\beta_2 m_1 p + \beta_2 m_1 m_2 - \beta_1\beta_2\Lambda - \beta_1 m_2 m_3 - \beta_1 m_3 p]^2 \\ &+ 4\beta_2 p(\beta_2 m_1 - \beta_1 m_3)(\beta_1\Lambda - m_1 p - m_1 m_2). \end{aligned}$$

Next, we turn to analyze the value of Δ. When

$$\Delta > 0, \quad \beta_1\Lambda - m_1 p - m_1 m_2 > 0, \tag{2.28}$$

which further gives

$$\frac{g_3}{g_1} = \frac{\beta_1 p\Lambda - m_1 m_2 p - m_1 p^2}{\beta_1\beta_2 m_3 - \beta_2^2 m_1} = \frac{p(\beta_1\Lambda - m_1 m_2 - m_1 p)}{\beta_2(\beta_1 m_3 - \beta_2 m_1)} < 0.$$

Thus, Eq (2.26) has a unique positive root. When

$$\Delta = 0, \quad (\beta_2 m_1 - \beta_1 m_3)(m_2 + p) + \beta_2(m_1 p - \beta_1\Lambda) > 0, \tag{2.29}$$

which gives that

$$\begin{aligned} -\frac{g_2}{2g_1} &= -\frac{2\beta_2 m_1 p + \beta_2 m_1 m_2 - \beta_1\beta_2\Lambda - \beta_1 m_2 m_3 - \beta_1 m_3 p}{2\beta_2(\beta_1 m_3 - \beta_2 m_1)} \\ &= \frac{(\beta_2 m_1 - \beta_1 m_3)(m_2 + p) + \beta_2(m_1 p - \beta_1\Lambda)}{2\beta_2(\beta_2 m_1 - \beta_1 m_3)} > 0, \end{aligned}$$

and thus Eq (2.26) has a unique positive root.

Case 3. When $g_1 > 0$, if the drug addicts under treatment $T(t)$ has a unique positive root, the value of β_1 will be very small, and the drug addicts $U(t)$ and susceptible individuals $S(t)$ are negative, so we omit this case.

Theorem 5. *If the conditions of Theorem* 4 *are satisfied, and*

$$\Lambda\beta_1^2 - m_3\beta_2 p > 0, \tag{2.30}$$

then, model (2.20) *possesses a probability density function*

$$\Phi(S,U,T) = (2\pi)^{-\frac{3}{2}}|\Sigma|^{-\frac{1}{2}}e^{-\frac{1}{2}(\ln\frac{S}{S^*},\ln\frac{U}{U^*},\ln\frac{T}{T^*})\Sigma^{-1}(\ln\frac{S}{S^*},\ln\frac{U}{U^*},\ln\frac{T}{T^*})^T},$$

and the positive definite matrix Σ *is presented later.*

Proof. Let $x_i = z_i - z_i^*$ for $i = 1, 2, 3$, and the linearized equation of model (2.24) is written as

$$\begin{cases} \mathrm{d}x_1 = (-a_{11}x_1 - a_{12}x_2 + a_{13}x_3)\mathrm{d}t + \sigma_1\mathrm{d}B_1(t), \\ \mathrm{d}x_2 = (a_{21}x_1 + a_{22}x_2 + a_{23}x_3)\mathrm{d}t + \sigma_2\mathrm{d}B_2(t), \\ \mathrm{d}x_3 = (a_{31}x_1 + a_{32}x_2 - a_{33}x_3)\mathrm{d}t + \sigma_3\mathrm{d}B_3(t), \end{cases} \tag{2.31}$$

where

$$\begin{aligned} &a_{11} = \Lambda e^{-z_1^*}, a_{12} = \beta_1 e^{z_2^*}, a_{13} = 0, \\ &a_{21} = \beta_1 e^{z_1^*}, a_{22} = 0, a_{23} = \beta_2 e^{z_3^*}, \\ &a_{31} = 0, a_{32} = pe^{z_2^* - z_3^*} - \beta_2 e^{z_2^*}, a_{33} = pe^{z_2^* - z_3^*}. \end{aligned}$$

Let $X = (x_1, x_2, x_3)^{\mathrm{T}}, B(t) = (B_1(t), B_2(t), B_3(t))^{\mathrm{T}}, M = \mathrm{diag}\{\sigma_1, \sigma_2, \sigma_3\}$ and

$$A = \begin{pmatrix} -a_{11} & -a_{12} & 0 \\ a_{21} & 0 & a_{23} \\ 0 & a_{32} & -a_{33} \end{pmatrix}.$$

Therefore, Eq (2.31) can be equally rewritten as

$$\mathrm{d}X(t) = AX(t)\mathrm{d}t + M\mathrm{d}B(t).$$

According to the relative theory in Gardiner [43], there is a unique density function $\Phi(X)$ around the positive equilibrium point P^* which satisfies the following equation (i.e., Fokker-Planck equation):

$$\begin{aligned} &-\sum_{r=1}^{3}\frac{\sigma_i^2}{2}\frac{\partial^2\Phi}{\partial x_i^2} + \frac{\partial}{x_1}[(-a_{11}x_1 - a_{12}x_2 + a_{13}x_3)\Phi] \\ &+ \frac{\partial}{x_2}[(a_{21}x_1 + a_{22}x_2 + a_{23}x_3)\Phi] + \frac{\partial}{x_3}[a_{31}x_1 + a_{32}x_2 - a_{33}x_3)\Phi] = 0. \end{aligned} \tag{2.32}$$

On the basis of Roozen [44], we can approximate it with a Gaussian distribution

$$\Phi(X) = \Phi(x_1, x_2, x_3) = C_0 e^{-\frac{1}{2}(x_1,x_2,x_3)Q(x_1,x_2,x_3)^T},$$

where C_0 is a positive constant, which is determined by

$$\int_{\mathbb{R}^3} \Phi(x_1, x_2, x_3)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 = 1.$$

Also, the real symmetric inverse matrix Q meets the subsequent algebraic equation

$$QM^2Q + QA + A^TQ = 0,$$

such that $\Sigma = Q^{-1}$, and then we derive

$$M^2 + A\Sigma + \Sigma A^T = 0. \tag{2.33}$$

Furthermore, we have $C_0 = (2\pi)^{-\frac{3}{2}}|\Sigma|^{-\frac{1}{2}}$.

According to the finite independent superposition principle, we express Eq (2.33) as the sum of the solutions of the following algebraic sub-equations:

$$M_k^2 + A\Sigma_k + \Sigma_k A^T = 0, k = 1, 2, 3, \tag{2.34}$$

where

$$M_1 = \mathrm{diag}(\sigma_1, 0, 0), M_2 = \mathrm{diag}(0, \sigma_2, 0), M_3 = \mathrm{diag}(0, 0, \sigma_3)$$

with

$$\Sigma = \Sigma_1 + \Sigma_2 + \Sigma_3, M^2 = M_1^2 + M_2^2 + M_3^2.$$

Obviously, the characteristic polynomial of matrix A is

$$\lambda^3 + p_1\lambda^2 + p_2\lambda + p_3 = 0,$$

with

$$\begin{aligned} p_1 &= a_{33} + a_{11}, \\ p_2 &= a_{11}a_{33} + a_{12}a_{21} - a_{23}a_{32}, \\ p_3 &= a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}. \end{aligned} \tag{2.35}$$

We find that

$$\begin{aligned} \Delta_1 &= p_1 = a_{33} + a_{11} > 0, \\ \Delta_2 &= p_1p_2 - p_3 = a_{11}^2a_{33} + a_{11}a_{12}a_{21} + a_{11}a_{33}^2 - a_{23}a_{32}a_{33}, \end{aligned} \tag{2.36}$$

due to $a_{11}^2a_{33} + a_{11}a_{33}^2 > 0$, and direct substitution gives that

$$a_{11}a_{12}a_{21} - a_{23}a_{32}a_{33} = \frac{m_3T^*(\Lambda\beta_1^2 - \beta_2m_3p)}{p - \beta_2T^*} > 0. \tag{2.37}$$

We derive that A is a Hurwitz matrix. Next, we will prove that Σ is positive definite.

Step 1. We consider the algebraic equation

$$M_1^2 + A\Sigma_1 + \Sigma_1 A^T = 0, \tag{2.38}$$

and our discussion will be separated into two cases according to the value of a_{32}.

Case 1.1. If $a_{32} \neq 0$, according to Li et al. [45], we select the standardized transformation matrix

$$H_1 = \begin{pmatrix} a_{32}a_{21} & -a_{32}a_{33} & a_{33}^2 + a_{32}a_{23} \\ 0 & a_{32} & -a_{33} \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.39}$$

such that $B_1 = H_1 A H_1^{-1}$. By direct calculation, we obtain

$$B_1 = \begin{pmatrix} -y_1 & -y_2 & -y_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

where

$$\begin{aligned} y_1 &= a_{11} + a_{33}, \\ y_2 &= a_{11}a_{33} + a_{12}a_{21} - a_{23}a_{32}, \\ y_3 &= a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}. \end{aligned}$$

Furthermore, algebraic Eq (2.38) can be converted to the equivalent

$$H_1 M_1^2 H_1^T + B_1 H_1 \Sigma_1 H_1^T + H_1 \Sigma_1 H_1^T B_1^T = 0,$$

letting

$$\Theta_1 = \varrho_1^{-2} H_1 \Sigma_1 H_1^T, \varrho_1 = a_{21}a_{32}\sigma_1,$$

and algebraic Eq (2.38) is converted as

$$G_0^2 + B_1\Theta_1 + \Theta_1 B_1^T = 0. \tag{2.40}$$

We notice that the real parts of the eigenvalues of A are all negative, so B_1 is a Hurwitz matrix. By Lemma 4, Θ_1 is positive definite and takes the form

$$\Theta_1 = \frac{1}{2(y_1y_2 - y_3)} \begin{pmatrix} y_2 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & \dfrac{y_1}{y_3} \end{pmatrix}.$$

Therefore, $\Sigma_1 = \varrho_1^2 H_1^{-1} \Theta_1 (H_1^T)^{-1}$.

Case 1.2. If $a_{32} = 0$, we choose $\hat{H}_1$ such that $\hat{B}_1 = \hat{H}_1 A \hat{H}_1^{-1}$ with

$$\hat{H}_1 = \begin{pmatrix} a_{21} & 0 & a_{23} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \hat{B}_1 = \begin{pmatrix} -b_1 & -b_2 & -b_3 \\ 1 & 0 & 0 \\ 0 & 0 & -a_{33} \end{pmatrix},$$

where

$$b_1 = a_{11}, b_2 = a_{12}a_{21}, b_3 = a_{23}a_{33} - a_{11}a_{23}.$$

One can equivalently transform (2.38) into

$$\hat{H}_1 M_1^2 \hat{H}_1^T + \hat{B}_1 \hat{H}_1 \Sigma_1 \hat{H}_1^T + \hat{H}_1 \Sigma_1 \hat{H}_1^T \hat{B}_1^T = 0,$$

letting

$$\hat{\Theta}_1 = \hat{\varrho}_1^{-2} \hat{H}_1 \Sigma_1 \hat{H}_1^T, \hat{\varrho}_1 = a_{21}\sigma_1.$$

The algebraic Eq (2.38) becomes

$$G_0^2 + \hat{B}_1 \hat{\Theta}_1 + \hat{\Theta}_1 \hat{B}_1^T = 0, \tag{2.41}$$

with

$$\hat{\Theta}_1 = \text{diag}\Big\{\frac{1}{2b_1}, \frac{1}{2b_1 b_2}, 0\Big\}. \tag{2.42}$$

Therefore, $\Sigma_1 = \hat{\varrho}_1^2 \hat{H}_1^{-1} \hat{\Theta}_1 (\hat{H}_1^T)^{-1}$.

Step 2. Let us consider the following algebraic equation

$$M_2^2 + A\Sigma_2 + \Sigma_2 A^T = 0, \tag{2.43}$$

we select the corresponding elimination matrix J_2 and let $A_2 = J_2 A J_2^{-1}$ with

$$J_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & \dfrac{a_{12}}{a_{32}} \end{pmatrix}, A_2 = \begin{pmatrix} 0 & -\dfrac{a_{12}a_{21}}{a_{32}} + a_{23} & a_{21} \\ a_{32} & -a_{33} & 0 \\ 0 & k_2 & -a_{11} \end{pmatrix},$$

with

$$k_2 = \frac{a_{11}a_{12}}{a_{32}} - \frac{a_{12}a_{33}}{a_{32}}.$$

Case 2.1. If $k_2 \neq 0$, we then let $B_2 = H_2 A_2 H_2^{-1}$ with

$$H_2 = \begin{pmatrix} k_2 a_{32} & -k_2(a_{11} + a_{33}) & a_{11}^2 \\ 0 & k_2 & -a_{11} \\ 0 & 0 & 1 \end{pmatrix}, B_2 = \begin{pmatrix} -q_1 & -q_2 & -q_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{2.44}$$

where

$$\begin{aligned} q_1 &= a_{11} + a_{33}, \\ q_2 &= a_{11}a_{33} + a_{12}a_{21} - a_{23}a_{32}, \\ q_3 &= a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}. \end{aligned}$$

Moreover, algebraic Eq (2.43) is equivalently transformed into

$$(H_2J_2)M_2^2(H_2J_2)^T + B_2[(H_2J_2)\Sigma_2(H_2J_2)^T] + [(H_2J_2)\Sigma_2(H_2J_2)^T]B_2^T = 0,$$

and letting

$$\Theta_2 = \varrho_2^{-2}(H_2J_2)\Sigma_2(H_2J_2)^T, \varrho_2 = k_2a_{32}\sigma_2,$$

algebraic Eq (2.43) is converted as

$$G_0^2 + B_2\Theta_2 + \Theta_2B_2^T = 0, \tag{2.45}$$

with

$$\Theta_2 = \frac{1}{2(q_1q_2 - q_3)}\begin{pmatrix} q_2 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & \frac{q_1}{q_3} \end{pmatrix}.$$

In other words, $\Sigma_2 = \varrho_2^2(H_2J_2)^{-1}\Theta_2[(H_2J_2)^T]^{-1}$.

Case 2.2. If $k_2 = 0$, then we select $\hat{H}_2$ and let $\hat{B}_2 = \hat{H}_2A_2\hat{H}_2^{-1}$ with

$$\hat{H}_2 = \begin{pmatrix} a_{32} & -a_{33} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \hat{B}_2 = \begin{pmatrix} -\omega_1 & -\omega_2 & -\omega_3 \\ 1 & 0 & 0 \\ 0 & 0 & -a_{11} \end{pmatrix},$$

where

$$\omega_1 = a_{33}, \omega_2 = a_{12}a_{21} - a_{23}a_{32}, \omega_3 = -a_{21}a_{32}.$$

One can equivalently transform (2.43) into

$$(\hat{H}_2J_2)M_2^2(\hat{H}_2J_2)^T + \hat{B}_2[(\hat{H}_2J_2)\Sigma_2(\hat{H}_2J_2)^T] + [(\hat{H}_2J_2)\Sigma_2(\hat{H}_2J_2)^T]\hat{B}_2^T = 0,$$

letting

$$\hat{\Theta}_2 = \hat{\varrho}_2^{-2}(\hat{H}_2J_2)\Sigma_2(\hat{H}_2J_2)^T, \hat{\varrho}_2 = a_{32}\sigma_2,$$

which by Lemma 5 is simplified as

$$G_0^2 + \hat{B}_2\hat{\Theta}_2 + \hat{\Theta}_2\hat{B}_2^T = 0, \tag{2.46}$$

with

$$\hat{\Theta}_2 = \text{diag}\Big\{\frac{1}{2\omega_1}, \frac{1}{2\omega_1\omega_2}, 0\Big\}.$$

Therefore, $\Sigma_2 = \hat{\varrho}_2^2(\hat{H}_2J_2)^{-1}\hat{\Theta}_2[(\hat{H}_2J_2)^T]^{-1}$.

Step 3. Let us consider the algebraic equation

$$M_3^2 + A\Sigma_3 + \Sigma_3A^T = 0, \tag{2.47}$$

and we select J_3 and let $A_3 = J_3 A J_3^{-1}$ with

$$J_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} -a_{33} & a_{32} & 0 \\ a_{23} & 0 & a_{21} \\ 0 & -a_{12} & -a_{11} \end{pmatrix}.$$

We find H_3 such that $B_3 = H_3 A_3 H_3^{-1}$ with

$$H_3 = \begin{pmatrix} -a_{12}a_{23} & a_{11}a_{12} & a_{11}^2 - a_{12}a_{21} \\ 0 & -a_{12} & -a_{11} \\ 0 & 0 & 1 \end{pmatrix}, B_3 = \begin{pmatrix} -s_1 & -s_2 & -s_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{2.48}$$

where

$$\begin{aligned} s_1 &= a_{11} + a_{33}, \\ s_2 &= a_{11}a_{33} + a_{12}a_{21} - a_{23}a_{32}, \\ s_3 &= a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}. \end{aligned}$$

So, (2.47) is equivalently transformed into

$$(H_3J_3)M_3^2(H_3J_3)^T + B_3[(H_3J_3)\Sigma_3(H_3J_3)^T] + [(H_3J_3)\Sigma_3(H_3J_3)^T]B_3^T = 0,$$

and letting

$$\Theta_3 = \varrho_3^{-2}(H_3J_3)\Sigma_3(H_3J_3)^T, \varrho_3 = a_{12}a_{23}\sigma_3,$$

algebraic Eq (2.47) is converted as

$$G_0^2 + B_3\Theta_3 + \Theta_3 B_3^T = 0, \tag{2.49}$$

with

$$\Theta_3 = \frac{1}{2(s_1s_2 - s_3)} \begin{pmatrix} s_2 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & \frac{s_1}{s_3} \end{pmatrix}.$$

In other words, $\Sigma_3 = \varrho_3^2(H_3J_3)^{-1}\Theta_3[(H_3J_3)^T]^{-1}$.

3. Examples and numerical experiments

We assume that the Markov chain $m(t)$ takes values in the state space $\mathbb{S} = \{1, 2\}$ with the generator

$$\Gamma = \begin{pmatrix} -0.80 & 0.80 \\ 0.20 & -0.20 \end{pmatrix}.$$

The initial value is $(S(0), U(0), T(0)) = (0.70, 0.50, 0.40)$, and the unique stationary distribution of $m(t)$ is $\pi = (\pi_1, \pi_2) = (0.20, 0.80)$, respectively. We next apply two methods to simulate the sample paths of model (1).

Milstein's higher order method (MHOM). The discretization equations of model (1.1) by MHOM in [46] are written as follows:

$$\begin{aligned} S_{i+1} =& S_i + (\Lambda(k) - \beta_1(k)S_iU_i - \mu(k)S_i)\,\Delta t + \sigma_1(k)S_i\sqrt{\Delta t}v_{k,i} \\ &+ 0.5\sigma_1^2(k)S_i^2(v_{k,i}^2 - 1)\Delta t, \\ U_{i+1} =& U_i + (\beta_1(k)S_iU_i + \beta_2(k)U_iT_i - (\mu(k) + p(k) + \delta_1(k))\,U_i)\Delta t \\ &+ \sigma_2(k)U_i\sqrt{\Delta t}v_{k,i} + 0.5\sigma_2^2(k)U_i^2(v_{k,i}^2 - 1)\Delta t, \\ T_{i+1} =& T_i + (p(k) - \beta_2(k)U_iT_i - (\mu(k) + \delta_2(k))\,T_i)\,\Delta t + \sigma_3(k)T_i\sqrt{\Delta t}v_{k,i} \\ &+ 0.5\sigma_3^2(k)T_i^2(v_{k,i}^2 - 1)\Delta t, \quad i = 0, 1, 2, \cdots. \end{aligned} \tag{3.1}$$

Partially truncated Euler-Maruyama method (PTEMM). The PTEMM in [47] is written as follows:

$$\Delta t = 10^{-4},\ \ h(\Delta t) = \Delta t^{-\frac{1}{3}},\ \ u^{-1}(r) = \sqrt{\frac{r}{3}},\ \ X_c = \frac{\sqrt{3}\times 10^{\frac{2}{3}}}{3\sqrt{S_i^2 + U_i^2 + T_i^2}},$$

and the discretization equations of model (1.1) are written in (3.2), so the verifications of assumptions in [48] are straightforward

$$\begin{aligned} S_{i+1} =& S_i + (\Lambda(k) - f_{1\Delta t,i} - \mu(k)S_i)\,\Delta t + g_{1\Delta t,i}\sqrt{\Delta t}v_{k,i}, \\ U_{i+1} =& U_i + (f_{2\Delta t,i} - (\mu(k) + p(k) + \delta_1(k))\,U_i)\Delta t + g_{2\Delta t,i}\sqrt{\Delta t}v_{k,i}, \\ T_{i+1} =& T_i + (p(k) - f_{3\Delta t,i} - (\mu(k) + \delta_2(k))\,T_i)\,\Delta t + g_{3\Delta t,i}\sqrt{\Delta t}v_{k,i}, \end{aligned} \tag{3.2}$$

where $i = 0, 1, 2, \cdots$ and

$$\begin{aligned} &f_{1\Delta t,i} = (1 \wedge X_c)\beta_1(k)S_iU_i, \quad f_{2\Delta t,i} = (1 \wedge X_c)(\beta_1(k)S_iU_i + \beta_2(k)U_iT_i), \\ &f_{3\Delta t,i} = (1 \wedge X_c)\beta_2(k)U_iT_i, \quad g_{1\Delta t,i} = (1 \wedge X_c)\sigma_1(k)S_i, \\ &g_{2\Delta t,i} = (1 \wedge X_c)\sigma_2(k)U_i, \quad g_{3\Delta t,i} = (1 \wedge X_c)\sigma_3(k)T_i. \end{aligned}$$

$v_{k,i}$ are the Gaussian random variables, which follow the standard normal distribution $\mathcal{N}(0, 1)$. Next, we use PTEMM to simulate the figures in Examples 1–3.

Table 1. Values of parameters to model (1.1).

Group	k	$\Lambda(k)$	$p(k)$	$\mu(k)$	$\delta_1(k)$	$\delta_2(k)$	$\beta_1(k)$	$\beta_2(k)$	$\sigma_1^2(k)$	$\sigma_2^2(k)$	$\sigma_3^2(k)$
(I)	1	0.60	0.65	0.35	0.20	0.25	0.25	0.35	0.200	0.450	0.100
	2	0.40	0.55	0.25	0.10	0.20	0.15	0.25	0.100	0.350	0.050
(II)	1	0.60	0.65	0.35	0.20	0.25	0.25	0.35	0.200	0.490	0.100
	2	0.40	0.55	0.25	0.10	0.20	0.15	0.25	0.100	0.400	0.050
(III)	1	0.60	0.30	0.25	0.20	0.25	0.55	0.65	0.002	0.003	0.003
	2	0.40	0.20	0.15	0.10	0.20	0.45	0.55	0.001	0.001	0.002

Example 1 We choose (I) and (II) of Table 1 to simulate the extinction in Theorem 2. By (I), we obtain

$$\hat{\mu} = 0.25 > 0.225 = \frac{1}{2}(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2), \frac{\max\{\check{\beta}_1, \check{\beta}_2\}}{\hat{\mu}}\check{\Lambda} - \Big(\hat{p} + \hat{\mu} + \hat{\delta}_1 + \frac{1}{2}\hat{\sigma}_2^2\Big) = -0.235 < 0.$$

By (II), we derive

$$\hat{\mu} = 0.25 > 0.245 = \frac{1}{2}(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2), \frac{\max\{\check{\beta}_1, \check{\beta}_2\}}{\hat{\mu}}\check{\Lambda} - \left(\hat{p} + \hat{\mu} + \hat{\delta}_1 + \frac{1}{2}\hat{\sigma}_2^2\right) = -0.260 < 0.$$

Compare the trajectories of solutions under conditions (I) and (II), and the time spent in Figure 2 under (II) is shorter than that in Figure 1 under (I) when the intensities of the white noises increase.

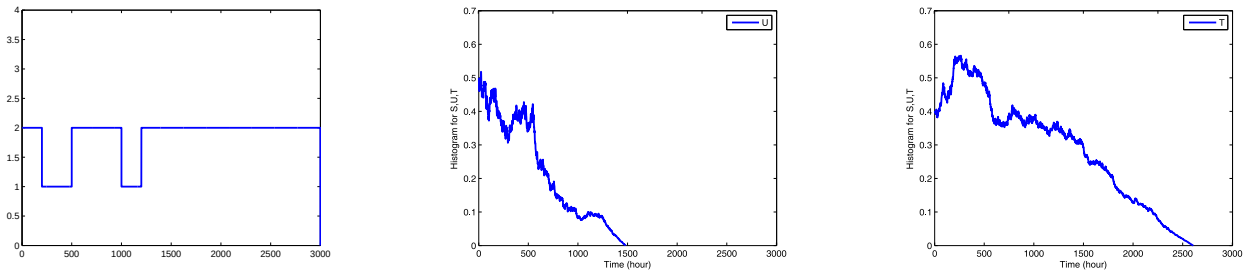

Figure 1. The extinction of the untreated drug addicts to model (1.1) under (I) with initial conditions $(S(0), U(0), T(0)) = (0.70, 0.50, 0.40)$ and $\sigma_1^2(1) = 0.2, \sigma_1^2(2) = 0.1, \sigma_2^2(1) = 0.45, \sigma_2^2(2) = 0.35, \sigma_3^2(1) = 0.1, \sigma_3^2(2) = 0.05$.

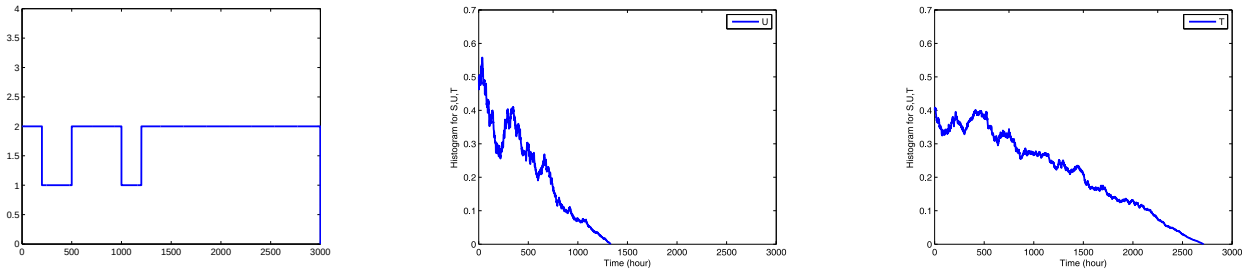

Figure 2. The extinction of the untreated drug addicts to model (1.1) under (II) with initial conditions $(S(0), U(0), T(0)) = (0.70, 0.50, 0.40)$ and $\sigma_1^2(1) = 0.2, \sigma_1^2(2) = 0.1, \sigma_2^2(1) = 0.49, \sigma_2^2(2) = 0.4, \sigma_3^2(1) = 0.1, \sigma_3^2(2) = 0.05$.

Example 2 We choose (III) of Table 1 to present the results in Theorem 3. In fact, the following condition is valid:

$$R_0^s = \sum_{k \in S} \pi_k R_{0k} = 0.707 > 0.$$

As shown in Figure 3, the densities of the susceptible, the untreated drug addicts, and the drug addicts under treatment are stationary over time. The related simulations are demonstrated by MHOM in the middle and by PTEMM on the right. Moreover, for 50000 sample paths in total, the distributions of frequency for the solution of model (1.1) are carried out in Figure 4.

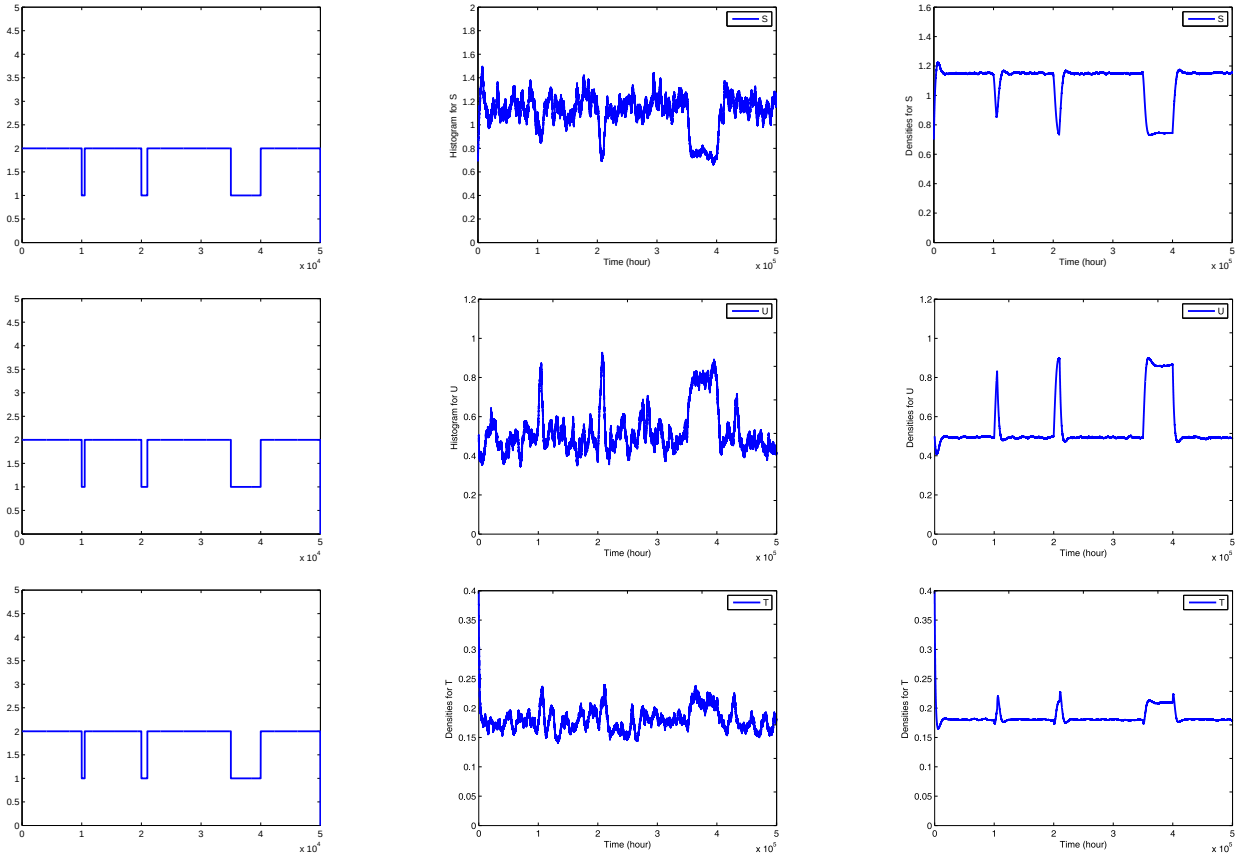

Figure 3. The stationary distributions with same Markov chain (left) under MHOM (middle) and PTEMM (right) respectively.

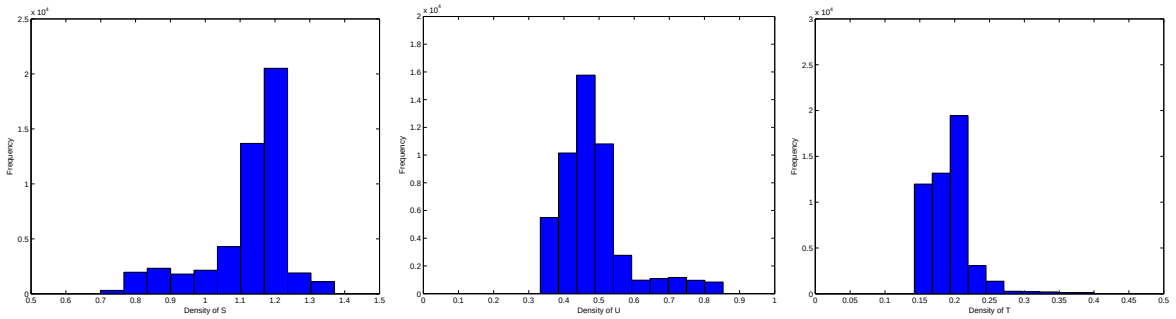

Figure 4. Histogram of $S(t)$, $U(t)$, $T(t)$ to model (1) with 50000 sample paths.

Example 3 We choose the data in Table 2 to verify the conditions of Theorem 4 and Theorem 5. By (IV), the conditions

$$1 + \frac{m_1 m_2}{m_1 p - \beta_1 \Lambda} = 0.618 > 0, \Lambda\beta_1^2 - m_3\beta_2 p = 0.043 > 0$$

hold, we derive the equilibrium point $P^* = (1.078, 0.861, 0.407)$ by Theorem 4. Meanwhile, the stochastic persistence of density function of model (2.20) is demonstrated in Figure 5.

Table 2. Values of parameters to model (2.20).

Group	Λ	p	μ	δ_1	δ_2	β_1	β_2	σ_1^2	σ_2^2	σ_3^2
(IV)	0.60	0.35	0.15	0.125	0.115	0.400	0.532	0.125	0.045	0.035
(V)	0.60	0.35	0.05	0.150	0.055	0.400	0.380	0.170	0.080	0.044
(VI)	0.60	0.35	0.17	0.050	0.070	0.261	0.450	0.188	0.045	0.035

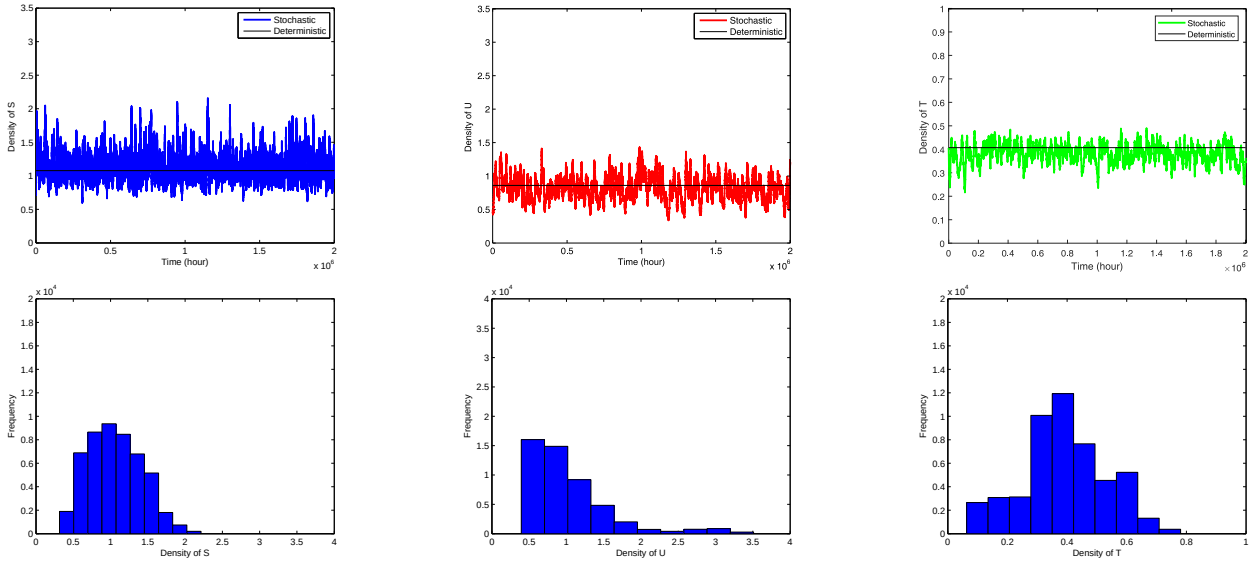

Figure 5. Persistence and density function of model (2.20) around (1.078, 0.861, 0.407).

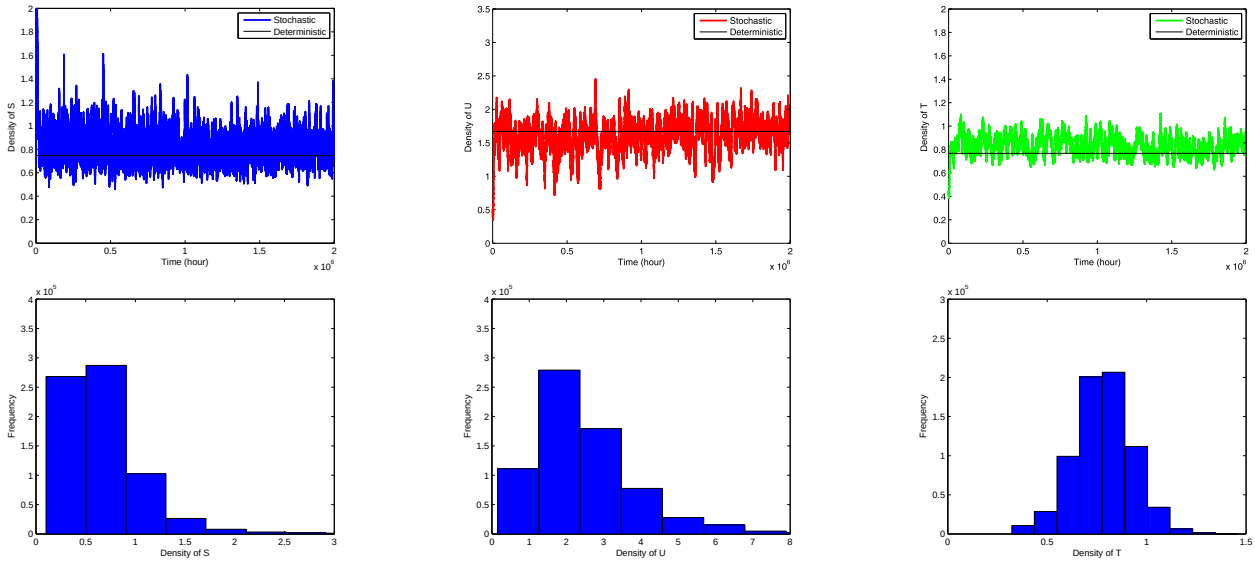

Figure 6. Persistence and density function of model (2.20) around (0.746,1.670,0.768).

Or, we take parameter (V) to compute the following conditions

$$\beta_1\Lambda - m_1 p - m_1 m_2 = 0.105 > 0,\ \Lambda\beta_1^2 - m_3\beta_2 p = 0.106 > 0$$

then, the equilibrium point $P^* = (0.746, 1.670, 0.768)$ is followed. Further, the stochastic persistence of density function of model (2.20) is shown in Figure 6. Or, by selecting parameter (VI), the following conditions

$$(\beta_2 m_1 - \beta_1 m_3)(m_2 + p) + \beta_2(m_1 p - \beta_1 \Lambda) = 0.160 > 0, \Lambda\beta_1^2 - m_3\beta_2 p = 0.079 > 0$$

hold, we obtain the positive equilibrium point $P^* = (0.967, 0.845, 0.679)$. So, the same dynamical properties appear, and we omit this case hereby.

4. Conclusions and discussion

Heroin is an addictive drug made from the various opium poppy plants around the world. The price and spreading of heroin depend on the flowering period (usually May–July for a year) and the fruiting period (usually June–August for a year). So, we give an SUT epidemic model with regime switching to describe the flowering period and fruiting period of opium poppy plants in this paper. We are motivated by the switching between flowering period and fruiting period of opium poppy plants in years, and the recent contributions [17, 30, 31] on epidemic models. We focus on the survival analysis of switching model (1.1) and its probability density function of constant model (2.20) for investigating their long-time dynamical properties.

For the switching SUT epidemic model (1.1), the existence and uniqueness is first derived with probability one in Theorem 1 by contradiction and stochastic analysis. Further, Theorem 2, Figures 1 and 2 verify the extinction of the switching SUT model under moderate conditions in theoretical and numerical aspects. The simulations therein also reveal that the larger intensities of the white noises make the time of extinction earlier. As a consequence of theoretical investigation, we derive the important index $R_0^s > 0$ of the existence and uniqueness of the ergodic stationary distribution in Theorem 3. The corresponding sample paths and histogram frequencies are demonstrated in Figures 3 and 4, respectively, in which Milstein's higher order method and partially truncated Euler-Maruyama method both verify well under the same Markovian chain.

For the constant SUT epidemic model (2.20), we aim at the existence of the positive equilibrium point in Theorem 4 and the existence of probability density function in Theorem 5, respectively. One of three types of sufficient conditions is required for determining a positive equilibrium point, and details could be found in Example 3. The sample paths of model (2.20) under distinct positive equilibrium points are demonstrated in Figure 5. Further, the expression of probability density function around the positive equilibrium point is obtained in Theorem 5 after we prove that coefficient matrix A is a Hurwitz matrix and diffusion matrix Σ is positive definite by using the Fokker-Planck equation.

Acknowledgments

The research is supported by the Natural Science Foundation of Fujian Province of China (2021J01621), Special Projects of the Central Government Guiding Local Science and Technology Development (2021L3018) and Education and Research Project for Middle and Young Teachers in Fujian Province (JAT220307).

Conflict of interest

The authors declare there is no conflict of interest.

References

1. S. A. Ochoa-Orozco, J. C. Gutiérrez-Segura, A. M. Coral-Leiton, E. A. Trejos-Orozco, I. Gutirrez-Sanjun, J. D. Carvajal-Guevara, Chasing the dragon: A fatal case report of toxic leucoencé phalopathie due to inhaled heroin, *Rev. Colomb. Psiquiat.*, **49** (2020), 289–292. https://doi.org/10.1016/j.rcp.2019.06.003
2. National, Institute on drug abuse. Available from: https://www.drugabuse.gov/drug-topics/opioids
3. E. White, C. Comiskey, Heroin epidemics, treatment and ODE modelling, *Math. Biosci.*, **208** (2007), 312–324. https://doi.org/10.1016/j.mbs.2006.10.008
4. S. Djilali, T. M. Touaoula, M. S. El-Hadi, A heroin epidemic model: Very general non-linear incidence treat-age and global stability, *Acta. Appl. Math.*, **152** (2017), 171–194. https://doi.org/10.1007/s10440-017-0117-2
5. J. Wang, J. Wang, T. Kuniya, Analysis of an age-structured multi-group heroin epidemic model, *Appl. Math. Comput.*, **347** (2019), 78–100. https://doi.org/10.1016/j.amc.2018.11.012
6. X. Duan, X. Li, M. Martcheva, Qualitative analysis on a diffusive age-structured heroin transmission model, *Nonlinear Anal.-Real World Appl.*, **54** (2020), 103105. https://doi.org/10.1016/j.nonrwa.2020.103105
7. J. Liu, T. Zhang, Global behaviour of a heroin epidemic model with distributed delays, *Appl. Math. Lett.*, **24** (2011), 1685–1692. https://doi.org/10.1016/j.aml.2011.04.019
8. G. Huang, A. Liu, A note on global stability for a heroin epidemic model with distributed delay, *Appl. Math. Lett.*, **26** (2013), 687–691. https://doi.org/10.1016/j.aml.2013.01.010
9. X. Abdurahman, Z. Teng, L. Zhang, Global dynamics in a heroin epidemic model with difff-ferent conscious stages and two distributed delays, *Int. J. Biomath.*, **12** (2019), 1950038. https://doi.org/10.1142/S1793524519500384
10. M. Ma, S. Liu, J. Li, Bifurcation of a heroin model with nonlinear incidence rate, *Nonlinear Dyn.*, **88** (2017), 555–565. https://doi.org/10.1007/s11071-016-3260-9
11. L. Chen, F. Wei, Study on a susceptible-exposed-infected-recovered model with nonlinear incidence rate, *Adv. Differ. Equations*, **2020** (2020), 206. https://doi.org/10.1186/s13662-020-02662-5
12. S. Djilali, S. Bentout, T. M. Touaoula, A. Tridane, S. Kumar, Global behavior of Heroin epidemic model with time distributed delay and nonlinear incidence function, *Results Phys.*, **31** (2021), 104953. https://doi.org/10.1016/j.rinp.2021.104953
13. S. Bentout, Y. Chen, S. Djilali, Global dynamics of an SEIR model with two age structures and a nonlinear incidence, *Acta. Appl. Math.*, **171** (2021), 1–27. https://doi.org/10.1007/s10440-020-00369-z

14. S. Liu, L. Zhang, Y. Xing, Dynamics of a stochastic heroin epidemic model, *J. Comput. Appl. Math.*, **351** (2019), 260–269. https://doi.org/10.1016/j.cam.2018.11.005

15. S. Liu, Z. Liang, X. Zhang, A. Li, Dynamics of a stochastic heroin epidemic model-with bilinear incidence and varying population size, *Int. J. Biomath.*, **12** (2019), 1950005. https://doi.org/10.1142/S1793524519500050

16. Y. Wei, Q. Yang, G. Li, Dynamics of the stochastically perturbed heroin epidemic model under non-degenerate noises, *Physica A*, **526** (2019), 120914. https://doi.org/10.1016/j.physa.2019.04.150

17. F. Wei, H. Jiang, Q. Zhu, Dynamical behaviors of a heroin population model with standard incidence rates between distinct patches, *J. Frankl. Inst.-Eng. Appl. Math.*, **358** (2021), 4994–5013. https://doi.org/10.1016/j.jfranklin.2021.04.024

18. J. Liu, S. Wang, Dynamics in a stochastic heroin model with seasonal variation, *Phys. A*, **532** (2019), 121873. https://doi.org/10.1016/j.physa.2019.121873

19. M. Jovanović, V. Jovanović, Stability of stochastic heroin model with two distributed delays, *Discrete Cont. Dyn. Sys.-B*, **25** (2020), 2407–2432. https://doi.org/10.3934/dcdsb.2020016

20. Q. Luo, X. Mao, Stochastic population dynamics under regime switching, *J. Math. Anal. Appl.*, **334** (2007), 69–84. https://doi.org/10.1016/j.jmaa.2006.12.032

21. M. Slatkin, The dynamics of a population in a Markovian environment, *Ecology*, **59** (1978), 249–256. https://doi.org/10.2307/1936370

22. X. Zou, K. Wang, The protection zone for biological population in random environment, *Math. Method. Appl. Sci.*, **36** (2013), 707–721. https://doi.org/10.1002/mma.2621

23. S. He, F. Liu, Optimal finite-time passive controller design for uncertain nonlinear Markovian jumping systems, *J. Frankl. Inst.-Eng. Appl. Math.*, **351** (2014), 3782–3796. https://doi.org/10.1016/j.jfranklin.2013.03.006

24. X. Zhang, D. Jiang, A. Alsaedi, T. Hayat, Stationary distribution of stochastic SIS epidemic model with vaccination under regime switching, *Appl. Math. Lett.*, **59** (2016), 87–93. https://doi.org/10.1016/j.aml.2016.03.010

25. D. Greenhalgh, Y. Liang, X. Mao, Modelling the effect of telegraph noise in the SIRS epidemic model using Markovian switching, *Phys. A*, **462** (2016), 684–704. https://doi.org/10.1016/j.physa.2016.06.125

26. Q. Lin, L. Chen, C. Wen, F. Wei, Asymptotic properties of a stochastic Lotka-Volterra model with infinite delay and regime switching, *Adv. Differ. Equations*, **2018** (2018), 155. https://doi.org/10.1186/s13662-018-1609-8

27. Q. Liu, D. Jiang, N. Shi, Threshold behavior in a stochastic SIQR epidemic model with standard incidence and regime switching, *Appl. Math. Comput.*, **316** (2018), 310–325. https://doi.org/27.10.1016/j.amc.2017.08.042

28. H. Wang, D. Jiang, T. Hayat, A. Alsaedi, A. Bashir, Stationary distribution of stochastic NP ecological model under regime switching, *Phys. A*, **549** (2020), 124064. https://doi.org/10.1016/j.physa.2019.124064

29. N. D. Phu, D. O'Regan, T. D. Tuong, Longtime characterization for the general stochastic epidemic SIS model under regime-switching, *Nonlinear Anal.-Hybrid Syst.*, **38** (2020), 100951. https://doi.org/10.1016/j.nahs.2020.100951

30. X. Zhang, H. Peng, Stationary distribution of a stochastic cholera epidemic model with vaccination under regime switching, *Appl. Math. Lett.*, **102** (2020), 106095. https://doi.org/10.1016/j.aml.2019.106095

31. B. Zhou, B. Han, D. Jiang, T. Hayat, A. Alsaedi, Ergodic stationary distribution and extinction of a hybrid stochastic SEQIHR epidemic model with media coverage, quarantine strategies and pre-existing immunity under discrete Markov switching, *Appl. Math. Comput.*, **410** (2021), 126388. https://doi.org/10.1016/j.amc.2021.126388

32. J. Xu, Y. Wang, Z. Cao, Dynamics of a stochastic SIRS epidemic model with standard incidence under regime switching, *Int. J. Biomath.*, **15** (2022), 2150074. https://doi.org/10.1142/S1793524521500741

33. G. Li, Q. Yang, Y. Wei, Dynamics of stochastic heroin epidemic model with Levy jumps, *J. Appl. Anal. Comput.*, **8** (2018), 998–1010. https://doi.org/10.11948/2018.99

34. F. Wei, C. Wang, Survival analysis of a single-species population model with fluctuations and migrations between patches, *Appl. Math. Model.*, **81** (2020), 113–127. https://doi.org/10.1016/j.apm.2019.12.023

35. F. Wei, L. Chen, Psychological effect on single-species population models in a polluted environment, *Math. Biosci.*, **290** (2017), 22–30. https://doi.org/10.1016/j.mbs.2017.05.011

36. L. Chen, F. Wei, Persistence and distribution of a stochastic susceptible-infected-removed epidemic model with varying population size, *Phys. A*, **483** (2017), 386–397. https://doi.org/10.1016/j.physa.2017.04.114

37. F. Wei, R. Xue, Stability and extinction of SEIR epidemic models with generalized nonlinear incidence, *Math. Comput. Simul.*, **170** (2020), 1–15. https://doi.org/10.1016/j.matcom.2018.09.029

38. R. Lu, F. Wei, Persistence and extinction for an age-structured stochastic SVIR epidemic model with generalized nonlinear incidence rate, *Phys. A*, **513** (2019), 572–587. https://doi.org/10.1016/j.physa.2018.09.016

39. F. Wei, J. Liu, Long-time behavior of a stochastic epidemic model with varying population size, *Phys. A*, **470** (2017), 146–153. https://doi.org/10.1016/j.physa.2016.11.031

40. X. Mao, C. Yuan, *Stochastic Differential Equations with Markovian Switching*, Imperial College Press, London, 2006. https://doi.org/10.1142/p473

41. H. Peng, X. Zhang, Dynamics of a stochastic rabies epidemic model with Markovian switching, *Int. J. Biomath.*, **14** (2021), 2150032. https://doi.org/10.1142/S1793524521500327

42. B. Zhou, X. Zhang, D. Jiang, Dynamics and density function analysis of a stochastic SVI epidemic model with half saturated incidence rate, *Chaos Solitons Fract.*, **137** (2020), 109865. https://doi.org/10.1016/j.chaos.2020.109865

43. C. W. Gardiner, *Handbook of stochastic methods for physics, chemistry and the natural sciences*, Springer, Berlin, 1986. https://doi.org/10.1007/978-3-662-02452-2

44. H. Roozen, An asymptotic solution to a two-dimensional exit problem arising in population dynamics, *SIAM J. Appl. Math.*, **49** (1989), 1793–1810. https://doi.org/10.2307/2101938

45. D. Li, F. Wei, X. Mao, Stationary distribution and density function of a stochastic SVIR epidemic model, *J. Frankl. Inst.-Eng. Appl. Math.*, **359** (2022), 9422–9449. https://doi.org/10.1016/j.jfranklin.2022.09.026

46. D. J. Higham, An algorithmic introduction to numerical simulation of stochastic differential equations, *SIAM Rev.*, **43** (2001), 525–546. https://doi.org/10.1137/S0036144500378302

47. X. Mao, F. Wei, T. Wiriyakraikul, Positivity preserving truncated Euler-Maruyama method for stochastic Lotka-Volterra competition model, *J. Comput. Appl. Math.*, **394** (2021), 113566. https://doi.org/10.1016/j.cam.2021.113566

48. Q. Guo, W. Liu, X. Mao, R. Yue, The partially truncated Euler-Maruyama method and its stability and boundedness, *Appl. Numer. Math.*, **115** (2017), 235–251. https://doi.org/10.1016/j.apnum.2017.01.010

AIMS Press

© 2023 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0)