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Research article

$\mathcal{L}_2 - \mathcal{L}_\infty$ control for memristive NNs with non-necessarily differentiable time-varying delay

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Abstract: This paper investigates $\mathcal{L}_2-\mathcal{L}_\infty$ control for memristive neural networks (MNNs) with a non-necessarily differentiable time-varying delay. The objective is to design an output-feedback controller to ensure the $\mathcal{L}_2-\mathcal{L}_\infty$ stability of the considered MNN. A criterion on the $\mathcal{L}_2-\mathcal{L}_\infty$ stability is proposed using a Lyapunov functional, the Bessel-Legendre inequality, and the convex combination inequality. Then, a linear matrix inequalities-based design scheme for the required output-feedback controller is developed by decoupling nonlinear terms. Finally, two examples are presented to verify the proposed $\mathcal{L}_2 - \mathcal{L}_\infty$ stability criterion and design method.

Keywords: memristive neural network (MNNs); $\mathcal{L}_2 - \mathcal{L}_\infty$ control; asymptotic stability; time-varying delay

1. Introduction

Over the past few decades, stability analysis and controller design for memristive neural networks (MNNs) with delays have drawn extensive attention from the automation community. Various noteworthy findings have been reported in the literature. For instance, Zhang et al. [1] explored the stabilizability of delayed complex-valued MNNs and proposed a new memory-based controller with distinguishable real-imaginary parts that can achieve state convergence to an equilibrium point in finite time. Wu and Zeng [2] proposed an optimal control law to minimize the general cost function and derived the required gain control matrix within the framework of Filippov's solution to achieve exponential stabilization of MNNs. In [3], Li et al. addressed the control issue for quaternion-valued fractional-order fuzzy MNNs by integrating quaternion algebra into fractional-order MNNs, where the states and connection weights were treated as quaternion values. Recently, several researchers have investigated MNNs with time-varying delays (TVDs) and reported important findings [4–8]. However, these studies imposed a condition of differentiability on the TVDs, which might be overly restrictive.

In dynamic systems, disturbances are likely to occur, and they can have a significant impact on the system's performance. To mitigate these effects, various robust control methods have been developed [9–14]. For MNNs, Cao et al. [15] discussed synchronization of MNNs with uncertain parameters and topologies, proposing an adaptive robust controller strategy. Ghous et al. [16] first proposed conditions on the $\mathcal{H}_\infty$ stability analysis, and then developed a state-feedback-based control scheme. Yan et al. [17] explored $\mathcal{H}_\infty$ control of MNNs with dynamic quantization by constructing two different time-dependent bilateral cyclic functions using differential inclusions, and introduced a design methodology for a quantized controller relied on linear matrix inequalities (LMIs). In recent years, numerous results of $\mathcal{L}_2-\mathcal{L}_\infty$ control for various dynamic systems have been reported. Unlike $\mathcal{H}_\infty$ control, $\mathcal{L}_2-\mathcal{L}_\infty$ control can ensure that the energy-to-peak (ETP) gain from disturbance to the output signal is less than a predetermined threshold for all energy-bounded disturbances. Nevertheless, as far as we know, there are no reports on $\mathcal{L}_2-\mathcal{L}_\infty$ control studies for MNNs, not to mention MNNs with TVDs.

Motivated by the above observations, this paper investigates the problem of $\mathcal{L}_2-\mathcal{L}_\infty$ control for MNNs with TVD. In contrast to the TVDs discussed in previous literature, such as [4–8], in this study, the delay factor under consideration is allowed to be non-differentiable. Due to the difficulty in obtaining the full state information of a dynamic system, the controller scheme employed in this work is based on output-feedback, as in [18–22]. The main objective is to design an output-feedback controller that ensures the $\mathcal{L}_2-\mathcal{L}_\infty$ stability of the closed-loop MNN (i.e., guaranteeing the asymptotic stability of the MNNs in the absence of disturbance and ensuring that the ETP gain from the disturbance to the output signal is less than a prescribed $\mathcal{L}_2-\mathcal{L}_\infty$ disturbance-suppression level when the disturbance is energy-bounded) [23]. We first propose a criterion for the $\mathcal{L}_2-\mathcal{L}_\infty$ stability by using a Lyapunov functional, the Bessel-Legendre inequality (BLI), and the convex combination inequality (CCI). Subsequently, we develop a design scheme for the required output-feedback controller using a nonlinear decoupling technique. The scheme is based on LMIs that can be easily verified using popular mathematical computing software. Finally, we apply two examples to validate the proposed $\mathcal{L}_2-\mathcal{L}_\infty$ stability criterion and design method.

2. Preliminaries

This paper adopts the same notations as [24] unless explicitly stated otherwise. The time-delayed MNN we consider is modeled as

$$\begin{aligned}
\dot{\phi}(t) &= -D\phi(t) + A(\phi(t)\sigma(\phi(t)) + A_d(\phi(t))\sigma(\phi(t-\rho(t))) + Bu(t) + Ew(t),\\
u(t) &= Ky(t),\\
y(t) &= C\phi(t),
\end{aligned} \tag{2.1}$$

where $\phi(t) = \mathrm{col}\{\phi_1(t), \phi_2(t), \ldots, \phi_n(t)\} \in \mathbb{R}^n$ denotes the state vector, $\rho(t)$ is a TVD satisfying $\rho_1 \le \rho(t) \le \rho_2$, and ρ_1, ρ_2 are constants. As in [25–29], the time delay under consideration is allowed to be non-differentiable. The activation function for a neuron is defined as $\sigma(\cdot) = \mathrm{col}\{\sigma_1(\cdot), \sigma_2(\cdot), \ldots, \sigma_n(\cdot)\} \in \mathbb{R}^n$ with $\sigma_j(0) = 0$, $j \in \{1, \ldots, n\}$. The positive self-feedback matrix is indicated by $D = \mathrm{diag}\{d_1, d_2, \ldots, d_n\}$, and $A(\phi(t)) = (a_{ij}(\phi_i(t)))_{n\times n}$ and $A_d(\phi(t)) = (a_{dij}(\phi_i(t)))_{n\times n}$ represent the memristive connection weights. Unlike [30–33], the controller

$u(t) \in \mathbb{R}^m$ to be designed is based on output-feedback, which is easier to implement. $w(t) \in \mathbb{R}^q$ is the disturbance that belongs to $L_2[0, \infty)$; $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{n\times q}$, and $E \in \mathbb{R}^{v\times n}$ are known system matrices. As in [34, 35], it is postulated that the state-dependent parameters related to (2.1) adhere to the subsequent condition:

$$a_{ij}(\phi_i(t)) = \begin{cases} \breve{a}_{ij}, & |\phi_i(t)| \le T_i, \\ \hat{a}_{ij}, & |\phi_i(t)| > T_i, \quad i, j \in \nu. \end{cases}$$
$$a_{dij}(\phi_i(t)) = \begin{cases} \breve{a}_{dij}, & |\phi_i(t)| \le T_i, \\ \hat{a}_{dij}, & |\phi_i(t)| > T_i, \quad i, j \in \nu, \end{cases}$$

where $T_i > 0$ represents the switching jumps, and $\breve{a}_{ij}$, $\hat{a}_{ij}$, $\breve{a}_{dij}$, and $\hat{a}_{dij}$, $i, j \in \nu$ are known constants. It is evident that the MNN model (2.1) can be regarded as a state-dependent switched system.

Let us define

$$\chi^a_{ij}(\phi_i(t)) = \begin{cases} \operatorname{sgn}\left(\breve{a}_{ij} - \hat{a}_{ij}\right), & |\phi_i(t)| \le T_i, \\ -\operatorname{sgn}\left(\breve{a}_{ij} - \hat{a}_{ij}\right), & |\phi_i(t)| > T_i, \end{cases}$$
$$\chi^b_{ij}(\phi_i(t)) = \begin{cases} \operatorname{sgn}\left(\breve{a}_{dij} - \hat{a}_{dij}\right), & |\phi_i(t)| \le T_i, \\ -\operatorname{sgn}\left(\breve{a}_{dij} - \hat{a}_{dij}\right), & |\phi_i(t)| > T_i. \end{cases}$$

The MNN (2.1) is a differential system with state-dependent parameters. The solutions of the system can be interpreted within Filippov's framework. By using differential inclusion and set-valued mapping theory, as in [16], the MNN can be transformed into an uncertain system as follows:

$$\begin{aligned} \dot{\phi}(t) = &- (D - BKC)\phi(t) + (A_0 + A_1\Delta(\phi(t))A_2)\,\sigma(\phi(t)) \\ &+ (A_{d0} + A_{d1}\Delta(\phi(t-\rho(t)))A_{d2})\,\sigma(\phi(t-\rho(t))) + Ew(t), \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} A_0 &= \frac{\left(\breve{a}_{ij}\right)_{n\times n} + \left(\hat{a}_{ij}\right)_{n\times n}}{2},\ A_{d0} = \frac{\left(\breve{a}_{dij}\right)_{n\times n} + \left(\hat{a}_{dij}\right)_{n\times n}}{2}, \\ v^a_{ij} &= \sqrt{\left(\left|\breve{a}_{ij} - \hat{a}_{ij}\right|\right)/2},\ \ v^b_{ij} = \sqrt{\left(\left|\breve{a}_{dij} - \hat{a}_{dij}\right|\right)/2}, \\ A_1 &= (v^a_{11}v_1, \ldots, v^a_{1n}v_1, \ldots, v^a_{n1}v_n, \ldots, v^a_{nn}v_n)_{n\times n^2}, \\ A_{d1} &= \left(v^b_{11}v_1, \ldots, v^b_{1n}v_1, \ldots, v^b_{n1}v_n, \ldots, v^b_{nn}v_n\right)_{n\times n^2}, \\ A_2 &= (v^a_{11}v_1, \ldots, v^a_{1n}v_n, \ldots, v^a_{n1}v_1, \ldots, v^a_{nn}v_n)^T_{n\times n^2}, \\ A_{d2} &= \left(v^b_{11}v_1, \ldots, v^b_{1n}v_n, \ldots, v^b_{n1}v_1, \ldots, v^b_{nn}v_n\right)^T_{n\times n^2}, \\ \Delta(\phi(t)) &\in \operatorname{co}\left[\chi^a(\phi(t))\right],\ \Delta(\phi(t-\rho(t))) \in \operatorname{co}\left[\chi^b(\phi(t))\right], \\ \chi^a(\phi(t)) &= \operatorname{diag}\{\chi^a_{11}(\phi_1(t)), \ldots, \chi^a_{1n}(\phi_1(t)), \ldots, \chi^a_{n1}(\phi_n(t)), \ldots, \chi^a_{nn}(\phi_n(t))\}, \\ \chi^b(\phi(t)) &= \operatorname{diag}\{\chi^b_{11}(\phi_1(t)), \ldots, \chi^b_{1n}(\phi_1(t)), \ldots, \chi^b_{n1}(\phi_n(t)), \ldots, \chi^b_{nn}(\phi_n(t))\} \end{aligned}$$

with co[·] indicating the convex hull, and the vector v_i denoting a column vector with a value of 1 in its i-th entry and 0 in all other entries.

Definition 1. *System (2.2) is said to be $\mathcal{L}_2-\mathcal{L}_\infty$ stable if it achieves asymptotic stability when $w(t)\equiv 0$, and satisfies*

$$\sup_{t\geq 0}\left\{y^T(t)y(t)\right\}\leq \gamma^2\int_0^\infty w^T(\beta)w(\beta)d\beta \tag{2.3}$$

under the zero-initial condition for a predefined constant $\gamma>0$ and all $w(t)\in L_2[0,\infty)$.

Lemma 1. *[36] (BLI) For any given matrix $Q\in\mathbb{S}_+^n$ and function $\phi: C[a,b]\rightarrow\mathbb{R}^n$, the inequality*

$$\int_a^b \dot{\phi}^T(r)Q\dot{\phi}(r)dr\geq\frac{1}{b-a}\Omega^T diag\{Q,3Q,5Q\}\Omega$$

holds, where

$$\Omega=\begin{bmatrix}\phi(b)-\phi(a)\\ \phi(b)+\phi(a)-\frac{2}{b-a}\int_a^b\phi(t)dt\\ \phi(b)-\phi(a))-\frac{6}{b-a}\int_a^b\delta_{a,b}(t)\phi(t)dt\end{bmatrix},$$

$$\delta_{a,b}(t)=2\left(\frac{t-a}{b-a}\right)-1.$$

Lemma 2. *[37] (CCI) For any given matrix $Q\in\mathbb{R}_+^n$, if there exists a matrix $X\in\mathbb{R}^{n\times n}$ such that $\begin{bmatrix}Q & X\\ X^T & Q\end{bmatrix}\geq 0$, then,*

$$\begin{bmatrix}\frac{1}{\alpha}Q & 0\\ 0 & \frac{1}{1-\alpha}Q\end{bmatrix}\geq\begin{bmatrix}Q & X\\ X^T & Q\end{bmatrix},\quad\forall\alpha\in(0,1).$$

Lemma 3. *[38] For any $x,\ y\in\mathbb{R}^n$, scalar $\iota>0$, and positive definite $n\times n$ matrix Υ, the following holds:*

$$2x^{\mathrm{T}}y\leq\frac{1}{\iota}x^{\mathrm{T}}\Upsilon x+\iota y^{\mathrm{T}}\Upsilon^{-1}y.$$

Lemma 4. *(Schur's complement) [39] For any given matrix $S\in\mathbb{S}_+^n$, $S=\begin{bmatrix}Q_a & Q_b\\ * & Q_c\end{bmatrix}<0$ is equivalent to*

$$Q_c<0,\ Q_a-Q_bQ_c^{-1}Q_b^T<0.$$

Lemma 5. *[40] For any natural number N, if there are a scalar $\mu>0$, and matrices Λ, V_i, U_i, $Y_i\,(i=1,\cdots,N)$ such that*

$$\begin{bmatrix}\Lambda & V_1+\mu U_1\cdots V_N+\mu U_N\\ * & \mathrm{diag}\left\{-\mu Y_1-\mu Y_1^T,\cdots,-\mu Y_N-\mu Y_N^T\right\}\end{bmatrix}<0,$$

then we can obtain that

$$\Lambda+\sum_{i=1}^N He\left(V_iY_i^{-1}U_i^T\right)<0.$$

Assumption 1. *[41] There is a positive matrix $F=\mathrm{diag}\{F_1,F_2,\ldots,F_n\}$ such that*

$$\left\|\sigma_j(a)-\sigma_j(b)\right\|\leq F_j\|a-b\|,\quad\forall a,b\in\mathbb{R}. \tag{2.4}$$

Now, the problem we investigate can be stated more precisely as follows: given the MNN (2.1) with TVD that is not necessarily differentiable, we design an output-feedback controller $u(t)=Ky(t)$ to ensure the $\mathcal{L}_2-\mathcal{L}_\infty$ stability of MNN (2.2) as defined in Definition 1.

3. $\mathcal{L}_2 - \mathcal{L}_\infty$ stability analysis

For the sake of clarity in the presentation, the notations listed below will be employed in this section:

$$
\begin{aligned}
\alpha_i &= \begin{bmatrix} 0_{n\times(i-1)n} & I_n & 0_{n\times(18-i)n} \end{bmatrix}, i = 1, \ldots, 18,\\
G_2 &= \begin{bmatrix} \alpha_1^T - \alpha_2^T & \alpha_1^T + \alpha_2^T - 2\alpha_5^T & \alpha_1^T - \alpha_2^T - 6\alpha_6^T \end{bmatrix}^T,\\
G_3 &= \begin{bmatrix} \alpha_2^T - \alpha_3^T & \alpha_2^T + \alpha_3^T - 2\alpha_7^T & \alpha_2^T - \alpha_3^T - 6\alpha_8^T \end{bmatrix}^T,\\
G_4 &= \begin{bmatrix} \alpha_3^T - \alpha_4^T & \alpha_3^T + \alpha_4^T - 2\alpha_9^T & \alpha_3^T - \alpha_4^T - 6\alpha_{10}^T \end{bmatrix}^T,\\
\Gamma &= \begin{bmatrix} G_3^T & G_4^T \end{bmatrix}^T, \quad \phi_t(r) = \phi(t+r), \quad \rho_{12} = \rho_2 - \rho_1,\\
G_0 &= \begin{bmatrix} \alpha_{15}^T & \alpha_1^T - \alpha_2^T & \alpha_1^T + \alpha_2^T - 2\alpha_5^T & \alpha_2^T - \alpha_4^T & \hat{G}_0^T \end{bmatrix}^T,\\
\hat{G}_0 &= \rho_{12}(\alpha_2 + \alpha_4) - 2(\alpha_{11} + \alpha_{13}),\\
G_1(\theta) &= \begin{bmatrix} \alpha_1^T & \rho_1\alpha_5^T & \rho_1\alpha_6^T & \alpha_{11}^T + \alpha_{13}^T & \hat{G}_1^T(\theta) \end{bmatrix}^T,\\
\hat{G}_1(\theta) &= (\rho_2 - \theta)(\alpha_{11} + \alpha_{14}) + (\theta - \rho_1)(\alpha_{12} - \alpha_{13}),\\
g_1(\theta) &= (\theta - \rho_1)\begin{bmatrix} \alpha_7 \\ \alpha_8 \end{bmatrix} - \begin{bmatrix} \alpha_{11} \\ \alpha_{12} \end{bmatrix},\\
g_2(\theta) &= (\rho_2 - \theta)\begin{bmatrix} \alpha_9 \\ \alpha_{10} \end{bmatrix} - \begin{bmatrix} \alpha_{13} \\ \alpha_{14} \end{bmatrix},\\
\hat{\eta}(t) &= \begin{bmatrix} \eta_0(t) & \cdots & \eta_5(t) & \dot{\phi}^T(t) & \sigma^T(\phi(t)) & \sigma^T(\phi(t-\rho(t)) & w^T(t) \end{bmatrix}^T,\\
\eta_0(t) &= \begin{bmatrix} \phi^T(t) & \phi^T(t-\rho_1) & \phi^T(t-\rho(t)) & \phi^T(t-\rho_2) \end{bmatrix}^T,\\
\eta_1(t) &= \frac{1}{\rho_1}\begin{bmatrix} \int_{-\rho_1}^{0} \phi_t^T(r)dr & \int_{-\rho_1}^{0} \delta_1(r)\phi_t^T(r)dr \end{bmatrix}^T,\\
\eta_2(t) &= \frac{1}{\rho(t)-\rho_1}\begin{bmatrix} \int_{-\rho(t)}^{-\rho_1} \phi_t^T(r)dr & \int_{-\rho(t)}^{-\rho_1} \delta_2(r)\phi_t^T(r)dr \end{bmatrix}^T,\\
\eta_3(t) &= \frac{1}{\rho_2-\rho(t)}\begin{bmatrix} \int_{-\rho_2}^{-\rho(t)} \phi_t^T(r)dr & \int_{-\rho_2}^{-\rho(t)} \delta_3(r)\phi_t^T(r)dr \end{bmatrix}^T,\\
\eta_4(t) &= (\rho(t)-\rho_1)\,\eta_2(t), \quad \eta_5(t) = (\rho_2 - \rho(t))\,\eta_3(t),\\
\eta_6(t) &= \begin{bmatrix} \int_{-\rho_2}^{-\rho_1} \phi_t^T(r)dr & \rho_{12}\int_{-\rho_2}^{-\rho_1} \delta_4(r)\phi_t^T(r)dr \end{bmatrix}^T
\end{aligned} \tag{3.1}
$$

where

$$
\begin{aligned}
\delta_1(r) &= 2\frac{r+\rho_1}{\rho_1} - 1,\ \delta_2(r) = 2\frac{r+\rho(t)}{\rho(t)-\rho_1} - 1,\\
\delta_3(r) &= 2\frac{r+\rho_2}{\rho_2-\rho(t)} - 1,\ \delta_4(r) = 2\frac{r+\rho_2}{\rho_{12}} - 1.
\end{aligned}
$$

We can establish the following criterion for MNN (2.2).

Theorem 1. *For given positive scalars ρ_1, ρ_2, if there exist matrices $P = \left(P_{jk}\right)_{5\times5} \in \mathbb{S}_+^{5n}$, $M_1, M_2, Q_1, Q_2 \in \mathbb{S}_+^n$, $W_1, W_2 \in \mathbb{R}^{n\times n}$ $N_1, N_2 \in \mathbb{R}^{18n\times2n}$, positive diagonal matrices*

$T_1, T_2, \Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4$, and a matrix $X \in \mathbb{R}^{3n\times 3n}$ such that

$$\Psi = \begin{bmatrix} \tilde{R}_2 & X \\ X^T & \tilde{R}_2 \end{bmatrix} \geq 0, \tag{3.2}$$

$$C^T C - P_{11} < 0, \tag{3.3}$$

$$\begin{bmatrix} \Theta_i(\theta) & \Xi_1 & \Xi_2 & \Xi_3 & \Xi_4 \\ * & -\Lambda_1 & 0 & 0 & 0 \\ * & * & -\Lambda_3 & 0 & 0 \\ * & * & * & -\Lambda_2 & 0 \\ * & * & * & * & -\Lambda_4 \end{bmatrix} < 0 \tag{3.4}$$

for $i = 1, 2$, and any θ in $\mathbb{R}$,

$$\begin{aligned}
\Theta_i(\theta) &= \Theta_0^i(\theta) + 2\alpha_1^T W_1^T E\alpha_{18} + 2\alpha_{15}^T W_2^T E\alpha_{18} + \alpha_{18}\gamma^2 I\alpha_{18}, \\
\Theta_0^i(\theta) &= He\left[\left(G_1^T(\theta)PG_0\right) + N_1 g_1(\theta) + N_2 g_2(\theta)\right] + \hat{H} + 2\alpha_1^T(-W_1^T - D_K^T W_2)\alpha_{15} \\
&\quad + 2\alpha_1^T(W_1^T A_0 + T_1 F)\alpha_{16} + 2\alpha_3^T T_2 F\alpha_{17} + 2\alpha_1^T W_1^T A_{d0}\alpha_{17} + 2\alpha_{15}^T W_2^T A_0\alpha_{16} \\
&\quad + 2\alpha_{15}^T W_2^T A_{d0}\alpha_{17} - G_2^T \tilde{R}_1 G_2 - \Gamma^T \Psi \Gamma + \hat{M} - \alpha_{18}\alpha_{18}, \\
\hat{M} &= \operatorname{diag}\{M_1, -M_1 + M_2, 0_{n\times n}, -M_2, 0_{14n\times 14n}\}, \\
\hat{H} &= \operatorname{diag}\{He(-W_1^T D_K), 0_{13n\times 13n}, He(-W_2^T) + \rho_1^2 Q_1 + \rho_{12}^2 Q_2, \\
&\qquad He(-T_1^T) + A_2^T(\Lambda_1 + \Lambda_2)A_2, He(-T_2^T) + A_{d2}^T(\Lambda_3 + \Lambda_4)A_{d2}, 0_{n\times n}\}, \\
\tilde{R}_i &= \operatorname{diag}\{R_i, 3R_i, 5R_i\} \quad i = 1, 2, \\
D_K &= D - BKC, \\
\Xi_1 &= [W_1^T A_1 \quad 0_{17n\times 2n}]^T, \\
\Xi_2 &= [W_1^T A_{d1} \quad 0_{17n\times 2n}]^T, \\
\Xi_3 &= [0_{14n\times 2n} \quad W_2^T A_1 \quad 0_{3n\times 2n}]^T, \\
\Xi_4 &= [0_{14n\times 2n} \quad W_2^T A_{d1} \quad 0_{3n\times 2n}]^T,
\end{aligned}$$

then, MNN (2.2) achieves $\mathcal{L}_2 - \mathcal{L}_\infty$ stability.

Proof. Construct the following Lyapunov functional:

$$\begin{aligned}
V(\phi_t, \dot{\phi}_t) &= V_1(\phi_t) + V_2(\phi_t) + V_3(\phi_t, \dot{\phi}_t), \\
V_1(\phi_t) &= \tilde{\phi}^T(t)P\tilde{\phi}(t), \\
V_2(\phi_t) &= \int_{t-\rho_1}^{t} \phi^T(r)M_1\phi(r)dr + \int_{t-\rho_2}^{t-\rho_1} \phi^T(r)M_2\phi(r)dr, \\
V_3(\phi_t, \dot{\phi}_t) &= \rho_1 \int_{-\rho_1}^{0}\int_{t+\theta}^{t} \dot{\phi}^T(r)Q_1\dot{\phi}(r)drd\theta + \rho_{12}\int_{-\rho_2}^{-\rho_1}\int_{t+\theta}^{t} \dot{\phi}^T(r)Q_2\dot{\phi}(r)drd\theta,
\end{aligned}$$

where

$$\tilde{\phi}(t) = col\left\{\phi(t), \rho_1\eta_1(t), \eta_6(t)\right\}.$$

Differentiating $V_1(\phi_t)$ along the trajectories of (2.2), we obtain

$$\dot{V}_1(\phi_t) \ = 2\tilde{\phi}^T(t)P\dot{\tilde{\phi}}(t).$$

If we let ϕ, $\hat{\eta}$ stand for $\phi(t)$ and $\hat{\eta}(t)$, then we get

$$\begin{aligned}\rho_1\dot{\eta}_1(t) &= \begin{bmatrix} \alpha_1^T - \alpha_2^T & \alpha_1^T + \alpha_2^T - 2\alpha_5^T \end{bmatrix}^T \hat{\eta},\\ \dot{\eta}_6(t) &= \begin{bmatrix} \alpha_2^T - \alpha_4^T & \rho_{12}\alpha_2^T + \rho_{12}\alpha_4^T - 2\alpha_{11}^T - 2\alpha_{13}^T \end{bmatrix}^T \hat{\eta}\\ &= \begin{bmatrix} \alpha_2^T - \alpha_4^T & \hat{G}_0^T \end{bmatrix}^T \hat{\eta},\end{aligned}$$

leading to

$$\dot{\tilde{\phi}}(t) = G_0\hat{\eta},$$

where G_0 is defined as shown in (3.1), and

$$\phi(t) = \alpha_1\hat{\eta}, \ \ \rho_1\eta_1(t) = \rho_1 \begin{bmatrix} \alpha_5^T & \alpha_6^T \end{bmatrix}^T \hat{\eta}.$$

Consider the last element of $\tilde{\phi}(t)$ (i.e., $\eta_6(t)$). We can get

$$\eta_6(t) = \begin{bmatrix} \int_{-\rho}^{-\rho_1} \phi_t(r)dr \\ \rho_{12} \int_{-\rho}^{-\rho_1} \delta_4(r)\phi_t(r)dr \end{bmatrix} + \begin{bmatrix} \int_{-\rho_2}^{-\rho} \phi_t(r)dr \\ \rho_{12} \int_{-\rho_2}^{-\rho} \delta_4(r)\phi_t(r)dr \end{bmatrix}. \tag{3.5}$$

From the given expression, we observe that the initial n elements can be represented as $(\alpha_{11} + \alpha_{13})\,\hat{\eta}$. For the remaining n components, it is necessary to determine two expressions of $\delta_4(s)$ based on $\delta_2(s)$ and $\delta_3(s)$, respectively. Some calculations show

$$\begin{aligned}\rho_{12}\delta_4(r) &= (\rho - \rho_1)\,\delta_2(r) + (\rho_2 - \rho)\\ &= (\rho_2 - \rho)\,\delta_3(r) - (\rho - \rho_1)\,.\end{aligned} \tag{3.6}$$

Reinjecting (3.6) into (3.5) leads to

$$\begin{aligned}&\rho_{12}\left(\int_{-\rho}^{-\rho_1} \delta_4(r)\phi_t(r)dr + \int_{-\rho_2}^{-\rho} \delta_4(r)\phi_t(r)dr\right)\\ &= \int_{-\rho}^{-\rho_1} \left[(\rho - \rho_1)\,\delta_2(r) + (\rho_2 - \rho)\right]\phi_t(r)dr + \int_{-\rho_2}^{-\rho} \left[(\rho_2 - \rho)\,\delta_3(r) - (\rho - \rho_1)\right]\phi_t(r)dr\\ &= \hat{G}_1(\rho)\hat{\eta}.\end{aligned}$$

Hence, we obtain that $\eta_6(t) = \begin{bmatrix} \alpha_{11}^T + \alpha_{13}^T & \hat{G}_1^T(\rho) \end{bmatrix}\hat{\eta}$ and

$$\tilde{\phi}(t) = G_1(\rho)\hat{\eta}.$$

Moreover, based on the definition of $\hat{\eta}$, it is apparent that

$$\begin{aligned}(\rho - \rho_1)\eta_2(t) - \eta_4(t) &=0,\\ (\rho_2 - \rho)\eta_3(t) - \eta_5(t) &=0.\end{aligned}$$

Therefore, using the matrices g_1 and g_2 defined in (3.1), the following equality holds for any matrices $N_1, N_2 \in \mathbb{R}^{18n\times 2n}$:

$$2\hat{\eta}^T\left(N_1 g_1(\rho) + N_2 g_2(\rho)\right)\hat{\eta} = 0.$$

The derivatives of $\dot{V}_1(\phi_t)$, $\dot{V}_2(\phi_t)$, and $\dot{V}_3(\phi_t)$ in MNN (2.2) can be computed as follows:

$$\begin{aligned}
\dot{V}_1(\phi_t) &= \hat{\eta}^T He\left[\left(G_1^T(\rho)PG_0\right) + N_1 g_1(\rho) + N_2 g_2(\rho))\right]\hat{\eta}, && (3.7)\\
\dot{V}_2(\phi_t) &= \phi^T(t)M_1\phi(t) - \phi^T(t-\rho_1)M_1\phi(t-\rho_1)\\
&\quad + \phi^T(t-\rho_1)M_2\phi(t-\rho_1) - \phi^T(t-\rho_2)M_2\phi(t-\rho_2)\\
&= \hat{\eta}^T\hat{M}\hat{\eta}, && (3.8)\\
\dot{V}_3(\phi_t,\dot{\phi}_t) &= \rho_{12}\left[\int_{-\rho_2}^{-\rho_1}\left(\dot{\phi}^T(t)Q_2\dot{\phi}(t) - \dot{\phi}^T(t+\theta)Q_2\dot{\phi}(t+\theta)\right)d\theta\right]\\
&\quad + \rho_1\left[\int_{-\rho_1}^{0}\left(\dot{\phi}^T(t)Q_1\dot{\phi}(t) - \dot{\phi}^T(t+\theta)Q_1\dot{\phi}(t+\theta)\right)d\theta\right]\\
&= \rho_{12}^2\dot{\phi}^T(t)Q_2\dot{\phi}(t) - \rho_{12}\int_{-\rho_2}^{-\rho_1}\dot{\phi}^T(t+\theta)Q_2\dot{\phi}(t+\theta)d\theta\\
&\quad + \rho_1^2\dot{\phi}^T(t)Q_1\dot{\phi}(t) - \rho_1\int_{-\rho_1}^{0}\dot{\phi}^T(t+\theta)Q_1\dot{\phi}(t+\theta)d\theta\\
&= \dot{\phi}^T(t)\left(\rho_1^2 Q_1 + \rho_{12}^2 Q_2\right)\dot{\phi}(t) - \rho_1\int_{t-\rho_1}^{t}\dot{\phi}^T(r)Q_1\dot{\phi}(r)dr - \rho_{12}\int_{t-\rho_2}^{t-\rho_1}\dot{\phi}^T(r)Q_2\dot{\phi}(r)dr. && (3.9)
\end{aligned}$$

Using Lemma 1, we can get

$$\begin{aligned}
-\rho_1\int_{t-\rho_1}^{t}\dot{\phi}^T(r)Q_1\dot{\phi}(r)dr &\le -\begin{bmatrix}\alpha_1-\alpha_2\\ \alpha_1+\alpha_2-2\alpha_5\\ \alpha_1-\alpha_2-6\alpha_6\end{bmatrix}^T\tilde{R}_1\begin{bmatrix}\alpha_1-\alpha_2\\ \alpha_1+\alpha_2-2\alpha_5\\ \alpha_1-\alpha_2-6\alpha_6\end{bmatrix}\\
&= -G_2^T\tilde{R}_1G_2,
\end{aligned}$$

and

$$\begin{aligned}
-\rho_{12}\int_{t-\rho_2}^{t-\rho_1}\dot{\phi}^T(r)Q_2\dot{\phi}(r)dr &\le -\frac{\rho_{12}}{\rho_2-\rho}\begin{bmatrix}\alpha_3-\alpha_4\\ \alpha_3+\alpha_4-2\alpha_9\\ \alpha_3-\alpha_4-6\alpha_{10}\end{bmatrix}^T\tilde{R}_2\begin{bmatrix}\alpha_3-\alpha_4\\ \alpha_3+\alpha_4-2\alpha_9\\ \alpha_3-\alpha_4-6\alpha_{10}\end{bmatrix}\\
&\quad -\frac{\rho_{12}}{\rho-\rho_1}\begin{bmatrix}\alpha_2-\alpha_3\\ \alpha_2+\alpha_3-2\alpha_7\\ \alpha_2-\alpha_3-6\alpha_8\end{bmatrix}^T\tilde{R}_2\begin{bmatrix}\alpha_2-\alpha_3\\ \alpha_2+\alpha_3-2\alpha_7\\ \alpha_2-\alpha_3-6\alpha_8\end{bmatrix}\\
&= -\frac{\rho_{12}}{\rho_2-\rho}G_4^T\tilde{R}_2G_4 - \frac{\rho_{12}}{\rho-\rho_1}G_3^T\tilde{R}_2G_3\\
&= -\begin{bmatrix}G_3^T & G_4^T\end{bmatrix}\begin{bmatrix}\frac{\rho_{12}}{\rho-\rho_1}\tilde{R}_2 & 0\\ 0 & \frac{\rho_{12}}{\rho_2-\rho}\tilde{R}_2\end{bmatrix}\begin{bmatrix}G_3\\ G_4\end{bmatrix}.
\end{aligned}$$

For the matrix $X \in \mathbb{R}^{3n\times 3n}$, and by using Lemma 2, the derivative of $V_3(\phi_t, \dot{\phi}_t)$ is given by

$$\begin{aligned}\dot{V}_3(\phi_t, \dot{\phi}_t) &\le \dot{\phi}^T(t)\left(\rho_1^2 Q_1 + \rho_{12}^2 Q_2\right)\dot{\phi}(t) - G_2^T \tilde{R}_1 G_2 \\ &\quad - \begin{bmatrix} G_3^T & G_4^T \end{bmatrix}\begin{bmatrix} \tilde{R}_2 & X \\ X^T & \tilde{R}_2 \end{bmatrix}\begin{bmatrix} G_3 \\ G_4 \end{bmatrix} \\ &= \hat{\eta}^T\left[\alpha_{15}^T\left(\rho_1^2 Q_1 + \rho_{12}^2 Q_2\right)\alpha_{15} + G_2^T \tilde{R}_1 G_2 - \Gamma^T \Psi \Gamma\right]\hat{\eta}.\end{aligned} \tag{3.10}$$

Then, we consider the free-weighting $n \times n$ matrices, W_1 and W_2, under the conditions of MNN (2.2), we can conclude that the following equation holds true:

$$\begin{aligned}0 =& 2\left[\phi^{\mathrm{T}}(t)W_1^{\mathrm{T}} + \dot{\phi}^{\mathrm{T}}(t)W_2^{\mathrm{T}}\right]\left[-\dot{\phi}(t) - D_K\phi(t) + A_0\sigma(\phi(t)) + A_{d0}\sigma(\phi(t-\rho(t))\right] \\ &+ 2\left[\phi^{\mathrm{T}}(t)W_1^{\mathrm{T}} + \dot{\phi}^{\mathrm{T}}(t)W_2^{\mathrm{T}}\right][A_1\Delta(\phi(t))A_2\sigma(\phi(t)) + A_{d1}\Delta(\phi(t-\rho(t)))A_{d2}\sigma(\phi(t-\rho(t)))].\end{aligned} \tag{3.11}$$

Given that $\Delta(\phi(t))\Delta(\phi(t)) \le I$ and $\Delta(\phi(t-\rho(t)))\Delta(\phi(t-\rho(t))) \le I$, we can derive the following set of four inequalities for positive diagonal matrices $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4$:

$$2\phi^{\mathrm{T}}(t)W_1^{\mathrm{T}}A_1\Delta(\phi(t))A_2\sigma(\phi(t)) \le \phi^{\mathrm{T}}(t)W_1^{\mathrm{T}}A_1\Lambda_1^{-1}A_1^{\mathrm{T}}W_1\phi(t) + \sigma^{\mathrm{T}}(\phi(t))A_2^{\mathrm{T}}\Lambda_1 A_2\sigma(\phi(t)), \tag{3.12}$$

$$2\dot{\phi}^{\mathrm{T}}(t)W_2^{\mathrm{T}}A_1\Delta(\phi(t))A_2\sigma(\phi(t)) \le \dot{\phi}^{\mathrm{T}}(t)W_2^{\mathrm{T}}A_1\Lambda_2^{-1}A_1^{\mathrm{T}}W_2\dot{\phi}(t) + \sigma^{\mathrm{T}}(\phi(t))A_2^{\mathrm{T}}\Lambda_2 A_2\sigma(\phi(t)), \tag{3.13}$$

$$\begin{aligned}&2\phi^{\mathrm{T}}(t)W_1^{\mathrm{T}}A_{d1}\Delta(\phi(t-\rho(t)))A_{d2}\sigma(\phi(t-\rho(t))) \\ &\quad \le \phi^{\mathrm{T}}(t)W_1^{\mathrm{T}}A_{d1}\Lambda_3^{-1}A_{d1}^{\mathrm{T}}W_1\phi(t) + \sigma^{\mathrm{T}}(\phi(t-\rho(t)))A_{d2}^{\mathrm{T}}\Lambda_3 A_{d2}\sigma(\phi(t-\rho(t))),\end{aligned} \tag{3.14}$$

$$\begin{aligned}&2\dot{\phi}^{\mathrm{T}}(t)W_2^{\mathrm{T}}A_{d1}\Delta(\phi(t-\rho(t)))A_{d2}\sigma(\phi(t-\rho(t))) \\ &\quad \le \dot{\phi}^{\mathrm{T}}(t)W_2^{\mathrm{T}}A_{d1}\Lambda_4^{-1}A_{d1}^{\mathrm{T}}W_2\dot{\phi}(t) + \sigma^{\mathrm{T}}(\phi(t-\rho(t)))A_{d2}^{\mathrm{T}}\Lambda_4 A_{d2}\sigma(\phi(t-\rho(t))).\end{aligned} \tag{3.15}$$

Under the condition specified in (2.4) for neuron activation functions, the following inequalities hold:

$$\begin{aligned}0 &\le 2\phi^T(t)T_1F\sigma(\phi(t)) - 2\sigma^T(\phi(t))T_1\sigma(\phi(t)), \\ 0 &\le 2\phi^T(t-\rho(t))T_2F\sigma(\phi(t-\rho(t))) - 2\sigma^T(\phi(t-\rho(t)))T_2\sigma(\phi(t-\rho(t))).\end{aligned} \tag{3.16}$$

Through a synthesis of Eqs (3.7)–(3.16), we arrive at the ensuing outcome:

$$\begin{aligned}\dot{V}(\phi_t, \dot{\phi}_t) \le\ & \hat{\eta}^T\Big(He[\left(G_1^T(\theta)PG_0\right) + N_1g_1(\theta) + N_2g_2(\theta)] + \alpha_{15}^T\left(\rho_1^2 Q_1 + \rho_{12}^2 Q_2\right)\alpha_{15} + \hat{H} + \hat{M} \\ & - G_2^T\tilde{R}_1G_2 - \Gamma^T\Psi\Gamma + 2\alpha_1^T(-W_1^T - D_K^T W_2)\alpha_{15} + 2\alpha_3^T T_2F\alpha_{17} + 2\alpha_1^T(W_1^T A_0 + T_1F)\alpha_{16} \\ & + 2\alpha_1^T W_1^T A_{d0}\alpha_{17} + 2\alpha_{15}^T W_2^T A_0\alpha_{16} + 2\alpha_{15}^T W_2^T A_{d0}\alpha_{17}\Big)\hat{\eta} + w^T(t)w(t) \\ =\ & \hat{\eta}^T\Theta_0^i(\theta)\hat{\eta} + w^T(t)w(t).\end{aligned} \tag{3.17}$$

Utilizing the convexity property of $\Theta_0^i(\cdot)$, we infer from (3.4) that

$$\Theta_0^i(\rho(t))) < 0 \tag{3.18}$$

for all $\rho(t) \in [\rho_1, \rho_2]$. After taking (3.18) into account, it can be inferred from (3.17) that

$$\dot{V}_i\left(\phi_t, \dot{\phi}_t\right) \le w^T(t)w(t), \tag{3.19}$$

In the case where $w(t) \equiv 0$, we can obtain the following expression from (3.19):

$$\dot{V}_i\left(\phi_t, \dot{\phi}_t\right) \le 0.$$

According to Lyapunov's theory, for all non-zero $\phi(t)$, MNN (2.2) is guaranteed to exhibit asymptotic stability. On the other side, when $w(t) \neq 0$, an index function can be introduced and defined as follows:

$$J(t) = y(t)^T y(t) - \gamma^2 \int_0^t w(r)^T w(r)dr.$$

Assuming zero initial condition, applying the Newton-Leibniz formula results in:

$$\begin{aligned} J(t) &= y(t)^T y(t) - \gamma^2 \int_0^t w(r)^T w(r)dr + \int_0^t \dot{V}(r)dr - (V(t) - V(0)). \\ &\leq \phi^T(t)\left(C^T C - P_{11}\right)\phi(t) + \int_0^t \left(\dot{V}(r) - \gamma^2 w(r)^T w(r)\right) ds \\ &\leq \phi^T(t)\left(C^T C - P_{11}\right)\phi(t) + \int_0^t \hat{\eta}^T \Theta_i(\theta)\hat{\eta} ds. \end{aligned}$$

From (3.3) and (3.4), it follows that for all $w(t) \neq 0$, we have $J(t) \leq 0$. Therefore, MNN (2.2) has $\mathcal{L}_2 - \mathcal{L}_\infty$ stability. □

4. $\mathcal{L}_2 - \mathcal{L}_\infty$ control

The objective of this section is to discuss the issue of $\mathcal{L}_2 - \mathcal{L}_\infty$ control concerning MNN (2.2) with TVD. The following theorem provides a design strategy based on LMIs:

Theorem 2. *For given positive scalars γ, ε, ρ_1, and ρ_2, if there exist matrices $P = \left(P_{jk}\right)_{5\times 5} \in \mathbb{S}_+^{5n}$, $M_1, M_2, Q_1, Q_2, L, U \in \mathbb{S}_+^n$, $W_1, W_2 \in \mathbb{R}^{n\times n}$, $N_1, N_2 \in \mathbb{R}^{18n\times 2n}$, positive diagonal matrices $T_1, T_2, \Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4$, and a matrix $X \in \mathbb{R}^{3n\times 3n}$. Such that (3.2), (3.3), and*

$$\begin{bmatrix} \tilde{\Theta}_i(\theta) & \Xi_1 & \Xi_2 & \Xi_3 & \Xi_4 & \Xi_5 \\ * & -\Lambda_1 & 0 & 0 & 0 & 0 \\ * & * & -\Lambda_3 & 0 & 0 & 0 \\ * & * & * & -\Lambda_2 & 0 & 0 \\ * & * & * & * & -\Lambda_4 & 0 \\ * & * & * & * & * & -\varepsilon(L + L^T) \end{bmatrix} < 0, \tag{4.1}$$

$$\begin{aligned} \tilde{\Theta}_i(\theta) &= \tilde{\Theta}_0^i(\theta) + 2\alpha_1^T BUC\alpha_1 + 2\alpha_1^T C^T U^T B^T \alpha_{15}, \\ \tilde{\Theta}_0^i(\theta) &= He[(G_1^T(\theta)PG_0) + N_1 g_1(\theta) + N_2 g_2(\theta)] - G_2^T \tilde{R}_1 G_2 - \Gamma^T \Psi \Gamma + \tilde{H} + \hat{M} \\ &\quad + 2\alpha_1^T(-W_1^T - D^T W_2)\alpha_{15} + 2\alpha_1^T(W_1^T A_0 + T_1 F)\alpha_{16} + 2\alpha_1^T W_1^T A_{d0}\alpha_{17} \\ &\quad + 2\alpha_3^T T_2 F\alpha_{17} + 2\alpha_{15}^T W_2^T A_0\alpha_{16} + 2\alpha_{15}^T W_2^T A_{d0}\alpha_{17} + 2\alpha_1^T W_1^T E\alpha_{18} + 2\alpha_{15}^T W_2^T E\alpha_{18}, \\ \hat{M} &= \mathrm{diag}(M_1, -M_1 + M_2, 0_{n\times n}, -M_2, 0_{14n\times 14n}), \\ \tilde{H} &= \mathrm{diag}\Big(He(-W_1^T D), 0_{13n\times 13n}, He(-W_2^T) + \rho_1^2 Q_1 + \rho_{12}^2 Q_2, \\ &\qquad He(-T_1^T) + A_2^T(\Lambda_1 + \Lambda_2)A_2, He(-T_2^T) + A_{d2}^T(\Lambda_3 + \Lambda_4)A_{d2}, -\gamma^2 I\Big), \\ \Xi_5 &= [W_1^T B - BL + \varepsilon U^T C^T \quad 0_{13n\times n} \quad W_2^T B - BL + \varepsilon U^T C^T \quad 0_{3n\times n}]^T, \end{aligned}$$

and the other notations used are the same as in Theorem 1, then, the control gain for the desired output-feedback controller can then be calculated by

$$K = L^{-1}U. \tag{4.2}$$

Proof. The inequality expressed in Eq (4.1) can be restated in the following form:

$$\Omega + He\,(\Delta_B K \Delta_C) < 0 \tag{4.3}$$

where

$$\begin{aligned} \Omega &= \tilde{\Theta}_0^i(\theta), \\ \Delta_B &= [W_1^T B \quad 0_{13n\times n} \quad W_2^T B \quad 0_{3n\times n}]^T, \\ \Delta_C &= [C \quad 0_{13n\times n} \quad C \quad 0_{3n\times n}]. \end{aligned}$$

Recognizing (4.2), the Eq (4.3) can be rewritten as:

$$\Omega + He\left(\Delta_J U \Delta_C + (\Delta_B - \Delta_J L)\, L^{-1} U \Delta_C\right) < 0 \tag{4.4}$$

where

$$\Delta_J = \left[\begin{array}{cccc} B & 0_{n\times 13n} & B & 0_{n\times_{3n}} \end{array}\right]^T.$$

In light of Lemma 5, it follows that the validity of inequality (4.4) is guaranteed by (4.1), culminating in the completion of the proof. □

5. Number examples

Two numerical examples are presented to demonstrate the effectiveness of the $\mathcal{L}_2 - \mathcal{L}_\infty$ stability analysis for the MNN with non-necessarily differentiable TVD, and the proposed output-feedback controller design method.

Example 1. In this context, we consider MNN (2.1) with $n = 2$ and activation functions $\sigma(\phi_j(t)) = \tanh(\phi_j(t))$ for $j = 1, 2$, which satisfies (2.4). The network parameters are selected in accordance with [34]:

$$\begin{aligned} D(\phi(t)) &= \begin{bmatrix} d_1(\phi_1(t)) & 0 \\ 0 & d_2(\phi_2(t)) \end{bmatrix}, \\ A(\phi(t)) &= \begin{bmatrix} a_{11}(\phi_1(t)) & 0.6 \\ 0.5 & a_{22}(\phi_2(t)) \end{bmatrix}, \\ A_d(\phi(t)) &= \begin{bmatrix} -0.1 & a_{d12}(\phi_1(t)) \\ a_{d21}(\phi_2(t)) & -0.2 \end{bmatrix}, \end{aligned}$$

where

$$d_1(\phi_1(t)) = \begin{cases} 1, & |\phi_1(t)| \le 0.5, \\ 0.9, & |\phi_1(t)| > 0.5, \end{cases} \qquad d_2(\phi_2(t)) = \begin{cases} 0.9, & |\phi_2(t)| \le 0.5, \\ 1, & |\phi_2(t)| > 0.5, \end{cases}$$

$$a_{11}(\phi_1(t)) = \begin{cases} 0, & |\phi_1(t)| \le 0.5, \\ -0.1, & |\phi_1(t)| > 0.5, \end{cases} \qquad a_{22}(\phi_2(t)) = \begin{cases} -0.1, & |\phi_2(t)| \le 0.5, \\ 0, & |\phi_2(t)| > 0.5, \end{cases}$$

$$a_{d12}(\phi_1(t)) = \begin{cases} 0.6, & |\phi_1(t)| \le 0.5, \\ 0.1, & |\phi_1(t)| > 0.5, \end{cases} \qquad a_{d21}(\phi_2(t)) = \begin{cases} 0.3, & |\phi_2(t)| \le 0.5, \\ 0.2, & |\phi_2(t)| > 0.5. \end{cases}$$

Then, we can obtain that

$$D=\begin{bmatrix}0.95 & 0\\ 0 & 0.95\end{bmatrix},\quad A_0=\begin{bmatrix}-0.05 & 0.6\\ 0.5 & -0.05\end{bmatrix},\quad A_{d0}=\begin{bmatrix}-0.1 & 0.35\\ 0.25 & -0.2\end{bmatrix},$$
$$A_1=\begin{bmatrix}\sqrt{0.05} & 0 & 0 & 0\\ 0 & 0 & 0 & \sqrt{0.05}\end{bmatrix},\quad A_2^T=\begin{bmatrix}\sqrt{0.05} & 0 & 0 & 0\\ 0 & 0 & 0 & \sqrt{0.05}\end{bmatrix},$$
$$A_{d1}=\begin{bmatrix}0 & \sqrt{0.25} & 0 & 0\\ 0 & 0 & \sqrt{0.05} & 0\end{bmatrix},\quad A_{d2}^T=\begin{bmatrix}0 & 0 & \sqrt{0.05} & 0\\ 0 & \sqrt{0.25} & 0 & 0\end{bmatrix}.$$

Table 1. Maximum allowable value ρ_2, ρ_{12} for given ρ_1.

ρ_1	0	0.5	1	1.5	2	2.5	3
ρ_2	3.04	3.46	3.85	4.21	4.59	5.01	5.46
ρ_{12}	3.04	2.96	2.85	2.71	2.59	2.51	2.46

The presented results in Table 1 demonstrate the upper delay bound and the allowable time delay in the non-differentiable TVD system governed by MNN (2.2) for different values of the parameter ρ_1. The analysis indicates that the upper bound of the time delay (ρ_2) increases as the lower bound of the time delay (ρ_1) increases, while the allowable time delay interval (ρ_{12}) decreases.

Example 2. Consider MNN (2.1) with $n = 2$, where the activation functions for neurons $j = 1, 2$ are given by $\sigma(\phi_j(t)) = \tanh(\phi_j(t))$ and $\sigma(\phi_j(t-\rho(t))) = \tanh(\phi_j(t-\rho(t)))$. These functions satisfy the condition in (2.4). The other parameters are as follows:

$$w(t)=\begin{bmatrix}1.8e^{-0.5t} & 1.8e^{-0.5t}\end{bmatrix},\quad \rho(t)=1+2.8\,|sin(t)|,$$
$$D=\begin{bmatrix}0.9 & 0\\ 0 & 0.9\end{bmatrix},\quad B=\begin{bmatrix}4 & 2\\ 1 & 3\end{bmatrix},\quad E=\begin{bmatrix}0.1 & 0.2\\ 0.3 & 0.4\end{bmatrix},\quad C=\begin{bmatrix}1 & 2\\ 1 & 1\end{bmatrix},$$
$$A(\phi(t))=\begin{bmatrix}a_{11}(\phi_1(t)) & a_{12}(\phi_1(t))\\ a_{21}(\phi_1(t)) & a_{22}(\phi_2(t))\end{bmatrix},\quad A_d(\phi(t))=\begin{bmatrix}-0.1 & a_{d12}(\phi_1(t))\\ a_{d21}(\phi_2(t)) & -0.2\end{bmatrix},$$

where

$$a_{11}(\phi_1(t))=\begin{cases}1, & |\phi_1(t)|\le 1,\\ 1.5, & |\phi_1(t)|>1,\end{cases}\qquad a_{12}(\phi_1(t))=\begin{cases}1.4, & |\phi_1(t)|\le 1,\\ 1.7, & |\phi_1(t)|>1,\end{cases}$$
$$a_{21}(\phi_1(t))=\begin{cases}-3, & |\phi_1(t)|\le 1,\\ -3.2, & |\phi_1(t)|>1,\end{cases}\qquad a_{22}(\phi_2(t))=\begin{cases}2, & |\phi_2(t)|\le 1,\\ 1.6, & |\phi_2(t)|>1,\end{cases}$$
$$a_{d12}(\phi_1(t))=\begin{cases}0.6, & |\phi_1(t)|\le 1,\\ 0.1, & |\phi_1(t)|>1,\end{cases}\qquad a_{d21}(\phi_2(t))=\begin{cases}0.3, & |\phi_2(t)|\le 1,\\ 0.2, & |\phi_2(t)|>1.\end{cases}$$

By excluding matrix C, the matrices are identical to these in [16], and MNN (2.2) can be acquired with the parameters listed below:

$$A_0 = \begin{bmatrix} 1.25 & 1.55 \\ -3.1 & 1.8 \end{bmatrix},\ A_1 = \begin{bmatrix} 0.5 & \sqrt{0.15} & 0 & 0 \\ 0 & 0 & \sqrt{0.1} & \sqrt{0.2} \end{bmatrix},$$

$$A_{d0} = \begin{bmatrix} -0.1 & 0.35 \\ 0.25 & -0.2 \end{bmatrix},\ A_2 = \begin{bmatrix} 0.5 & 0 & \sqrt{0.1} & 0 \\ 0 & \sqrt{0.15} & 0 & \sqrt{0.2} \end{bmatrix}^T,$$

$$A_{d1} = \begin{bmatrix} 0 & \sqrt{0.25} & 0 & 0 \\ 0 & 0 & \sqrt{0.05} & 0 \end{bmatrix},\ A_{d2} = \begin{bmatrix} 0 & 0 & \sqrt{0.05} & 0 \\ 0 & \sqrt{0.25} & 0 & 0 \end{bmatrix}^T.$$

Under the parameters specified above, Figure 1 illustrates the state trajectories of the system (2.2), where the initial condition is set to $\phi(t) = [2 \quad 4]^T$ and $u(t) \equiv 0$. The figure demonstrates that the system trajectories fail to converge to zero without any control input.

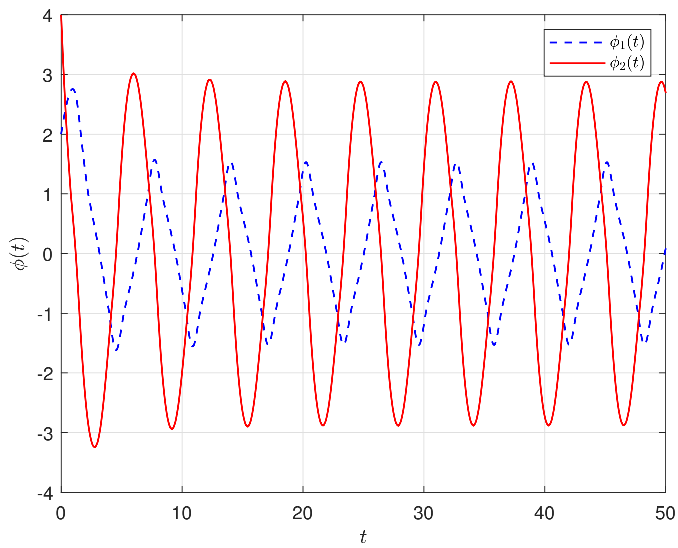

Figure 1. The state trajectories of MNN system (2.1) without control.

For this example, the control methods presented in [4–8] are not applicable since the time delay is not differentiable. However, the present control method is suitable for stabilizing MNN (2.2). The determination of the required gain matrices can be achieved by utilizing Theorem 2, which yields the following result:

$$K = \begin{bmatrix} 13.5374 & -15.4456 \\ -15.2271 & 13.2160 \end{bmatrix}. \tag{5.1}$$

In the simulation, we set

$$\gamma = 1,\ \varepsilon = 0.9,\ \phi(t) = [2 \quad 5]^T,$$

$$L(t) = \frac{\left(\sup_{0\le t}\left\{y^T(t)y(t)\right\}\right)^{1/2}}{\left(\int_0^t w^T(\beta)w(\beta)d\beta\right)^{1/2}},\ F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Figure 2 presents MNN (2.2) state trajectories in response to control inputs. It is apparent that under the controller (5.1) specified by Theorem 2, MNN (2.2) states achieve $\mathcal{L}_2 - \mathcal{L}_\infty$ stablility. Additionally, Figure 3 depicts the evolution of $L(t)$. Controller design efficacy is confirmed through simulation results.

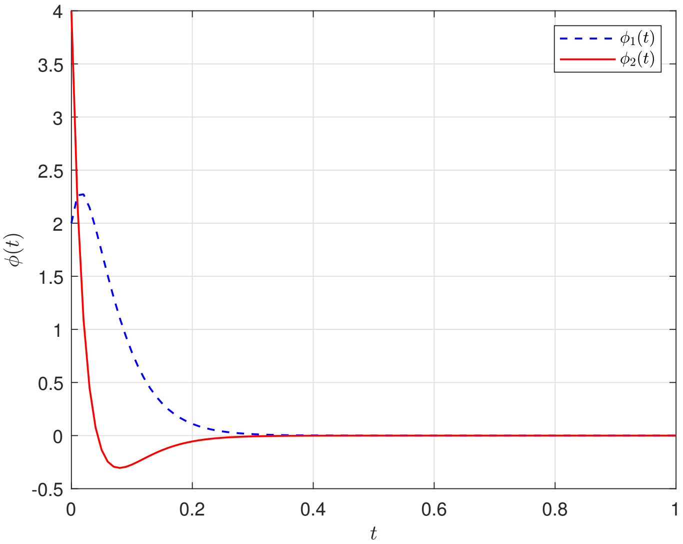

Figure 2. The state trajectories of MNN (2.1) with control.

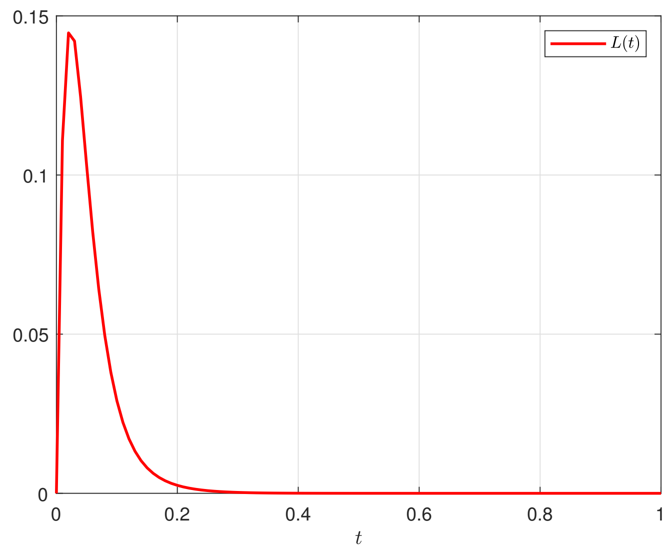

Figure 3. $L(t)$.

6. Conclusions

This paper has investigated $\mathcal{L}_2 - \mathcal{L}_\infty$ control for MNNs a with non-necessarily differentiable TVD. A criterion on the $\mathcal{L}_2 - \mathcal{L}_\infty$ stability was proposed using a Lyapunov functional, the BLI, and the

CCI. Then, a LMIs-based design scheme for the required output-feedback controller was developed by decoupling nonlinear terms. Finally, two examples were presented to verify the proposed $\mathcal{L}_2 - \mathcal{L}_\infty$ stability criterion and design method.

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Conflict of interest

The authors declare there is no conflict of interest.

References

1. Z. Zhang, X. Liu, D. Zhou, C. Lin, J. Chen, H. Wang, Finite-time stabilizability and instabilizability for complex-valued memristive neural networks with time delays, *IEEE Trans. Syst. Man Cybern. Syst.*, **48** (2017), 2371–2382. https://doi.org/10.1109/TSMC.2017.2754508

2. A. Wu, Z. Zeng, Exponential stabilization of memristive neural networks with time delays, *IEEE Trans. Neural Networks Learn. Syst.*, **23** (2012), 1919–1929. https://doi.org/10.1109/TNNLS.2012.2219554

3. R. Li, X. Gao, J. Cao, Quasi-state estimation and quasi-synchronization control of quaternion-valued fractional-order fuzzy memristive neural networks: Vector ordering approach, *Appl. Math. Comput.*, **362** (2019), 124572. https://doi.org/10.1016/j.amc.2019.124572

4. J. Zhang, X. Ma, Y. Li, Q. Gan, C. Wang, Synchronization in fixed/preassigned-time of delayed fully quaternion-valued memristive neural networks via non-separation method, *Commun. Nonlinear Sci. Numer. Simul.*, **113** (2022), 106581. https://doi.org/10.1016/j.cnsns.2022.106581

5. N. Yang, Y. Yu, S. Zhong, X. Wang, K. Shi, J. Cai, Exponential synchronization of stochastic delayed memristive neural networks via a novel hybrid control, *Neural Networks*, **131** (2020), 242–250. https://doi.org/10.1016/j.neunet.2020.07.034

6. Z. Yan, X. Huang, J. Cao, Variable-sampling-period dependent global stabilization of delayed memristive neural networks based on refined switching event-triggered control, *Sci. China Inf. Sci.*, **63** (2020), 212201.

7. X. Wu, S. Liu, H. Wang, Asymptotic stability and synchronization of fractional delayed memristive neural networks with algebraic constraints, *Commun. Nonlinear Sci. Numer. Simul.*, **114** (2022), 106694.

8. Z. Yan, X. Huang, Y. Fan, J. Xia, H. Shen, Threshold-function-dependent quasi-synchronization of delayed memristive neural networks via hybrid event-triggered control, *IEEE Trans. Syst. Man Cybern. Syst.*, **51** (2021), 6712–6722. https://doi.org/10.1109/TSMC.2020.2964605

9. S. Murugesan, Y. C. Liu, Finite-time resilient control for networked control systems with multiple cyber-attacks: Memory/adaptive event-triggered scheme, *Int. J. Adapt. Control Signal Process.*, **36** (2022), 901–925.

10. W. Tai, D. Gao, A. Zhao, J. Zhou, X. Wang, Weight learning for $\mathcal{H}_\infty$ stabilization of uncertain switched neural networks with external disturbance and reaction-diffusion, *Int. J. Adapt. Control Signal Process.*, **37** (2023), 1015–1029. https://doi.org/10.1002/acs.3558

11. M. Sathishkumar, R. Sakthivel, F. Alzahrani, B. Kaviarasan, Y. Ren, Mixed $\mathcal{H}_\infty$ and passivity-based resilient controller for nonhomogeneous Markov jump systems, *Nonlinear Anal. Hybrid Syst.*, **31** (2019), 86–99. https://doi.org/10.1016/j.mattod.2019.06.003

12. J. Zhou, J. Dong, S. Xu, Asynchronous dissipative control of discrete-time fuzzy Markov jump systems with dynamic state and input quantization, *IEEE Trans. Fuzzy Syst.*, **2023** (2023), forthcoming. https://doi.org/10.1109/TFUZZ.2023.3271348

13. X. Meng, G. Zhang, Q. Zhang, Robust adaptive neural network integrated fault-tolerant control for underactuated surface vessels with finite-time convergence and event-triggered inputs, *Math. Biosci. Eng.*, **20** (2022), 2131–2156. https://doi.org/10.3934/mbe.2023099

14. R. Saravanakumar, Y. Hoon Joo, Network-based robust exponential fuzzy control for uncertain systems, *Math. Methods Appl. Sci.*, **2023** (2023), forthcoming. https://doi.org/10.1002/mma.8943

15. Y. Cao, N. Liu, C. Zhang, T. Zhang, Z.-F. Luo, Synchronization of multiple reaction–diffusion memristive neural networks with known or unknown parameters and switching topologies, *Knowl. Based Syst.*, **254** (2022), 109595. https://doi.org/10.1016/j.knosys.2022.109595

16. I. Ghous, J. Lu, Z. Duan, $\mathcal{H}_\infty$ stabilization problem for memristive neural networks with time-varying delays, *Inf. Sci.*, **607** (2022), 27–43.

17. Z. Yan, D. Zuo, T. Guo, J. Zhou, Quantized $\mathcal{H}_\infty$ stabilization for delayed memristive neural networks, *Neural Comput. Appl.*, **2023** (2023), forthcoming. https://doi.org/10.1007/s00521-023-08510-3

18. W. Tai, X. Li, J. Zhou, S. Arik, Asynchronous dissipative stabilization for stochastic Markov-switching neural networks with completely-and incompletely-known transition rates, *Neural Networks*, **161** (2023), 55–64. https://doi.org/10.1016/j.neunet.2023.01.039

19. C. Deng, D. Zhang, G. Feng, Resilient practical cooperative output regulation for mass with unknown switching exosystem dynamics under DoS attacks, *Automatica*, **139** (2022), 110172. https://doi.org/10.1016/j.automatica.2022.110172

20. S. Dong, L. Liu, G. Feng, M. Liu, Z. G. Wu, R. Zheng, Cooperative output regulation quadratic control for discrete-time heterogeneous multiagent Markov jump systems, *IEEE Trans. Cybern.*, **52** (2022), 9882–9892.

21. J. Gu, H. Wang, W. Li, Output-feedback stabilization for stochastic nonlinear systems with Markovian switching and time-varying powers, *Math. Biosci. Eng.*, **19** (2022), 11 071–11 085.

22. T. Yu, L. Liu, Y. J. Liu, Observer-based adaptive fuzzy output feedback control for functional constraint systems with dead-zone input, *Math. Biosci. Eng.*, **20** (2023), 2628–2650. https://doi.org/10.3934/mbe.2023123

23. Z. Yan, C. Sang, M. Fang, J. Zhou, Energy-to-peak consensus for multi-agent systems with stochastic disturbances and Markovian switching topologies, *Trans. Inst. Measure. Control*, **40** (2018), 4358–4368.

24. J. Zhou, D. Xu, W. Tai, C. K. Ahn, Switched event-triggered $\mathcal{H}_\infty$ security control for networked systems vulnerable to aperiodic DoS attacks, *IEEE Trans. Network Sci. Eng.*, **2023** (2023), forthcoming. https://doi.org/10.1109/TNSE.2023.3243095

25. Y. Liu, J. Li, F. Fang, J. H. Park, Further studies on sampled-data consensus of multi-agent systems with communication delays, *IEEE Trans. Signal Inf. Process. Networks*, **8** (2022), 920–931. https://doi.org/10.1109/TSIPN.2022.3220153

26. E. Arslan, G. Narayanan, M. S. Ali, S. Arik, S. Saroha, Controller design for finite-time and fixed-time stabilization of fractional-order memristive complex-valued bam neural networks with uncertain parameters and time-varying delays, *Neural Networks*, **130** (2020), 60–74. https://doi.org/10.1016/j.neunet.2020.06.021

27. R. Saravanakumar, G. Rajchakit, M. S. Ali, Z. Xiang, Y. H. Joo, Robust extended dissipativity criteria for discrete-time uncertain neural networks with time-varying delays, *Neural Comput. Appl.*, **30** (2018), 3893–3904. https://doi.org/10.1007/s00521-017-2974-z

28. L. He, W. Wu, J. Zhou, G. Yao, Input-to-state stable synchronization for delayed Lurie systems via sampled-data control, *Discrete Contin. Dyn. Syst.*, **28** (2023), 1553–1570. https://doi.org/10.3934/dcdsb.2022135

29. X. Huang, J. Jia, Y. Fan, Z. Wang, J. Xia, Interval matrix method based synchronization criteria for fractional-order memristive neural networks with multiple time-varying delays, *J. Franklin Inst.*, **357** (2020), 1707–1733.

30. L. Yao, Z. Wang, X. Huang, Y. Li, Q. Ma, H. Shen, Stochastic sampled-data exponential synchronization of Markovian jump neural networks with time-varying delays, *IEEE Trans. Neural Networks Learn. Syst.*, **34** (2023), 909–920. https://doi.org/10.1109/TNNLS.2021.3103958

31. R. Vadivel, P. Hammachukiattikul, N. Gunasekaran, R. Saravanakumar, H. Dutta, Strict dissipativity synchronization for delayed static neural networks: An event-triggered scheme, *Chaos Solitons Fractals*, **150** (2021), 111212. https://doi.org/10.1016/j.chaos.2021.111212

32. Y. Ni, Z. Wang, Y. Fan, X. Huang, H. Shen, Memory-based event-triggered control for global synchronization of chaotic Lur'e systems and its application, *IEEE Trans. Syst. Man Cybern. Syst.*, **53** (2023), 1920–1931.

33. G. Zhang, Z. Zeng, D. Ning, Novel results on synchronization for a class of switched inertial neural networks with distributed delays, *Inf. Sci.*, **511** (2020), 114–126. https://doi.org/10.1016/j.ins.2019.09.048

34. Z. Wang, S. Ding, Z. Huang, H. Zhang, Exponential stability and stabilization of delayed memristive neural networks based on quadratic convex combination method, *IEEE Trans. Neural Networks Learn. Syst.*, **27** (2015), 2337–2350. https://doi.org/10.1109/TNNLS.2015.2485259

35. X. Yang, D. W. Ho, Synchronization of delayed memristive neural networks: robust analysis approach, *IEEE Trans. Cybern.*, **46** (2015), 3377–3387. https://doi.org/10.1109/TCYB.2015.2505903

36. A. Seuret, F. Gouaisbaut, Hierarchy of LMI conditions for the stability analysis of time-delay systems, *Syst. Control Lett.*, **81** (2015), 1–7. https://doi.org/10.1016/j.sysconle.2015.03.007

37. P. Park, J. W. Ko, C. Jeong, Reciprocally convex approach to stability of systems with time-varying delays, *Automatica*, **47** (2011), 235–238. https://doi.org/10.1016/j.automatica.2010.10.014

38. S. Xu, J. Lam, A survey of linear matrix inequality techniques in stability analysis of delay systems, *Int. J. Syst. Sci.*, **39** (2008), 1095–1113. https://doi.org/10.1080/00207720802300370

39. S. Boyd, L. El Ghaoui, E. Feron, V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, SIAM, 1994.

40. J. Zhou, J. H. Park, Q. Ma, Non-fragile observer-based $\mathcal{H}_\infty$ control for stochastic time-delay systems, *Appl. Math. Comput.*, **291** (2016), 69–83. https://doi.org/10.1016/j.amc.2016.06.024

41. M. Solak, O. Faydasicok, S. Arik, A general framework for robust stability analysis of neural networks with discrete time delays, *Neural Networks*, **162** (2023), 186–198. https://doi.org/10.1016/j.neunet.2023.02.040

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