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Research article

Modeling the fear effect in the predator-prey dynamics with an age structure in the predators

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Abstract: We incorporate the fear effect and the maturation period of predators into a diffusive predator-prey model. Local and global asymptotic stability for constant steady states as well as uniform persistence of the solution are obtained. Under some conditions, we also exclude the existence of spatially nonhomogeneous steady states and the steady state bifurcation bifurcating from the positive constant steady state. Hopf bifurcation analysis is carried out by using the maturation period of predators as a bifurcation parameter, and we show that global Hopf branches are bounded. Finally, we conduct numerical simulations to explore interesting spatial-temporal patterns.

Keywords: predator-prey dynamics; fear effect; reaction-diffusion system; maturation period; stability; hopf bifurcation

1. Introduction

Interactions between the predators and the preys are diverse and complex in ecology. The predators increase the mortality rate of preys by direct predation. In existing literatures, many predator-prey models only involve direct predation for predator-prey interactions [1–4]. However, besides the population loss caused by direct predation, the prey will modify their behavior, psychology and physiology in response to the predation risk. This is defined as the fear effect by Cannon [5]. Zanette et al. [6] investigated the variation of song sparrows offspring reproduction when the sounds and calls of predators were broadcasted to simulate predation risk. They discovered that the fear effect alone can cause a significant reduction of song sparrows offspring reproduction. Inspired by this experimental result, Wang et al. [7] incorporated the fear effect into the predator-prey model. In their theoretical analysis, linear and Holling type II functional response are chosen respectively. According to their results, the structure of the equilibria will not be affected by the fear effects, but the stability of equilibria and Hopf bifurcation are slightly different from models with no fear effects.

Soon afterward, Wang and Zou [8] considered a model with the stage structure of prey (juvenile prey and adult prey) and a maturation time delay. Additionally, Wang and Zou [9] pointed out anti-predation behaviors will not only decrease offspring reproduction of prey but also increase the difficulty of the prey being caught. Based on this assumption, they derived an anti-predation strategic predator-prey model.

Recently, the aforementioned ordinary or delayed differential equation models were extended to reaction diffusion equation [10–12]. Wang et al. [11] used several functional responses to study the effect of the degree of prey sensitivity to predation risk on pattern formation. Following this work, Wang et al. [10] introduced spatial memory delay and pregnancy delay into the model. Their numerical simulation presented the effect of some biological important variables, including the level of fear effect, memory-related diffusion, time delay induced by spatial memory and pregnancy on pattern formation. Moreover, Dai and Sun [13] incorporated chemotaxis and fear effect into predator-prey model, and investigated the Turing-Hopf bifurcation by selecting delay and chemotaxis coefficient as two analysis parameters.

Denote by $u_1(x,t)$ and $u_2(x,t)$, the population of the prey and adult predator at location x and time t, respectively. We suppose juvenile predators are unable to prey on. By choosing simplest linear functional response in the model of Wang and Zou [7], the equation for prey population is given by

$$\partial_t u_1 - d_1 \Delta u_1 = u_1 \left(r g(K, u_2) - \mu_1 - m u_1 \right) - p u_1 u_2,$$

where d_1 is a diffusion coefficient for prey, μ_1 is the mortality rate for prey, m is the intraspecies competition coefficient, p is predation rate, r is reproduction rate for prey and $g(K,u_2)$ represents the cost of anti-predator defense induced by fear with K reflecting response level. We assume $g(K,u_2)$ satisfies the following conditions.

($\mathbf{H}_1$) $g(0,u_2) = g(K,0) = 1$, $\lim_{u_2\to\infty} g(K,u_2) = \lim_{K\to\infty} g(K,u_2) = 0$, $\frac{\partial g(K,u_2)}{\partial K} < 0$ and $\frac{\partial g(K,u_2)}{\partial u_2} < 0$.

Considering the maturation period of the predator, we set $b(x,t,a_1)$ be the density of the predator at age a_1, location x and time t. Establish the following population model with spatial diffusion and age-structure

$$\begin{aligned} &(\partial_{a_1} + \partial_t) b(x,t,a_1) = d(a_1)\Delta b(x,t,a_1) - \mu(a_1) b(x,t,a_1), \\ &b(x,t,0) = c p u_1(x,t) u_2(x,t), \end{aligned} \tag{1.1}$$

where $x \in \Omega$, a bounded spatial habitat with the smooth boundary $\partial\Omega$, $t, a_1 > 0$, c is the conversion rate of the prey to predators, $\tau > 0$ be the maturation period of predator and age-specific functions

$$d(a_1) = \begin{cases} d_0, \ a_1 \le \tau, \\ d_2, \ a_1 > \tau, \end{cases} \quad \mu(a_1) = \begin{cases} \gamma, \ a_1 \le \tau, \\ \mu_2, \ a_1 > \tau, \end{cases}$$

represent the diffusion rate and mortality rate at age a_1, respectively. We introduce the total population of the matured predator as $u_2 = \int_\tau^\infty b(x,t,a_1)da_1$. Thus, (1.1) together with $b(x,t,\infty) = 0$ yields

$$\partial_t u_2 = d_2 \Delta u_2 + b(x,t,\tau) - \mu_2 u_2. \tag{1.2}$$

Let $s_1 = t - a_1$ and $w(x, t, s_1) = b(x, t, t - s_1)$. Along the characteristic line, solving (1.1) yields

$$\partial_t w(x,t,s_1) = \begin{cases} d_0 \Delta w(x,t,s_1) - \gamma w(x,t,s_1), & x \in \Omega,\ 0 \le t - s_1 \le \tau, \\ d_2 \Delta w(x,t,s_1) - \mu_2 w(x,t,s_1), & x \in \Omega,\ t - s_1 > \tau, \end{cases}$$
$$w(x,s_1,s_1) = b(x,s_1,0) = cpu_1(x,s_1)u_2(x,s_1),\ s_1 \ge 0;\ \ w(x,0,s_1) = b(x,0,-s_1),\ s_1 < 0.$$

Assume linear operator $d_0\Delta - \gamma$ with Neumann boundary conditions yields the C_0 semigroups $T_1(t)$. Therefore,

$$b(x,t,\tau) = w(x,t,t-\tau) = \begin{cases} T_1(\tau) b(\cdot, t-\tau, 0), & t > \tau, \\ T_1(t) b(\cdot, 0, \tau - t), & t \le \tau. \end{cases}$$

In particular, $\mathcal{G}(x, y, t)$ denotes the kernel function corresponding to $T_1(t)$. Thus

$$b(x,t,\tau) = T_1(\tau) b(\cdot, t-\tau, 0) = cp \int_\Omega \mathcal{G}(x,y,\tau) u_1(y,t-\tau) u_2(y,t-\tau) dy,\ \ t > \tau.$$

The above equation together with (1.2) yields a nonlocal diffusive predator-prey model with fear effect and maturation period of predators

$$\begin{aligned} \frac{\partial u_1}{\partial t} &= d_1 \Delta u_1 + u_1(x,t)\left[rg(K, u_2(x,t)) - \mu_1 - mu_1(x,t)\right] - pu_1(x,t)u_2(x,t), \\ \frac{\partial u_2}{\partial t} &= d_2 \Delta u_2 + cp \int_\Omega \mathcal{G}(x,y,\tau) u_1(y,t-\tau) u_2(y,t-\tau) dy - \mu_2 u_2(x,t). \end{aligned} \tag{1.3}$$

Since the spatial movement of mature predators is much bigger than that of juvenile predators, we assume that diffusion rate of juvenile predators d_0 approaches zero. Hence, the kernel function becomes $\mathcal{G} = e^{-\gamma\tau} f(x-y)$ with a Dirac-delta function f. Thus, the equation of $u_2(x,t)$ in (1.3) becomes

$$\frac{\partial u_2}{\partial t} = d_2 \Delta u_2 + cpe^{-\gamma\tau} \int_\Omega f(x-y) u_1(y,t-\tau) u_2(y,t-\tau) dy - \mu_2 u_2(x,t).$$

It follows from the properties of Dirac-delta function that

$$\begin{aligned} \int_\Omega f(x-y) u_1(y,t-\tau) u_2(y,t-\tau) dy &= \lim_{\epsilon \to 0} \int_{B_\epsilon(x)} f(x-y) u_1(y,t-\tau) u_2(y,t-\tau) dy \\ &= u_1(x,t-\tau) u_2(x,t-\tau) \lim_{\epsilon \to 0} \int_{B_\epsilon(x)} f(x-y) dy \\ &= u_1(x,t-\tau) u_2(x,t-\tau) \end{aligned}$$

where $B_\epsilon(x)$ is the open ball of radius ϵ centered at x. Therefore, model (1.3) equipped with nonnegative initial conditions and Neumann boundary conditions is

$$\begin{aligned} &\frac{\partial u_1(x,t)}{\partial t} = d_1 \Delta u_1(x,t) + u_1(x,t)\left[rg(K, u_2(x,t)) - \mu_1 - mu_1(x,t)\right] - pu_1(x,t)u_2(x,t),\ \ x \in \Omega, t > 0, \\ &\frac{\partial u_2(x,t)}{\partial t} = d_2 \Delta u_2(x,t) + cpe^{-\gamma\tau} u_1(x,t-\tau) u_2(x,t-\tau) - \mu_2 u_2(x,t),\ \ x \in \Omega, t > 0, \\ &\frac{\partial u_1(x,t)}{\partial \nu} = \frac{\partial u_2(x,t)}{\partial \nu} = 0, \quad x \in \partial\Omega, t > 0, \\ &u_1(x,\vartheta) = u_{10}(x,\vartheta) \ge 0, u_2(x,\vartheta) = u_{20}(x,\vartheta) \ge 0, \quad x \in \Omega,\ \vartheta \in [-\tau, 0]. \end{aligned} \tag{1.4}$$

If $r \leq \mu_1$, then as $t \to \infty$, we have $(u_1, u_2) \to (0, 0)$ for $x \in \Omega$, namely, two species will extinct. Throughout this paper, suppose $r > \mu_1$, which ensures that the prey and predator will persist.

This paper is organized as follows. We present results on well-posedness and uniform persistence of solutions and prove the global asymptotic stability of predator free equilibrium in Section 2. The nonexistence of nonhomogeneous steady state and steady state bifurcation are proven in Section 3. Hopf bifurcation analysis is carried out in Section 4. In Section 5, we conduct numerical exploration to illustrate some theoretical conclusions and further explore the dynamics of the nonlocal model numerically. We sum up our paper in Section 6.

2. Preliminary

Denote by $\mathbf{C} := C([-\tau, 0], X^2)$ the Banach space of continuous maps from $[-\tau, 0]$ to X^2 equipped with supremum norm, where $X = L^2(\Omega)$ is the Hilbert space of integrable function with the usual inner product. $\mathbf{C}_+$ is the nonnegative cone of $\mathbf{C}$. Let $u_1(x, t)$ and $u_2(x, t)$ be a pair of continuous function on $\Omega \times [-\tau, \infty)$ and $(u_{1t}, u_{2t}) \in C$ as $(u_{1t}(\vartheta), u_{2t}(\vartheta)) = (u_1(\cdot, t + \vartheta), u_2(\cdot, t + \vartheta))$ for $\vartheta \in [-\tau, 0]$. By using [14, Corollary 4], we can prove model (1.4) exists a unique solution. Note (1.4) is mixed quasi-monotone [15], together with comparison principle implies that the solution of (1.4) is nonnegative.

Lemma 2.1. *For any initial condition* $(u_{10}(x, \vartheta), u_{20}(x, \vartheta)) \in \mathbf{C}_+$*, model* (1.4) *possesses a unique solution* $(u_1(x, t), u_2(x, t))$ *on the maximal interval of existence* $[0, t_{max})$*. If* $t_{max} < \infty$*, then* $\limsup_{t \to t_{max}^-}(\|u_1(\cdot, t)\| + \|u_2(\cdot, t)\|) = \infty$*. Moreover,* u_1 *and* u_2 *are nonnegative for all* $(x, t) \in \overline{\Omega} \times [-\tau, t_{max})$*.*

We next prove u_1 and u_2 are bounded which implies that $t_{max} = \infty$.

Theorem 2.2. *For any initial condition* $\varphi = (u_{10}(x, \vartheta), u_{20}(x, \vartheta)) \in \mathbf{C}_+$*, model* (1.4) *possesses a global solution* $(u_1(x, t), u_2(x, t))$ *which is unique and nonnegative for* $(x, t) \in \overline{\Omega} \times [0, \infty)$*. If* $u_{10}(x, \vartheta) \geq 0(\not\equiv 0), u_{20}(x, \vartheta) \geq 0(\not\equiv 0)$*, then this solution remains positive for all* $(x, t) \in \overline{\Omega} \times (0, \infty)$*. Moreover, there exists a positive constant* ξ *independent of* φ *such that* $\limsup_{t\to\infty} u_1 \leq \xi$*,* $\limsup_{t\to\infty} u_2 \leq \xi$ *for all* $x \in \Omega$*.*

Proof. Consider

$$\begin{aligned}
&\frac{\partial w_1}{\partial t} = d_1 \Delta w_1 + r w_1 - \mu_1 w_1 - m w_1^2, \ x \in \Omega, t > 0, \\
&\frac{\partial w_1}{\partial \nu} = 0, \ x \in \partial\Omega, t > 0, \ \ w_1(x, 0) = \sup_{\vartheta \in [-\tau, 0]} u_{10}(x, \vartheta), \ x \in \Omega.
\end{aligned} \tag{2.1}$$

Clearly, $w_1(x, t)$ of (2.1) is a upper solution to (1.4) due to $\partial u_1/\partial t \leq d_1 \Delta u_1 + r u_1 - \mu_1 u_1 - m u_1^2$. By using Lemma 2.2 in [16], $(r - \mu_1)/m$ of (2.1) is globally asymptotically stable in $C(\overline{\Omega}, \mathbb{R}^+)$. This together with comparison theorem indicates

$$\limsup_{t\to\infty} u_1 \leq \lim_{t\to\infty} w_1 = \frac{r - \mu_1}{m} \ \text{ uniformly for } \ x \in \overline{\Omega}. \tag{2.2}$$

Thus, there exists $\widetilde{\xi} > 0$ which is not dependent on initial condition, such that $\|u_1\| \leq \widetilde{\xi}$ for all $t > 0$. $T_2(t)$ denotes the C_0 semigroups yielded by $d_2\Delta - \mu_2$ with the Neumann boundary condition. Then from (1.4),

$$u_2 = T_2(t) u_{20}(\cdot, 0) + c p e^{-\gamma\tau} \int_{-\tau}^{t-\tau} T_2(t - \tau - a) u_1(\cdot, a) u_2(\cdot, a) da.$$

Let $-\delta < 0$ be the principle eigenvalue of $d_2\Delta - \mu_2$ with the Neumann boundary condition. Then, $\|T_2(t)\| \leq e^{-\delta t}$. The above formula yields

$$\|u_2(\cdot,t)\| \leq e^{-\delta t}\|u_{20}(\cdot,0)\| + cpe^{-\gamma\tau}\widetilde{\xi}\int_{-\tau}^{t-\tau} e^{-\delta(t-\tau-a)}\|u_2(\cdot,a)\|da,$$

$$\leq \widetilde{B}_1 + \int_0^t \widetilde{B}_2\|u_2(\cdot,a)\|da,$$

by choosing constants $\widetilde{B}_1 \geq \|u_2(\cdot,0)\| + cp\tau\widetilde{\xi}\sup_{a\in[-\tau,0]}\|u_2(\cdot,a)\|$ and $\widetilde{B}_2 \geq cp\widetilde{\xi}$. Using Gronwall's inequality yields $\|u_2(\cdot,t)\| \leq \widetilde{B}_1 e^{\widetilde{B}_2 t}$ for all $0 \leq t < t_{max}$. Lemma 2.1 implies $t_{max} = \infty$. So (u_1,u_2) is a global solution. Moreover, if $u_{10}(x,\vartheta) \geq 0(\not\equiv 0), u_{20}(x,\vartheta) \geq 0(\not\equiv 0)$, then by [17, Theorem 4], this solution is positive for all $t > 0$ and $x \in \overline{\Omega}$.

We next prove (u_1,u_2) is ultimately bounded by a constant which is not dependent on the initial condition. Due to (2.2), there exist $t_0 > 0$ and $\xi_0 > 0$ such that $u_1(x,t) \leq \xi_0$ for any $t > t_0$ and $x \in \overline{\Omega}$. Let $z(x,t) = cu_1(x,t-\tau) + u_2(x,t)$, $\mu = \min\{\mu_1,\mu_2\}$ and $I_1 = \int_\Omega z(x,t)dx$. We integrate both sides of (1.4) and add up to obtain

$$I_1'(t) \leq \int_\Omega (c(r-\mu_1)u_1(x,t-\tau) - \mu_2 u_2(x,t))\,dx \leq cr\xi_0|\Omega| - \mu I_1(t),\ t \geq t_0 + \tau.$$

Comparison principle implies

$$\limsup_{t\to\infty}\|u_2\|_1 \leq \limsup_{t\to\infty} I_1 \leq cr\xi_0|\Omega|/\mu.$$

Especially, there exist $t_1 > t_0$ and $\xi_1 > 0$ such that $\|u_2\|_1 \leq \xi_1$ for all $t \geq t_1$.

Now, we define $V_l(t) = \int_\Omega (u_2(x,t))^l dx$ with $l \geq 1$, and estimate the upper bound of $V_2(t)$. For $t > t_1$, the second equation of (1.4) and Young's inequality yield

$$\frac{1}{2}V_2'(t) \leq -d_2\int_\Omega |\nabla u_2|^2 dx + cp\xi_0\int_\Omega u_2(x,t-\tau)u_2(x,t)dx - \mu_2 V_2(t)$$

$$\leq -d_2\|\nabla u_2\|_2^2 + \frac{cp\xi_0}{2}V_2(t-\tau) + \frac{cp\xi_0}{2}V_2(t).$$

The Gagliardo-Nirenberg inequality states:

$$\forall\,\epsilon > 0, \exists\,\hat{c} > 0,\ \text{s.t.}\ \|\mathcal{P}\|_2^2 \leq \epsilon\|\nabla\mathcal{P}\|_2^2 + \hat{c}\epsilon^{-n/2}\|\mathcal{P}\|_1^2,\ \forall\,\mathcal{P} \in W^{1,2}(\Omega).$$

We obtain

$$V_2'(t) \leq C_1 + C_2V_2(t-\tau) - (C_2 + C_3)V_2(t),$$

where $C_1 = \hat{c}\epsilon^{-n/2-1}2d_2\xi_1^2 > 0$, $C_2 = cp\xi_0 > 0$ and $C_3 = 2(d_2/\epsilon - C_2) > 0$ with small $\epsilon \in (0, d_2/C_2)$. Using comparison principle again yields $\limsup_{t\to\infty} V_2(t) \leq C_1/C_3$, which implies there exist $t_2 > t_1$ and $\xi_2 > 0$ such that $V_2 \leq \xi_2$ for $t \geq t_2$.

Let $L_l = \limsup_{t\to\infty} V_l(t)$ with $l \geq 1$, we want to estimate L_{2l} with the similar method of estimation for L_2. Multiply the second equation in (1.4) by $2lv^{2l-1}$ and integrate on Ω. Young's inequality implies

$$V_{2l}'(t) \leq -2d_2\int_\Omega |\nabla u_2^l|^2 dx + cp\xi_0 V_{2l}(t-\tau) + (2l-1)cp\xi_0 V_{2l}(t).$$

Then

$$V_{2l}'(t) \leq 2d_2\hat{c}\epsilon^{-n/2-1}V_l^2(t) - \frac{2d_2}{\epsilon}V_{2l}(t) + cp\xi_0(V_{2l}(t-\tau) + (2l-1)V_{2l}(t)),$$

via Gagliardo-Nirenberg inequality. Since $L_l = \limsup_{t\to\infty} V_l(t)$, there exists $t_l > 0$ such that $V_l \leq 1 + L_l$ when $t > t_l$. Hence,

$$V_{2l}'(t) \leq 2d_2\hat{c}\epsilon^{-(n/2+1)}(1+L_l)^2 - \frac{2d_2}{\epsilon}V_{2l}(t) + lC_4(V_{2l}(t) + V_{2l}(t-\tau))$$

with $C_4 = 2cp\xi_0$. We choose $\epsilon^{-1} = (2C_4+1)l/(2d_2)$ and $C_5 = 2d_2\hat{c}[(2C_4+1)/(2d_2)]^{n/2+1}$. Then for $t > t_l$, we obtain

$$V_{2l}'(t) \leq C_5 l^{n/2+1}(1+L_l)^2 + lC_4V_{2l}(t-\tau) - (lC_4+l)V_{2l}(t).$$

By comparison principle, the above inequality yields $L_{2l} \leq C_5 l^{n/2}(1+L_l)^2$, with a constant C_5 which is not dependent on l and initial conditions. Finally, prove $L_{2^s} < \infty$ for all $s \in \mathbb{N}_0$. Let $B = 1 + C_5$ and $\{b_s\}_{s=0}^{\infty}$ be an infinite sequence denoted by $b_{s+1} = B^{(1/2)^{(s+1)}}2^{sn((1/2)^{(s+2)})}b_s$ with the first term $b_0 = L_1 + 1$. Clearly, $L_{2^s} \leq (b_s)^{2^s}$ and

$$\lim_{s\to\infty} \ln b_s = \ln b_0 + \ln B + \frac{n}{2}\ln 2.$$

Therefore,

$$\limsup_{s\to\infty}(L_{2^s})^{(1/2)^s} \leq \lim_{s\to\infty} b_s = B(1+L_1)2^{n/2} \leq B(1+\xi_1)2^{n/2} \leq \xi := \max\{B(\xi_1+1)2^{n/2}, (r-\mu_1)/m\}.$$

The above inequality leads to $\limsup_{t\to\infty} u_1 \leq \xi$ and $\limsup_{t\to\infty} u_2 \leq \xi$ for all $x \in \Omega$.

In Theorem 2.2, we proved that the solution of model (1.4) is uniformly bounded for any nonnegative initial condition, this implies the boundedness of the population of two species. Clearly model (1.4) exists two constant steady states $E_0 = (0,0)$ and $E_1 = ((r-\mu_1)/m, 0)$, where E_0 is a saddle. Define the basic reproduction ratio [18] by

$$R_0 = \frac{cpe^{-\gamma\tau}(r-\mu_1)}{m\mu_2}. \tag{2.3}$$

Thus model (1.4) possesses exactly one positive constant steady state $E_2 = (u_1^*, u_2^*)$ if and only if $R_0 > 1$, which is equivalent to $cp(r-\mu_1) > m\mu_2$ and $0 \leq \tau < \tau_{max} := \frac{1}{\gamma}\ln\frac{cp(r-\mu_1)}{m\mu_2}$. Here,

$$u_1^* = \frac{\mu_2 e^{\gamma\tau}}{pc}, \;\; u_2^* \text{ satisfies } \; rg(K, u_2) - pu_2 = \mu_1 + mu_1^*.$$

The linearization of (1.4) at the positive constant steady state $(\tilde{u}_1, \tilde{u}_2)$ gives

$$\partial W/\partial t = \mathcal{D}\Delta W + \mathcal{L}(W_t), \tag{2.4}$$

where domain $Y := \{(u_1,u_2)^T : u_1, u_2 \in C^2(\Omega) \cap C^1(\bar{\Omega}), (u_1)_\nu = (u_2)_\nu = 0 \text{ on } \partial\Omega\}$, $W = (u_1(x,t), u_2(x,t))^T$, $\mathcal{D} = \text{diag}(d_1, d_2)$ and a bounded linear operator $\mathcal{L} : \mathbf{C} \to X^2$ is

$$\mathcal{L}(\varphi) = M\varphi(0) + M_\tau\varphi(-\tau), \text{ for } \varphi \in \mathbf{C},$$

with

$$M = \begin{pmatrix} rg(K,\tilde{u}_2) - \mu_1 - 2m\tilde{u}_1 - p\tilde{u}_2 & \tilde{u}_1(rg'_{u_2}(K,\tilde{u}_2) - p) \\ 0 & -\mu_2 \end{pmatrix},\ M_\tau = \begin{pmatrix} 0 & 0 \\ cpe^{-\gamma\tau}\tilde{u}_2 & cpe^{-\gamma\tau}\tilde{u}_1 \end{pmatrix}.$$

The characteristic equation of (2.4) gives

$$\rho\eta - \mathcal{D}\Delta\eta - \mathcal{L}(e^{\rho\cdot}\eta) = 0,\ \text{for some } \eta \in Y\backslash\{0\},$$

or equivalently

$$\det(\rho I + \sigma_n\mathcal{D} - M - e^{-\rho\tau}M_\tau) = 0,\ \text{for } n \in \mathbb{N}_0. \tag{2.5}$$

Here, σ_n is the eigenvalue of $-\Delta$ in Ω with Neumann boundary condition with respect to eigenfunction ψ_n for all $n \in \mathbb{N}_0$, and

$$0 = \sigma_0 < \sigma_1 \le \sigma_2 \le \cdots \le \sigma_n \le \sigma_{n+1} \le \cdots \ \text{and}\ \lim_{n\to\infty}\sigma_n = \infty. \tag{2.6}$$

Theorem 2.3. *(i) The trivial constant steady state $E_0 = (0,0)$ is always unstable.*

(ii) If $R_0 > 1$, then $E_1 = ((r-\mu_1)/m, 0)$ is unstable, and model (1.4) *possesses a unique positive constant steady state $E_2 = (u_1^*, u_2^*)$.*

(iii) If $R_0 \le 1$, then E_1 is globally asymptotically stable in $\mathbf{C}_+$.

Proof. (i) Note, (2.5) at E_0 takes form as $(\rho + \sigma_n d_1 - r + \mu_1)(\rho + \sigma_n d_2 + \mu_2) = 0$ for all $n \in \mathbb{N}_0$. Then $r - \mu_1 > 0$ is a positive real eigenvalue, namely, E_0 is always unstable.

(ii) The characteristic equation at E_1 gives

$$(\rho + \sigma_n d_1 + r - \mu_1)(\rho + \sigma_n d_2 + \mu_2 - cpe^{-\gamma\tau}\frac{r-\mu_1}{m}e^{-\rho\tau}) = 0\ \text{for } n \in \mathbb{N}_0. \tag{2.7}$$

Note that one eigenvalue $\rho_1 = -\sigma_n d_1 - r + \mu_1$ remains negative. Hence, we only need to consider the root distribution of the following equation

$$\rho + \sigma_n d_2 + \mu_2 - cpe^{-\gamma\tau}\frac{r-\mu_1}{m}e^{-\rho\tau} = 0\ \text{for } n \in \mathbb{N}_0. \tag{2.8}$$

According to [19, Lemma 2.1], we obtain that (2.8) exists an eigenvalue $\rho > 0$ when $R_0 > 1$, namely, E_1 is unstable when $R_0 > 1$.

(iii) By using [19, Lemma 2.1] again, any eigenvalue ρ of (2.8) satisfies $Re(\rho) < 0$ when $R_0 < 1$, namely, E_1 is locally asymptotically stable when $R_0 < 1$. Now, consider $R_0 = 1$. 0 is an eigenvalue of (2.7) for $n = 0$ and all other eigenvalues satisfy $Re(\rho) < 0$. To prove the stability of E_1, we shall calculate the normal forms of (1.4) by the algorithm introduced in [20]. Set

$$\Upsilon = \{\rho \in \mathbb{C}, \rho \text{ is the eigenvalue of equation (2.7) and } Re\rho = 0\}.$$

Obviously, $\Upsilon = \{0\}$ when $R_0 = 1$. System (1.4) satisfies the non-resonance condition relative to Υ. Denote $\overline{u}_1 = (r-\mu_1)/m$, and let $\mathbf{w} = (w_1, w_2)^T = (\overline{u}_1 - u_1, u_2)^T$ and (1.4) can be written as

$$\dot{\mathbf{w}}_t = \mathcal{A}_0\mathbf{w}_t + \mathcal{F}_0(\mathbf{w}_t)\ \text{ on }\ \mathbf{C}.$$

Here linear operator $\mathcal{A}_0$ is given by $(\mathcal{A}_0\varphi)(\vartheta) = (\varphi(\vartheta))'$ when $\vartheta \in [-\tau, 0)$ and

$$(\mathcal{A}_0\varphi)(0) = \begin{pmatrix} d_1\Delta & 0 \\ 0 & d_2\Delta \end{pmatrix}\varphi(0) + \begin{pmatrix} -r+\mu_1 & (p - rg'_{u_2}(K,0))\overline{u}_1 \\ 0 & -\mu_2 \end{pmatrix}\varphi(0) + \begin{pmatrix} 0 & 0 \\ 0 & \mu_2 \end{pmatrix}\varphi(-\tau),$$

and the nonlinear operator $\mathcal{F}_0$ satisfies $[\mathcal{F}_0(\varphi)](\vartheta) = 0$ for $-\tau \le \vartheta < 0$. By Taylor expansion, $[\mathcal{F}_0(\varphi)](0)$ can be written as

$$[\mathcal{F}_0(\varphi)](0) = \begin{pmatrix} m\varphi_1^2(0) - r\overline{u}_1 g''_{u_2}(K,0)\varphi_2^2(0)/2 + (rg'_{u_2}(K,0) - p)\varphi_1(0)\varphi_2(0) \\ -cpe^{-\gamma\tau}\varphi_1(-\tau)\varphi_2(-\tau) \end{pmatrix} + h.o.t. \tag{2.9}$$

Define a bilinear form

$$\langle\beta, \alpha\rangle = \int_\Omega \left[\alpha_1(0)\beta_1(0) + \alpha_2(0)\beta_2(0) + \mu_2\int_{-\tau}^0 \beta_2(\vartheta+\tau)\alpha_2(\vartheta)d\vartheta\right]dx,\ \beta \in C([0,\tau], X^2),\ \alpha \in \mathbf{C}.$$

Select $\alpha = (1, m/(p - rg'_{u_2}(K,0))$ and $\beta = (0,1)^T$ to be the right and left eigenfunction of $\mathcal{A}_0$ relative to eigenvalue 0, respectively. Decompose $\mathbf{w}_t$ as $\mathbf{w}_t = h\alpha + \delta$ and $\langle\beta,\delta\rangle = 0$. Notice $\mathcal{A}_0\alpha = 0$ and $\langle\beta, \mathcal{A}_0\delta\rangle = 0$. Thus,

$$\langle\beta, \dot{\mathbf{w}}_t\rangle = \langle\beta, \mathcal{A}_0\mathbf{w}_t\rangle + \langle\beta, \mathcal{F}_0(\mathbf{w}_t)\rangle = \langle\beta, \mathcal{F}_0(\mathbf{w}_t)\rangle.$$

Moreover,

$$\langle\beta, \dot{\mathbf{w}}_t\rangle = \dot{h}\langle\beta,\alpha\rangle + \langle\beta,\dot{\delta}\rangle = \dot{h}\langle\beta,\alpha\rangle.$$

It follows from the above two equations that

$$\dot{h}\frac{m(1+\mu_2\tau)|\Omega|}{p - rg'_{u_2}(K,0)} = \langle\beta, \mathcal{F}_0(h\alpha+\delta)\rangle = \int_\Omega \beta^T[\mathcal{F}_0(h\alpha+\delta)](0)dx = \int_\Omega [\mathcal{F}_0(h\alpha+\delta)]_2(0)dx.$$

When the initial value is a small perturbation of E_1, then $\delta = O(h^2)$, together with Taylor expansion yields

$$[\mathcal{F}_0(h\alpha+\delta)]_2(0) = -cpe^{-\gamma\tau}(h\alpha_1(-\tau) + \delta_1(-\tau))(h\alpha_2(-\tau) + \delta_2(-\tau)) = -\frac{cpe^{-\gamma\tau}m}{p - rg'_{u_2}(K,0)}h^2 + O(h^3).$$

Therefore, we obtain the norm form of (1.4) as follows

$$\dot{h} = \frac{-cpe^{-\gamma\tau}}{1+\mu_2\tau}h^2 + O(h^3). \tag{2.10}$$

Then for any positive initial value, the stability of zero solution of (2.10) implies E_1 is locally asymptotically stable when $R_0 = 1$.

Next, it suffices to show the global attractivity of E_1 in $\mathbf{C}_+$ when $R_0 \le 1$. Establish a Lyapunov functional $V : \mathbf{C}_+ \to \mathbb{R}$ as

$$V(\phi_1,\phi_2) = \int_\Omega \phi_2(0)^2dx + cpe^{-\gamma\tau}\frac{r-\mu_1}{m}\int_\Omega\int_{-\tau}^0 \phi_2(\theta)^2d\theta dx\ \text{ for } (\phi_1,\phi_2) \in \mathbf{C}_+.$$

Along solutions of (1.4), taking derivative of $V(\phi_1, \phi_2)$ with respect to t yields

$$\begin{aligned}\frac{dV}{dt} &\leq -2d_2 \int_\Omega |\nabla u_2|^2 dx + \int_\Omega 2cpe^{-\gamma\tau}\frac{r-\mu_1}{m}u_2(x,t-\tau)u_2(x,t) \\ &- 2\mu_2 u_2^2(x,t) + cpe^{-\gamma\tau}\frac{r-\mu_1}{m}[u_2^2(x,t) - u_2^2(x,t-\tau)]dx \\ &\leq \int_\Omega 2\mu_2(R_0 - 1)u_2^2(x,t)dx \leq 0 \text{ if } R_0 \leq 1.\end{aligned}$$

Note $\{E_1\}$ is the maximal invariant subset of $dV/dt = 0$, together with LaSalle-Lyapunov invariance principle [21, 22] implies E_1 is globally asymptotically stable if $R_0 \leq 1$.

In Theorem 2.3, E_0 is always unstable which suggests that at leat one species will persist eventually. Moreover, if $R_0 \leq 1$, then E_1 is globally asymptotically stable in $\mathbf{C}_+$, which implies that when the basic reproduction ratio is no more than one, the predator species will extinct and only the prey species can persist eventually. Next, we will prove the solution is uniformly persistent. Θ_t denotes the solution semiflow of (1.4) mapping $\mathbf{C}_+$ to $\mathbf{C}_+$; namely, $\Theta_t\varphi := (u_1(\cdot, t+\cdot), u_2(\cdot, t+\cdot)) \in \mathbf{C}_+$. Set $\zeta^+(\varphi) = \cup_{t\geq 0}\{\Theta_t\varphi\}$ be the positive orbit and $\varpi(\varphi)$ be the omega limit set of $\zeta^+(\varphi)$. Denote

$$Z := \{(\varphi_1, \varphi_2) \in \mathbf{C}_+ : \varphi_1 \not\equiv 0 \text{ and } \varphi_2 \not\equiv 0\},\ \partial Z := \mathbf{C}_+\backslash Z = \{(\varphi_1, \varphi_2) \in \mathbf{C}_+ : \varphi_1 \equiv 0 \text{ or } \varphi_2 \equiv 0\},$$

Γ_∂ as the largest positively invariant set in ∂Z, and $W^s((\tilde{u}_1, \tilde{u}_2))$ as the stable manifold associated with $(\tilde{u}_1, \tilde{u}_2)$. We next present persistence result of model (1.4).

Theorem 2.4. *Suppose $R_0 > 1$. Then there exists $\kappa > 0$ such that $\liminf_{t\to\infty} u_1(x,t) \geq \kappa$ and $\liminf_{t\to\infty} u_2(x,t) \geq \kappa$ for any initial condition $\varphi \in Z$ and $x \in \overline{\Omega}$.*

Proof. Note Θ_t is compact, and Theorem 2.2 implies Θ_t is point dissipative. Then Θ_t possesses a nonempty global attractor in $\mathbf{C}_+$ [23]. Clearly, $\Gamma_\partial = \{(\varphi_1, \varphi_2) \in \mathbf{C}_+ : \varphi_2 \equiv 0\}$, and $\varpi(\varphi) = \{E_0, E_1\}$ for all $\varphi \in \Gamma_\partial$. Define a generalized distance function ψ mapping $\mathbf{C}_+$ to $\mathbb{R}^+$ by

$$\psi(\varphi) = \min_{x\in\bar{\Omega}}\{\varphi_1(x,0), \varphi_2(x,0)\},\ \forall\varphi = (\varphi_1, \varphi_2) \in \mathbf{C}_+.$$

Following from strong maximum principle [24], $\psi(\Theta_t\varphi) > 0$ for all $\varphi \in Z$. Due to $\psi^{-1}(0,\infty) \subset Z$, assumption (P) in [25, Section 3] holds. Then verify rest conditions in [25, Theorem 3].

First, prove $W^s(E_0) \cap \psi^{-1}(0,\infty) = \emptyset$. Otherwise, there exists an initial condition $\varphi \in \mathbf{C}_+$ with $\psi(\varphi) > 0$, such that $(u_1, u_2) \to E_0$ as $t \to \infty$. Thus, for any sufficiently small $\varepsilon_1 > 0$ satisfying $rg(K, \varepsilon_1) - \mu_1 > p\varepsilon_1$, there exists $t_1 > 0$ such that $0 < u_1, u_2 < \varepsilon_1$ for all $x \in \Omega$ and $t > t_1$. Note that $rg(K, 0) - \mu_1 > 0$ and $\partial g(K, u_2)/\partial u_2 < 0$ ensure the existence of small $\varepsilon_1 > 0$. Then the first equation in (1.4) and $(\mathbf{H}_1)$ lead to

$$\partial_t u_1 > d_1\Delta u_1 + u_1\left[(rg(K,\epsilon_1) - \mu_1 - p\epsilon_1) - mu_1\right], t > t_1.$$

Notice

$$\begin{aligned}&\partial_t\hat{u}_1 - d_1\Delta\hat{u}_1 = \hat{u}_1\left[(rg(K,\epsilon_1) - p\epsilon_1 - \mu_1) - m\hat{u}_1\right],\ x \in \Omega, t > t_1, \\ &\partial_\nu\hat{u}_1 = 0,\ x \in \partial\Omega, t > t_1,\end{aligned}$$

has a globally asymptotically stable positive steady state $(rg(K,\epsilon_1)-\mu_1-p\epsilon_1)/m$ due to [16, Lemma 2.2], together with comparison principle yields $\lim_{t\to\infty} u_1 \geq \lim_{t\to\infty} \hat{u}_1 > 0$. A contradiction is derived, so $W^s(E_0)\cap\psi^{-1}(0,\infty)=\emptyset$.

Next check $W^s(E_1)\cap\psi^{-1}(0,\infty)=\emptyset$. If not, there exists $\varphi\in\mathbf{C}_+$ with $\psi(\varphi)>0$ such that (u_1,u_2) converges to E_1 as $t\to\infty$. According to (2.2), for any small $\varepsilon_2>0$ satisfying $cpe^{-\gamma\tau}((r-\mu_1)/m-\epsilon_2)>\mu_2$, there exists $t_2>0$ such that $u_1>(r-\mu_1)/m-\varepsilon_2$ for all $x\in\overline{\Omega}$ and $t>t_2-\tau$. Note that $R_0>1$ ensures the existence of $\varepsilon_2>0$. Thus, the second equation of (1.4) yields

$$\partial_t u_2 > d_2\Delta u_2 + cpe^{-\gamma\tau}(\frac{r-\mu_1}{m}-\varepsilon_2)u_2(x,t-\tau)-\mu_2 u_2(x,t),\ t>t_2.$$

In a similar manner, we derive $\lim_{t\to\infty} u_2(x,t)>0$ by above inequality, $cpe^{-\gamma\tau}((r-\mu_1)/m-\epsilon_2)>\mu_2$ and comparison principle. A contradiction yields again. Hence, it follows from Theorem 3 in [25] that, for any $\varphi\in\mathbf{C}_+$, there exists $\kappa>0$ such that $\liminf_{t\to\infty}\psi(\Theta_t\varphi)\geq\kappa$ uniformly for any $x\in\overline{\Omega}$.

Theorem 2.4 implies that when the basic reproduction ratio is bigger than one, both the predator species and prey species will persist eventually. We next investigate the stability of E_2. The corresponding characteristic equation at E_2 gives

$$\rho^2+a_{1,n}\rho+a_{0,n}+(b_{1,n}\rho+b_{0,n})e^{-\rho\tau}=0,\ n\in\mathbb{N}_0, \tag{2.11}$$

with

$$\begin{aligned}
&a_{1,n}=\sigma_n(d_1+d_2)+\mu_2+mu_1^*>0,\ a_{0,n}=(\sigma_n d_1+au_1^*)(\sigma_n d_2+\mu_2)>0,\\
&b_{1,n}=-\mu_2<0,\ \ b_{0,n}=-\mu_2(\sigma_n d_1+mu_1^*+u_2^*(rg'_{u_2}(K,u_2^*)-p)).
\end{aligned}$$

Characteristic equation (2.11) with $\tau=0$ is

$$\rho^2+(a_{1,n}+b_{1,n})\rho+a_{0,n}+b_{0,n}=0,\ n\in\mathbb{N}_0. \tag{2.12}$$

We observe that $a_{0,n}+b_{0,n}=\sigma_n d_2(\sigma_n d_1+mu_1^*)-\mu_2 u_2^*(rg'_{u_2}(K,u_2^*)-p)>0$, and $a_{1,n}+b_{1,n}=\sigma_n(d_1+d_2)+mu_1^*>0$ for all integer $n\geq 0$, which yields that any eigenvalue ρ of (2.12) satisfies $Re(\rho)<0$. Then, local asymptotic stability of E_2 is derived when $\tau=0$ which implies Turing instability can not happen for the non-delay system of (1.4). In addition, $a_{0,n}+b_{0,n}>0$ for any $n\in\mathbb{N}_0$ leads to that (2.11) can not have an eigenvalue 0 for any $\tau\geq 0$. This suggests we look for the existence of simple $\rho=\pm i\delta$ ($\delta>0$) for some $\tau>0$. Substitute $\rho=i\delta$ into (2.11) and then

$$G_n(\delta,\tau)=\delta^4+(a_{1,n}^2-2a_{0,n}-b_{1,n}^2)\delta^2+a_{0,n}^2-b_{0,n}^2=0,\ \ n\in\mathbb{N}_0, \tag{2.13}$$

with

$$\begin{aligned}
&a_{1,n}^2-2a_{0,n}-b_{1,n}^2=(\sigma_n d_1+mu_1^*)^2+(\sigma_n d_2)^2+2\mu_2\sigma_n d_2>0,\\
&a_{0,n}+b_{0,n}=\sigma_n d_2(\sigma_n d_1+mu_1^*)-\mu_2 u_2^*(rg'_{u_2}(K,u_2^*)-p)>0,\\
&a_{0,n}-b_{0,n}=\sigma_n^2 d_1 d_2+\sigma_n(2\mu_2 d_1+mu_1^* d_2)+\mu_2(2mu_1^*+u_2^*(rg'_{u_2}(K,u_2^*)-p)).
\end{aligned}$$

Thus, $a_{0,n}-b_{0,n}\geq 0$ for all $n\in\mathbb{N}_0$ is equivalent to

$$(\mathbf{A}_0):\ 2mu_1^*\geq u_2^*(p-rg'_{u_2}(K,u_2^*)).$$

If ($\mathbf{A}_0$) holds, then (2.13) admits no positive roots, together with for $\tau = 0$, any eigenvalues ρ of (2.11) satisfies $Re(\rho) < 0$, yields the next conclusion.

Theorem 2.5. *Suppose $R_0 > 1$. Then, E_2 is locally asymptotically stable provided that* ($\mathbf{A}_0$) *holds.*

3. Nonhomogeneous steady states analyses

Now, we consider positive nonhomogeneous steady states. The steady state $(u_1(x), u_2(x))$ of (1.4) satisfies the elliptic equation

$$\begin{aligned} -d_1\Delta u_1 &= rg(K, u_2)u_1 - \mu_1 u_1 - mu_1^2 - pu_1u_2, \ x \in \Omega, \\ -d_2\Delta u_2 &= cpe^{-\gamma\tau}u_1u_2 - \mu_2 u_2, \ x \in \Omega, \\ \partial_\nu u_1 &= \partial_\nu u_2 = 0, \ x \in \partial\Omega. \end{aligned} \tag{3.1}$$

From Theorem 2.3, all the solutions converge to E_1 when $R_0 \le 1$ and the positive nonhomogeneous steady state may exist only if $R_0 > 1$. Throughout this section, we assume that $R_0 > 1$. In what follows, the positive lower and upper bounds independent of steady states for all positive solutions to (3.1) are derived.

Theorem 3.1. *Assume that $R_0 > 1$. Then any nonnegative steady state of* (3.1) *other than* $(0, 0)$*, and* $((r - \mu_1)/m, 0)$ *should be positive. Moreover, there exist constants $\overline{\mathcal{B}}, \underline{\mathcal{B}} > 0$ which depend on all parameters of* (3.1) *and* Ω*, such that $\underline{\mathcal{B}} \le u_1(x), u_2(x) \le \overline{\mathcal{B}}$ for any positive solution of* (3.1) *and $x \in \overline{\Omega}$.*

Proof. We first show any nonnegative solution (u_1, u_2) other than E_0 and E_1, should be $u_1 > 0$ and $u_2 > 0$ for all $x \in \overline{\Omega}$. To see this, suppose $u_2(x_0) = 0$ for some $x_0 \in \overline{\Omega}$, then $u_2(x) \equiv 0$ via strong maximum principle and

$$0 \le d_1 \int_\Omega |\nabla(u_1 - \frac{r - \mu_1}{m})|^2 dx = \int_\Omega -mu_1(x)(u_1(x) - \frac{r - \mu_1}{m})^2 dx \le 0.$$

Thus the above inequality implies $u_1(x) \equiv 0$ or $u_1(x) \equiv (r - \mu_1)/m$. Now, we assume $u_2 > 0$ for all $x \in \overline{\Omega}$. Strong maximum principle yields $u_1 > 0$ for all $x \in \overline{\Omega}$. Hence, $u_1 > 0$ and $u_2 > 0$ for all $x \in \overline{\Omega}$.

We now prove u_1 and u_2 have a upper bound which is a positive constant. Since $-d_1\Delta u_1(x) \le (r - \mu_1 - mu_1(x))u_1(x)$, we then obtain from Lemma 2.3 in [26] that $u_1(x) \le (r - \mu_1)/m$ for any $x \in \overline{\Omega}$.

By two equations in (3.1), we obtain

$$-\Delta(d_1cu_1 + d_2u_2) \le rc(r - \mu_1)/m - \min\{\frac{\mu_1}{d_1}, \frac{\mu_2}{d_2}\}(d_1cu_1 + d_2u_2).$$

By using [26, Lemma 2.3] again, we conclude

$$u_1(x), u_2(x) \le \overline{\mathcal{B}} = \frac{rc(r - \mu_1)}{m \min\{c\mu_1, \mu_2, \mu_1 d_2/d_1, \mu_2 d_1 c/d_2\}}.$$

Next, we only need to prove $\|u_1(x)\|$ and $\|u_2(x)\|$ have a positive lower bound which is not dependent on the solution. Otherwise, there exists a positive steady states sequence $(u_{1,n}(x), u_{2,n}(x))$ such that either $\lim_{n\to\infty} \|u_{1,n}\|_\infty = 0$ or $\lim_{n\to\infty} \|u_{2,n}\|_\infty = 0$. Integrating second equation of (3.1) gives

$$0 = \int_\Omega u_{2,n}(cpe^{-\gamma\tau}u_{1,n} - \mu_2)dx. \tag{3.2}$$

If $\|u_{1,n}(x)\|_\infty \to 0$ as $n \to \infty$, then $cpe^{-\gamma\tau}u_{1,n}(x) - \mu_2 < -\mu_2/2$ for sufficiently large n, which yields $u_{2,n}(cpe^{-\gamma\tau}u_{1,n} - \mu_2) < 0$, a contradiction derived. Thus, $\|u_{2,n}(x)\|_\infty \to 0$ as $n \to \infty$ holds. We then assume that $(u_{1,n}, u_{2,n}) \to (u_{1,\infty}, 0)$, as $n \to \infty$ where $u_{1,\infty} \geq 0$. Similarly, we obtain that either $u_{1,\infty} \equiv 0$ or $u_{1,\infty} \equiv (r - \mu_1)/m$. Obviously, $u_{1,\infty} \not\equiv 0$ based on the above argument, thus $u_{1,\infty} \equiv (r - \mu_1)/m$ and $\lim_{n\to\infty} cpe^{-\gamma\tau}u_{1,n}(x) - \mu_2 = \mu_2(R_0 - 1) > 0$. This again contradicts (3.2). Hence, we have shown $\|u_1(x)\|_\infty$ and $\|u_2(x)\|_\infty$ have a positive constant lower bound independent on the solution. Therefore, $u_1(x)$ and $u_2(x)$ have a uniform positive constant lower bound independent on the solution of (3.1) via Harnack's inequality [26, Lemma 2.2]. This ends the proof.

Theorem 3.2. *Suppose $R_0 > 1$. There exists a constant $\chi > 0$ depending on $r, \mu_1, p, c, \gamma, \tau, \mu_2, g$ and σ_1, such that if* $\min\{d_1, d_2\} > \chi$ *then model* (1.4) *admits no positive spatially nonhomogeneous steady states, where σ_1 is defined in* (2.6).

Proof. Denote the averages of the positive solution (u_1, u_2) of system (3.1) on Ω by

$$\widetilde{u_1} = \frac{\int_\Omega u_1(x)dx}{|\Omega|} \text{ and } \widetilde{u_2} = \frac{\int_\Omega u_2(x)dx}{|\Omega|}.$$

By (2.2), we have $u_1(x) \leq \overline{u}_1$, where $\overline{u}_1 = (r - \mu_1)/m$, which implies $\widetilde{u_1} \leq \overline{u}_1$. Multiplying the first equation by $ce^{-\gamma\tau}$ and adding two equations of (3.1) lead to

$$-(d_1ce^{-\gamma\tau}\Delta u_1 + d_2\Delta u_2) = ce^{-\gamma\tau}\left(rg(K, u_2)u_1 - \mu_1u_1 - mu_1^2\right) - \mu_2u_2.$$

The integration of both sides for the above equation yields

$$\widetilde{u_2} = \frac{ce^{-\gamma\tau}}{\mu_2|\Omega|}\int_\Omega \left(rg(K, u_2)u_1 - \mu_1u_1 - mu_1^2\right)dx \leq (r - \mu_1)\overline{u}_1\frac{ce^{-\gamma\tau}}{\mu_2} := M_v.$$

It is readily seen that $\int_\Omega(u_1-\widetilde{u_1})dx = \int_\Omega(u_2-\widetilde{u_2})dx = 0$. Note u_1 and u_2 are bounded by two constants $\overline{u}_1 > 0$ and $\overline{u}_2 := rc\overline{u}_1/\min\{\mu_1d_2/d_1, \mu_2\} > 0$ by Theorem 3.1. Denote $M_f = \max_{u_2\in[0,\overline{u}_2]} |g'_{u_2}(K, u_2)|$ and we then obtain

$$\begin{aligned}
d_1\int_\Omega |\nabla(u_1 - \widetilde{u_1})|^2dx &= \int_\Omega(u_1 - \widetilde{u_1})(rg(K, u_2)u_1 - \mu_1u_1 - mu_1^2)dx - \int_\Omega pu_1u_2(u_1 - \widetilde{u_1})dx \\
&= \int_\Omega(u_1 - \widetilde{u_1})\left(rg(K, u_2)u_1 - \mu_1u_1 - mu_1^2 - (rg(K, \widetilde{u_2})\widetilde{u_1} - \mu_1\widetilde{u_1} - m\widetilde{u_1}^2)\right)dx \\
&\quad + \int_\Omega p(\widetilde{u_1}\widetilde{u_2} - u_1u_2)(u_1 - \widetilde{u_1})dx \\
&\leq \left(r - \mu_1 + (rM_f + p)\frac{\overline{u}_1}{2}\right)\int_\Omega(u_1 - \widetilde{u_1})^2dx + (rM_f + p)\frac{\overline{u}_1}{2}\int_\Omega(u_2 - \widetilde{u_2})^2dx,
\end{aligned}$$

$$d_2\int_\Omega |\nabla(u_2-\widetilde{u_2})|^2dx = \int_\Omega (cpe^{-\gamma\tau}u_1u_2-\mu_2u_2)(u_2-\widetilde{u_2})dx$$
$$=cpe^{-\gamma\tau}\int_\Omega (u_1u_2-\widetilde{u_1}\widetilde{u_2})(u_2-\widetilde{u_2})dx - \int_\Omega \mu_2(u_2-\widetilde{u_2})^2dx$$
$$\le cpe^{-\gamma\tau}\int_\Omega \frac{M_v}{2}(u_1-\widetilde{u_1})^2dx + cpe^{-\gamma\tau}(\overline{u}_1+\frac{M_v}{2})\int_\Omega (u_2-\widetilde{u_2})^2dx.$$

Set $A_1 = (r-\mu_1) + \left((rM_f+p)\overline{u}_1 + cpe^{-\gamma\tau}M_v\right)/2$, and $A_2 = \left((rM_f+p)\overline{u}_1 + cpe^{-\gamma\tau}(2\overline{u}_1+M_v)\right)/2$. Then, the above inequalities and Poincar*é* inequality yield that

$$d_1\int_\Omega |\nabla(u_1-\widetilde{u_1})|^2dx + d_2\int_\Omega |\nabla(u_2-\widetilde{u_2})|^2dx \le \chi\int_\Omega \left(|\nabla(u_1-\widetilde{u_1})|^2+|\nabla(u_2-\widetilde{u_2})|^2\right)dx,$$

with a positive constant $\chi = \max\{A_1/\sigma_1, A_2/\sigma_1\}$ depending on $r, f, \mu_1, p, c, \gamma, \tau, \mu_2$ and σ_1. Hence, if $\chi < \min\{d_1, d_2\}$, then $\nabla(u_1-\widetilde{u_1}) = \nabla(u_2-\widetilde{u_2}) = 0$, which implies (u_1, u_2) is a constant solution.

Select $u_1^* := \nu$ as the bifurcation parameter and study nonhomogeneous steady state bifurcating from E^*. Let $u_{2,\nu}$ satisfy $rg(K,u_2) - pu_2 = \mu_1 + m\nu$, $E_2 = (\nu, u_{2,\nu})$, and $(\widehat{u_1}, \widehat{u_2}) = (u_1-\nu, u_2-u_{2,\nu})$. Drop $\widehat{\cdot}$. System (3.1) becomes

$$\mathcal{H}(\nu,u_1,u_2) = \begin{pmatrix} d_1\Delta u_1 + (u_1+\nu)(rg(K,u_2+u_{2,\nu}) - \mu_1 - m(u_1+\nu) - p(u_2+u_{2,\nu})) \\ d_2\Delta u_2 + cpe^{-\gamma\tau}(u_1+\nu)(u_2+u_{2,\nu}) - \mu_2(u_2+u_{2,\nu}) \end{pmatrix} = 0,$$

for $(\nu,u_1,u_2) \in \mathbb{R}^+\times\mathcal{Y}$ with $\mathcal{Y} = \{(u_1,u_2) : u_1,u_2 \in H^2(\Omega), (u_1)_\nu = (u_2)_\nu = 0, \text{ on } \partial\Omega\}$. Calculating Fr*é*chet derivative of $\mathcal{H}$ gives

$$D_{(u_1,u_2)}\mathcal{H}(\nu,0,0) = \begin{pmatrix} d_1\Delta - m\nu & \nu(rg'_{u_2}(K,u_{2,\nu})-p) \\ cpe^{-\gamma\tau}u_{2,\nu} & d_2\Delta \end{pmatrix}.$$

Then the characteristic equation follows

$$\rho^2 + P_i(\nu)\rho + Q_i(\nu) = 0 \ \text{ for } \ i \in \mathbb{N}_0, \tag{3.3}$$

where

$$P_i(\nu) = m\nu + (d_1+d_2)\sigma_i, \ \ Q_i(\nu) = d_1d_2\sigma_i^2 + d_2m\nu\sigma_i - \mu_2u_{2,\nu}(rg'_{u_2}(K,u_{2,\nu})-p).$$

Obviously, $Q_i > 0$ and $P_i > 0$ for all $\nu \in \mathbb{R}^+$ and $i \in \mathbb{N}_0$. Therefore, (3.3) does not have a simple zero eigenvalue. According to [4], we obtain the nonexistence of steady state bifurcation bifurcating at E_2.

Theorem 3.3. *Model* (1.4) *admits no positive nonhomogeneous steady states bifurcating from* E_2.

4. Hopf bifurcation analyses

Next, the stability switches at E_2 and existence of periodic solutions of (1.4) bifurcating from E_2 are studied. Suppose $R_0 > 1$, namely, $cp(r-\mu_1) > m\mu_2$ and $0 \le \tau < \tau_{max} := \frac{1}{\gamma}\ln\frac{cp(r-\mu_1)}{m\mu_2}$ to guarantee the existence of E_2.

4.1. Local Hopf bifurcation analyses

Recall the stability of E_2 for $\tau = 0$ is proved and 0 is not the root of (2.11) for $\tau \geq 0$. So, we only consider eigenvalues cross the imaginary axis to the right which corresponds to the stability changes of E_2. Now, we shall consider the positive root of $G_n(\delta, \tau)$. Clearly, there exists exactly one positive root of $G_n(\delta, \tau) = 0$ if and only if $a_{0,n} < b_{0,n}$ for $n \in \mathbb{N}_0$. More specifically,

$$(\mathbf{A}_1): \ 2mu_1^* < u_2^*(p - rg'_{u_2}(K, u_2^*))$$

is the sufficient and necessary condition to ensure $G_0(\delta, \tau)$ has exactly one positive zero. For some integer $n \geq 1$, the assumption $(\mathbf{A}_1)$ is a necessary condition to guarantee $G_n(\delta, \tau)$ exists positive zeros. Set

$$J_n = \{\tau : \tau \in [0, \tau_{max}) \text{ satisfies } a_{0,n}(\tau) < b_{0,n}(\tau)\}, \quad n \in \mathbb{N}_0. \tag{4.1}$$

Implicit function theorem implies $G_n(\delta, \tau)$ has a unique zero

$$\delta_n(\tau) = \sqrt{\left(\left[b_{1,n}^2 + 2a_{0,n} - a_{1,n}^2 + \sqrt{(b_{1,n}^2 + 2a_{0,n} - a_{1,n}^2)^2 - 4(a_{0,n}^2 - b_{0,n}^2)}\right]/2\right)}$$

which is a C^1 function for $\tau \in J_n$. Hence, $i\delta_n(\tau)$ is an eigenvalue of (2.11), and $\delta_n(\tau)$ satisfies

$$\begin{aligned} \sin(\delta_n(\tau)\tau) &= \frac{\delta_n(-\mu_2\delta_n^2 + \mu_2 a_{0,n} + b_{0,n}a_{1,n})}{\mu_2^2\delta_n^2 + b_{0,n}^2} := h_{1,n}(\tau), \\ \cos(\delta_n(\tau)\tau) &= \frac{b_{0,n}(\delta_n^2 - a_{0,n}) + a_{1,n}\mu_2\delta_n^2}{\mu_2^2\delta_n^2 + b_{0,n}^2} := h_{2,n}(\tau), \end{aligned} \tag{4.2}$$

for $n \in \mathbb{N}_0$. Let

$$\vartheta_n(\tau) = \begin{cases} \arccos h_{2,n}(\tau), \ \text{if } \delta_n^2 < a_{0,n} + b_{0,n}a_{1,n}/\mu_2, \\ 2\pi - \arccos h_{2,n}(\tau), \ \text{if } \delta_n^2 \geq a_{0,n} + b_{0,n}a_{1,n}/\mu_2, \end{cases}$$

which is the unique solution of $\sin\vartheta_n = h_{1,n}$ and $\cos\vartheta_n = h_{2,n}$ and satisfies $\vartheta_n(\tau) \in (0, 2\pi]$ for $\tau \in I_n$.

According to [3, 27], we arrive at the next properties.

Lemma 4.1. *Suppose that $R_0 > 1$ and $(\mathbf{A}_1)$ holds.*

(i) There exists a nonnegative integer M_1 such that $J_n \neq \emptyset$ for $0 \leq n \leq M_1$, with $J_{M_1} \subset J_{M_1-1} \subset \cdots \subset J_1 \subset J_0$, and $J_n = \emptyset$ for $n \geq 1 + M_1$, where J_n is defined in (4.1).

(ii) Define

$$\mathcal{S}_n^k(\tau) = \delta_n(\tau)\tau - \vartheta_n(\tau) - 2k\pi \ \text{ for integer } 0 \leq n \leq M_1, \ k \in \mathbb{N}_0, \text{ and } \tau \in J_n. \tag{4.3}$$

Then, $\mathcal{S}_0^0(0) < 0$; for $0 \leq n \leq M_1$ and $k \in \mathbb{N}_0$, we have $\lim_{\tau\to\widehat{\tau}_n} \mathcal{S}_n^k(\tau) = -(2k+1)\pi$, where $\widehat{\tau}_n = \sup J_n$; $\mathcal{S}_n^{k+1}(\tau) < \mathcal{S}_n^k(\tau)$ and $\mathcal{S}_n^k(\tau) > \mathcal{S}_{n+1}^k(\tau)$.

(iii) For each integer $n \in [0, N_1]$ and some $k \in \mathbb{N}_0$, $\mathcal{S}_n^k(\tau)$ has one positive zero $\overline{\tau}_n \in J_n$ if and only if (2.11) *has a pair of eigenvalues $\pm i\delta_n(\overline{\tau}_n)$. Moreover,*

$$Sign(Re\lambda'(\overline{\tau}_n)) = Sign((\mathcal{S}_n^k)'(\overline{\tau}_n)). \tag{4.4}$$

When $(\mathcal{S}_n^k)'(\overline{\tau}_n) < 0$, $\pm i\delta_n(\overline{\tau}_n)$ cross the imaginary axis from right to left at $\tau = \overline{\tau}_n$; when $(\mathcal{S}_n^k)'(\overline{\tau}_n) > 0$, $\pm i\delta_n(\overline{\tau}_n)$ cross the imaginary axis from left to right.

If $\sup_{\tau\in I_0} \mathcal{S}_0^0 \le 0$, then $\mathcal{S}_n^k < 0$ in J_n holds for any $0 \le n \le M_1$ and $k \in \mathbb{N}_0$; or only $\mathcal{S}_0^0$ has a zero with even multiplicity in J_0 and $\mathcal{S}_n^k < 0$ for any positive integers n and k. Therefore, E_2 is locally asymptotically stable for $\tau \in [0, \tau_{max})$. The following assumption ensures Hopf bifurcation may occur at E_2.

($\mathbf{A}_2$) $\sup_{\tau\in J_0} \mathcal{S}_0^0(\tau) > 0$ and $\mathcal{S}_n^k(\tau)$ has at most two zeros (counting multiplicity) for integer $0 \le n \le M_1$ and $k \in \mathbb{N}_0$.

Note $\sup_{\tau\in J_n} \mathcal{S}_n^0$ is strictly decreasing in n due to Lemma 4.1. It then follows from ($\mathbf{A}_2$), and $\mathcal{S}_n^k(0) < \mathcal{S}_0^0(0) < 0$, $\lim_{\tau\to\widehat{\tau}_n^-} \mathcal{S}_n^k(\tau) < 0$ for any integer $0 \le n \le M_1$ and $k \in \mathbb{N}_0$, that we can find two positive integers

$$M = \{n \in [0, M_1] : \ \sup \mathcal{S}_n^0 > 0 \text{ and } \sup \mathcal{S}_{n+1}^0 \le 0\} \ge 0, \tag{4.5}$$

and

$$K_n = \{j \ge 1 : \ \sup \mathcal{S}_n^{j-1} > 0 \text{ and } \sup \mathcal{S}_n^j \le 0\} \ge 1, \text{ for any integer } 0 \le n \le M. \tag{4.6}$$

Then $\mathcal{S}_n^k(\tau)$ admits two simple zeros τ_n^k and $\tau_n^{2K_n-k-1}$ for $k \in [0, K_n - 1]$ and no zeros for $k \ge K_n$. The above analysis, together with Lemma 4.1(iii), yields the next result.

Lemma 4.2. *Suppose $R_0 > 1$ and* ($\mathbf{A}_1$) *and* ($\mathbf{A}_2$) *hold. Let $\mathcal{S}_n^k(\tau)$, M and K_n be defined in* (4.3), (4.5) *and* (4.6).

(i) For integer $n \in [0, M]$, there are $2K_n$ simple zeros τ_n^j $(0 \le j \le 2K_n - 1)$ of $\mathcal{S}_n^i(\tau)$ $(0 \le i \le K_n - 1)$, $0 < \tau_n^0 < \tau_n^1 < \tau_n^2 < \cdots < \tau_n^{2K_n-1} < \widehat{\tau}_n$, and $d\mathcal{S}_n^i(\tau_n^i)/d\tau > 0$ and $d\mathcal{S}_n^i(\tau_n^{2K_n-i-1})/d\tau < 0$ for each $0 \le i \le K_n - 1$.

(ii) If there exist exactly two bifurcation values $\tau_{n_1}^j = \tau_{n_2}^i := \tau_*$ with $n_1 \ne n_2$ and $(n_1, j), (n_2, i) \in [0, M] \times [0, 2K_n - 1]$, then the double Hopf bifurcation occurs at E_2 when $\tau = \tau_*$.

Collect all values τ_n^j with $0 \le n \le M$ and $0 \le j \le 2K_n - 1$ in a set. To ensure Hopf bifurcation occurs, remove values which appear more than once. The new set becomes

$$\Sigma = \{\tau_0, \tau_1, \cdots, \tau_{2L-1}\}, \text{ with } \tau_i < \tau_j \text{ if } i < j \text{ and } 1 \le L \le \sum_{n=0}^{M} K_n. \tag{4.7}$$

Lemma 4.1(ii) implies $\mathcal{S}_0^0(\tau)$ exists two simple zeros $\tau_0 < \tau_{2L-1}$. When $\tau = \tau_i$ with $0 \le i \le 2L-1$, the Hopf bifurcation occurs at E_2. Moreover, E_2 is locally asymptotically stable for $\tau \in [0, \tau_0) \cup (\tau_{2L-1}, \tau_{max})$ and unstable for $\tau \in (\tau_0, \tau_{2L-1})$. Define

$$\Sigma_0 = \{\tau \in \Sigma : \mathcal{S}_0^j(\tau) = 0 \text{ for integer } 0 \le j \le K_0\}. \tag{4.8}$$

Theorem 4.3. *Suppose $R_0 > 1$. Let J_n, $\mathcal{S}_n^k(\tau)$, Σ and Σ_0 be defined in* (4.1), (4.3), (4.7) *and* (4.8), *respectively.*

(i) E_2 is locally asymptotically stable for all $\tau \in [0, \tau_{max})$ provided that either $J_0 = \emptyset$ or $\sup_{\tau \in J_0} \mathcal{S}_0^0(\tau) \le 0$.

(ii) If ($\mathbf{A}_1$) *and* ($\mathbf{A}_2$) *hold, then a Hopf bifurcation occurs at E_2 when $\tau \in \Sigma$. E_2 is locally asymptotically stable for $\tau \in [0, \tau_0) \cup (\tau_{2L-1}, \tau_{max})$, and unstable for $\tau \in (\tau_0, \tau_{2L-1})$. Further, for $\tau \in \Sigma \backslash \Sigma_0$, the bifurcating periodic solution is spatially nonhomogeneous; for $\tau \in \Sigma_0$, the bifurcating periodic solution is spatially homogeneous.*

4.2. Global Hopf bifurcation analyses

Next investigate the properties of bifurcating periodic solutions by global Hopf bifurcation theorem [28]. Set $y(t) = (y_1(t), y_2(t))^T = (u_1(\cdot, \tau t) - u_1^*, u_2(\cdot, \tau t) - u_2^*)^T$ and write (1.4) as

$$y'(t) = Ay(t) + Z(y_t, \tau, Q), \ (t, \tau, Q) \in \mathbb{R}^+ \times [0, \tau_{max}) \times \mathbb{R}^+, \ y_t \in C([-1, 0], X^2), \tag{4.9}$$

where $y_t(\nu) = y(t + \nu)$ for $\nu \in [-1, 0]$, $A = \text{diag}(\tau d_1 \Delta - \tau \mu_1, \tau d_2 \Delta - \tau \mu_2)$ and

$$Z(y_t) = \tau \begin{pmatrix} (y_{1t}(0) + u_1^*) \left(rg(K, y_{2t}(0) + u_2^*) - m(y_{1t}(0) + u_1^*) - p(y_{2t}(0) + u_2^*) \right) - \mu_1 u_1^* \\ cpe^{-\gamma\tau}(y_{1t}(-1) + u_1^*)(y_{2t}(-1) + u_2^*) - \mu_2 u_2^* \end{pmatrix}.$$

$\{\Psi(t)\}_{t \ge 0}$ denotes the semigroup yielded by A in Ω, with Neumann boundary condition. Clearly, $\lim_{t \to \infty} \Psi(t) = 0$. The solution of (4.9) can be denoted by

$$y(t) = \Psi(t) y(0) + \int_0^t \Psi(t - \sigma) Z(y_\sigma) d\sigma. \tag{4.10}$$

If $y(t)$ is a a−periodic solution of (4.9), then (4.10) yields

$$y(t) = \int_{-\infty}^t \Psi(t - s) Z(y_\sigma) d\sigma, \tag{4.11}$$

since $\Psi(t + na) y(0) \to 0$ as $n \to \infty$. Hence we only need to consider (4.11). The integral operator of (4.11) is differentiable, completely continuous, and G-equivariant by [24, Chapter 6.5]. The condition $\min\{d_1, d_2\} > \chi$ ensures (1.4) admits exactly one positive steady state E_2. Using a similar argument as in [29, Section 4.2], $(H1)$–$(H4)$ in [24, Chapter 6.5] hold and we shall study the periodic solution.

Lemma 4.4. *Suppose $R_0 > 1$, then all nonnegative periodic solutions of* (4.9) *satisfies $\kappa \le u_1(x, t), u_2(x, t) \le \xi$ for all $(x, t) \in \overline{\Omega} \times \mathbb{R}^+$, where ξ and κ are defined in Theorems 2.2 and 2.4, respectively.*

Lemma 4.4 can be obtained by Theorems 2.2 and 2.4. We further assume

$$(\mathbf{A}_3): \ \frac{g(K, u_2) - g(K, u_2^*)}{u_1 - u_1^*} - \frac{m}{r} \le 0 \ \text{ for } \ u_1 \in [\kappa, \frac{r - \mu_1}{m}] \ \text{ and } \ u_2 \in [\kappa, \xi].$$

This technical condition is used to exclude the τ−periodic solutions. Note assumption ($\mathbf{A}_3$) holds when $K = 0$. This, together with that $(g(K, u_2) - g(K, u_2^*))/(u_1 - u_1^*)$ is continuous in K, implies there exists $\varepsilon > 0$ such that ($\mathbf{A}_3$) holds for $0 \le K < \varepsilon$, that is, ($\mathbf{A}_3$) holds for the model (1.4) with weak fear effect.

Lemma 4.5. *Assume that $R_0 > 1$ and $(\mathbf{A}_3)$ holds, then model (1.4) admits no nontrivial $\tau-$periodic solution.*

Proof. Otherwise, let (u_1, u_2) be the nontrivial $\tau-$periodic solution, that is, $(u_1(x, t-\tau), u_2(x, t-\tau)) = (u_1(x, t), u_2(x, t))$. Thus, we have

$$\begin{aligned}
&\partial_t u_1 = d_1\Delta u_1 + u_1 (rg(K, u_2) - \mu_1 - mu_1 - pu_2), \ x \in \Omega, t > 0,\\
&\partial_t u_2 = d_2\Delta u_2 + cpe^{-\gamma\tau} u_1 u_2 - \mu_2 u_2, \ x \in \Omega, t > 0,\\
&\partial_\nu u_1 = \partial_\nu u_2 = 0, x \in \partial\Omega, t > 0,\\
&u_1(x, \vartheta) = u_{10}(x, \vartheta) \geq 0, u_2(x, \vartheta) = u_{20}(x, \vartheta) \geq 0, x \in \Omega, \vartheta \in [-\tau, 0].
\end{aligned} \tag{4.12}$$

Claim

$$(u_1, u_2) \to E_2 \ \text{ as } \ t \to \infty.$$

To see this, establish the Lyapunov functional $\mathbb{L}_1 : C(\bar{\Omega}, \mathbb{R}^+ \times \mathbb{R}^+) \to \mathbb{R}$,

$$\mathbb{L}_1(\phi_1, \phi_2) = \int_\Omega (ce^{-\gamma\tau}(\phi_1 - u_1^* \ln \phi_1) + (\phi_2 - u_2^* \ln \phi_2))\, dx \text{ for } (\phi_1, \phi_2) \in C(\bar{\Omega}, \mathbb{R}^+ \times \mathbb{R}^+).$$

Along the solution of system (4.12), the time derivative of $\mathbb{L}_1(\phi_1, \phi_2)$ is

$$\frac{d\mathbb{L}_1}{dt} = \int_\Omega \left[-\frac{d_1\mu_2|\nabla u_1|^2}{pu_1^2} - d_2 u_2^* \frac{|\nabla u_2|^2}{u_2^2} + r(u_1 - u_1^*)^2 \left(\frac{g(K, u_2) - g(K, u_2^*)}{u_1 - u_1^*} - \frac{m}{r} \right) \right] dx.$$

The assumption $(\mathbf{A}_3)$ ensures $d\mathbb{L}_1/dt \leq 0$ for all $(u_1, u_2) \in C(\bar{\Omega}, \mathbb{R}^+ \times \mathbb{R}^+)$. The maximal invariant subset of $d\mathbb{L}_1/dt = 0$ is $\{E_2\}$. Therefore, E_2 attracts all positive solution of (4.12) by LaSalle-Lyapunov invariance principle [21, 22] which excludes the nonnegative nontrivial $\tau-$periodic solution.

To obtain the nonexistence of $\tau-$periodic solution for model (1.4), we must use the condition $(\mathbf{A}_3)$ which is very restrictive. However, in numerical simulations, Lemma 4.5 remains true even $(\mathbf{A}_3)$ is violated. Thus, we conjecture the nonexistence of τ-periodic solution for (1.4).

In the beginning of this section, when $\tau = \tau_i$ with $0 \leq i \leq 2L - 1$, $\pm i\delta_n(\tau_i)$ are a pair of eigenvalues of (2.11). Give the next standard notations:

(*i*) For $0 \leq i \leq 2L - 1$, $(E_2, \tau_i, 2\pi/(\delta_n(\tau_i)\tau_i))$ is an isolated singular point.

(*ii*) $\Gamma = Cl\{(y, \tau, Q) \in X^2 \times \mathbb{R}^+ \times \mathbb{R}^+ : \ y$ is the nontrivial Q-periodic solution of (4.9)$\}$ is a closed subset of $X^2 \times \mathbb{R}^+ \times \mathbb{R}^+$.

(*iii*) For $0 \leq i \leq 2L - 1$, $C_i(E_2, \tau_i, Q_i)$ is the connected component of (E_2, τ_i, Q_i) in Γ.

(*iv*) For integer $0 \leq k \leq \max\limits_{n \in [0,M]} K_n - 1$, let $\Sigma_H^k = \{\tau \in \Sigma : \mathcal{S}_n^k(\tau) = 0$ for integer $0 \leq n \leq M\}$.

We are ready to present a conclusion on the global Hopf branches by a similar manner in [30, Theorem 4.12].

Theorem 4.6. *Assume $R_0 > 1$, $\min\{d_1, d_2\} > \chi$ and $(\mathbf{A}_1)$–$(\mathbf{A}_3)$ hold. Then we have the following results.*

(i) The global Hopf branch $C_i(E_2, \tau_i, Q_i)$ is bounded for $\tau_i \in \Sigma_H^k$ with $k \geq 1$ and $i \in [0, 2L - 1]$.

(ii) For any $\tau \in (\min\limits_k \Sigma_H^k, \max\limits_k \Sigma_H^k)$, model (1.4) possesses at least one periodic solution.

(iii) For $\tau_{i_1} \in \Sigma_H^{k_1}$, $\tau_{i_2} \in \Sigma_H^{k_2}$ with $i_1, i_2 \in [0, 2L - 1]$ and $k_1, k_2 \in [0, \max\limits_{n \in [0,M]} K_n - 1]$, we have $C_{i_1}(E_2, \tau_{i_1}, Q_{i_1}) \cap C_{i_2}(E_2, \tau_{i_2}, Q_{i_2}) = \emptyset$ if $k_1 \neq k_2$.

5. Numerical simulation

To verify obtained theoretical results, the numerical simulation is presented in this part. We choose the fear effect function as $g(K, u_2) = e^{-Ku_2}$ and let

$$\Omega = (0, 2\pi), d_1 = d_2 = 1, r = 8, \mu_1 = 0.1, a = 0.2, p = 1, \gamma = 0.3, \mu_2 = 0.2, c = 0.1.$$

Figure 1 shows the existence and stability of E_0, E_1 and E_2, and Hopf bifurcation curve of model (1.4) in $K - \tau$ plane. Above the line $\tau = \tau_{max}$, E_2 does not exist, E_1 is globally asymptotically stable and E_0 is unstable; Below the line $\tau = \tau_{max}$, there exist three constant steady states E_0, E_1 and E_2. In the region which is bounded by $\tau = \tau_{max}$ and $2mu_1^* + u_2^*(rg'_{u_2}(K, u_2^*) - p) = 0$, no Hopf bifurcation occurs and E_2 is stable. In the region which is bounded by $\tau = \tau_0$ and $\tau = \tau_1$, there exist periodic solutions through Hopf bifurcation bifurcating at E_2

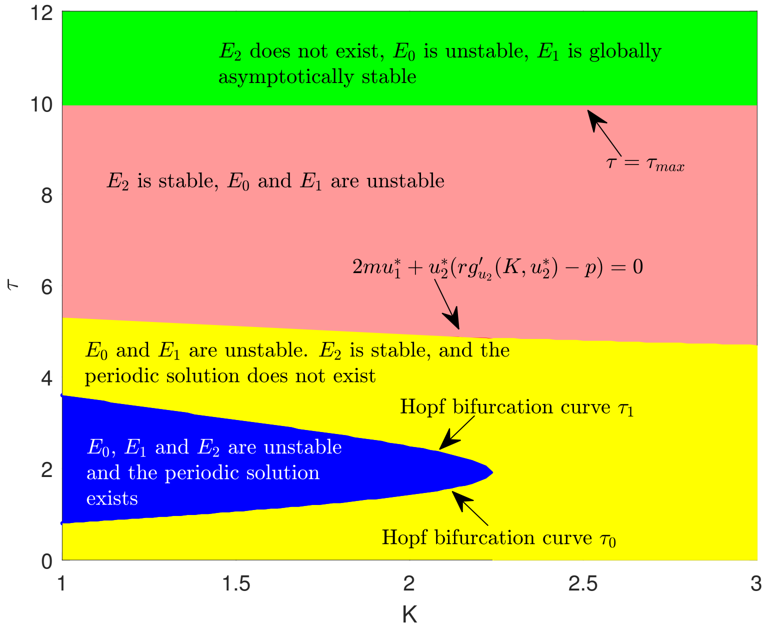


Figure 1. Basic dynamics of model (1.4) in different regions with $K - \tau$ plane.

Fix $K = 1.07$, then by simple calculation, we have $\tau_{max} \approx 9.944$, $\tau_0 \approx 0.85$, $\tau_1 \approx 3.55$, $J_0 = [0, 5.25]$, $J_1 = [0, 3.05]$, $J_n = \emptyset$ for $n \geq 2$, $\sup_{\tau \in J_0} \mathcal{S}_0^0 > 0$ and $\sup_{\tau \in J_1} \mathcal{S}_1^0 < 0$. Thus, all Hopf bifurcation values τ_0 and τ_1 are the all zeros of $S_0^k(\tau)$ for integer $k \geq 0$. We summarize the dynamics of model (1.4) as follows.

- (i) For $\tau \in [\tau_{max}, \infty)$, we obtain E_1 is globally asymptotically stable and E_0 is unstable, see Figure 2(a).
- (ii) For $\tau \in (0, \tau_0) \cup (\tau_1, \tau_{max})$, we obtain E_2 is locally asymptotically stable, and two constant steady state E_0 and E_1 are unstable, as shown in Figure 2(b).
- (iii) For $\tau \in (\tau_0, \tau_1)$, we obtain E_0, E_1 and E_2 are unstable, a periodic solution bifurcates from E_2, as shown in Figure 2(c). Further, a Hopf bifurcation occurs at E_2 when $\tau = \tau_0$, and τ_1.

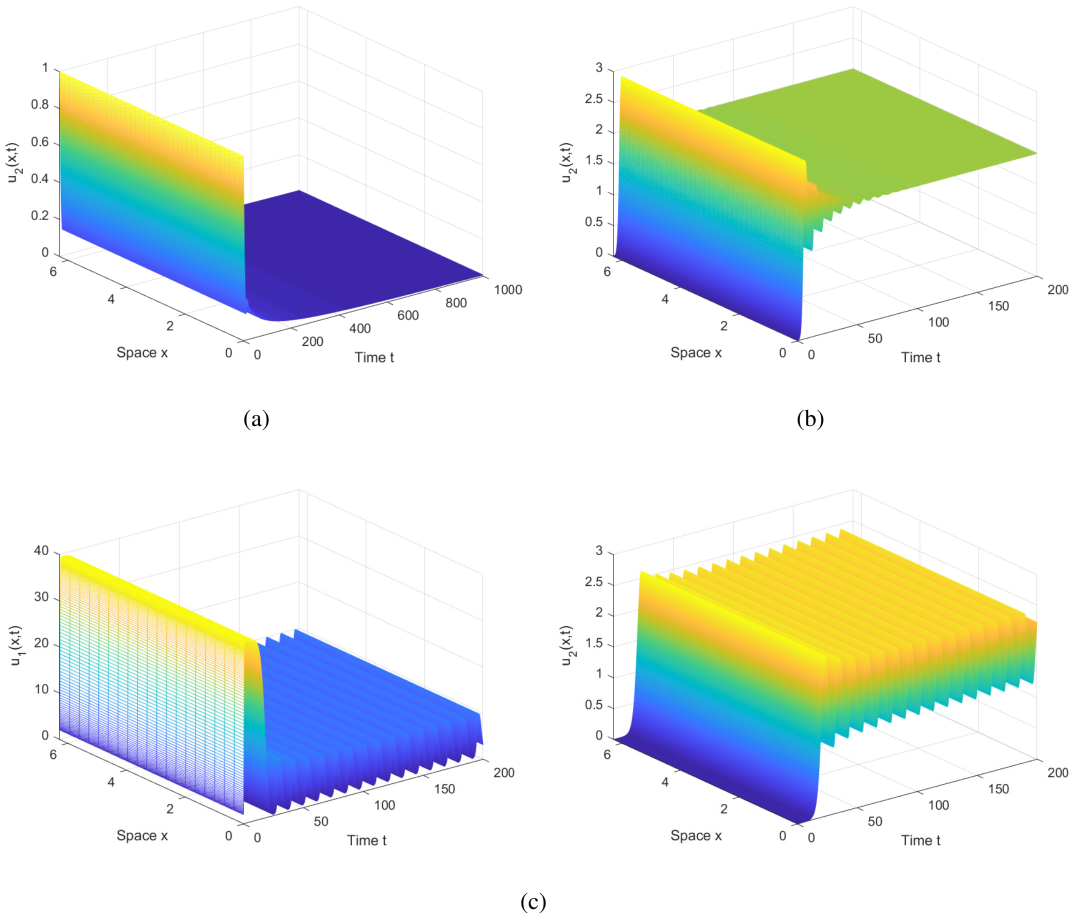


Figure 2. (a) $\tau = 10 > \tau_{max}$, E_1 is globally asymptotically stable. (b) $\tau = 0.5 \in (0, \tau_0)$, E_2 is locally asymptotically stable. (c) $\tau = 2 \in (\tau_0, \tau_1)$, a homogeneous periodic solution emerges.

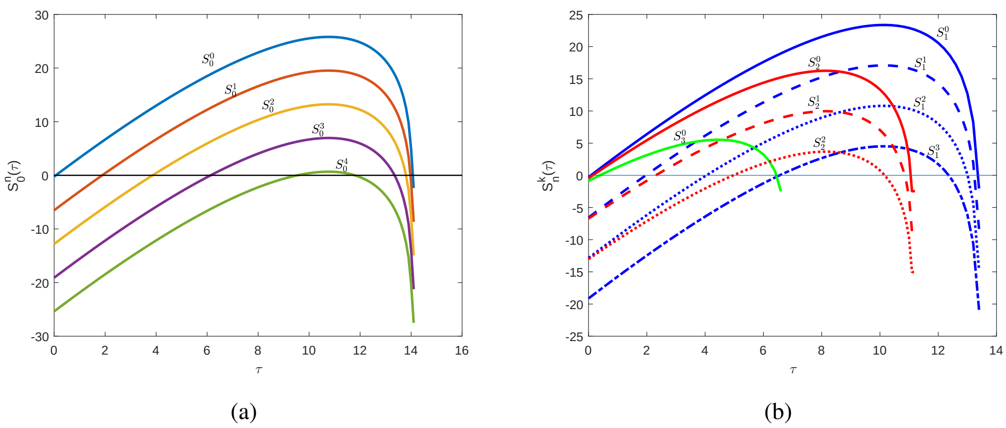


Figure 3. The graphs of $\mathcal{S}_n^k(\tau)$ with integers $0 \leq n \leq 3$ and $0 \leq k \leq 4$.

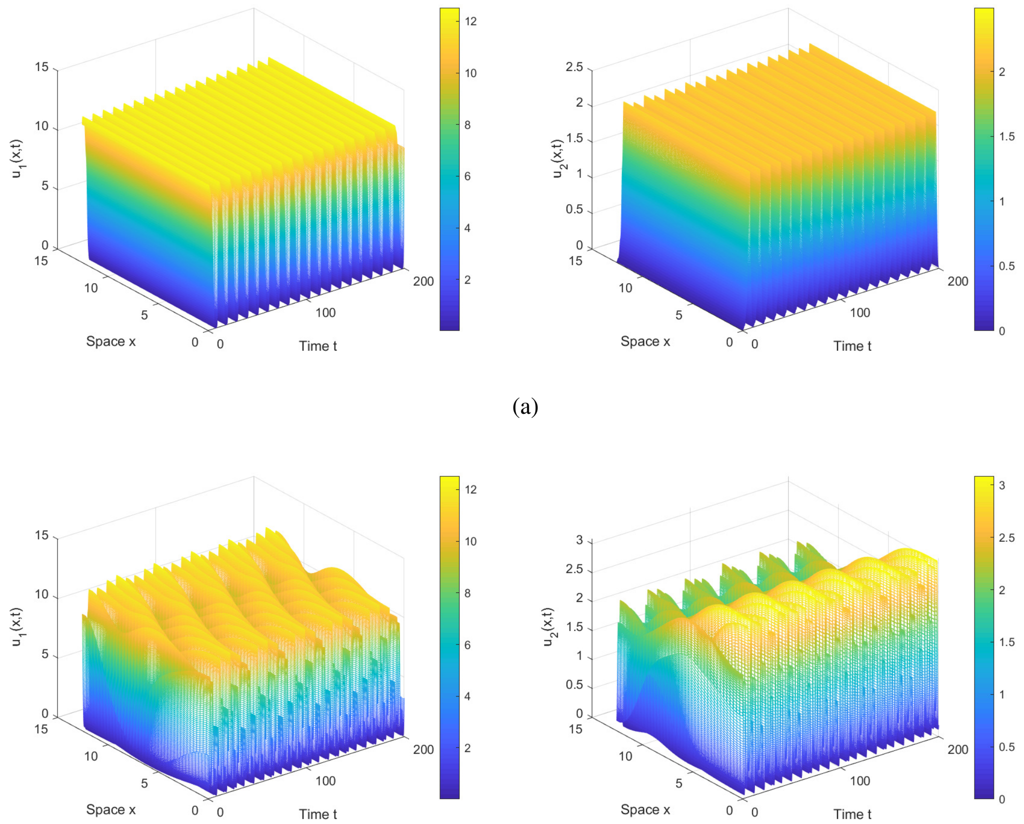
Next, we explore the global Hopf branches by choosing $\Omega = (0, 4\pi)$ and

$$d_1 = 1, d_2 = 1, r = 10, \mu_1 = 5, a = 0.4, p = 1, \gamma = 0.05, \mu_2 = 4.75, c = 2.5, K = 0.4. \qquad (5.1)$$

As shown in Figure 3, we collect all zeros of $\mathcal{S}_n^k(\tau)$ for nonnegative n, k in set B_i with $i = 0, 1, 2, 3$, namely,

$$B_0 = \{0.06, 1.88, 3.86, 6.14, 9.54, 11.8, 13.36, 13.8, 13.98, 14.04\},$$
$$B_1 = \{0.08, 1.96, 4.04, 6.56, 12.36, 13.01, 13.25, 13.35\},$$
$$B_2 = \{0.15, 2.31, 4.96, 10.16, 10.85, 11.07\}, \; B_3 = \{0.39, 6.45\}.$$

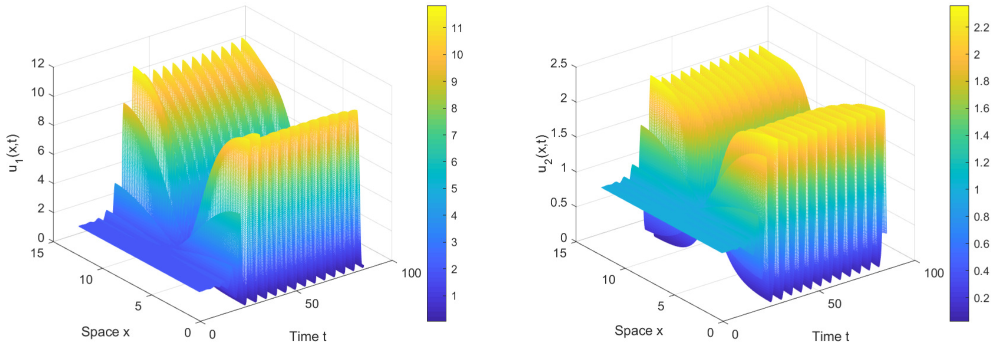
From Theorem 4.3, E_2 is locally asymptotically stable when $\tau \in (0, 0.06) \cup (14.04, \tau_{max})$ and unstable when $\tau \in (0.06, 14.04)$, at least one periodic solution emerges for $\tau \in (0.06, 14.04)$. Moreover, a spatially homogeneous periodic solution bifurcates from $\tau \in B_0$, see Figure 4(a); a spatially nonhomogeneous periodic solution bifurcates from $\tau \in B_1 \cup B_2 \cup B_3$, see Figure 4(b).



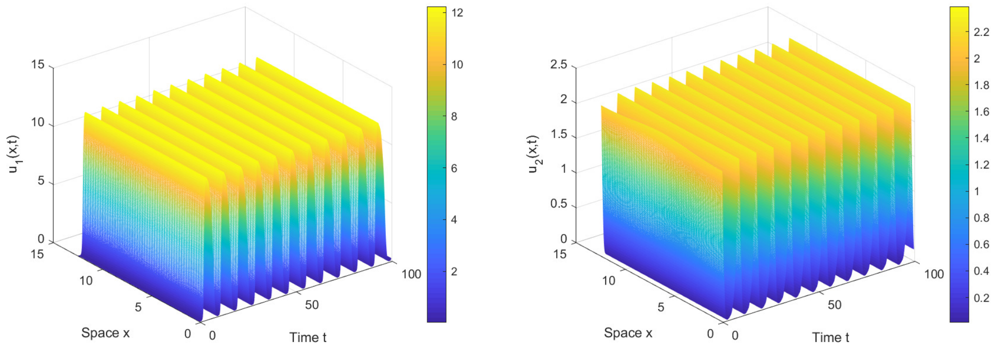
(a)

(b)

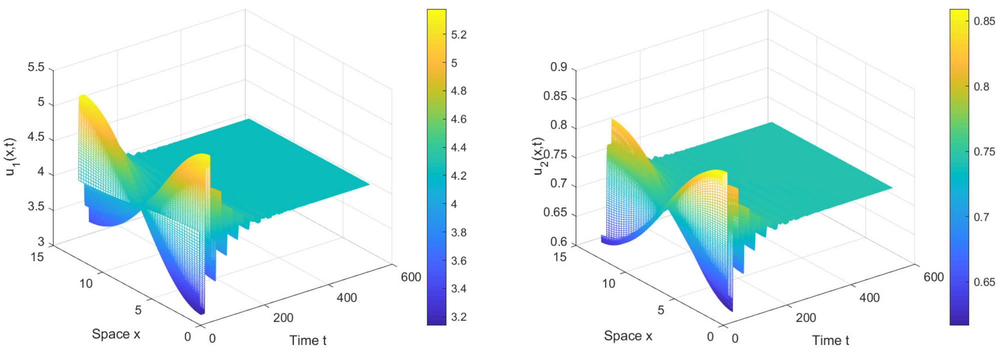
Figure 4. (a) $\tau = 1.88 \in B_0$, the bifurcating periodic solution is spatially homogeneous. (b) $\tau = 2.68 \in (\tau_4, \tau_5)$, the bifurcating periodic solution is spatially nonhomogeneous.



(a) When $\tau = 1.03$, the periodic solution is stable and spatially nonhomogeneous.



(b) When $\tau = 1.5$, the periodic solution is stable and spatially homogeneous.



(c) When $\tau = 16$, the positive constant steady state is locally asymptotically stable.

Figure 5. For the nonlocal model (1.3) with $\mathcal{G}$ defined in (5.2), delay τ induces different dynamics.

In model (1.3), the kernel function takes form as $\mathcal{G} = e^{-\gamma\tau} f(x-y)$ by reasonable assumptions and our theoretical results are derived by choosing $f(x-y)$ as Dirac-delta function. Next, we choose

$$f = \frac{e^{-2|x-y|^2}}{\int_\Omega e^{-2|x-y|^2} dy}. \tag{5.2}$$

Here, f is the truncated normal distribution. Clearly, $\int_{\Omega} f(x-y)dy = 1$. Let $\Omega = (0, 4\pi)$ and the parameter values chosen according to (5.1). As shown in Figure 5, when $\tau = 1.03$, a stable nonhomogeneous periodic solution emerges; when $\tau = 1.5$, a homogeneous periodic solution emerges; when $\tau = 16$, the positive constant steady state is stable. Numerical simulation suggests nonlocal interaction can produce more complex dynamics.

6. Summary

We formulate an age-structured predator-prey model with fear effect. For $R_0 \leq 1$, the global asymptotic stability for predator-free constant steady state is proved via Lyapunov-LaSalle invariance principle. For $R_0 > 1$, we prove the nonexistence of spatially nonhomogeneous steady states and exclude steady state bifurcation. Finally, we carry out Hopf bifurcation analyses and prove global Hopf branches are bounded.

Our theoretical results are obtained by choosing a special kernel function in model (1.3). However, in numerical results, we explore rich dynamics when the nonlocal interaction is incorporated into the delayed term. The theoretical results concerning the nonlocal model are left as an open problem.

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Conflict of interest

The authors declare no conflicts of interest in this paper.

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