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Research article

On the offensive alliance number for the zero divisor graph of $\mathbb{Z}_n$

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Abstract: A nonempty subset D of vertices in a graph $\Gamma = (V, E)$ is said is an *offensive alliance*, if every vertex $v \in \partial(D)$ satisfies $\delta_D(v) \geq \delta_{\overline{D}}(v) + 1$; the cardinality of a minimum offensive alliance of Γ is called the *offensive alliance number* $\alpha^o(\Gamma)$ of Γ. An offensive alliance D is called *global*, if every $v \in V - D$ satisfies $\delta_D(v) \geq \delta_{\overline{D}}(v) + 1$; the cardinality of a minimum global offensive alliance of Γ is called the *global offensive alliance number* $\gamma^o(\Gamma)$ of Γ. For a finite commutative ring with identity R, $\Gamma(R)$ denotes the zero divisor graph of R. In this paper, we compute the offensive alliance (global, independent, and independent global) numbers of $\Gamma(\mathbb{Z}_n)$, for some cases of n.

Keywords: Offensive alliance; global offensive alliance; alliance number; Zero divisor graph; independent offensive alliance

1. Introduction

We consider a finite and simple graphs $\Gamma = (V, E)$ with vertices V and edge set E. The degree of a vertex $v \in V$ is denoted by $\delta(v)$. For a nonempty subset $D \subseteq V$ and a vertex $v \in V$, $N_D(v) = \{u \in D : u \sim v\}$ is the set of neighbors of v in D, and the degree of v in D will be denoted by $\delta_D(v) = |N_D(v)|$. $\overline{D} = V - D$ is the complement of D in V, and the *boundary* of $D \subseteq V$, denoted by $\partial(D)$, is defined as

$$\partial(D) = \bigcup_{v \in D} N_{\overline{D}}(v).$$

An *independent set* in a graph Γ is a subset D of the vertex set of Γ such that no two vertices of D are adjacent. The *independence number* of Γ, denoted by $\alpha(\Gamma)$, is defined as the cardinality of a maximum independent set of Γ.

In 2004 P. Kristiansen *et al.* [1] studied alliances in graphs, and in their work, alliances of different types were proposed and studied. Since then, alliances such as defensive alliances [2–4], offensive alliances [5–7] and powerful alliances [8–10] have been studies. Alliance has been extensively studied since a couple of decades ago. Generalizations of these, called *k-alliances*, were introduced by Shafique and Dutton [11]; and inspired by that research, other researchers have dedicated time to the study of k-alliances [12–14]. It is known that problems of finding small defensive and offensive alliances are NP-complete [15–17]. We are interested in the study of the mathematical properties of offensive (global, independent and independent global) alliances. Odile Favaron *et al.* [18] in 2004, derived several bounds on the offensive alliance number and the strong offensive alliance number. Recall that given a nonempty subset $D \subseteq V$, D is an *offensive alliance* of Γ if it satisfies

$$\delta_D(v) \geq \delta_{\overline{D}}(v) + 1, \qquad \forall v \in \partial(D). \tag{1.1}$$

The *offensive alliance number* $\alpha^o(\Gamma)$ is the cardinality of a minimum offensive alliance. D is called a *global offensive alliance* if it satisfies

$$\delta_D(v) \geq \delta_{\overline{D}}(v) + 1 \qquad \forall v \in \overline{D}. \tag{1.2}$$

The *global offensive alliance number* $\gamma^o(\Gamma)$ is the cardinality of a minimum global offensive alliance. We say that a (global) offensive alliance D is independent if D is an independent set. The *independent offensive alliance number* is denoted by $\alpha^i(\Gamma)$, and the independent global offensive alliance number is denoted by $\gamma^i(\Gamma)$.

Throughout this paper, R denotes a finite commutative ring with identity. The *zero divisor graph* of R is the simple graph $\Gamma(R)$ with the vertex set being vertices set the proper zero-divisors of R, *i.e.*, $Z(R)^* = Z(R) - \{0\}$, and for different $u, v \in Z(R)^*$ they are adjacent if and only if $uv = 0$. For any real number t, $\lceil t \rceil$ (resp., $\lfloor t \rfloor$) denotes the ceiling of t, that is, the least integer greater than or equal to t (resp., the floor of t, that is the greatest integer less than or equal to t).
Istvan Beck in 1988 introduced the concept of a ring associated graph. This idea establishes a connection between graph theory and commutative rings [19]. In [20], Anderson and Livingston studied the zero divisor graph with a slight modification. In [21], Muthana and Mamouni studied the global defensive alliance number of the zero divisor graphs. In [22] Raúl Juárez *et al.* introduced the global offensive alliances of zero divisor graph. Later, Driss Bennis et al. in [23] generalized these results to global defensive k-alliances. From now on, only $\mathbb{Z}_n$ rings will be considered.

2. Preliminaries

In this section, we give an explanation of how the set of zero divisors is organized for each case of n. Let p and q be distinct prime numbers, and n, k and r are positive integers.

Case 1. If $n = p^k$, one can divide the zero divisors into $k-1$ sets. These sets are:

$$S_{p^i} = \{sp^i : gcd(s, p^{k-i}) = 1\},$$

for $i \in \{1, 2, \cdots, k-1\}$. Each vertex of S_{p^i} is adjacent to every vertex of S_{p^j}, when $i + j \geq k$. Moreover, $|S_{p^i}| = (p-1)p^{k-i-1}$. For more details, see [24].

Case 2. If $n = p^k q^r$, where p and q are distinct primes, and k and r are positive integers, the zero divisors set can be separated into three families as follows:

$$S_{p^i} = \{sp^i : gcd(s, p^{k-i}q^r) = 1\},$$
$$S_{q^j} = \{sq^j : gcd(s, p^k q^{r-j}) = 1\},$$
$$S_{p^i q^j} = \{sp^i q^j : gcd(s, p^{k-i}q^{r-j}) = 1\},$$

for $i \in \{1, 2, \cdots, k\}$, $j \in \{1, 2, \cdots, r\}$, and it is not possible to have $i = k$ and $j = r$ simultaneously.

In [25] we have the following result.

Lemma 1. *Let p and q be distinct prime numbers, and k and r are positive integers*

1. $|S_{p^i}| = q^{r-1}p^{k-i-1}(p-1)(q-1)$, *for* $i \in \{1, \cdots, k-1\}$ *and* $|S_{p^k}| = q^{r-1}(q-1)$.
2. $|S_{q^i}| = p^{k-1}q^{r-i-1}(p-1)(q-1)$, *for* $i \in \{1, \cdots, r-1\}$ *and* $|S_{q^r}| = p^{k-1}(p-1)$.
3. $|S_{p^i q^j}| = p^{k-i-1}q^{r-j-1}(p-1)(q-1)$, *for* $i \in \{1, \cdots, k-1\}$ *and* $j \in \{1, \cdots, r-1\}$. $|S_{p^k q^j}| = q^{r-j-1}(q-1)$, *and* $|S_{p^i q^r}| = p^{k-i-1}(p-1)$.

3. The offensive alliance number for $\Gamma(\mathbb{Z}_n)$

In this section, we calculate the offensive alliance number of a zero divisor graph over the ring $\mathbb{Z}_n$, for $n = p^k$, $n = p^k q$ with $p^k < q$, and $n = pq^k$ with $p < q$, where p and q are distinct prime numbers, and $k \geq 2$ is an integer number.

Theorem 2. *Let p and q be distinct prime numbers such that* $p^k < q$, *where* $k \geq 2$ *is an integer. Then,*

$$\alpha^o(\Gamma(\mathbb{Z}_{p^k q})) = \left\lfloor \frac{p^k - 1}{2} \right\rfloor + 1.$$

Proof. Consider $D \subseteq V$, the set defined by $D = \bigcup_{i=1}^{k-1} S_{p^i q} \cup X$, with $X \subseteq S_q$ such that $|X| = \left\lfloor \frac{p^k-1}{2} \right\rfloor - p^{k-1} + 2$. Observe that $|D| = \left\lfloor \frac{p^k-1}{2} \right\rfloor + 1$, and $\partial(D) = \bigcup_{i=1}^{k} S_{p^i}$. To verify that D is an offensive alliance, take $v \in \partial(D)$. If $v \in \bigcup_{i=1}^{k-1} S_{p^i}$, then $\delta_D(v) \geq p - 1$, and $\delta_{\overline{D}}(v) = 0$. On the other hand, if $v \in S_{p^k}$, we get $\delta_D(v) = \left\lfloor \frac{p^k-1}{2} \right\rfloor + 1$ while $\delta_{\overline{D}}(v) = \left\lceil \frac{p^k-3}{2} \right\rceil$. In any case, v satisfies the offensive alliance condition, and hence $\alpha^o(\Gamma(\mathbb{Z}_{p^k q})) \leq \left\lfloor \frac{p^k-1}{2} \right\rfloor + 1$.

We claim that any offensive alliance A with $|A| \leq \left\lfloor \frac{p^k-1}{2} \right\rfloor + 1$ is such that $S_{p^k} \subseteq \partial(A)$. Indeed, if there is a vertex $v \in S_{p^k}$ and $v \notin \partial(A)$, then $v \in A$, or $v \in (V - A) - \partial(A)$. We proceed by cases: If $v \in A$, we get two subcases. If there is a vertex $u \in S_q$ such that $u \in \partial(A)$, it should satisfy the offensive alliance condition, yielding $|A| \geq \left\lfloor \frac{q-1}{2} \right\rfloor + 1$; and if S_q does not contain vertices of $\partial(A)$, then $S_q \subseteq A$, yielding $|A| \geq p^{k-1}(p-1) + 1$. In both cases, we get a contradiction. Now, if $v \in (V - A) - \partial(A)$, then $A \subseteq \bigcup_{i=1}^{k} S_{p^i}$, and $S_{p^{k-1}q} \subseteq \partial(A)$, yielding $|A| \geq \left\lfloor \frac{qp^{k-1}-1}{2} \right\rfloor + 1$, a contradiction.

Finally, taking $v \in S_{p^k} \subseteq \partial(A)$, we get $\delta_A(v) \geq \left\lfloor \frac{p^k-1}{2} \right\rfloor + 1$, which implies $|A| \geq \left\lfloor \frac{p^k-1}{2} \right\rfloor + 1$. Therefore, $\alpha^o(\Gamma(\mathbb{Z}_{p^k q})) = \left\lfloor \frac{p^k-1}{2} \right\rfloor + 1$. □

Theorem 3. *Let p be a prime number and $k \geq 2$ is an integer. Then,*

$$\gamma^o(\Gamma(\mathbb{Z}_{p^k})) = \begin{cases} p^{\frac{k-1}{2}} - 1 & if\ k\ is\ odd, \\ \left\lceil \frac{p^{\frac{k}{2}}-1}{2} \right\rceil & if\ k\ is\ even. \end{cases}$$

Proof. Let $k \geq 2$ be an integer. Now, we analyze the following two cases. First, suppose k is odd, and let $D = \bigcup_{i=\frac{k+1}{2}}^{k-1} S_{p^i}$, with $|D| = p^{\frac{k-1}{2}} - 1$. Notice that $\overline{D} = \bigcup_{i=1}^{\frac{k-1}{2}} S_{p^i}$ is an independent set and that any $v \in \overline{D}$ is adjacent to every element of $S_{p^{k-1}}$, yielding $\delta_D(v) \geq p - 1$ and $\delta_{\overline{D}}(v) = 0$, that is, D is a global offensive alliance, and $\gamma^o(\Gamma(\mathbb{Z}_{p^k})) \leq p^{\frac{k-1}{2}} - 1$.

Now, observe that any global offensive alliance A, with $|A| < |D|$, is contained in D. Indeed, if $v \in D - A$ implies that $v \in \overline{A}$ the global offensive alliance condition will be satisfied yielding $|A| \geq \left\lceil \frac{p^{\frac{k+1}{2}}-1}{2} \right\rceil$, which is a contradiction. Thus, the said alliance does not exist. Hence, $\gamma^o(\Gamma(\mathbb{Z}_{p^k})) = p^{\frac{k-1}{2}} - 1$.

Now, suppose k is even, and let $D \subseteq V$ be the set given by $D = \bigcup_{i=\frac{k}{2}+1}^{k-1} S_{p^i} \cup X$, with $X \subseteq S_{p^{\frac{k}{2}}}$ and $|X| = (p-1)p^{\frac{k}{2}-1} - \left\lceil \frac{p^{\frac{k}{2}}-1}{2} \right\rceil$. Notice that $|D| = \left\lceil \frac{p^{\frac{k}{2}}-1}{2} \right\rceil$. We affirm that D is a global offensive alliance. In effect, with an analysis similar to the odd case, we ensure that vertices of $\bigcup_{i=1}^{\frac{k}{2}-1} S_{p^i}$ satisfy the global offensive alliance condition. If $v \in S_{p^{\frac{k}{2}}} - X$, then $\delta_D(v) = \left\lceil \frac{p^{\frac{k}{2}}-1}{2} \right\rceil$ and $\delta_{\overline{D}}(v) = \left\lceil \frac{p^{\frac{k}{2}}-1}{2} \right\rceil - 1$. Thus,

$$\delta_D(v) \geq \delta_{\overline{D}}(v) + 1,$$

that is, D is a global offensive alliance and $\gamma^o(\Gamma(\mathbb{Z}_{p^k})) \leq \left\lceil \frac{p^{\frac{k}{2}}-1}{2} \right\rceil$.

Observe that any global offensive alliance A, with $|A| < |D|$, $\bigcup_{i=\frac{k}{2}+1}^{k-1} S_{p^i} \subseteq A$. Indeed, if there is $v \in \bigcup_{i=\frac{k}{2}+1}^{k-1} S_{p^i}$ such that $v \in \overline{A}$, the global offensive alliance condition will be satisfied yielding $|A| \geq \left\lceil \frac{p^{\frac{k}{2}+1}-1}{2} \right\rceil$, which is a contradiction. Thus, the said alliance does not exist. Hence, $\gamma^o(\Gamma(\mathbb{Z}_{p^k})) = \left\lceil \frac{p^{\frac{k}{2}}-1}{2} \right\rceil$. □

Corollary 4. *Let p be a prime number and $k \geq 2$ is an integer. Then,*

$$\alpha^o(\Gamma(\mathbb{Z}_{p^k})) = \gamma^o(\Gamma(\mathbb{Z}_{p^k})).$$

Proof. First, note that the set D considered in the last proof turns out to be an offensive alliance, and thus $\alpha^o(\Gamma(\mathbb{Z}_{p^k})) \leq p^{\frac{k-1}{2}} - 1$.

Now, suppose that A is an offensive alliance contained in $\overline{D}$, with $|A| \leq |D|$. Then, $v \in \partial(A)$ for each vertex $v \in S_{p^{k-1}}$, and this implies $|A| \geq \frac{p^{k-1}-1}{2} > |D|$, which is a contradiction. Consequently, A contains elements of D. Moreover, $|A| < |D|$ implies that there exists a vertex $v \in D - A$, that is $v \in \partial(A)$, since v the global offensive alliance condition will be satisfied, we have $\delta_A(v) \geq \delta_{\overline{A}}(v) + 1$, getting $|A| \geq \left\lfloor \frac{p^{\frac{k+1}{2}}-1}{2} \right\rfloor > p^{\frac{k-1}{2}} - 1$, which is not possible. Hence, $\alpha^o(\Gamma(\mathbb{Z}_{p^k})) = p^{\frac{k-1}{2}} - 1$.

The proof for even k is analogous. □

Theorem 5. *Let p and q be distinct prime numbers such that $p^k < q$, and $k \geq 2$ is an integer. Then,*

$$\gamma^o(\Gamma(\mathbb{Z}_{p^k q})) = p^k - 1.$$

Proof. Let $D = \bigcup_{i=1}^{k-1} S_{p^i q} \cup S_q$, and note that $|D| = p^k - 1$. Observe that $\overline{D} = \bigcup_{i=1}^{k} S_{p^i}$ is an independent set and that each $v \in \overline{D}$ is adjacent to every element of $S_{p^{k-1}q}$, and $|S_{p^{k-1}q}| = (p-1) \geq 1$. Thus, $\delta_D(v) \geq 1$, and $\delta_{\overline{D}}(v) = 0$, which implies

$$\delta_D(v) \geq \delta_{\overline{D}}(v) + 1,$$

that is, D is a global offensive alliance, and $\gamma^o(\Gamma(\mathbb{Z}_{p^k q})) \leq p^k - 1$.

Now, observe that any global offensive alliance A, with $|A| < |D|$, is contained in D. Indeed, if there is a vertex $v \in \bigcup_{i=1}^{k-1} S_{p^i q} \cup S_q$ such that $v \in \overline{A}$, it should satisfy the global offensive alliance condition. If $v \in \bigcup_{i=1}^{k-1} S_{p^i q}$, then $|A| \geq \left\lfloor \frac{pq-1}{2} \right\rfloor + 1$; and if $v \in S_q$, we have $|A| \geq \frac{q-1}{2} + \frac{p^k-1}{2} + 2$. In both cases, we get a contradiction. Hence, $\gamma^o(\Gamma(\mathbb{Z}_{p^k q})) = p^k - 1$. □

Theorem 6. *Let $p < q$ be prime numbers and $k \geq 2$ is an integer. Then,*

$$\gamma^o(\Gamma(\mathbb{Z}_{pq^k})) = \begin{cases} pq^{\frac{k-1}{2}} - 1 & if\ k\ is\ odd, \\ (p-1)q^{\frac{k}{2}-1} + q^{\frac{k}{2}} - 1 & if\ k\ is\ even. \end{cases}$$

Proof. First, suppose k is odd, and let $D = \bigcup_{i=\frac{k+1}{2}}^{k-1} S_{pq^i} \cup \bigcup_{i=\frac{k+1}{2}}^{k} S_{q^i}$. Then $|D| = pq^{\frac{k-1}{2}} - 1$. Observe that $\overline{D} = \bigcup_{i=1}^{\frac{k-1}{2}} S_{pq^i} \cup \bigcup_{i=1}^{\frac{k-1}{2}} S_{q^i} \cup S_p$ is an independent set and that each $v \in \overline{D}$ is adjacent to every element of $S_{pq^{k-1}}$ or S_{q^k}. By cases: If $v \in \bigcup_{i=1}^{\frac{k-1}{2}} S_{pq^i} \cup \bigcup_{i=1}^{\frac{k-1}{2}} S_{q^i}$, note that v is adjacent to every element of $S_{pq^{k-1}}$, and $|S_{pq^{k-1}}| = (q-1) \geq 2$. Thus, $\delta_D(v) \geq (q-1)$, and $\delta_{\overline{D}}(v) = 0$, which implies

$$\delta_D(v) \geq \delta_{\overline{D}}(v) + 1.$$

if $v \in S_p$ is adjacent to every element S_{q^k}, then $\delta_D(v) = (p-1)$, and $\delta_{\overline{D}}(v) = 0$. Thus,

$$\delta_D(v) \geq \delta_{\overline{D}}(v) + 1,$$

that is, D is a global offensive alliance, and $\gamma^o(\Gamma(\mathbb{Z}_{pq^k})) \leq pq^{\frac{k-1}{2}} - 1$.
Now, observe that any global offensive alliance E, with $|E| < |D|$, is contained in D. Indeed, if there is a vertex $v \in \bigcup_{i=\frac{k+1}{2}}^{k-1} S_{pq^i} \cup \bigcup_{i=\frac{k+1}{2}}^{k} S_{q^i}$ such that $v \in \overline{E}$, it should satisfy the global offensive alliance condition. If $v \in \bigcup_{i=\frac{k+1}{2}}^{k-1} S_{pq^i}$, then $|E| \geq \left\lceil \frac{pq^{\frac{k+1}{2}}-1}{2} \right\rceil$; and if $v \in \bigcup_{i=\frac{k+1}{2}}^{k} S_{q^i}$, we have $|E| \geq \frac{q^{\frac{k-1}{2}}(q+p-2)}{2} + 2$. In both cases, we get a contradiction. Thus, there is $v \in D$ such that $v \notin E$ is adjacent to every element of $S_{pq^{\frac{k-1}{2}}}$ and $\delta_E(v) < pq^{\frac{k-1}{2}} - 2$, while $\delta_{\overline{E}}(v) \geq q^{\frac{k-1}{2}}(q-1)$, yielding

$$\delta_E(v) \leq pq^{\frac{k-1}{2}} - 2 \leq q^{\frac{k-1}{2}}(q-1) \leq \delta_{\overline{E}}(v) + 1,$$

which implies that E is not a global offensive alliance. Hence $\gamma^o(\Gamma(\mathbb{Z}_{pq^k})) = pq^{\frac{k-1}{2}} - 1$.
Finally, note that for even k the proof is analogous to the odd case, taking $D = \bigcup_{i=\frac{k}{2}}^{k-1} S_{pq^i} \cup \bigcup_{i=\frac{k}{2}+1}^{k} S_{q^i}$. □

4. The independent offensive alliance number for $\Gamma(\mathbb{Z}_n)$

In this section we give closed formulas for the independent offensive alliance number of $\Gamma(\mathbb{Z}_n)$ for the cases of $n = p^k$, $p^k q$ with $p^k < q$ and pq^k with $p < q$.

Theorem 7. *Let p be a prime number and $k \geq 2$ is an integer. Then,*

$$\alpha^i(\Gamma(\mathbb{Z}_{p^k})) = \left\lceil \frac{p^{k-1}-1}{2} \right\rceil.$$

Proof. Let $D \subseteq S_p$ be an independent set, with $|D| = \left\lceil \frac{p^{k-1}-1}{2} \right\rceil$. Notice that $\partial(D) = S_{p^{k-1}}$ and that $v \in \partial(D)$ satisfies $\delta_D(v) = \left\lceil \frac{p^{k-1}-1}{2} \right\rceil$ and $\delta_{\overline{D}}(v) = \left\lfloor \frac{p^{k-1}-1}{2} \right\rfloor - 1$. Thus,

$$\delta_D(v) \geq \delta_{\overline{D}}(v) + 1,$$

which implies that D is an independent offensive alliance and that $\alpha^i(\Gamma(\mathbb{Z}_{p^k})) \leq \left\lceil \frac{p^{k-1}-1}{2} \right\rceil$.
Now, suppose k is even and observe that any independent offensive alliance A, with $|A| < |D|$, is contained in $\bigcup_{i=1}^{\frac{k}{2}-1} S_{p^i}$. Indeed, if there is $v \in \bigcup_{i=\frac{k}{2}}^{k-1} S_{p^i}$ such that $v \in A$, then there exists $u \in S_{p^{\frac{k}{2}}}$ and $u \in \partial(A)$. Hence, $\delta_A(u) = 1$, and $\delta_{\overline{A}}(u) = p^{\frac{k}{2}} - 3$, a contradiction. The situation is analogous if k is odd. Finally, we have $S_{p^{k-1}} \subseteq \partial(A)$. Thus, any vertex should satisfy the independent offensive alliance condition, that is, $\delta_A(v) \geq \left\lceil \frac{p^{k-1}-1}{2} \right\rceil$, which is a contradiction. Hence, $\alpha^i(\Gamma(\mathbb{Z}_{p^k})) = \left\lceil \frac{p^{k-1}-1}{2} \right\rceil$. □

Theorem 8. *Let p and q be distinct prime numbers such that $p^k < q$, and $k \geq 2$ is an integer. Then,*

$$\alpha^i(\Gamma(\mathbb{Z}_{p^k q})) = \left\lfloor \frac{p^k - 1}{2} \right\rfloor + 1.$$

Proof. Consider $D \subseteq S_q$, with $|D| = \left\lfloor \frac{p^k-1}{2} \right\rfloor + 1$, and observe that it is an independent set. Notice that $\partial(D) = S_{p^k}$, and that every $v \in \partial(D)$ satisfies $\delta_D(v) = \left\lfloor \frac{p^k-1}{2} \right\rfloor + 1$, and $\delta_{\overline{D}}(u) = \left\lfloor \frac{p^k-1}{2} \right\rfloor - 1$. Thus,

$$\delta_D(v) > \delta_{\overline{D}}(v) + 1,$$

which implies that D is an independent offensive alliance, and $\alpha^o(\Gamma(\mathbb{Z}_{p^k q})) \leq \left\lfloor \frac{p^k-1}{2} \right\rfloor + 1$.
Now, suppose k is even, and $p > 2$, or $k > 2$. Observe that any independent offensive alliance A, with $|A| < |D|$, is contained in $\bigcup_{i=1}^{\frac{k}{2}-1} S_{p^i q} \cup S_q$. Indeed, if there is a vertex $v \in \bigcup_{i=1}^{k} S_{p^i} \cup \bigcup_{i=\frac{k}{2}}^{k-1} S_{p^i q}$ such that $v \in A$, then we proceed by cases: If $v \in \bigcup_{i=1}^{k} S_{p^i}$, then $S_{p^{k-1}q} \subseteq \partial(A)$, and each vertex should satisfy the offensive alliance condition, yielding $|A| \geq \left\lfloor \frac{qp^{k-1}-1}{2} \right\rfloor + 1$, a contradiction. If $v \in \bigcup_{i=\frac{k}{2}}^{k-1} S_{p^i q}$, then there is $u \in S_{p^{\frac{k}{2}}} \subseteq \partial(A)$ such that $\delta_A(u) = 1$, while $\delta_{\overline{A}}(u) = p^{\frac{k}{2}} - 2$, for $p > 2$ or $k > 2$. In this case, the independent offensive alliance does not exist. If $p = 2 = k$, we have two cases: The alliance consists of vertices in S_{pq} and $v \in S_q$, yielding $\delta_A(u) = 1$ and $\delta_{\overline{A}}(u) = 0$ for $u \in S_{p^{k-1}}$ and yielding $\delta_A(u) = 2$ and $\delta_{\overline{A}}(u) = 1$ for $u \in S_{p^k}$. If the alliance consists of vertices in S_q, we have $S_{p^k} = \partial(A)$, yielding $\delta_A(u) = 2$ and $\delta_{\overline{A}}(u) = 1$ for $u \in \partial(A)$. The situation is analogous if k is odd. On the other hand, observe that $S_{p^k} \subseteq \partial(A)$. Thus, $u \in S_{p^k}$ satisfies $\delta_A(u) \geq \left\lfloor \frac{p^k-1}{2} \right\rfloor + 1$, which is a contradiction. Therefore, $\alpha^i(\Gamma(\mathbb{Z}_{p^k q})) = \left\lfloor \frac{p^k-1}{2} \right\rfloor + 1$. □

Lemma 9. *In $\mathbb{Z}_{pq^k}$, if $p > 2$ and $p < q$, then*

$$|S_q| \geq \sum_{i=2}^{k} |S_{q^i}| + \sum_{i=1}^{k-1} |S_{pq^i}| + 1.$$

Proof. By Lemma 1, we have $\sum_{i=2}^{k} |S_{q^i}| = (p-1)\left[q^{k-2} - 1\right] + p - 1$ and $\sum_{i=1}^{k-1} |S_{pq^i}| = q^{k-1} - 1$, which implies

$$\sum_{i=2}^{k} |S_{q^i}| + \sum_{i=1}^{k-1} |S_{pq^i}| = q^{k-1} + (p-1)q^{k-2} - 1. \tag{4.1}$$

We also have

$$
\begin{aligned}
|S_q| &= q^{k-2}(p-1)(q-1) \\
&= (p-1)q^{k-1} - (p-1)q^{k-2} \\
&= \overbrace{q^{k-1} + \cdots + q^{k-1}}^{(p-1)\text{-times}} - \overbrace{\left(q^{k-2} + \cdots + q^{k-2}\right)}^{(p-1)\text{-times}} \\
&= q^{k-1} + q\overbrace{\left(q^{k-2} + \cdots + q^{k-2}\right)}^{(p-2)\text{-times}} - \overbrace{\left(q^{k-2} + \cdots + q^{k-2}\right)}^{(p-2)\text{-times}} - q^{k-2} \\
&= q^{k-1} + (q-1)(p-2)q^{k-2} - q^{k-2} \\
&= q^{k-1} + (q-2)(p-2)q^{k-2}.
\end{aligned}
\tag{4.2}
$$

□

Theorem 10. *Let $p < q$ be prime numbers and $k \geq 2$ is an integer. Then,*

$$\alpha^i(\Gamma(\mathbb{Z}_{pq^k})) = \begin{cases} q^{k-1} & if\ p = 2, \\ \frac{pq^{k-1}-1}{2} & if\ p > 2. \end{cases}$$

Proof. First, suppose $p > 2$. Lemma 9 ensures the existence of $D \subseteq S_q$, with $|D| = \frac{pq^{k-1}-1}{2}$, which is an independent set. Note that $\partial(D) = S_{pq^{k-1}}$ and that every vertex $v \in \partial(D)$ satisfies $\delta_{\overline{D}}(v) = \frac{pq^{k-1}-1}{2} - 1$, and $\delta_D(v) = \frac{pq^{k-1}-1}{2}$. Thus,

$$\delta_D(v) = \delta_{\overline{D}}(v) + 1,$$

which implies that D is an independent offensive alliance, and $\alpha^i(\Gamma(\mathbb{Z}_{pq^k})) \leq \frac{pq^{k-1}-1}{2}$.

Now, observe that any independent offensive alliance A, with $|A| < |D|$, is contained in $\bigcup_{i=1}^{k} S_{q^i}$. Indeed, if there is a vertex $v \in \bigcup_{i=1}^{k-1} S_{pq^i} \cup S_p$ such that $v \in A$, then $S_{q^k} \subseteq \partial(A)$, and each vertex should satisfy the independent offensive alliance condition yielding $|A| \geq \frac{q^k-1}{2}$, a contradiction. Analogously, an independent offensive alliance of minimal cardinality cannot contain elements of S_p. Note also that $S_{pq^{k-1}} \subseteq \partial(A)$ and that any vertex should satisfy the independent offensive alliance condition yielding $\delta_A(v) \geq \frac{pq^{k-1}-1}{2}$, which is a contradiction. Hence, $\alpha^i(\Gamma(\mathbb{Z}_{pq^k})) = \frac{pq^{k-1}-1}{2}$.

Now, suppose $p = 2$, and consider $D = \bigcup_{i=1}^{k} S_{q^i}$. Observe that $|D| = q^{k-1}$ and $\partial(D) = \bigcup_{i=1}^{k-1} S_{pq^i} \cup S_p$. It is not difficult to verify that D is an independent set. To show that D is an offensive alliance note, that for every vertex $v \in \partial(D)$, we have

$$\delta_D(v) = \delta_{\overline{D}}(v) + 1.$$

Hence, $\alpha^i(\Gamma(\mathbb{Z}_{pq^k})) \leq q^{k-1}$. Finally, the proof of the other inequality is analogous to the case $p > 2$. □

In [25] we have the following result.

Lemma 11. *Let p be a prime number and let $k \geq 2$ be an integer. Then,*

1. *If $k = 2$, then $\alpha(\Gamma(\mathbb{Z}_{p^k})) = 1$.*

2. *If k is an odd integer that is greater than two, then we have* $\alpha(\Gamma(\mathbb{Z}_{p^k})) = p^{\frac{k-1}{2}(p^{\frac{k-1}{2}}-1)}$.
3. *If k is an even integer that is greater than two, then we have* $\alpha(\Gamma(\mathbb{Z}_{p^k})) = p^{k-1} - p^{\frac{k}{2}} + 1$.

Let p be a prime number and k is an even integer, with $p \geq 3$ or $k \geq 4$. By the previous lemma, $\alpha(\Gamma(\mathbb{Z}_{p^k})) = p^{k-1} - p^{\frac{k}{2}} + 1$. Observe that $I = \bigcup_{i=1}^{\frac{k}{2}-1} S_{p^i} \cup \{u\}$ is the only independent set of maximal cardinality, where $u \in S_{p^{\frac{k}{2}}}$. Consider $J \subseteq I$ and notice that for every $v \in (S_{p^{\frac{k}{2}}} - u) \subseteq \overline{J}$, we have $\delta_{\overline{J}}(v) = p^{\frac{k}{2}} - 2$, while $\delta_J(v) = 0$. Therefore, there are no independent global offensive alliances in these graphs. Observe that for $p = 2$ and $k = 2$, the graph of $\mathbb{Z}_4$ consists of just one vertex. Therefore $\gamma^i(\Gamma(\mathbb{Z}_{p^k})) = 1$.

Theorem 12. *Let p be a prime number and $k \geq 3$ is an odd integer. Then,*

$$\gamma^i(\Gamma(\mathbb{Z}_{p^k})) = p^{k-1} - p^{\frac{k-1}{2}}.$$

Proof. Let $D = \bigcup_{i=1}^{\frac{k-1}{2}} S_{p^i}$ and observe that $|D| = p^{k-1} - p^{\frac{k-1}{2}}$. Notice that $\overline{D} = \bigcup_{\frac{k+1}{2}}^{k-1} S_{p^i}$, and $|\overline{D}| = p^{\frac{k-1}{2}} - 1$. Moreover, for every vertex $v \in \overline{D}$, we have $\delta_{\overline{D}}(v) = p^{\frac{k-1}{2}} - 2$, and $\delta_D(v) \geq (p-1)p^{\frac{k-1}{2}}$. Thus

$$\delta_D(v) > \delta_{\overline{D}}(v) + 1.$$

Hence, $\gamma^i(\Gamma(\mathbb{Z}_{p^k})) \leq p^{k-1} - p^{\frac{k-1}{2}}$.
Now, observe that any independent global offensive alliance A, with $|A| < |D|$, is contained in D. Indeed, if there is a vertex $v \in \bigcup_{i=\frac{k+1}{2}}^{k-1} S_{p^i}$ such that $v \in A$, then any vertex $u \in S_{p^{\frac{k-1}{2}}} \subseteq \overline{A}$ should satisfy the global offensive alliance condition, yielding $\delta_A(u) = 1$ and $\delta_{\overline{A}}(u) \geq p^{\frac{k-1}{2}} - 1$. Thus, there is $v \in D$ such that $v \notin A$, and $\delta_{\overline{A}}(v) \geq p - 1$, while $\delta_A(v) = 0$, which implies that A is not an independent global offensive alliance. Therefore $\gamma^i(\Gamma(\mathbb{Z}_{p^k})) = p^{k-1} - p^{\frac{k-1}{2}}$. □

Theorem 13. *Let p and q be distinct prime numbers such that $p^k < q$ and $k \geq 2$ is an integer. Then,*

$$\gamma^i(\Gamma(\mathbb{Z}_{p^k q})) = \begin{cases} 3 & if\ k = 2\ \ and\ p = 2, \\ (q-1)(p^{k-1} - p^{\frac{k-2}{2}}) + p^k - p^{\frac{k}{2}} & if\ k \geq 4\ and\ even, \\ q(p^{k-1} - p^{\frac{k-1}{2}}) + p^{k-1}(p-1) & if\ k\ is\ odd. \end{cases}$$

Proof. First, suppose $p = 2$, and $k = 2$. Let $D = S_{2q} \cup S_q$, with $|D| = 3$. It is not difficult to verify that D is an independent set, so we just show that D is a global offensive alliance. Note that every vertex $v \in \overline{D} = S_2 \cup S_4$ satisfies $\delta_D(v) \geq 1$ and $\delta_{\overline{D}}(v) = 0$. Thus $\gamma^i(\Gamma(\mathbb{Z}_{p^k q})) \leq 3$. Now, observe that any independent global offensive alliance A, with $|A| < |D|$, is contained in D. Indeed, if there is a vertex $v \in S_2 \cup S_4$ such that $v \in A$, then any vertex $v \in S_{2q} \subseteq \overline{A}$ satisfies the global offensive alliance condition, yielding $|A| \geq q$, a contradiction. Thus, there is $v \in D$ such that $v \notin A$. If $v \in S_{2q}$, then

$\delta_A(v) = 0$ while $\delta_{\overline{A}}(v) = 2(q-1)$; and if $v \in S_q$, then $\delta_A(v) = 0$ while $\delta_{\overline{A}}(v) = q-1$. In both cases, the global offensive alliance condition is not satisfied. Hence, $\gamma^i(\Gamma(\mathbb{Z}_{p^k q})) = 3$.

Next, suppose that $k \geq 4$ is an even integer, and let $D = \bigcup_{i=1}^{\frac{k}{2}-1} S_{p^i q} \cup \bigcup_{i=1}^{\frac{k}{2}} S_{p^i} \cup S_q$, with $|D| = (q-1)(p^{k-1} - p^{\frac{k-2}{2}}) + p^k - p^{\frac{k}{2}}$. We may note that D is an independent set, so we just show that D is a global offensive alliance. Let $v \in \overline{D}$ and proceed by cases: If $v \in \bigcup_{i=\frac{k}{2}}^{k-1} S_{p^i q}$, then $\delta_{\overline{D}}(v) = (q-1)p^{\frac{k-2}{2}} - p^{\frac{k}{2}} - 2$, while $\delta_D(v) \geq (q-1)(p-1)p^{\frac{k-2}{2}}$; If $v \in \bigcup_{i=\frac{k}{2}+1}^{k-1} S_{p^i}$, then $\delta_{\overline{D}}(v) = p^{\frac{k}{2}} - 1$, and $\delta_D(v) \geq p^{\frac{k}{2}}(p-1)$. If $v \in S_{p^k}$, then $\delta_{\overline{D}}(v) = p^{\frac{k}{2}} - 1$, while $\delta_D(v) = p^{\frac{k}{2}}(p^{\frac{k}{2}} - 1)$. In any case, the condition $\delta_D(v) \geq \delta_{\overline{D}}(v) + 1$ is satisfied, that is, $\gamma^i(\Gamma(\mathbb{Z}_{p^k q})) \leq (q-1)(p^{k-1} - p^{\frac{k-2}{2}}) + p^k - p^{\frac{k}{2}}$.

Now, notice that any global offensive alliance A, with $|A| < |D|$, is contained in D. Indeed, suppose that there is a vertex $v \in \bigcup_{i=\frac{k}{2}}^{k-1} S_{p^i q} \cup \bigcup_{i=\frac{k}{2}+1}^{k} S_{p^i}$ such that $v \in A$. By cases: If $v \in \bigcup_{i=\frac{k}{2}}^{k-1} S_{p^i q}$, then for any vertex $u \in S_{p^{\frac{k}{2}}} \subseteq \overline{A}$ we have $\delta_A(u) = 1$, while $\delta_{\overline{A}}(u) = p^{\frac{k}{2}} - 2$. If $v \in S_{p^{\frac{k}{2}+i}}$, with $1 \leq i \leq \frac{k}{2}$, and $v \in A$, then $S_{p^j} \subseteq A$, for $j = 1, \ldots, \frac{k}{2} + i$. Otherwise, if $u \in S_{p^{\frac{k}{2}+i}}$ and $u \notin A$, then $\delta_E(u) = 0 < \delta_{\overline{E}}(u) + 1$, which is not possible. Thus $\bigcup_{j=1}^{\frac{k}{2}+i} S_{p^j} \subseteq A$, yielding $|A| \geq (q-1)(p^{k-1} - p^{\frac{k}{2}-2}) + p^k - p^{\frac{k}{2}+1}$, a contradiction. Now, if there is $v \in D - A$, then $\delta_{\overline{A}}(v) \geq p - 1$, while $\delta_A(v) = 0$, which implies that A is not an independent global offensive alliance. Hence, $\gamma^i(\Gamma(\mathbb{Z}_{p^k q})) = (q-1)(p^{k-1} - p^{\frac{k-2}{2}}) + p^k - p^{\frac{k}{2}}$.

Finally, we may observe that for odd k the proof is analogous to the even case, taking $D = \bigcup_{i=1}^{\frac{k-1}{2}} S_{p^i q} \cup \bigcup_{i=1}^{\frac{k-1}{2}} S_{p^i} \cup S_q$. ☐

5. Conclusion

Since a couple of decades ago, when alliances in graphs were introduced for the first time [1], a lot of researchers have focused on studying various parameters of different types of alliances in graphs. In this paper, we computed the number of offensive alliances (global, independent and independent global) of the zero-divisor graph of the ring $\mathbb{Z}_n$, for $n = p^k$, $n = p^k q$, with $p^k < q$, and $n = pq^k$, with $p < q$, where p and q are distinct prime numbers, and $k > 2$ is an integer number. Among the open problems raised by our results, the following are of particular interest.

1. Generalize these results for offensive (defensive, powerful) k-alliances of the zero-divisor graph of the ring $\mathbb{Z}_n$.
2. Explore upper offensive (defensive) alliances on these graphs.
3. Since offensive alliances can be used to model real-world situations, it is worthwhile to find algo-

rithms (which could even be non-polynomial for some values of k) together with some heuristics that allow the making of some implementations.

Conflict of interest

The authors declare there is no conflict of interest.

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