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Research article

Threshold dynamics of a stochastic general SIRS epidemic model with migration

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Abstract: In this study, a stochastic SIRS epidemic model that features constant immigration and general incidence rate is investigated. Our findings show that the dynamical behaviors of the stochastic system can be predicted using the stochastic threshold R_0^S. If $R_0^S < 1$, the disease will become extinct with certainty, given additional conditions. Conversely, if $R_0^S > 1$, the disease has the potential to persist. Moreover, the necessary conditions for the existence of the stationary distribution of positive solution in the event of disease persistence is determined. Our theoretical findings are validated through numerical simulations.

Keywords: immigration; general incidence rate; SIRS epidemic model; threshold dynamics; ergodicity

1. Introduction

Many well-known epidemic models [1–7] have been proposed and discussed over the years. For instance, De la Sen et al. [8] in their study analyzed an epidemic model that incorporates delayed, distributed disease transmission and a general vaccination policy. Weera et al. conducted a numerical investigation of a nonlinear computer virus epidemic model with time delay effects [9]. Li et al. [3] examined an SIRS epidemic model with a general incidence rate and constant immigration, which took the following form

$$\begin{cases} \dot{S} = aA - \beta f(N)SI - \mu S + \delta R, \\ \dot{I} = bA + \beta f(N)SI - (\mu + \gamma + \alpha)I, \\ \dot{R} = cA + \gamma I - (\mu + \delta)R, \end{cases} \tag{1.1}$$

where $N == S + I + R$ and the biological implications are shown in Table 1, and the infectious force $\beta f(N)$ is a continuous and twice differentiable function of total population and $\beta > 0$ is adequate contact rate. Furthermore, f satisfies the following hypotheses

1) $f \in C^2((0, \infty); (0, \infty))$.
2) $f'(N) \leq 0$ for any $N > 0$.
3) $[f(N)N]' \geq 0$ for any $N > 0$.

Table 1. Variables in model (1.1).

Variables	Biological implications
S	Numbers of susceptible individuals
I	Numbers of infectious individuals
R	Numbers of removed individuals
N	Total population
A	Rate of input to the total population
a	Fraction of input to susceptible class
b	Fraction of input to infectious class
c	Fraction of input to removed class
μ	Natural death rate
γ	Recovery rate
α	Mortality due to virulence
δ	Rate of losing immunity

Their research [3] found that

$$R_0 = \beta f(\frac{A}{\mu}) \frac{A(\delta + (1 - c)\mu)}{(\mu + \gamma + \alpha)(\mu + \delta)\mu}$$

is the basic reproduction number. Furthermore, one gets

- If $b = 0$ and $R_0 < 1$, then system (1.1) has a disease-free equilibrium $E^0 = (S^0, I^0, R^0) = (\frac{A}{\mu} - \frac{cA}{\mu+\delta}, 0, \frac{cA}{\mu+\delta})$, which is globally asymptotically stable (GAS).
- If $R_0 > 1$ and $b = 0$, there exists a unique endemic equilibrium $E^* = (S^*, I^*, R^*)$ which is GAS.
- Otherwise if $b > 0$, there is no disease-free equilibrium in system (1.1) and there exists a unique endemic equilibrium $P^* = (S_1^*, I_1^*, R_1^*)$ which is locally asymptotically stable. In addition, when $\alpha \leq \mu + 2\delta$, the endemic equilibrium P^* is GAS.

However, in reality, variations in environmental factors affect the transmission coefficients of infectious diseases. As a result, stochastic modelling is an appropriate way to model epidemics in a variety of situations. For example, stochastic models can account for the randomness of infectious contacts that may occur during potential and infectious periods [10]. In comparison to deterministic models, stochastic epidemic models can provide more realism. A growing number of authors have recently focused on stochastic epidemic models [4–7, 11–22]. Cai et al. [7] discovered that the global dynamics of a general SIRS epidemic model can determine the existence of either a unique stationary distribution free of disease or a unique stationary distribution with endemic disease. Liu et al. [18] found that in a stochastic SIRS epidemic model with standard incidence, in which two threshold parameters R_0^S and $\hat{R_0^S}$ exist.

Inspired by Mao et al. [23], this paper posits that fluctuations in the environment primarily manifest as fluctuations in the transmission coefficient,

$$\beta \to \beta + \sigma \dot{B}(t),$$

where $B(t)$ is a standard Brownian motion and $\sigma^2 > 0$ indicates its intensity. Then we have

$$\begin{cases} dS(t) = [aA - \beta f(N)S(t)I(t) - \mu S(t) + \delta R(t)]dt - \sigma f(N)S(t)I(t)dB(t), \\ dI(t) = [bA + \beta f(N)S(t)I(t) - (\mu + \gamma + \alpha)I(t)]dt + \sigma f(N)S(t)I(t)dB(t), \\ dR(t) = [cA + \gamma I(t) - (\mu + \delta)R(t)]dt. \end{cases} \tag{1.2}$$

Our study is based on the deterministic SIRS epidemic model, which has proven to be an effective tool for investigating the spread of infectious diseases. Our approach incorporates two crucial elements: constant immigration and a general incidence rate, which are essential for understanding the impact of environmental fluctuations on disease dynamics.

One of the main strengths of our study lies in the fact that we have integrated these essential components into a stochastic framework. This has enabled us to analyze the effects of random fluctuations in disease transmission and immigration rates, which are significant factors that can profoundly influence the dynamics of infectious diseases. By examining these effects, we can obtain a more comprehensive understanding of the factors that contribute to the spread and persistence of diseases. Furthermore, our research has established the necessary conditions for the existence of a stationary distribution of positive solutions in the case of disease persistence. This novel contribution to the field has significant implications for the development of effective strategies for managing and controlling infectious diseases. Ultimately, our study provides valuable insights that can inform public health policies and initiatives aimed at reducing the impact of infectious diseases on global health.

The purpose of this paper is to explore the impact of environmental fluctuations on disease dynamics by analyzing the global dynamics of the stochastic SIRS epidemic model (1.2). The paper is structured as follows: In Section 2, we provide some preliminaries. Section 3 outlines the necessary conditions for disease extinction and persistence. We determine sufficient conditions for the existence of stationary distributions for persistent solutions of the model in Section 4. The paper concludes with numerical simulations and conclusions.

2. Preliminaries

In this paper, unless specified otherwise, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ denote a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ that satisfies the regular conditions. Let $B(t)$ be defined on this complete probability space. Denote $a \vee b = \max\{a, b\}$ for any $a, b \in \mathbb{R}$, and $\mathbb{X} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 > 0, x_2 > 0, x_3 > 0\}$.

Lemma 1. *[24] (Strong Law of Large Numbers) Let $M = \{M\}_{t\geq 0}$ be a real-valued continuous local martingale vanishing at $t = 0$. Then*

$$\lim_{t\to\infty}\langle M, M\rangle_t = \infty,\ a.s. \Rightarrow \lim_{t\to\infty}\frac{M_t}{\langle M, M\rangle_t} = 0\ a.s.,$$

and

$$\limsup_{t\to\infty}\frac{\langle M, M\rangle_t}{t} < \infty\ a.s. \Rightarrow \lim_{t\to\infty}\frac{M_t}{t} = 0\ a.s.$$

Theorem 1. *For any $(S(0), I(0), R(0)) \in \mathbb{X}$, there is a unique solution $(S(t), I(t), R(t))$ of system (1.2) that remain in $\mathbb{X}$ with probability one.*

The proof is standard and hence is omitted here.

Remark 1. *From Theorem 2.1, we have*

$$[A - (\alpha + \mu)N]dt \leq dN \leq [A - \mu N]dt, \quad dR \geq [cA - (\mu + \delta)R]dt, \quad t \in [0, \infty), \quad a.s.$$

This implies that

$$\Gamma = \{(S, I, R) \in \mathbb{X} : \frac{A}{\alpha + \mu} < N < \frac{A}{\mu}, R > \frac{cA}{\mu + \delta}\}$$

is a positively invariant set of system (1.2). Hence throughout this paper we always assume that the initial value $(S(0), I(0), R(0)) \in \Gamma$.

3. Extinction

In contrast to the deterministic system (1.1), the purpose of this section is to study the dynamics of the system (1.2) when $b = 0$ holds. Denote

$$R_0^s := \beta f(\frac{A}{\mu})\frac{A(\delta + (1 - c)\mu)}{(\mu + \gamma + \alpha)(\mu + \delta)\mu} - \frac{\sigma^2 f^2(\frac{A}{\mu})A^2}{2(\mu + \gamma + \alpha)\mu^2} = R_0 - \frac{\sigma^2 f^2(\frac{A}{\mu})A^2}{2(\mu + \gamma + \alpha)\mu^2}.$$

Theorem 2. *Let $b = 0$ and $(S(t), I(t), R(t))$ be a solution of system (1.2). If*

$$\sigma^2 > \max\{\frac{\beta^2}{2(\mu + \gamma + \alpha)}, \frac{\beta\mu}{f(\frac{A}{\mu})A}\} \tag{3.1}$$

or

$$R_0^s < 1 \text{ and } \sigma^2 < \frac{\beta\mu}{f(\frac{A}{\mu})A}, \tag{3.2}$$

then

$$\limsup_{t\to\infty}\frac{\ln I(t)}{t} < 0,\ \lim_{t\to\infty} S(t) = \frac{A}{\mu} - \frac{cA}{\mu + \delta},\ \lim_{t\to\infty} R(t) = \frac{cA}{\mu + \delta}\ a.s.$$

Proof. Making the use of Itô's formula [24] to $\ln I$, we have

$$d\ln I = (\beta f(N)S - (\mu+\gamma+\alpha) - \frac{\sigma^2}{2}f^2(N)S^2)dt + \sigma f(N)S\,dB(t).$$

Integrating the above equality from 0 to t and then dividing by t on both sides, one obtains

$$\frac{\ln I(t) - \ln I(0)}{t} = \frac{\int_0^t \phi(\tau)d\tau}{t} + \frac{G(t)}{t}, \tag{3.3}$$

where

$$\phi(\tau) = \beta f(N(\tau))S(\tau) - (\mu+\gamma+\alpha) - \frac{\sigma^2}{2}f^2(N(\tau))S^2(\tau),\ G(t) = \int_0^t \sigma f(N(\tau))S(\tau)dB(\tau).$$

Noting that $G(t)$ is a local martingale (since it is a right continuous adapted process defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$) whose quadratic variation is

$$\langle G, G\rangle_t = \int_0^t \sigma^2 f^2(N(\tau))S^2(\tau)d\tau \leq \sigma^2 f^2(\frac{A}{\mu+\alpha})\frac{A^2}{\mu^2}t.$$

Making the use of Lemma 2.1 leads to $\lim_{t\to\infty}\frac{G(t)}{t} = 0$ a.s. Combining (3.1), we have

$$\phi(\tau) = -\frac{\sigma^2}{2}(f(N(\tau))S(\tau) - \frac{\beta}{\sigma^2})^2 + \frac{\beta^2}{2\sigma^2} - (\mu+\gamma+\alpha) \leq \frac{\beta^2}{2\sigma^2} - (\mu+\gamma+\alpha).$$

Substituting the above inequality into (3.3) and taking the limit on both sides, we obtain

$$\lim_{t\to\infty}\frac{\ln I(t)}{t} \leq \frac{\beta^2}{2\sigma^2} - (\mu+\gamma+\alpha) < 0 \text{ a.s.} \tag{3.4}$$

Consider the case $\sigma^2 < \frac{\beta\mu}{f(\frac{A}{\mu})A}$, we get

$$\begin{aligned}\phi(\tau) &= \beta f(N(\tau))N(\tau)\frac{S(\tau)}{N(\tau)} - (\mu+\gamma+\alpha) - \frac{\sigma^2}{2}f^2(N(\tau))S^2(\tau)\\ &\leq \beta f(\frac{A}{\mu})\frac{A}{\mu}(1 - \frac{R(\tau)}{N(\tau)}) - \frac{\sigma^2}{2}f^2(\frac{A}{\mu})\frac{A^2}{\mu^2} - (\mu+\gamma+\alpha).\end{aligned} \tag{3.5}$$

Noting that $\frac{A}{\alpha+\mu} < N < \frac{A}{\mu}$, $R > \frac{cA}{\mu+\delta}$ and substituting them into (3.5), we have

$$\phi(\tau) \leq \beta f(\frac{A}{\mu})\frac{A(\delta + (1-c)\mu)}{\mu(\mu+\delta)} - \frac{\sigma^2}{2}f^2(\frac{A}{\mu})\frac{A^2}{\mu^2} - (\mu+\gamma+\alpha) = (R_0^s - 1)(\mu+\gamma+\alpha).$$

From (3.2) and (3.3), we get

$$\lim_{t\to\infty}\frac{\ln I(t)}{t} \leq (\mu+\gamma+\alpha)(R_0^s - 1) < 0 \text{ a.s.} \tag{3.6}$$

Then we have

$$\lim_{t\to\infty} I(t) = 0, \text{ a.s.}, \tag{3.7}$$

which means that for arbitrary small $\varepsilon > 0$ there are t_0 and Ω_ε such that $\mathbb{P}(\Omega_\varepsilon) \geq 1 - \varepsilon$ and $\alpha I \leq \varepsilon$ for $t \geq t_0$ and $\omega \in \Omega_\varepsilon$. In view of system (1.2), we have

$$\frac{A-\varepsilon}{\mu} \leq \lim_{t\to\infty} N(t) \leq \frac{A}{\mu} \text{ a.s.}$$

Due to the arbitrariness of ε, one has

$$\lim_{t\to\infty} N(t) = \frac{A}{\mu} \text{ a.s.} \tag{3.8}$$

Similarly as getting equality (3.8), we have

$$\lim_{t\to\infty} R(t) = \frac{cA}{\mu+\delta} \text{ a.s.} \tag{3.9}$$

In view of (3.7)–(3.9), we have

$$\lim_{t\to\infty} S(t) = \frac{A}{\mu} - \frac{cA}{\mu+\delta} \text{ a.s.}$$

□

Remark 2. *According to Theorem 3.1, if $R_0^s < 1$ and σ is not large, the disease will inevitably die out. It is worth noting that the expressions R_0^s and R_0 reveal that $R_0^s < R_0$. Furthermore, if $\sigma = 0$, $R_0^s = R_0$. In simpler terms, the conditions for the disease to die out in system* (1.2) *are considerably easier than those in the corresponding deterministic system* (1.1).

4. Asymptotic stability

In this section, we will prove that if $b = 0$ and $R_0^s > 1$ or $b > 0$, the densities of the distributions of the solutions to system (1.2) can converge in L^1 to an invariant density.

Theorem 3. *The distribution of $(S(t), I(t), R(t))$ has a density $U(t, x, y, z)$ for $t > 0$. If $b = 0$ and $R_0^s > 1$ or $b > 0$, then there is a unique density $U_*(x, y, z)$ such that*

$$\lim_{t\to\infty} \iiint_\Gamma |U(t, x, y, z) - U_*(x, y, z)| dxdydz = 0.$$

The following steps constitute the proof of Theorem 4.1 above:

- First, the kernel function of $(S(t), I(t), R(t))$ is absolutely continuous.
- We demonstrate that the kernel function is positive on $\mathbb{X}$.
- The Markov semigroup is either sweeping with respect to compact sets or asymptotically stable.
- Due to the presence of Khasminskiǐ function, we exclude sweeping.

For definitions related to Markov semigroups and their asymptotic properties, the reader is referred to the papers [25–31]. We will show this by Lemmas 4.1–4.5.

Lemma 2. *For $t > 0$ and any initial value $(x_0, y_0, z_0) \in \mathbb{X}$, the transition probability function $P(t, x_0, y_0, z_0, B)$ has a continuous density $k(t, x, y, z; x_0, y_0, z_0)$.*

Proof. Similar to the proof method in [31], the Lie bracket is given by

$$\left[\vec{a}, \vec{b}\right]_j (u) = \sum_{i=1}^{3} \left(a_i \frac{\partial b_j}{\partial u_i}(u) - b_i \frac{\partial a_j}{\partial u_i}(u) \right), \quad j = 1, 2, 3.$$

Let $a_0(S, I, R) = \begin{pmatrix} aA - \beta f(N)SI - \mu S + \delta R \\ bA + \beta f(N)SI - (\mu + \gamma + \alpha)I \\ cA + \gamma I - (\mu + \delta)R \end{pmatrix}$ and $a_1(S, I, R) = \begin{pmatrix} -\sigma f(N)SI \\ \sigma f(N)SI \\ 0 \end{pmatrix}$. Direct calculation leads to

$$a_2 = [a_0, a_1] = \begin{pmatrix} a_{21} \\ a_{22} \\ -\sigma\gamma f(N)SI \end{pmatrix},$$

with

$$\begin{aligned} a_{21} =& -(A - \mu N - \alpha I)\sigma SI f'(N) - \sigma f(N)(\beta f'(N)S^2 I^2 + (aA + \delta R)I + (bA - (\mu + \gamma + \alpha)I)S), \\ a_{22} =& \sigma f'(N)SI(A - \mu N - \alpha I) + \sigma f(N)(I(aA - (2\mu + \gamma + \alpha)S + \delta R) + bAS), \end{aligned}$$

and

$$a_3 = [a_1, a_2] = \begin{pmatrix} a_{31} \\ a_{32} \\ \sigma^2 \gamma SI f^2(N)(I - S) \end{pmatrix},$$

where

$$\begin{aligned} a_{31} =& \sigma^2 \beta S^2 I^2 f^2(N) f'(N)(I - 2S) + \sigma^2 f^2(N)(-\mu S^2 I - (aA + \delta R)I^2 + bAS^2) \\ & - \sigma^2 f(N) f'(N)SI(\beta f'(N)S^2 I^2 + \mu SI), \\ a_{32} =& \sigma^2 f^2(N)(\mu SI + \beta f'(N)S^2 I^3 + (aA + \delta R)I^2 + \mu S^2 I - bAS^2) + \sigma^2 f(N) f'(N)SI(\beta f'(N)S^2 I^2 + \mu SI). \end{aligned}$$

Therefore, we have

$$\begin{vmatrix} a_1 & a_2 & a_3 \end{vmatrix} = -\sigma\{\sigma^2 \gamma SI f^2(N)(I - S)(a_{21} + a_{22}) + \sigma\gamma f(N)SI(a_{31} + a_{32})\} < 0.$$

According to Hörmander Theorem [23], one obtains that $P(t, x_0, y_0, z_0, B)$ has a continuous density $k(t, x, y, z; x_0, y_0, z_0)$.

Next, fixing a point $(x_0, y_0, z_0) \in \mathbb{X}$ and a function $\psi \in L^2([0, T]; \mathbb{R})$, we have

$$\begin{cases} x_\psi(t) = x_0 + \int_0^t (f_1(x_\psi(\tau), y_\psi(\tau), z_\psi(\tau)) - \sigma\psi x_\psi(\tau) y_\psi(\tau) f(N_\psi(\tau)))d\tau, \\ y_\psi(t) = y_0 + \int_0^t (f_2(x_\psi(\tau), y_\psi(\tau), z_\psi(\tau)) - \sigma\psi x_\psi(\tau) y_\psi(\tau) f(N_\psi(\tau)))d\tau, \\ z_\psi(t) = z_0 + \int_0^t f_3(x_\psi(\tau), y_\psi(\tau), z_\psi(\tau))d\tau, \end{cases} \tag{4.1}$$

where

$$\begin{aligned} N_\psi &= x_\psi + y_\psi + z_\psi, \\ f_1 &= aA - \beta f(x + y + z)xy - \mu x + \delta z, \\ f_2 &= bA + \beta f(x + y + z)xy - (\mu + \gamma + \alpha)y, \\ f_3 &= cA + \gamma y - (\mu + \delta)z. \end{aligned}$$

Let $D_{X_0;\psi}$ be the Fr$\acute{e}$chet derivative. If for some $\psi \in L^2([0,T];\mathbb{R})$, the rank of $D_{X_0;\psi}$ is 3, then $k(T,x,y,x;x_0,y_0,z_0) > 0$ for $X = X_\psi(T)$. Let

$$\Psi(t) = f'(X_\psi(t)) + \psi g'(X_\psi(t)),$$

where f' and g' are the Jacobians of

$$f = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}, \quad g = \begin{pmatrix} -\sigma xyf(x+y+z) \\ \sigma xyf(x+y+z) \\ 0 \end{pmatrix}.$$

For $T \geq t \geq t_0 \geq 0$, let $Q(t,t_0)$ be a matrix function such that $Q(t_0,t_0) = Id$, $\frac{\partial Q(t,t_0)}{\partial t} = \Psi(t)Q(t,t_0)$. Then

$$D_{X_0;\psi}h = \int_0^T Q(T,\tau)g(\tau)h(\tau)d\tau.$$

□

Lemma 3. *For each* $(x_0,y_0,z_0),(x,y,z) \in \Gamma$*, there is* $T > 0$ *satistying* $k(T,x,y,z;x_0,y_0,z_0) > 0$.

Proof. Since we only need to find a continuous control function ψ, system (4.1) can be rewritten as follows

$$\begin{cases} x'_\psi(t) = f_1(x_\psi(t),y_\psi(t),z_\psi(t)) - \sigma\psi x_\psi(t)y_\psi(t)f(N_\psi(t)), \\ y'_\psi(t) = f_2(x_\psi(t),y_\psi(t),z_\psi(t)) - \sigma\psi x_\psi(t)y_\psi(t)f(N_\psi(t)), \\ z'_\psi(t) = f_3(x_\psi(t),y_\psi(t),z_\psi(t)), \end{cases} \tag{4.2}$$

First, we verify that the rank of $D_{X_0;\psi}$ is 3. Let $\varepsilon \in (0,T)$ and

$$h(t) = \frac{\chi_{[T-\varepsilon,T]}}{x_\psi(t)y_\psi(t)f(N_\psi(t))},\ t \in [0,T],$$

where χ denotes the indicator function of the interval $[T-\varepsilon,T]$. Since

$$Q(T,\tau) = Id + \Psi(T)(\tau - T) + \frac{1}{2}\Psi^2(T)(\tau-T)^2 + \circ((\tau-T)^2),$$

we have

$$D_{X_0;\psi}h = \varepsilon v - \frac{\varepsilon^2}{2}\Psi(T)v + \frac{\varepsilon^3}{6}\Psi^2(T)v + \circ(\varepsilon^3),$$

where $v = \begin{pmatrix} -\sigma \\ \sigma \\ 0 \end{pmatrix}$. Direct calculation leads to

$$\Psi(T)v = \sigma \begin{pmatrix} (\beta+\psi\sigma)f(N)(I-S) + \mu \\ (\beta+\psi\sigma)f(N)(I-S) - (\mu+\gamma+\alpha) \\ \gamma \end{pmatrix},$$

$$\Psi^2(T)v = \sigma \begin{pmatrix} c_{11} \\ c_{21} \\ \sigma\gamma(\beta+\psi\sigma)(S-I)f(N) - \sigma\gamma(2\mu+\gamma+\alpha+\delta) \end{pmatrix},$$

where

$$\begin{aligned}c_{11} =& -\sigma(S-I)^2(\beta+\psi\sigma)^2 f^2(N) + \sigma(\beta+\psi\sigma)(2\mu(S-I)+(\alpha+\gamma)S)f(N)\\ &+\sigma(\gamma\delta-\mu^2)+\sigma(\beta+\psi\sigma)(\alpha+\gamma)SIf'(N),\\ c_{21} =& \sigma(S-I)^2(\beta+\psi\sigma)^2 f^2(N)+\sigma(\beta+\psi\sigma)(-2(\alpha S-\mu I)+(\alpha+\gamma)I-2S(\mu+\gamma))f(N)\\ &-\sigma(\beta+\psi\sigma)(\alpha+\gamma)SIf'(N)+\sigma(\mu+\gamma+\alpha)^2.\end{aligned}$$

Thus the rank of $D_{X_0;\psi}$ is 3.

Then let $w_\psi = x_\psi + y_\psi + z_\psi$, (4.2) becomes

$$\begin{cases} x'_\psi(\tau) = g_1(x_\psi(\tau), w_\psi(\tau), z_\psi(\tau)) - \sigma\psi x_\psi(\tau)(w_\psi(\tau) - x_\psi(\tau) - z_\psi(\tau))f(w_\psi(\tau)),\\ w'_\psi(\tau) = g_2(x_\psi(\tau), w_\psi(\tau), z_\psi(\tau)),\\ z'_\psi(\tau) = g_3(x_\psi(\tau), w_\psi(\tau), z_\psi(\tau)),\end{cases} \tag{4.3}$$

where

$$\begin{aligned} g_1 &= aA - \beta f(w)x(w-x-z) - \mu x + \delta z,\\ g_2 &= bA - (\mu+\alpha)w + \alpha(x+z),\\ g_3 &= cA + \gamma(w-x) - (\gamma+\mu+\delta)z.\end{aligned} \tag{4.4}$$

Let

$$\Gamma_0 = \left\{(x,w,z)\in\mathbb{X} : 0 < x < w, \frac{cA}{\mu+\delta} < z < w \text{ and } \frac{A}{\alpha+\mu} < w < \frac{A}{\mu}\right\}. \tag{4.5}$$

First, we find a positive constant T and a differentiable function

$$w_\psi : [0,T] \to \left(\frac{A}{\alpha+\mu}, \frac{A}{\mu}\right)$$

such that $w_\psi(0) = w_0$, $w_\psi(T) = w_1$, $w'_\psi(0) = g_2(x_0, w_0, z_0) = w_0^d$, $w'_\psi(T) = g_2(x_1, w_1, z_1) = w_T^d$ and

$$A - (\alpha+\mu)w_\psi(t) < w'_\psi(t) < A - \mu w_\psi(t), \quad t \in [0,T].$$

We split the construction of the function w_ψ on three intervals $[0,\tau]$, $[\tau, T-\tau]$ and $[T-\tau, T]$, where $0 < \tau < T/2$. Let

$$\xi = \frac{1}{2}\min\left\{w_0 - \frac{A}{\alpha+\mu}, w_1 - \frac{A}{\alpha+\mu}, \frac{A}{\mu} - w_0, \frac{A}{\mu} - w_1\right\}.$$

If $w_\psi \in \left(\frac{A}{\alpha+\mu}+\theta, \frac{A}{\mu}-\theta\right)$, we have

$$A - (\alpha+\mu)w_\psi(t) < -(\alpha+\mu)\theta < 0,\ 0 < \mu\theta < A - \mu w_\psi(t), \quad t\in[0,T].$$

Then we construct a C^2-function w_ψ: $[0,\tau] \to \left(\frac{A}{\alpha+\mu}+\theta, \frac{A}{\mu}-\theta\right)$ such that

$$w_\psi(0) = w_0,\ w'_\psi(0) = w_0^d,\ w'_\psi(\tau) = 0,$$

and for $t \in [0,\tau]$, w_ψ satisfies

$$A - (\alpha+\mu)w_\psi(t) < w'_\psi(t) < A - \mu w_\psi(t).$$

Analogously, we can construct a C^2-function w_ψ: $[T-\tau, T] \to \left(\frac{A}{\alpha+\mu}+\theta, \frac{A}{\mu}-\theta\right)$ such that

$$w_\psi(T) = w_1,\ w'_\psi(T) = w^d_T,\ w'_\psi(T-\tau) = 0,$$

and for $t \in [T-\tau, T]$, w_ψ satisfies

$$A - (\alpha+\mu)w_\psi(t) < w'_\psi(t) < A - \mu w_\psi(t).$$

Taking T sufficiently large, then we can extend the function w_ψ: $[0,\tau]\bigcup[T-\tau, T] \to \left(\frac{A}{\alpha+\mu}+\theta, \frac{A}{\mu}-\theta\right)$ to a C^2-function w_ψ defined on the whole interval $[0, T]$ such that

$$A - (\alpha+\mu)w_\psi(t) < w'_\psi(t) < A - \mu w_\psi(t).$$

Thus, we can find C^1-functions x_ψ and z_ψ that satisfy (4.3). Finally we can determine a continuous function ψ. and $T > 0$ such that $x_\psi(0) = x_0$, $w_\psi(0) = w_0$, $z_\psi(0) = z_0$, $x_\psi(T) = x$, $w_\psi(T) = w$, $z_\psi(T) = z$. This completes the proof. □

Lemma 4. *If* $b = 0$ *and* $R^s_0 > 1$ *or* $b > 0$. *For* $\{P(t)\}_{t\geq 0}$ *and every density g, one has*

$$\lim_{t\to\infty}\iiint_\Gamma P(t)g(x,y,z)dxdydz = 1.$$

Proof. System (1.2) can be rewriten as

$$\begin{cases} dS = g_1(S,N,R)dt - \sigma S(N-S-R)f(N)dB(t),\\ dN = g_2(S,N,R)dt,\\ dR = g_3(S,N,R)dt. \end{cases} \tag{4.6}$$

From Remark 2.1, we get

$$A - (\alpha+\mu)N < \frac{dN}{dt} < A - \mu N \text{ and } \frac{dR}{dt} > cA - (\mu+\delta)R \text{ for } t \in (0,\infty) \text{ a.s.} \tag{4.7}$$

For almost every $w \in \Omega$, there is $t_0 \in t_0(w)$ such that

$$\frac{A}{\alpha+\mu} < N(t,w) < \frac{A}{\mu} \text{ and } R(t,w) > \frac{cA}{\mu+\delta} \text{ for } t > t_0.$$

As a matter of fact, there exist three possible cases:

1) $N(0,w) \in \left(\frac{A}{\alpha+\mu}, \frac{A}{\mu}\right)$. In this case, our statement is obvious from (4.7).

2) $N(0,w) \in \left(0, \frac{A}{\alpha+\mu}\right)$. Assume that our claim is not satisfied. Then there is $\Omega' \subset \Omega$ with $\mathbb{P}(\Omega') > 0$ such that $N(t,w) \in (0, \frac{A}{\alpha+\mu}), w \in \Omega'$. By (4.7), we obtain that for any $w \in \Omega'$, $N(t,w)$ is strictly increasing on $[0,\infty)$ and

$$\lim_{t\to\infty} N(t,w) = \frac{A}{\alpha+\mu},\ w \in \Omega'.$$

According to system (4.6), we get that $\lim_{t\to\infty} S(t,w) = \lim_{t\to\infty} R(t,w) = 0$, $w \in \Omega'$ and thus, $\lim_{t\to\infty} I(t,w) = \frac{A}{\alpha+\mu}$, $w \in \Omega'$.

Consider the case $b = 0$, making the use of Itô's formula, we have

$$d\ln I = \left(\beta f(N)S - (\mu+\gamma+\alpha) - \frac{\sigma^2}{2}S^2 f^2(N)\right)dt + \sigma S f(N)dB(t).$$

Thus

$$\begin{aligned}\frac{\ln I(t) - \ln I(0)}{t} &= \frac{1}{t}\int_0^t \left(\beta f(N(\tau))S(\tau) - (\mu+\gamma+\alpha) - \frac{\sigma^2}{2}S^2(\tau)f^2(N(\tau))\right)d\tau + \frac{1}{t}\int_0^t \sigma S(\tau)f(N(\tau))dB(\tau)\\ &= \frac{1}{t}\int_0^t \left(\beta f(N(\tau))S(\tau) - (\mu+\gamma+\alpha) - \frac{\sigma^2}{2}S^2(\tau)f^2(N(\tau))\right)d\tau + \frac{G(t)}{t},\end{aligned}$$

where $G(t) := \frac{1}{t}\int_0^t \sigma S(\tau)f(N(\tau))dB(\tau)$. Applying Lemma 2.1, we have

$$\lim_{t\to\infty}\frac{G(t)}{t} = 0 \text{ a.s.}$$

Thus, due to $S(t)$, $I(t)$, $f(N(t))$ are continuous,

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \left(\beta f(N(\tau))S(\tau) - (\mu+\gamma+\alpha) - \frac{\sigma^2}{2}S^2(\tau)f^2(N(\tau))\right)d\tau + \lim_{t\to\infty}\frac{G(t)}{t} = -(\mu+\gamma+\alpha).$$

This contradicts the assumption

$$\lim_{t\to\infty}\frac{\ln I(t) - \ln I(0)}{t} = 0 \; a.s.$$

Then let us consider the case $b > 0$. Since $\lim_{t\to\infty} N(t,w) = \frac{A}{\alpha+\mu}$ and $\lim_{t\to\infty} S(t,w) = \lim_{t\to\infty} R(t,w) = 0$ for $w \in \Omega'$, which contradicts that $R(t,w) > 0$ for $w \in \Omega'$, $t \in (0,\infty)$ and the claim follows.

3) $N(0,w) \in (\frac{A}{\mu}, \infty)$. We suppose, by contradiction, and analogous arguments to 2), that there is $\Omega' \subset \Omega$ with $\mathbb{P}(\Omega') > 0$ such that

$$\lim_{t\to\infty} N(t,w) = \frac{A}{\mu}, w \in \Omega'.$$

Firstly, consider the case $b = 0$, by the second and third equations of (4.6), for any $w \in \Omega'$, one gets

$$N(t,w) = e^{-(\mu+\alpha)t}\left(N(0,w) + \int_0^t e^{(\mu+\alpha)\tau}(A + \alpha(S(\tau,w) + R(\tau,w)))d\tau\right),$$

$$R(t,w) = e^{-(\mu+\delta)t}\left(R(0,w) + \int_0^t e^{(\mu+\delta)\tau}(cA + \gamma I(\tau,w))d\tau\right).$$

For all $w \in \Omega'$, one has

$$\lim_{t\to\infty} S(t,w) = \frac{A}{\mu} - \frac{cA}{\mu+\delta}, \quad \lim_{t\to\infty} I(t,w) = 0, \quad \lim_{t\to\infty} R(t,w) = \frac{cA}{\mu+\delta} \text{ a.s.}$$

Therefore

$$
\begin{aligned}
\lim_{t\to\infty}\frac{\ln I(t)-\ln I(0)}{t} &= \lim_{t\to\infty}\frac{1}{t}\int_0^t\left(\beta f(N(\tau))S(\tau)-(\mu+\gamma+\alpha)-\frac{\sigma^2}{2}S^2(\tau)f^2(N(\tau))\right)d\tau+\lim_{t\to\infty}\frac{G(t)}{t}\\
&= \lim_{t\to\infty}\frac{1}{t}\int_0^t\left(\beta f(N(\tau))S(\tau)-(\mu+\gamma+\alpha)-\frac{\sigma^2}{2}S^2(\tau)f^2(N(\tau))\right)d\tau\\
&= \beta f(\frac{A}{\mu})\frac{A(\delta+(1-c)\mu)}{\mu(\mu+\delta)}-(\mu+\gamma+\alpha)-\frac{\sigma^2}{2}\left(\frac{A}{\mu}-\frac{cA}{\mu+\delta}\right)^2 f^2(\frac{A}{\mu})\\
&> \beta f(\frac{A}{\mu})\frac{A(\delta+(1-c)\mu)}{\mu(\mu+\delta)}-(\mu+\gamma+\alpha)-\frac{\sigma^2}{2}\frac{A^2}{\mu^2}f^2(\frac{A}{\mu})\\
&= (\mu+\gamma+\alpha)(R_0^s-1)\\
&> 0 \ \text{ a.s. on } \Omega'.
\end{aligned}
$$

This contradicts the assumption $\lim_{t\to\infty} I(t) = 0$ a.s. In other words, for almost all $w \in \Omega$, there is $t_0 = t_0(w)$ such that

$$\frac{A}{\alpha+\mu} < N(t,w) < \frac{A}{\mu} \text{ for } t > t_0.$$

When $b > 0$, we get that $I(t,w) > 0$ for $t \in (0,\infty)$ and $w \in \Omega'$. This contradicts the assumption $\lim_{t\to\infty} N(t,w) = \frac{A}{\mu}$, $w \in \Omega'$ and the claim holds.

Similar to the proof of 2) and 3), one obtains that for almost all $w \in \Omega$, there is $t_1 = t_1(w)$ such that

$$R(t,w) > \frac{cA}{\mu+\delta} \text{ for } t > t_1.$$

□

Lemma 5. *$\{P(t)\}_{t\geq 0}$ is asymptotically stable or is sweeping with respect to compact sets.*

Proof. In view of Lemma 4.1, $\{P(t)\}_{t\geq 0}$ is an integral Markov semigroup with kernel $k(t,x,y,z;x_0,y_0,z_0)$. According to Lemma 4.3, it suffices to consider the restriction of $\{P(t)\}_{t\geq 0}$ to the space $L^1(\Gamma)$. By Lemma 4.2, one gets

$$\int_0^\infty P(t)g dt > 0 \ a.s.$$

on Γ, for every $g \in \mathbb{D}$. Then $\{P(t)\}_{t\geq 0}$ is asymptotically stable or is sweeping with respect to compact sets. □

Lemma 6. *Assume that $b = 0$ and $R_0^s > 1$ or $b > 0$, then $\{P(t)\}_{t\geq 0}$ is asymptotically stable.*

Proof. From Lemma 4.4, $\{P(t)\}_{t\geq 0}$ satisfies the Foguel alternative. In order to exclude sweeping it is sufficient to construct a nonnegative C^2-Khasminskiǐ function V and a closed set $D_\epsilon \in \Sigma$ such that

$$\sup_{(S,I,R)\in\mathbb{X}\setminus D_\epsilon}\mathscr{A}^*V(S,I,R) < 0.$$

First of all, we consider the case $b = 0$ and $R_0^s > 1$. Define

$$
\begin{aligned}
H =& M(-\ln I - \ell_1 N + \ell_2 R) - \ln S - \ln(\frac{A}{\mu} - N) - \ln(N - \frac{A}{\mu+\alpha}) - \ln(R - \frac{cA}{\mu+\delta}) \\
:=& MV_1 + V_2 + V_3 + V_4 + V_5,
\end{aligned}
$$

where $V_1 = -\ln I - \ell_1 N + \ell_2 R$, $V_2 = -\ln S$, $V_3 = -\ln(\frac{A}{\mu} - N)$, $V_4 = -\ln(N - \frac{A}{\mu+\alpha})$, $V_5 = -\ln(R - \frac{cA}{\mu+\delta})$, $\ell_1 = \frac{\beta f(\frac{A}{\mu})}{\mu}$, $\ell_2 = \frac{\beta f(\frac{A}{\mu})}{\mu+\delta}$ and M is a positive constant satisfying

$$
\frac{\beta A}{\mu} f(\frac{A}{\alpha+\mu}) + 4\mu + \alpha + \delta + \frac{\sigma^2}{2}\frac{A^2}{\mu^2} f^2(\frac{A}{\alpha+\mu}) \leq M(\mu+\gamma+\alpha)(R_0^s - 1) - 2. \tag{4.8}
$$

It is easy to find that H reaches a minimum at (S_*, I_*, R_*). Then we define

$$
V = MV_1 + V_2 + V_3 + V_4 + V_5 - H(S_*, I_*, R_*).
$$

Thus we have

$$
\begin{aligned}
\mathscr{A}^* V_1 =& -\beta S f(N) + (\mu+\gamma+\alpha) + \frac{\sigma^2}{2} S^2 f^2(N) - \ell_1 A + \ell_1 \mu N + \ell_1 \alpha I + \ell_2 cA + \ell_2 \gamma I - \ell_2(\mu+\delta) R \\
\leq& -\beta f(\frac{A}{\mu}) S + \ell_1 \mu S + \ell_1 \mu R - \ell_2(\mu+\delta) R + (\mu+\gamma+\alpha) + \frac{\sigma^2}{2}\frac{A^2}{\mu^2} f^2(\frac{A}{\mu}) \\
& - \ell_1 A + \ell_2 cA + (\ell_1 \mu + \ell_1 \alpha + \ell_2 \gamma) I \\
=& (\mu+\gamma+\alpha) + \frac{\sigma^2}{2}\frac{A^2}{\mu^2} f^2(\frac{A}{\mu}) - \ell_1 A + \ell_2 cA + (\ell_1 \mu + \ell_1 \alpha + \ell_2 \gamma) I \\
=& -\beta f(\frac{A}{\mu}) \frac{A(\delta + (1-c)\mu)}{\mu(\mu+\delta)} + (\mu+\gamma+\alpha) + \frac{\sigma^2}{2}\frac{A^2}{\mu^2} f^2(\frac{A}{\mu}) + (\ell_1 \mu + \ell_1 \alpha + \ell_2 \gamma) I \\
=& -(\mu+\gamma+\alpha)(R_0^s - 1) + (\ell_1 \mu + \ell_1 \alpha + \ell_2 \gamma) I.
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
\mathscr{A}^* V_2 =& -(\frac{aA}{S} - \beta I f(N) - \mu + \frac{\delta R}{S} - \frac{\sigma^2}{2} I^2 f^2(N)) \\
\leq& -\frac{aA}{S} + \beta \frac{A}{\mu} f(\frac{A}{\alpha+\mu}) + \mu + \frac{\sigma^2}{2}\frac{A^2}{\mu^2} f^2(\frac{A}{\alpha+\mu}),
\end{aligned}
$$

$$
\mathscr{A}^* V_3 = \mu - \frac{\alpha I}{\frac{A}{\mu} - N},
$$

$$
\mathscr{A}^* V_4 = -\frac{A - \mu N - \alpha I}{N - \frac{A}{\mu+\alpha}} \leq \mu + \alpha - \frac{\alpha I}{N - \frac{A}{\mu+\alpha}}
$$

and

$$
\mathscr{A}^* V_5 = -\frac{cA + \gamma I - (\mu+\delta) R}{R - \frac{cA}{\mu+\delta}} = \mu + \delta - \frac{\gamma I}{R - \frac{cA}{\mu+\delta}}.
$$

Therefore

$$\mathscr{A}^*V \leq -M(\mu+\gamma+\alpha)(R_0^s-1)+M(\ell_1\mu+\ell_1\alpha+\ell_2\gamma)I-\frac{aA}{S}-\frac{\alpha I}{\frac{A}{\mu}-N}-\frac{\alpha I}{N-\frac{A}{\mu+\alpha}}-\frac{\gamma I}{R-\frac{cA}{\mu+\delta}}$$
$$+\frac{\beta A}{\mu}f(\frac{A}{\alpha+\mu})+4\mu+\alpha+\delta+\frac{\sigma^2}{2}\frac{A^2}{\mu^2}f^2(\frac{A}{\alpha+\mu}).$$

Define

$$D_\epsilon=\{(S,I,R)\in\Gamma:\epsilon\leq S,\epsilon\leq I,\frac{cA}{\mu+\delta}+\epsilon^2\leq R,\frac{A}{\mu+\alpha}+\epsilon^2\leq N\leq\frac{A}{\mu}-\epsilon^2\},$$

where $\epsilon\in(0,1)$ is sufficiently small satisfying

$$-\frac{aA}{\epsilon}+M(\ell_1\mu+\ell_1\alpha+\ell_2\gamma)\frac{A}{\mu}+\beta\frac{A}{\mu}f(\frac{A}{\alpha+\mu})+4\mu+\alpha+\delta+\frac{\sigma^2}{2}\frac{A^2}{\mu^2}f^2(\frac{A}{\alpha+\mu})<-1, \tag{4.9}$$

$$\epsilon<\frac{1}{M(\ell_1\mu+\ell_1\alpha+\ell_2\gamma)}, \tag{4.10}$$

$$-\frac{\gamma}{\epsilon}+M(\ell_1\mu+\ell_1\alpha+\ell_2\gamma)\frac{A}{\mu}+\beta\frac{A}{\mu}f(\frac{A}{\alpha+\mu})+4\mu+\alpha+\delta+\frac{\sigma^2}{2}\frac{A^2}{\mu^2}f^2(\frac{A}{\alpha+\mu})<-1. \tag{4.11}$$

$$-\frac{\alpha}{\epsilon}+M(\ell_1\mu+\ell_1\alpha+\ell_2\gamma)\frac{A}{\mu}+\beta\frac{A}{\mu}f(\frac{A}{\alpha+\mu})+4\mu+\alpha+\delta+\frac{\sigma^2}{2}\frac{A^2}{\mu^2}f^2(\frac{A}{\alpha+\mu})<-1. \tag{4.12}$$

Denote

$$D_1=\{(S,I,R)\in\Gamma:S<\epsilon\},\ D_2=\{(S,I,R)\in\Gamma:I<\epsilon\},\ D_3=\{(S,I,R)\in\Gamma:I\geq\epsilon,R<\frac{cA}{\mu+\delta}+\epsilon^2\},$$

$$D_4=\{(S,I,R)\in\Gamma:I\geq\epsilon,\frac{A}{\mu}-\epsilon^2<N\},\ D_5=\{(S,I,R)\in\Gamma:I\geq\epsilon,N<\frac{A}{\mu+\alpha}+\epsilon^2\}.$$

Then we prove that $\mathscr{A}^*V(S,I,R)<-1$ for any $(S,I,R)\in\Gamma\setminus D_\epsilon=D_1\bigcup D_2\bigcup D_3\bigcup D_4\bigcup D_5$.

Case 1. For any $(S,I,R)\in D_1$, from (4.9),

$$\begin{aligned}\mathscr{A}^*V\leq&-\frac{aA}{S}+M(\ell_1\mu+\ell_1\alpha+\ell_2\gamma)\frac{A}{\mu}+\beta\frac{A}{\mu}f(\frac{A}{\alpha+\mu})+4\mu+\alpha+\delta+\frac{\sigma^2}{2}\frac{A^2}{\mu^2}f^2(\frac{A}{\alpha+\mu})\\<&-\frac{aA}{\epsilon}+M(\ell_1\mu+\ell_1\alpha+\ell_2\gamma)\frac{A}{\mu}+\beta\frac{A}{\mu}f(\frac{A}{\alpha+\mu})+4\mu+\alpha+\delta+\frac{\sigma^2}{2}\frac{A^2}{\mu^2}f^2(\frac{A}{\alpha+\mu})\\<&-1.\end{aligned}$$

Thus

$$\mathscr{A}^*V<-1 \text{ for any } (S,I,R)\in D_1.$$

Case 2. For any $(S,I,R)\in D_2$, from (4.8) and (4.10),

$$\begin{aligned}\mathscr{A}^*V\leq&-M(\mu+\gamma+\alpha)(R_0^s-1)+M(\ell_1\mu+\ell_1\alpha+\ell_2\gamma)\epsilon+\beta\frac{A}{\mu}f(\frac{A}{\alpha+\mu})+4\mu+\alpha+\delta+\frac{\sigma^2}{2}\frac{A^2}{\mu^2}f^2(\frac{A}{\alpha+\mu})\\<&-2+1\\=&-1.\end{aligned}$$

Therefore

$$\mathscr{A}^*V < -1 \text{ for any } (S, I, R) \in D_2.$$

Case 3. For any $(S, I, R) \in D_3$, from (4.11),

$$\begin{aligned}
\mathscr{A}^*V \leq & -\frac{\gamma I}{R - \frac{cA}{\mu+\delta}} + M(\ell_1\mu + \ell_1\alpha + \ell_2\gamma)I + \beta\frac{A}{\mu}f(\frac{A}{\alpha+\mu}) + 4\mu + \alpha + \delta + \frac{\sigma^2}{2}\frac{A^2}{\mu^2}f^2(\frac{A}{\alpha+\mu}) \\
< & -\frac{\gamma\epsilon}{\epsilon^2} + M(\ell_1\mu + \ell_1\alpha + \ell_2\gamma)\frac{A}{\mu} + \beta\frac{A}{\mu}f(\frac{A}{\alpha+\mu}) + 4\mu + \alpha + \delta + \frac{\sigma^2}{2}\frac{A^2}{\mu^2}f^2(\frac{A}{\alpha+\mu}) \\
= & -\frac{\gamma}{\epsilon} + M(\ell_1\mu + \ell_1\alpha + \ell_2\gamma)\frac{A}{\mu} + \beta\frac{A}{\mu}f(\frac{A}{\alpha+\mu}) + 4\mu + \alpha + \delta + \frac{\sigma^2}{2}\frac{A^2}{\mu^2}f^2(\frac{A}{\alpha+\mu}) \\
< & -1.
\end{aligned}$$

Hence

$$\mathscr{A}^*V < -1 \text{ for any } (S, I, R) \in D_3.$$

Case 4. For any $(S, I, R) \in D_4$, from (4.12),

$$\begin{aligned}
\mathscr{A}^*V \leq & -\frac{\alpha I}{\frac{A}{\mu} - N} + M(\ell_1\mu + \ell_1\alpha + \ell_2\gamma)I + \beta\frac{A}{\mu}f(\frac{A}{\alpha+\mu}) + 4\mu + \alpha + \delta + \frac{\sigma^2}{2}\frac{A^2}{\mu^2}f^2(\frac{A}{\alpha+\mu}) \\
\leq & -\frac{\alpha}{\epsilon} + M(\ell_1\mu + \ell_1\alpha + \ell_2\gamma)\frac{A}{\mu} + \beta\frac{A}{\mu}f(\frac{A}{\alpha+\mu}) + 4\mu + \alpha + \delta + \frac{\sigma^2}{2}\frac{A^2}{\mu^2}f^2(\frac{A}{\alpha+\mu}) \\
< & -1,
\end{aligned}$$

Then

$$\mathscr{A}^*V < -1 \text{ for any } (S, I, R) \in D_4.$$

Case 5. For any $(S, I, R) \in D_5$, from (4.12),

$$\begin{aligned}
\mathscr{A}^*V \leq & -\frac{\alpha I}{N - \frac{A}{\mu+\alpha}} + M(\ell_1\mu + \ell_1\alpha + \ell_2\gamma)I + \beta\frac{A}{\mu}f(\frac{A}{\alpha+\mu}) + 4\mu + \alpha + \delta + \frac{\sigma^2}{2}\frac{A^2}{\mu^2}f^2(\frac{A}{\alpha+\mu} \\
\leq & -\frac{\alpha}{\epsilon} + M(\ell_1\mu + \ell_1\alpha + \ell_2\gamma)\frac{A}{\mu} + \beta\frac{A}{\mu}f(\frac{A}{\alpha+\mu}) + 4\mu + \alpha + \delta + \frac{\sigma^2}{2}\frac{A^2}{\mu^2}f^2(\frac{A}{\alpha+\mu}) \\
< & -1.
\end{aligned}$$

Thus

$$\mathscr{A}^*V < -1 \text{ for any } (S, I, R) \in D_5.$$

In summary,

$$\sup_{(S,I,R)\in\Gamma\setminus D_\epsilon} \mathscr{A}^*V(S, I, R) < -1.$$

Using similar arguments to those in [27], we can obtain that $\{P(t)\}_{t\geq 0}$ is asymptotically stable.

Next, we consider the case $b > 0$, define

$$E = -\ln S - \ln I - \ln(R - \frac{cA}{\mu+\delta}) - \ln(N - \frac{A}{\mu+\alpha}) - \ln(\frac{A}{\mu} - N).$$

Obviously, E has a minimum point (S_{1*}, I_{1*}, R_{1*}) in the interior of Γ. Then we define

$$W = -\ln S - \ln I - \ln(R - \frac{cA}{\mu+\delta}) - \ln(N - \frac{A}{\mu+\alpha}) - \ln(\frac{A}{\mu} - N) - E(S_{1*}, I_{1*}, R_{1*}).$$

Then we have

$$\mathscr{A}^* W \leq -\frac{aA}{S} - \frac{bA}{I} - \frac{\gamma I}{R - \frac{cA}{\mu+\delta}} - \frac{\alpha I}{\frac{A}{\mu} - N} - \frac{\alpha I}{N - \frac{A}{\mu+\alpha}} + 5\mu + 2\alpha + \gamma + \delta + \frac{\beta A}{\mu} f(\frac{A}{\mu+\alpha}) + \frac{\sigma^2 A^2}{\mu^2} f^2(\frac{A}{\mu+\alpha}).$$

Similarly, define

$$U_{\epsilon_1} = \{(S, I, R) \in \Gamma : \epsilon_1 \leq S, \epsilon_1 \leq I, \frac{cA}{\mu+\delta} + \epsilon_1^2 \leq R, \frac{A}{\mu+\alpha} + \epsilon_1^2 \leq N \leq \frac{A}{\mu} - \epsilon_1^2\},$$

where $\epsilon_1 \in (0, 1)$ is sufficiently small satisfying

$$-\frac{aA}{\epsilon_1} + 5\mu + 2\alpha + \gamma + \delta + \frac{\beta A}{\mu} f(\frac{A}{\mu+\alpha}) + \frac{\sigma^2 A^2}{\mu^2} f^2(\frac{A}{\mu+\alpha}) < -1,$$

$$-\frac{bA}{\epsilon_1} + 5\mu + 2\alpha + \gamma + \delta + \frac{\beta A}{\mu} f(\frac{A}{\mu+\alpha}) + \frac{\sigma^2 A^2}{\mu^2} f^2(\frac{A}{\mu+\alpha}) < -1,$$

$$-\frac{\gamma}{\epsilon_1} + 5\mu + 2\alpha + \gamma + \delta + \frac{\beta A}{\mu} f(\frac{A}{\mu+\alpha}) + \frac{\sigma^2 A^2}{\mu^2} f^2(\frac{A}{\mu+\alpha}) < -1,$$

$$-\frac{\alpha}{\epsilon_1} + 5\mu + 2\alpha + \gamma + \delta + \frac{\beta A}{\mu} f(\frac{A}{\mu+\alpha}) + \frac{\sigma^2 A^2}{\mu^2} f^2(\frac{A}{\mu+\alpha}) < -1.$$

For convenience, we divide $\Gamma \setminus U_{\epsilon_1}$ as

$$U_1 = \{(S, I, R) \in \Gamma : S < \epsilon_1\},\ U_2 = \{(S, I, R) \in \Gamma : I < \epsilon_1\},\ U_3 = \{(S, I, R) \in \Gamma : I \geq \epsilon_1, R < \frac{cA}{\mu+\delta} + \epsilon_1^2\},$$

$$U_4 = \{(S, I, R) \in \Gamma : I \geq \epsilon_1, \frac{A}{\mu} - \epsilon_1^2 < N\},\ U_5 = \{(S, I, R) \in \Gamma : I \geq \epsilon_1, N < \frac{A}{\mu+\alpha} + \epsilon_1^2\}.$$

The rest of the proof is omitted here due to it is similar to the case of $b = 0$. This completes the proof. □

Remark 4.1. The stationary distribution of the correct solution refers to the long-term behavior of a stochastic system when the probability of the disease persisting is not zero. In other words, if the random threshold R_0^s is greater than 1, the disease may not be eradicated and will persist in the population. In this case, the stable distribution of the correct solution refers to the probability distribution of infected individuals in the population over time once the system has reached a steady state. This distribution is said to be stationary because it does not change over time, while the correct solution refers to the non-zero probability of individuals being infected.

Remark 4.2. According to Theorems 3.1 and 4.1, if $R_0^s < 1$, the disease will become extinct under mild additional conditions, whereas if $R_0^s > 1$, the disease will be stochastically persistent. The value of R_0^s can determine the extinction of the disease or not, and thus it can be considered as a threshold for the stochastic system (1.2).

5. Numerical simulations

In this section, we give several numerical examples to support our results. Employing Milstein's high-order method [32], the discretized system is

$$\begin{cases} S^{k+1} = S^k + [aA - \beta f(N^k)S^k I^k - \mu S^k + \delta R^k]\Delta t - \sigma f(N^k)S^k I^k \sqrt{\Delta t}\varrho_k \\ \qquad + \frac{1}{2}\sigma^2 f(N^k)S^k (I^k)^2 (f'(N^k)S^k + f(N^k))(\varrho_k^2 - 1)\Delta t, \\ I^{k+1} = I^k + [bA + \beta f(N^k)S^k I^k - (\mu + \gamma + \alpha)I^k]\Delta t + \sigma f(N^k)S^k I^k \sqrt{\Delta t}\varrho_k \\ \qquad + \frac{1}{2}\sigma^2 f(N^k)I^k (S^k)^2 (f'(N^k)I^k + f(N^k))(\varrho_k^2 - 1)\Delta t, \\ R^{k+1} = R^k + [cA + \gamma I - (\mu + \delta)R^k]\Delta t, \end{cases} \tag{5.1}$$

where the time increment $\Delta t > 0$, ϱ_k for $k = 1, 2, ..., n$ are Gaussian random variables following the standard normal distribution.

5.1. Threshold dynamics with the standard incidence

In this part, we focus on the dynamical behavior of system (1.2) with standard incidence. Let

$$f(N) = \frac{\lambda}{N}.$$

Assume

$$\begin{aligned} &A = 6, \;\; a = 0.9, \;\; \beta = 0.1, \;\; \alpha = 0.2, \;\; \mu = 0.02, \;\; \delta = 0.1, \\ &\lambda = 10, \;\; \gamma = 0.5, \;\; S(0) = 500, \;\; I(0) = 1, \;\; R(0) = 1, \end{aligned} \tag{5.2}$$

Parameters b, c and σ will take different values in different examples.

Example 1. (Stationary distribution) Let $b = 0$ and $c = 0.1$, then we obtain $R_0 = 1.3657 > 1$. From [3], the disease of the deterministic system (1.1) will persist in a long term (Figure 1).

For system (1.2), let $\sigma = 0.01$ and one obtains

$$R_0^s = R_0 - \frac{\sigma^2 f^2(\frac{A}{\mu})A^2}{2(\mu + \gamma + \alpha)\mu^2} = 1.3588 > 1.$$

From Theorem 4.1, system (1.2) admits an ergodic stationary distribution (Figure 1).

For the case with $b = 0.1$ and $c = 0$, we choose $\sigma = 0.075$ such that $R_0^s = R_0 - \frac{\sigma^2 f^2(\frac{A}{\mu})A^2}{2(\mu+\gamma+\alpha)\mu^2} = 0.9983 < 1$. From Theorem 4.1, system (1.2) admits an ergodic stationary distribution (Figure 2).

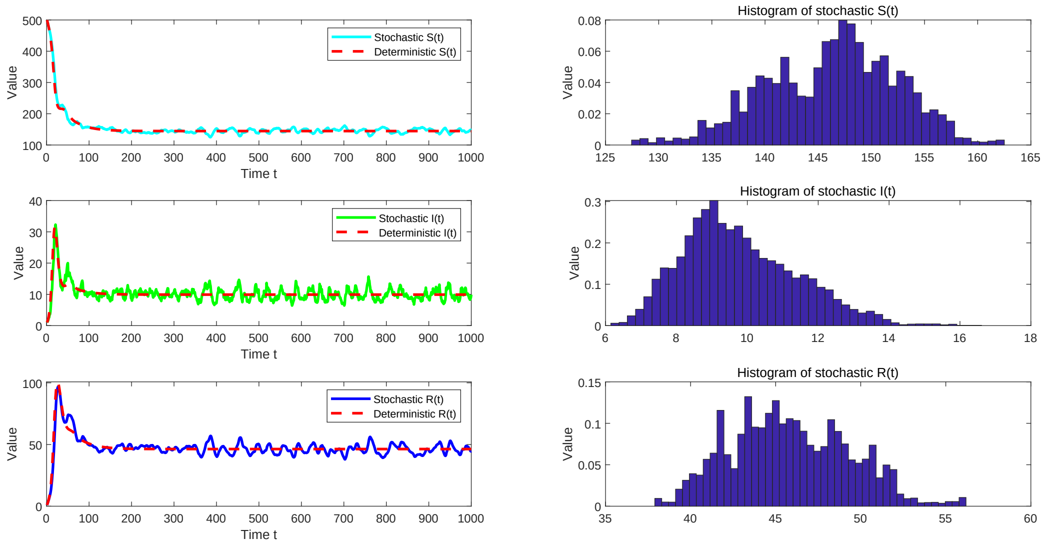

Figure 1. The pictures on the left present the numbers of S, I and R of system (1.2) with $b = 0$ and $R_0^s = 1.3588$, and its deterministic system (1.1) with $R_0 = 1.3657$. The pictures on the right show the corresponding frequency histogram of S, I and R with 50,000 iteration points, respectively. The run time of our code is about 1.6488 seconds on a standard computer with a 2.0 GHz processor and 8 GB of RAM.

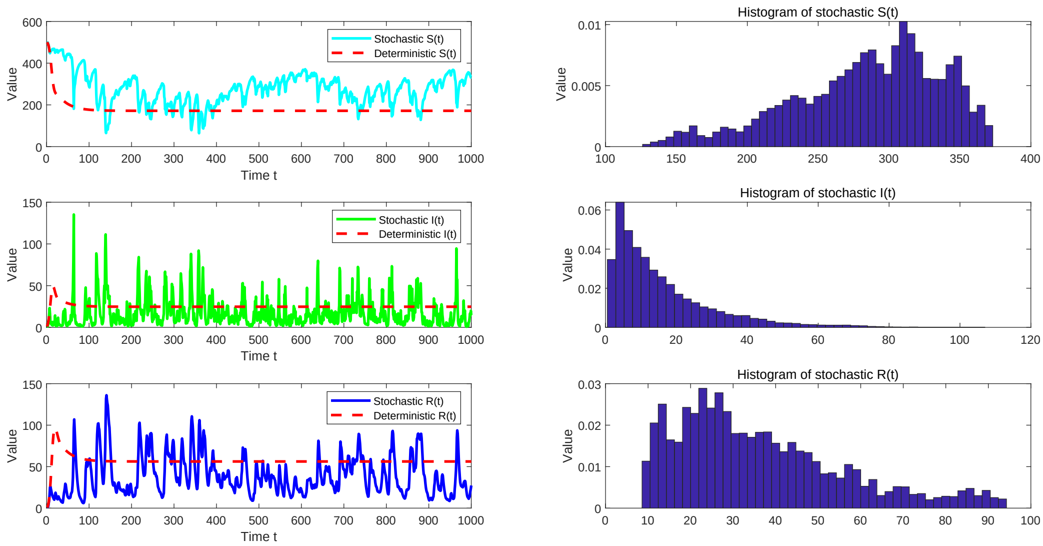

Figure 2. The pictures on the left present the numbers of S, I and R of system (1.2) with $b = 0.1$ and $R_0^s = 0.9983$, and its deterministic system (1.1) with $R_0 = 1.3657$. The pictures on the right show the corresponding frequency histogram of S, I and R with 50,000 iteration points, respectively. The run time of our code is about 1.7667 seconds.

Example 2. (Extinction) Let $b = 0$, $c = 0.1$, $\sigma = 0.1$, and the other parameters are shown in (5.2) such that

$$\sigma^2 - \max\{\frac{\beta^2}{2(\mu+\gamma+\alpha)}, \frac{\beta\mu}{f(\frac{A}{\mu})A}\} = 1.7347 \times 10^{-18} > 0,$$

then from Theorem 3.1, the disease of system (1.2) will become extinct, see Figure 3.

Let $b = 0$, $c = 0.1$ and $\sigma = 0.08$ and the other parameters are shown in (5.2) such that

$$R_0^s = R_0 - \frac{\sigma^2 f^2(\frac{A}{\mu})A^2}{2(\mu+\gamma+\alpha)\mu^2} = 0.9318 < 1,$$

and $\sigma^2 - \frac{\beta\mu}{f(\frac{A}{\mu})A} = -0.0036 < 0$. According to Theorem 3.1, the disease of system (1.2) will be extinct (Figure 4).

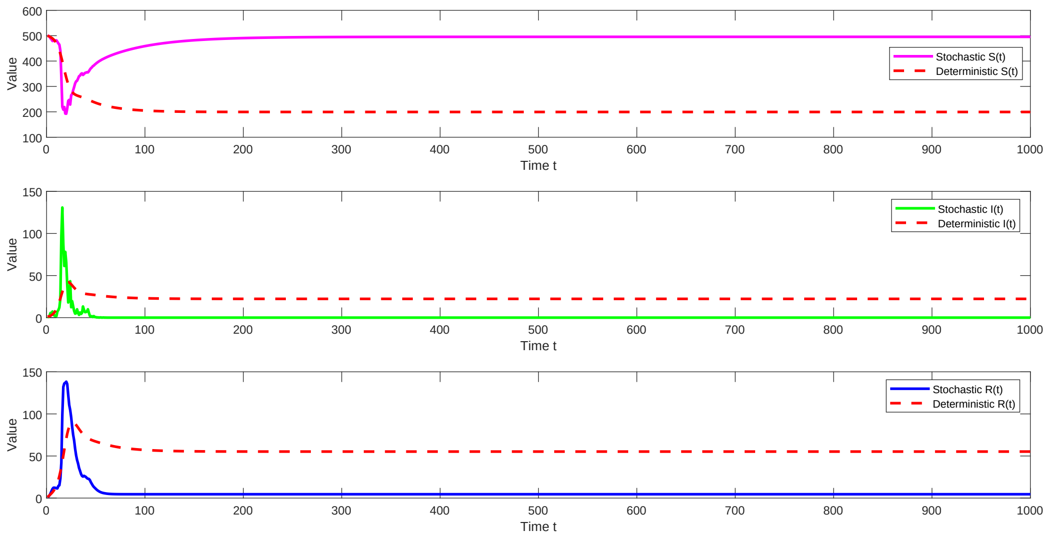

Figure 3. The pictures present the numbers of S, I and R of system (1.2) with $b = 0$ and $\sigma = 0.1$, and its deterministic system (1.1) with $R_0 = 1.3763 > 1$.

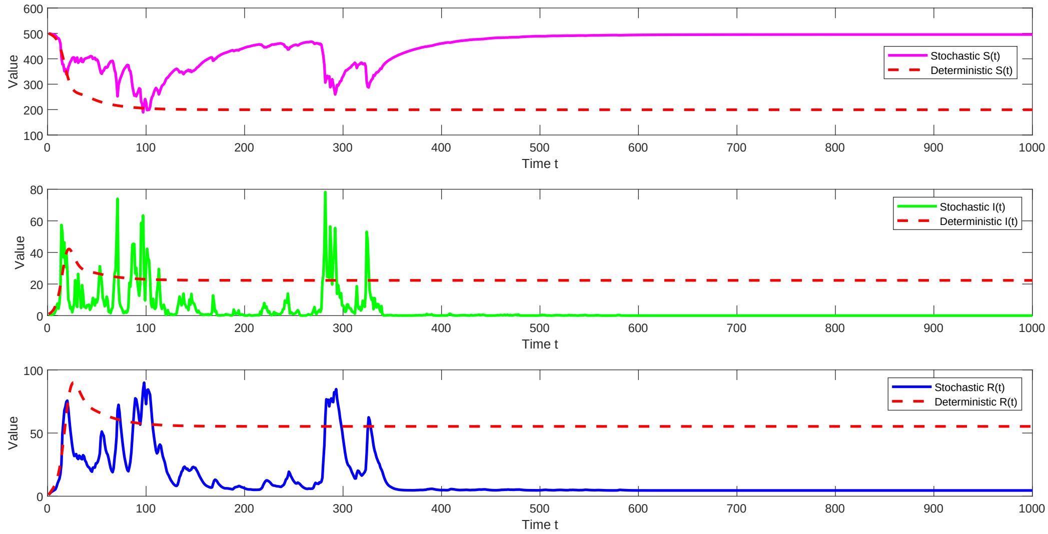

Figure 4. The phase diagram presents the numbers of S, I and R of system (1.2) with $b = 0$, $\sigma = 0.08$ and $R_0^s = 0.9318$, and its deterministic system (1.1) with $R_0 = 1.3763 > 1$.

5.2. *Threshold dynamics with the mass action incidence*

In this part, we investigate the threshold dynamics of deterministic system (1.1) and stochastic system (1.2) with mass action incidence. Let

$$f(N) = \lambda.$$

where λ is a positive constant. Assume

$$A = 10, \quad a = 0.9, \quad \alpha = 0.2, \quad \mu = 0.02, \quad \delta = 0.2, \quad \lambda = 1, \quad S(0) = 500, \quad I(0) = 1, \quad R(0) = 1. \tag{5.3}$$

Parameters β, b, c and σ will take different values in different examples.

Example 3. (Stationary distribution) First, consider the persistence of the disease of system (1.2) with $\beta = 0.002$, $b = 0$ and $c = 0.1$. Then we obtain $R_0 = 1.3763 > 1$. From [3], the disease of the deterministic system (1.1) will persist in a long term, see Figure 1.

For the stochastic system (1.2), let $\sigma = 0.0002$ and we obtain

$$R_0^s = R_0 - \frac{\sigma^2 f^2(\frac{A}{\mu})A^2}{2(\mu + \gamma + \alpha)\mu^2} = 1.37626 > 1.$$

From Theorem 4.1, the stochastic system (1.2) admits an ergodic stationary distribution. See Figure 5.

For the case with $b = 0.1$ and $c = 0$, we choose $\beta = 0.001$ and $\sigma = 0.0002$ such that $R_0^s = R_0 - \frac{\sigma^2 f^2(\frac{A}{\mu})A^2}{2(\mu+\gamma+\alpha)\mu^2} = 0.6944 < 1$. From Theorem 4.1, system (1.2) admits an ergodic stationary distribution. See Figure 6.

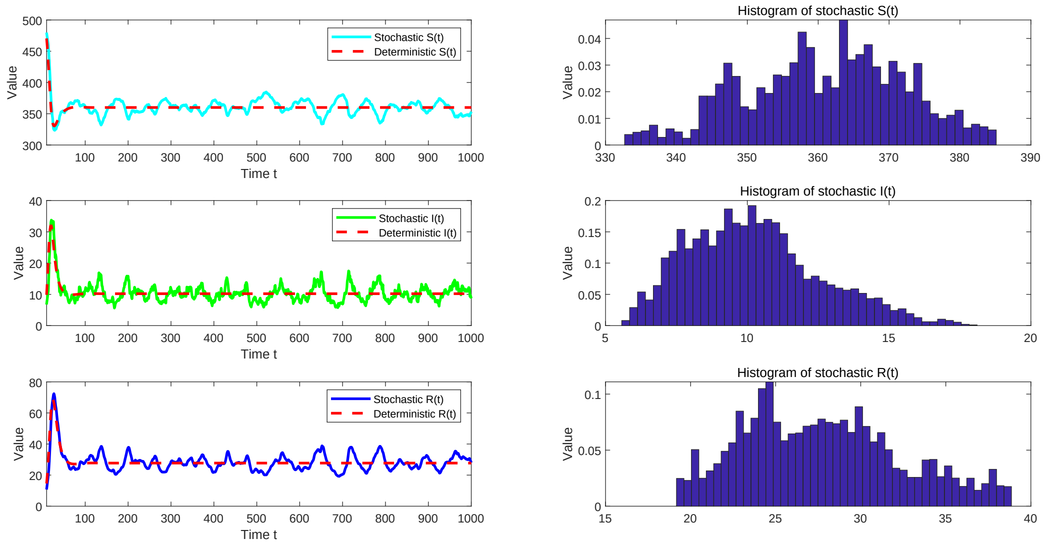

Figure 5. The pictures on the left present the numbers of S, I and R of system (1.2) with $b = 0$ and $R_0^s = 1.37626$, and its deterministic system (1.1) with $R_0 = 1.3763$. The pictures on the right show the corresponding frequency histogram of S, I and R with 50,000 iteration points, respectively. The run time of our code is about 1.6803 seconds.

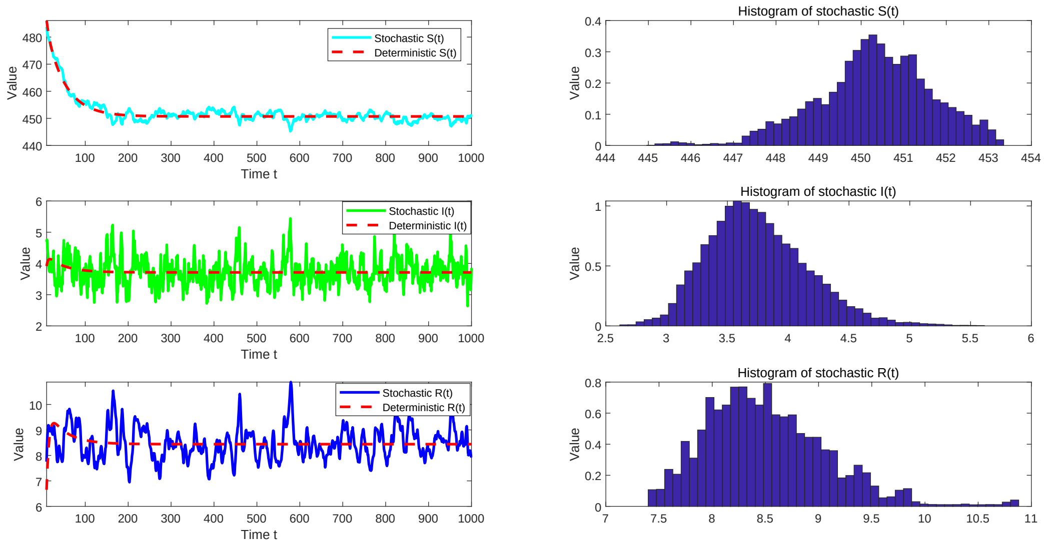

Figure 6. The pictures on the left present the numbers of S, I and R of the stochastic system (1.2) with $b = 0.1$ and $R_0^s = 0.6944$, and its deterministic system (1.1) with $R_0 = 1.3763$. The pictures on the right show the corresponding frequency histogram of S, I and R with 50,000 iteration points, respectively. The run time of our code is about 1.7259 seconds.

Example 4. (Extinction) Let $b = 0$, $c = 0.1$, $\beta = 0.0015$ and $\sigma = 0.002$, and the other parameters are shown in (5.3) such that

$$\sigma^2 - \max\{\frac{\beta^2}{2(\mu+\gamma+\alpha)}, \frac{\beta\mu}{f(\frac{A}{\mu})A}\} = 10^{-6} > 0,$$

thus from Theorem 3.1, the disease of system (1.2) will be extinct exponentially in a long term (Figure 7).

Let $b = 0$, $c = 0.1$, $\beta = 0.0014$ and $\sigma = 0.0015$ and the other parameters are shown in (5.3) such that

$$R_0^s = R_0 - \frac{\sigma^2 f^2(\frac{A}{\mu})A^2}{2(\mu+\gamma+\alpha)\mu^2} = 0.9634 < 1,$$

and $\sigma^2 - \frac{\beta\mu}{f(\frac{A}{\mu})A} = -5.5 \times 10^{-7} < 0$. According to Theorem 3.1, the disease of system (1.2) will be extinct exponentially in a long term (Figure 8).

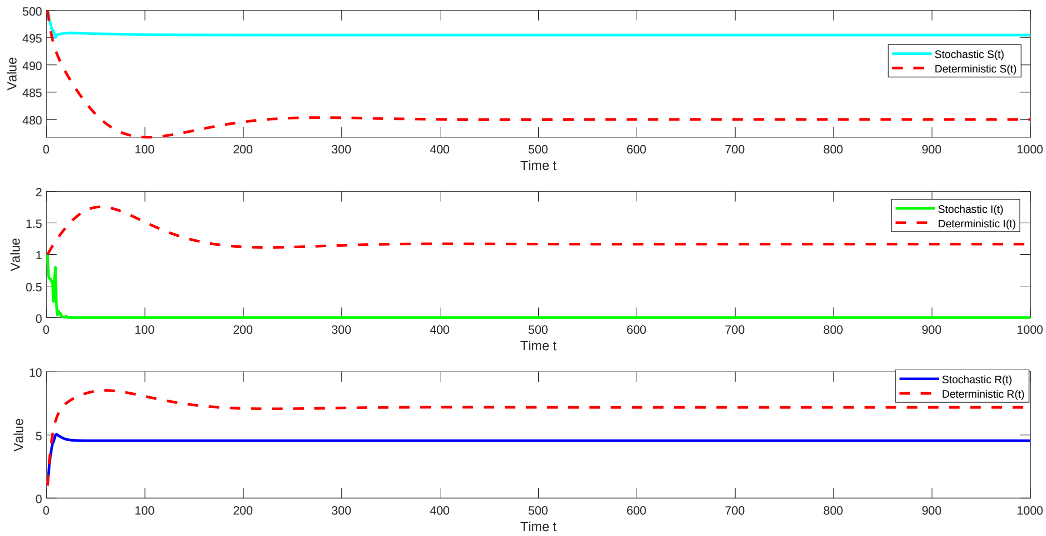

Figure 7. The phase diagram presents the numbers of S, I and R of system (1.2) with $b = 0$, $c = 0.1$, $\beta = 0.0015$ and $\sigma = 0.002$, and its deterministic system (1.1) with $R_0 = 1.0322$.

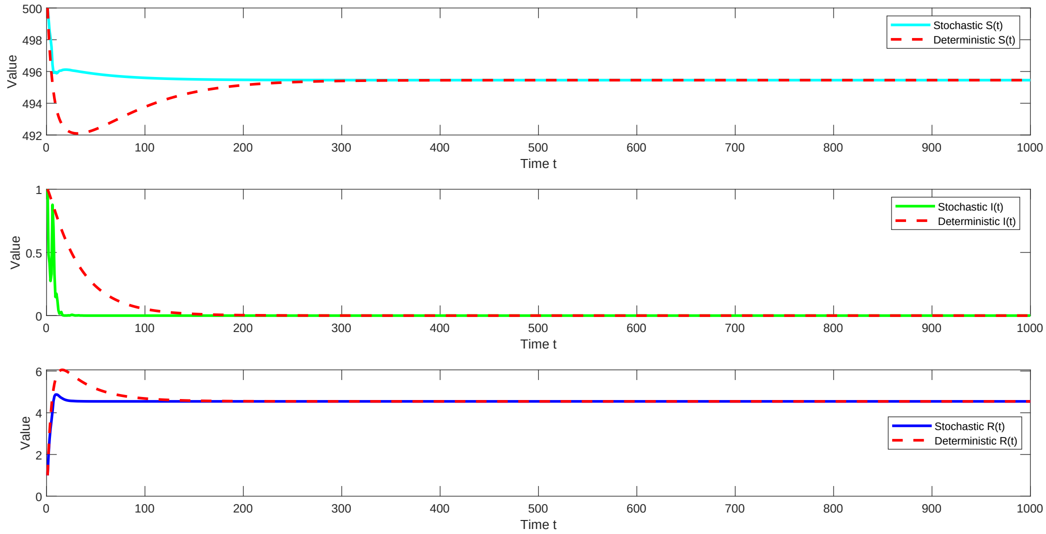

Figure 8. The phase diagram presents the numbers of S, I and R of system (1.2) with $b = 0$, $c = 0.1$, $\beta = 0.0014$, $\sigma = 0.0015$ and $R_0^s = 0.9634$, and its deterministic system (1.1) with $R_0 = 0.9634$.

6. Conclusions

In this study, we present a stochastic SIRS epidemic model with constant immigration and general incidence rate. Our results show that the threshold parameter

$$R_0^s = R_0 - \frac{\sigma^2 f^2(\frac{A}{\mu})A^2}{2(\mu+\gamma+\alpha)\mu^2}$$

for this model is lower than its deterministic counterpart ($R_0^s < 1 < R_0$). In this scenario, the deterministic system may have an endemic state, while the stochastic system leads to disease extinction with probability one (Theorem 3.1). On the other hand, if $R_0^s > 1$, the distribution of solution converge in L^1 to an invariant density (Theorem 4.1), indicating that environmental fluctuations can positively impact the control of infectious diseases. Moreover, if there is a constant influx of infected population, i.e. $b > 0$, the stationary distribution will always exist and the disease will persist. We contend that conducting a comprehensive analysis of the influence of migration on the dynamics of our model will yield valuable insights into the intricate interplay between migration and disease transmission.

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Conflict of interest

The authors declare there is no conflict of interest.

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