Mathematical Biosciences and Engineering

http://www.aimspress.com/journal/mbe

MBE, 20(6): 9891–9922.
DOI: 10.3934/mbe.2023434
Received: 15 January 2023
Revised: 17 February 2023
Accepted: 06 March 2023
Published: 27 March 2023

Research article

Simulations and fractional modeling of dengue transmission in Bangladesh

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Abstract: Dengue is one of the most infectious diseases in the world. In Bangladesh, dengue occurs nationally and has been endemic for more than a decade. Therefore, it is crucial that we model dengue transmission in order to better understand how the illness behaves. This paper presents and analyzes a novel fractional model for the dengue transmission utilizing the non-integer Caputo derivative (CD) and are analysed using q-homotopy analysis transform method (q-HATM). By using the next generation method, we derive the fundamental reproduction number R_0 and show the findings based on it. The global stability of the endemic equilibrium (EE) and the disease-free equilibrium (DFE) is calculated using the Lyapunov function. For the proposed fractional model, numerical simulations and dynamical attitude are seen. Moreover, A sensitivity analysis of the model is performed to determine the relative importance of the model parameters to the transmission.

Keywords: fractional model; q-Homotopy analysis method; stability analysis; basic reproduction number; sensitivity analysis; numerical simulations

1. Introduction

Dengue is a debilitating viral infection spread by the bite of Aedes mosquitoes carrying any one of the four dengue viral serotypes. It is common in urban and semi-urban areas of tropical and subtropical climates around the world. These serotypes of the virus that can cause dengue are closely related but antigenically distinct (DEN-1, DEN-2, DEN-3, DEN-4). While recovery from one virus confers lifetime immunity against that virus, [1] notes that recovery from the other three viruses offers no protection against infection. The majority of the world's population, particularly those who live in tropical and subtropical regions like Bangladesh, are at risk. About 390 million dengue infections are estimated to occur annually, of which a quarter of the cases (67–136 million) will manifest clinically [2], with the overall incidence of dengue having increased 30-fold over the past 50

years [3]. In Asia, a severe dengue outbreak was reported in 1950 in the Philippines and Thailand [3], and in 1964 Bangladesh experienced a dengue outbreak for the first time. Between 1964 and 1999, sporadic cases and small outbreaks clinically suggestive of dengue occurred across the country but were not officially reported, even though 5551 people were affected with 93 accounted fatalities [4].

Dengue Fever (DF) is marked by an onset of sudden high fever, severe headache, pain behind the eyes, and pain in muscles and joints. Some may also have a rash and varying degrees of bleeding from various parts of the body. Dengue has a wide spectrum of infection outcomes (asymptomatic to symptomatic). Symptomatic illness can vary from DF to more serious dengue hemorrhagic fever (DHF) [5, 6]. DHF is a more severe form, seen only in a small proportion of those infected. DHF is a stereotypic illness characterized by 3 phases; the febrile phase with high continuous fever usually lasting for less than 7 days; the critical phase lasting 1–2 days usually apparent when the fever comes down, leading to shock if not detected and treated early; convalescence phase lasting 2–5 days with an improvement of appetite. Dengue Shock Syndrome (DSS) is a dangerous complication of dengue infection and is associated with high mortality. Severe dengue occurs as a result of secondary infection with a different virus serotype. Statistics show that Bangladesh reported its first case of mosquito-borne virus infection in 2000, and about 100 people died of the disease from 2000 to 2003.

The first official outbreak of DF in Bangladesh was in 2000, and since then the number of hospitalized patients have exceeded 3000 patients six times: 6232 in 2002, 3934 in 2004, 3162 in 2015, 6060 in 2016, 10,148 in 2018, and 1,00107 as of Nov 30, 2019, with estimated projections of more than 1,12,000 cases by the end of 2019 (appendix) [4, 7, 8]. In parallel with the major epidemics in 2018 and 2019's outbreak, 26 deaths and 129 deaths, respectively, have been officially documented by the government surveillance systems with a clear predominance of cases and fatalities during the summer months (July to November), even if the death tally is likely to be much higher because of actual under-reporting (appendix) [8, 9].

In reality, Bangladesh is suffering from the pain of mosquitoes. On Saturday 15 October 2022, A total of 1,916 people are admitted to different government and private hospitals in Dhaka and 799 outside Dhaka. A total of 23,592 patients were admitted to the hospital from January 1 to October 14 this year. Of them, 20,794 people left the hospital after recovering. The number of dengue patients is increasing every day. The list of deaths is getting longer. So far this year, 83 people have died of dengue. Of these, 28 this month. In two weeks of this month, 7,500 patients were admitted to the hospital with the infection. Experts said pre-monsoon, post-monsoon, and monsoon surveillance are being done every year to see the mosquito situation.

A month-wise patient admission analysis showed that 126 patients were admitted to the hospital in January 2022, 20 in February, 20 in March, 23 in April, 163 in May, 737 in June, 1,571 in July, and 3,521 in August. The highest number of dengue patients were admitted in September and 9,911 people were admitted, 34 people died. In 2019, 1,01,354 people were admitted to the hospital with dengue. This is the highest number of patients admitted in a year in the country so far. That year, 164 people died. In 2018, 10,148 people were admitted to hospitals with dengue and 26 died. Although dengue infection was not seen much during the corona epidemic in 2020, 28,429 people were infected and 105 people died of dengue across the country in 2021 [10]. DF is the fastest-growing infectious disease in the world, causing an average of more than 500,000 potentially fatal infections and about 20,000 deaths each year. As of 17 November, 26,000 dengue cases were reported with 98 deaths [11]. So far, the case fatality rate seems higher in this outbreak (98/26,000) as compared

to 2019 massive outbreak (179/101,354) (Table 1).

Table 1. Number of reported dengue cases by months and yearly data between 2012 and 2022.

Months	2012	2013	2014	2015	2016	2017	2018	2019	2020	2021	2022
January	0	0	15	0	13	92	26	38	199	32	126
February	0	0	7	0	3	58	7	18	45	9	20
March	0	0	2	2	17	36	19	17	27	13	20
April	0	0	0	6	38	73	29	58	25	3	23
May	0	4	8	10	70	134	52	193	10	43	163
June	16	44	9	28	254	267	295	1884	20	272	737
July	108	220	82	171	926	286	946	16253	23	2286	1571
August	138	353	80	765	1451	346	1796	53636	68	7698	3521
September	262	495	76	965	1544	430	3087	16856	47	7841	9911
October	90	363	63	869	1077	512	2406	8143	164	5458	21932
November	57	212	22	271	522	409	1192	4011	546	3567	3457
December	0	58	11	75	145	126	293	1247	231	1207	–
Total cases	671	1749	375	3162	6060	2769	10148	101354	1405	28429	41481

Recently, fractional calculus has gained much interest from researchers. Because it has been widely used in different fields of science, and engineering, and also in different real-world problems. Several scholars have used different types of operator approaches and applied them to many linear and non-linear diseases and complex models [12–16]. In this paper, the proposed dengue epidemic model is derived using the Caputo fractional derivatives. In recent years, fractional-order calculus is found to be more appealing in modeling for a real-world problem in comparison to a classical integer order as it provides a tool for the description of memory effects and genetic properties of various materials. Recent entomological studies revealed that mosquitoes did not feed randomly on human blood, but they use their prior experience with human location and human defensiveness to select the host to feed on [17]. Thus, in dengue transmission, a future state does depend on the history of the transmission. Hence, the fractional-order differential equation is found to be the best approach to model the transmission. The purpose of this study is to propose and study a more accurate mathematical model of dengue transmission using the fractional-order derivative than those previously presented in the literature [18–20].

This paper is organized as follows: In Section 2, preliminaries and notations; Section 3 describes the model formulation for the fractional-order derivative; Section 4 explains the existence and uniqueness of a non-negative solution; Section 5, presents model equilibria and basic reproduction number; Section 6, stability analysis of equilibrium (DFE and EE state) presents, where the Jacobian matrix uses for disease-free and Lyapunov function uses for endemic equilibria; Section 7 basic reproduction number is presented through sensitivity analysis by calculating relevant parameters; Section 8, the numerical simulations; Section 9, the discussion and conclusion.

2. Preliminary definitions

The study of derivatives and integrals of any real number, even those of complex order, can be done very well using fractional calculus. Even though we use the CD in our proposed fractional dengue model, there are some fundamental concepts concerning the CD that we must understand first. Other relevant properties are listed below [21–25].

Definition 1([25]) Suppose $\alpha > 0$ and $f \in L^1([0, b], \mathbb{R})$ where $[0, b] \subset \mathbb{R}_+$. The fractional integral of order α of function f in the sense of Riemann-Liouville is defined as follows:

$$I_{0^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^t (x - t)^{\alpha-1} f(t)dt,$$

where $\Gamma(\cdot)$ is the classical gamma function defined by

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx.$$

The initial value problem for Caputo fractional differential equation is

$${}_a^C D^{\alpha} u(t) = f(t, u(t)), u(t_0) = u_0, t_0 \leq t \leq T.$$

and the corresponding fractional Volterra integral Eq [26] is

$$u(t) = u_0 + \frac{1}{\Gamma(q)} \int_{t_0}^t (t - s)^{q-1} f(s, u(s))ds. \tag{2.1}$$

Definition 2 [22] The Caputo fractional derivative of order $\alpha \in (n - 1, n]$ of $f(x)$ is defined as

$${}_a^C D_x^{\alpha} f(x) = \frac{1}{\Gamma(n - \alpha)} \int_0^t (x - t)^{n-\alpha-1} f^{(n)}(t)dt,$$

where $n = [\alpha] + 1$ and $[\alpha]$ represent the largest integer that is less or equal to α.

Definition 3 [27] The Laplace transform of an n-th derivative operator is obtained as

$$L\{f^n(t)\} = S^n L\{f(t)\} - \sum_{k=0}^{n-1} S^{n-k-1} f^{(k)}(t_0),$$

Similarly for $\alpha \in (n - 1, n]$ we obtain the Laplace transform of the Caputo fractional operator as

$$L\{{}_{t_0}^C D_t^{\alpha}\} = S^{\alpha} L\{N(t)\} - \sum_{k=0}^{n-1} S^{\alpha-k-1} N^{(k)}(t_0).$$

Definition 4 [23]. An entire function called Mittag-Leffler is defined in the form of power series as

$$E_{\alpha,\beta}(Z) = \sum_{n=0}^{\infty} \frac{Z^k}{\Gamma(\alpha k + \beta)}, \alpha > 0, \beta > 0,$$

and

$$E_{\alpha,1}(Z) = E_{\alpha}(Z) = \sum_{n=0}^{\infty} \frac{Z^k}{\Gamma(\alpha k + 1)}, \beta = 1.$$

The notion of convergence of mittag-Leffler function is fully discussed in [21].

Theorem 1 [28, 29]. Let us consider a fractional non-autonomous system like (4) with x^* say an equilibrium point and $X \in R^n$ a domain containing x^* and let $H : [0, \infty) \times X \to R$ be a continuous and differentiable function, such that

$$V_1(x) \leq H(t, x(t)) \leq V_2(x) \tag{2.2}$$

and

$${}^{C}D_0^{\alpha}(H(t, x(t)) \leq -V_3(x), \tag{2.3}$$

$\forall \alpha \in (0, 1)$ and all $x \in X$, where $V_1(x)$, $V_2(x)$ and $V_3(x)$ are positive definite continuous functions of X, then the equilibrium point of system (4) is uniformly asymtotically stable [30].

The Lyapunov function described above will be used to investigate the global stability of the proposed fractional dengue model.

Lemma 1 [31]. For a continuous and differentiable function $H(t) \in R_+$ and $\alpha \in (0, 1)$, then for any time $t \geq t_0$ we have

$${}^{C}D_t^{\alpha}\left[H(t) - H^* - H^* \ln \frac{H(t)}{H^*}\right] \leq \left[1 - \frac{H^*}{H(t)}\right] {}^{C}D_t^{\alpha} H(t), H^* \in R_+.$$

3. The model formulation

In this paper, we investigate the $S_hI_hH_hR_hS_mI_m$ human-mosquito fractional model, which comprises of two distinct populations, including human populations and mosquito populations. Three epidemiological states of humans are included in the proposed fractional-order dengue model: $S_h(t)$ susceptible (individuals who can contract the virus), $I_h(t)$ infected (individuals who can transmit the virus to others), $H_h(t)$ hospitalized human (the compartment of people who are hospitalized after infection), and $R_h(t)$ recovered (individuals who have required immunity). Since N_h is assumed to be constant, $N_h(t) = S_h(t) + I_h(t) + H_h(t) + R_h(t)$. For the mosquito model, we only consider the susceptible and infected mosquitoes, since the mosquito does not enter the recovery phase after being infected due to its shortened lifespan. On the other hand, the mosquito population is divided into two compartments, susceptible (S_m), and infectious (I_m) with $N_m(t) = S_m(t) + I_m(t)$, where $\alpha \in (0, 1]$ is the order of the fractional derivative. All model parameters are assumed to be non-negative. The following Table 1 lists the parameters that are used in our model. The fractional derivatives use in model (3.1) are all in the Caputo sense. So the model diagram of human-mosquito transmission dynamics of the disease is given below in Figure 1:

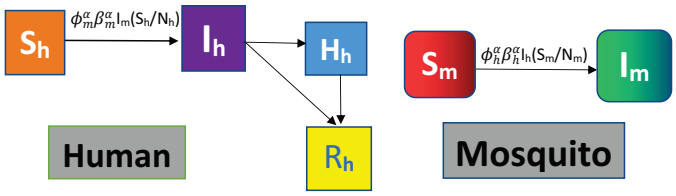


Figure 1. The flow chart of the considered dengue model.

Now we present a fractional model with CD given by,

$$\begin{cases} {}^{C}D_t^{\alpha} S_h(t) &= \Lambda_h^{\alpha} - \phi_m^{\alpha}\beta_m^{\alpha} I_m \frac{S_h}{N_h} - \gamma_h^{\alpha} S_h, \\ {}^{C}D_t^{\alpha} I_h(t) &= \phi_m^{\alpha}\beta_m^{\alpha} I_m \frac{S_h}{N_h} - (\mu_h^{\alpha} + \tau_h^{\alpha} + \gamma_h^{\alpha}) I_h, \\ {}^{C}D_t^{\alpha} H_h(t) &= \mu_h^{\alpha} I_h - (\epsilon_h^{\alpha} + \upsilon_h^{\alpha} + \gamma_h^{\alpha}) H_h, \\ {}^{C}D_t^{\alpha} R_h(t) &= \tau_h^{\alpha} I_h + \upsilon_h^{\alpha} H_h - \gamma_h^{\alpha} R_h, \\ {}^{C}D_t^{\alpha} S_m(t) &= \Lambda_m^{\alpha} - \phi_h^{\alpha}\beta_h^{\alpha} I_h \frac{S_m}{N_m} - \gamma_m^{\alpha} S_m, \\ {}^{C}D_t^{\alpha} I_m(t) &= \phi_h^{\alpha}\beta_h^{\alpha} I_h \frac{S_m}{N_m} - \gamma_m^{\alpha} I_m. \end{cases} \tag{3.1}$$

With the following non-negative initial conditions

$$S_h(0) \geq 0, I_h(0) \geq 0, H_h(0) \geq 0, R_h(0) \geq 0, S_m(0) \geq 0, I_m(0) \geq 0, \forall t \in \mathbb{R}, \tag{3.2}$$

where $0 < \alpha \leq 1$ and ${}^{C}D_t^{\alpha}$ is the Caputo fractional derivative of order α. The size of the entire human population and mosquito population is represented by the N_h and N_m so that, $N_h(t) = S_h(t) + I_h(t) + H_h(t) + R_h(t)$ and $N_m(t) = S_m(t) + I_m(t)$. The birth rate of human and mosquito populations are denoted as Λ_h and Λ_m respectively. The natural death rate for humans and mosquitoes is described by the parameters γ_h and γ_m and disease related death rate of human is denoted by ϵ_h. We assume that biting rate for humans and mosquitoes are ϕ_h and ϕ_m, respectively. β_h is the transmission probability from infected human to susceptible mosquito, β_m is the transmission probability from infected mosquito to susceptible human, τ_h represents natural recovery rate of infected human, μ_h is the rate of hospitalized infected humans and υ_h is the recovery rate of hospitalized infected human.

The different parameters used in this fractional model with their values and references are given below in Table 2:

Table 2. Description of parametric values for the dengue model of Bangladesh.

Parameter	Interpretation	Values	Reference
Λ_h	birth rate of human	2278130.05	Fitted
Λ_m	birth rate of mosquito	11874069.5	Fitted
ϕ_h	infected humans for the biting rate	0.68	Estimated
ϕ_m	infected mosquitoes for the biting rate	0.50	Estimated
γ_h	natural death rate of human	0.0137	Fitted
γ_m	natural death rate of mosquito	0.0238	[32, 33]
β_h	transmission probability from infected human to susceptible mosquito	0.0824	Fitted
β_m	transmission probability from infected mosquito to susceptible human	0.1648	Assumed
τ_h	natural recovery rate of infected human	0.01	Fitted
δ_h	incubation period of human	0.2599	Estimated
δ_m	incubation period of mosquito	0.1	Fitted
ϵ_h	disease related death rate of human	0.0001452	Fitted
μ_h	rate of hospitalized infected human	0.1	Fitted
υ_h	recovery rate of hospitalized infected human	0.440	[34]

4. Fundamental solution procedure of q-HATM

In this section, we present the solution procedure of q-HATM [35] by using LT with q-HAM. Soon after, it is employed by number of authors to evaluate the solution for numerous families of differential

equations exemplifying diverse phenomena including economic growth, biological models, human disease, chaotic behaviour, chemical reaction, optics, fluid mechanics and others [35–37] and further derived some fascinating consequences in comparison of other modified and classical algorithms.

Here, to present the procedure of q-HATM we hire the following differential equation of fractional order

$$ {}^{C}D_t^{\alpha} v(x,t) + Rv(x,t) + Nv(x,t) = f(x,t), \tag{4.1} $$

with the initial condition

$$ v(x,0) = g(x), \tag{4.2} $$

where ${}^{C}D_t^{\alpha} v(x,t)$ denotes the CD of v(x,t). On employing LT on Eq (4.1), we obtain

$$ L[v(x,t)] - \frac{g(x)}{s} + \frac{1}{s^{\alpha}} L[Rv(x,t)] + L[Nv(x,t)] - L[f(x,t)] = 0. \tag{4.3} $$

For $\phi(x,t;q)$, N is contracted as follows

$$ N[\phi(x,t;q)] = L[\phi(x,t;q)] - g(x)s + \frac{1}{s^{\alpha}} L[R\phi(x,t;q)] + L[N\phi(x,t;q)] - L[f(x,t)]. \tag{4.4} $$

where $q \in [0, \frac{1}{n}]$. Then, we present homotopy with the embedding parameter q and non-zero auxiliary parameter by HAM as

$$ (1 - nq)L[\phi(x,t;q) - v_0(x,t)] = \hbar q N[\phi(x,t;q)], \tag{4.5} $$

where L is signifying LT. For $q = 0$ and $q = \frac{1}{n}$, the following conditions satisfies

$$ \phi(x,t;0) = v_0(x,t), \phi(x,t;\frac{1}{n}) = v(x,t). \tag{4.6} $$

With the help of Taylor theorem, we have

$$ \phi(x,t;q) = v_0(x,t) + \sum_{m=1}^{\infty} v_m(x,t) q^m, \tag{4.7} $$

where

$$ v_m(x,t) = \frac{1}{m!} \left[\frac{\delta^m \phi(x,t;q)}{\delta q^m} \right]_{q=0}. \tag{4.8} $$

After differentiating Eq (4.7) m-times with q and multiplying by $\frac{1}{m!}$ and substituting $q = 0$, one can get

$$ L[v_m(x,t) - k_m v_{m-1}(x,t)] = \hbar R_m(\bar{v}_{m-1}), \tag{4.9} $$

where the vectors are defined as

$$ \bar{v}_m = v_0(x,t), v_1(x,t), ..., v_m(x,t). \tag{4.10} $$

Eq (4.9) reduces after employing inverse LT to

$$v_m(x,t) = k_m v_{m-1}(x,t) + \hbar L^{-1}[R_m(\bar{v}_{m-1})], \tag{4.11}$$

where

$$R_m(\bar{v}_{m-1}) = L[v_{m-1}(x,t)] + \frac{1}{s^\alpha}L[Rv_{m-1} + H_{m-1}] - \left(1 - \frac{k_m}{n}\right)\left(\frac{g(x)}{s} + \frac{1}{s^\alpha}L[f(x,t)]\right), \tag{4.12}$$

and

$$k_m = \begin{cases} 0, & m \leq 1 \\ n, & m > 1 \end{cases} \tag{4.13}$$

Here, H_m is homotopy polynomial and presented as

$$H_m = \frac{1}{m!}\left[\frac{\delta^m \phi(x,t;q)}{\delta q^m}\right]_{q=0}, \phi(x,t;q) = \phi_0 + q\phi_1 + q^2\phi_2 + ... \tag{4.14}$$

By using Eqs (4.11) and (4.12), we get

$$v_m(x,t) = (k_m + \hbar)v_{m-1}(x,t) - (1 - \frac{k_m}{n})L^{-1}\left[\frac{g(x)}{s} + \frac{1}{s^\alpha}L[f(x,t)]\right] + \hbar L^{-1}\left[\frac{1}{s^\alpha}L[Rv_{m-1} + H_{m-1}]\right]. \tag{4.15}$$

The series solution by projected algorithm is defined as

$$v(x,t) = v_0(x,t) + \sum_{m=1}^{\infty} v_m(x,t). \tag{4.16}$$

5. q-HATM solution for considered model

Here, we evaluate the solutions for model (3.1) with different parameters. Consider the system of the equation describing the fractional-order $S_h I_h H_h R_h S_m I_m$ epidemic model of dengue disease

$$\begin{cases} {}^C D_t^\alpha S_h(t) &= \Lambda_h^\alpha - \phi_m^\alpha \beta_m^\alpha I_m \frac{S_h}{N_h} - \gamma_h^\alpha S_h, \\ {}^C D_t^\alpha I_h(t) &= \phi_m^\alpha \beta_m^\alpha I_m \frac{S_h}{N_h} - (\mu_h^\alpha + \tau_h^\alpha + \gamma_h^\alpha) I_h, \\ {}^C D_t^\alpha H_h(t) &= \mu_h^\alpha I_h - (\epsilon_h^\alpha + \upsilon_h^\alpha + \gamma_h^\alpha) H_h, \\ {}^C D_t^\alpha R_h(t) &= \tau_h^\alpha I_h + \upsilon_h^\alpha H_h - \gamma_h^\alpha R_h, \\ {}^C D_t^\alpha S_m(t) &= \Lambda_m^\alpha - \phi_h^\alpha \beta_h^\alpha I_h \frac{S_m}{N_m} - \gamma_m^\alpha S_m, \\ {}^C D_t^\alpha I_m(t) &= \phi_h^\alpha \beta_h^\alpha I_h \frac{S_m}{N_m} - \gamma_m^\alpha I_m. \end{cases} \tag{5.1}$$

With the following initial conditions

$$S_h(0) \geq 0, I_h(0) \geq 0, H_h(0) \geq 0, R_h(0) \geq 0, S_m(0) \geq 0, I_m(0) \geq 0,$$

Taking LT on Eq (5.1) and then using the initial conditions, we get

$$
\begin{aligned}
&L[S_h(t)] - \frac{1}{s}(S_{h_0}) - \frac{1}{s^\alpha}L[\Lambda_h^\alpha - \phi_m^\alpha\beta_m^\alpha I_m\frac{S_h}{N_h} - \gamma_h^\alpha S_h] = 0,\\
&L[I_h(t)] - \frac{1}{s}(I_{h_0}) - \frac{1}{s^\alpha}L[\phi_m^\alpha\beta_m^\alpha I_m\frac{S_h}{N_h} - (\mu_h^\alpha + \tau_h^\alpha + \gamma_h^\alpha)I_h] = 0,\\
&L[H_h(t)] - \frac{1}{s}(H_{h_0}) - \frac{1}{s^\alpha}L[\mu_h^\alpha I_h - (\epsilon_h^\alpha + \upsilon_h^\alpha + \gamma_h^\alpha)H_h] = 0,\\
&L[R_h(t)] - \frac{1}{s}(R_{h_0}) - \frac{1}{s^\alpha}L[\tau_h^\alpha I_h + \upsilon_h^\alpha H_h - \gamma_h^\alpha R_h] = 0,\\
&L[S_m(t)] - \frac{1}{s}(S_{m_0}) - \frac{1}{s^\alpha}L[\Lambda_m^\alpha - \phi_h^\alpha\beta_h^\alpha I_h\frac{S_m}{N_m} - \gamma_m^\alpha S_m] = 0,\\
&L[I_m(t)] - \frac{1}{s}(I_{m_0}) - \frac{1}{s^\alpha}L[\phi_h^\alpha\beta_h^\alpha I_h\frac{S_m}{N_m} - \gamma_m^\alpha I_m] = 0.
\end{aligned}
\tag{5.2}
$$

Now, the non-linear operator N presented as:

$$
\begin{aligned}
&N^1[\pi_1(t;p),\pi_2(t;p),\pi_3(t;p),\pi_4(t;p),\pi_5(t;p),\pi_6(t;p)] = L[\pi_1(t;p)] - \frac{1}{s}(S_{h_0}) - \frac{1}{s^\alpha}\\
&\qquad L[\Lambda_h^\alpha - \phi_m^\alpha\beta_m^\alpha\pi_5(t;p)\frac{\pi_1(t;p)}{N_h} - \gamma_h^\alpha\pi_1(t;p)] = 0,\\
&N^2[\pi_1(t;p),\pi_2(t;p),\pi_3(t;p),\pi_4(t;p),\pi_5(t;p),\pi_6(t;p)] = L[\pi_2(t;p)] - \frac{1}{s}(I_{h_0}) - \frac{1}{s^\alpha}\\
&\qquad L[\phi_m^\alpha\beta_m^\alpha\pi_6(t;p)\frac{\pi_1(t;p)}{N_h} - (\mu_h^\alpha + \tau_h^\alpha + \gamma_h^\alpha)\pi_2(t;p)],\\
&N^3[\pi_1(t;p),\pi_2(t;p),\pi_3(t;p),\pi_4(t;p),\pi_5(t;p),\pi_6(t;p)] = L[\pi_3(t;p)] - \frac{1}{s}(H_{h_0}) - \frac{1}{s^\alpha}\\
&\qquad L[\mu_h^\alpha\pi_2(t;p) - (\epsilon_h^\alpha + \upsilon_h^\alpha + \gamma_h^\alpha)\pi_3(t;p)],\\
&N^4[\pi_1(t;p),\pi_2(t;p),\pi_3(t;p),\pi_4(t;p),\pi_5(t;p),\pi_6(t;p)] = L[\pi_4(t;p)] - \frac{1}{s}(R_{h_0}) - \frac{1}{s^\alpha}\\
&\qquad L[\tau_h^\alpha\pi_2(t;p) + \upsilon_h^\alpha\pi_3(t;p) - \gamma_h^\alpha\pi_4(t;p)] = 0,\\
&N^5[\pi_1(t;p),\pi_2(t;p),\pi_3(t;p),\pi_4(t;p),\pi_5(t;p),\pi_6(t;p)] = L[\pi_5(t;p)] - \frac{1}{s}(S_{m_0}) - \frac{1}{s^\alpha}\\
&\qquad L[\Lambda_m^\alpha - \phi_h^\alpha\beta_h^\alpha\pi_2(t;p)\frac{\pi_5(t;p)}{N_m} - \gamma_m^\alpha\pi_5(t;p)],\\
&N^6[\pi_6(t;p),\pi_2(t;p),\pi_3(t;p),\pi_4(t;p),\pi_5(t;p),\pi_6(t;p)] = L[\pi_1(t;p)] - \frac{1}{s}(I_{m_0}) - \frac{1}{s^\alpha}\\
&\qquad L[\phi_h^\alpha\beta_h^\alpha\pi_2(t;p)\frac{\pi_5(t;p)}{N_m} - \gamma_m^\alpha\pi_6(t;p)].
\end{aligned}
\tag{5.3}
$$

By applying the considered method and for $H(t) = 1$, the n-th order deformation equation is presented as

$$
\begin{aligned}
L[S_{h_n}(t) - k_n S_{h_{n-1}}(t)] &= \hbar(\mathbf{B}_{1,n}[\bar{S}_{h_{n-1}}, \bar{I}_{h_{n-1}}, \bar{H}_{h_{n-1}}, \bar{R}_{h_{n-1}}, \bar{S}_{m_{n-1}}, \bar{I}_{m_{n-1}}]),\\
L[I_{h_n}(t) - k_n I_{h_{n-1}}(t)] &= \hbar(\mathbf{B}_{2,n}[\bar{S}_{h_{n-1}}, \bar{I}_{h_{n-1}}, \bar{H}_{h_{n-1}}, \bar{R}_{h_{n-1}}, \bar{S}_{m_{n-1}}, \bar{I}_{m_{n-1}}]),\\
L[H_{h_n}(t) - k_n H_{h_{n-1}}(t)] &= \hbar(\mathbf{B}_{3,n}[\bar{S}_{h_{n-1}}, \bar{I}_{h_{n-1}}, \bar{H}_{h_{n-1}}, \bar{R}_{h_{n-1}}, \bar{S}_{m_{n-1}}, \bar{I}_{m_{n-1}}]),\\
L[R_{h_n}(t) - k_n R_{h_{n-1}}(t)] &= \hbar(\mathbf{B}_{4,n}[\bar{S}_{h_{n-1}}, \bar{I}_{h_{n-1}}, \bar{H}_{h_{n-1}}, \bar{R}_{h_{n-1}}, \bar{S}_{m_{n-1}}, \bar{I}_{m_{n-1}}]),\\
L[S_{m_n}(t) - k_n S_{m_{n-1}}(t)] &= \hbar(\mathbf{B}_{5,n}[\bar{S}_{h_{n-1}}, \bar{I}_{h_{n-1}}, \bar{H}_{h_{n-1}}, \bar{R}_{h_{n-1}}, \bar{S}_{m_{n-1}}, \bar{I}_{m_{n-1}}]),\\
L[I_{m_n}(t) - k_n I_{m_{n-1}}(t)] &= \hbar(\mathbf{B}_{6,n}[\bar{S}_{h_{n-1}}, \bar{I}_{h_{n-1}}, \bar{H}_{h_{n-1}}, \bar{R}_{h_{n-1}}, \bar{S}_{m_{n-1}}, \bar{I}_{m_{n-1}}]).
\end{aligned} \tag{5.4}
$$

where

$$
\begin{aligned}
\mathbf{B}_{1,n}[\bar{S}_{h_{n-1}}, \bar{I}_{h_{n-1}}, \bar{H}_{h_{n-1}}, \bar{R}_{h_{n-1}}, \bar{S}_{m_{n-1}}, \bar{I}_{m_{n-1}}] &= L[S_{h_{n-1}}(t)] - (1 - \frac{k_n}{n})\frac{1}{s}(S_{h_0}) - \frac{1}{s^\alpha}\\
&L[\Lambda_h^\alpha - \phi_m^\alpha \beta_m^\alpha \sum_{i=0}^{n-1} I_{m_{n-1-i}} \frac{S_{h_i}}{N_h} - \gamma_h^\alpha S_{h_{n-1}}]\\
\mathbf{B}_{2,n}[\bar{S}_{h_{n-1}}, \bar{I}_{h_{n-1}}, \bar{H}_{h_{n-1}}, \bar{R}_{h_{n-1}}, \bar{S}_{m_{n-1}}, \bar{I}_{m_{n-1}}] &= L[I_{h_{n-1}}(t)] - (1 - \frac{k_n}{n})\frac{1}{s}(I_{h_0}) - \frac{1}{s^\alpha}\\
&L[\phi_m^\alpha \beta_m^\alpha \sum_{i=0}^{n-1} I_{m_{n-1-i}} \frac{S_{h_i}}{N_h} - (\mu_h^\alpha + \tau_h^\alpha + \gamma_h^\alpha) I_{h_{n-1}}]\\
\mathbf{B}_{3,n}[\bar{S}_{h_{n-1}}, \bar{I}_{h_{n-1}}, \bar{H}_{h_{n-1}}, \bar{R}_{h_{n-1}}, \bar{S}_{m_{n-1}}, \bar{I}_{m_{n-1}}] &= L[H_{h_{n-1}}(t)] - (1 - \frac{k_n}{n})\frac{1}{s}(H_{h_0}) - \frac{1}{s^\alpha}\\
&L[\mu_h^\alpha I_{h_{n-1}} - (\epsilon_h^\alpha + \upsilon_h^\alpha + \gamma_h^\alpha) H_{h_{n-1}}],\\
\mathbf{B}_{4,n}[\bar{S}_{h_{n-1}}, \bar{I}_{h_{n-1}}, \bar{H}_{h_{n-1}}, \bar{R}_{h_{n-1}}, \bar{S}_{m_{n-1}}, \bar{I}_{m_{n-1}}] &= L[R_{h_{n-1}}(t)] - (1 - \frac{k_n}{n})\frac{1}{s}(R_{h_0}) - \frac{1}{s^\alpha}\\
&L[\tau_h^\alpha I_{h_{n-1}} + \upsilon_h^\alpha H_{h_{n-1}} - \gamma_h^\alpha R_{h_{n-1}}],\\
\mathbf{B}_{5,n}[\bar{S}_{h_{n-1}}, \bar{I}_{h_{n-1}}, \bar{H}_{h_{n-1}}, \bar{R}_{h_{n-1}}, \bar{S}_{m_{n-1}}, \bar{I}_{m_{n-1}}] &= L[S_{m_{n-1}}(t)] - (1 - \frac{k_n}{n})\frac{1}{s}(S_{m_0}) - \frac{1}{s^\alpha}\\
&L[\Lambda_m^\alpha - \phi_h^\alpha \beta_h^\alpha \sum_{i=0}^{n-1} I_{h_{n-1-i}} \frac{S_{m_i}}{N_m} - \gamma_m^\alpha S_{m_{n-1}}],\\
\mathbf{B}_{6,n}[\bar{S}_{h_{n-1}}, \bar{I}_{h_{n-1}}, \bar{H}_{h_{n-1}}, \bar{R}_{h_{n-1}}, \bar{S}_{m_{n-1}}, \bar{I}_{m_{n-1}}] &= L[I_{m_{n-1}}(t)] - (1 - \frac{k_n}{n})\frac{1}{s}(I_{m_0}) - \frac{1}{s^\alpha}\\
&L[\phi_h^\alpha \beta_h^\alpha I_{h_{n-1-i}} \frac{S_{m_i}}{N_m} - \gamma_m^\alpha I_{m_{n-1}}].
\end{aligned} \tag{5.5}
$$

Now using inverse LT on Eq (5.4), we get

$$
\begin{aligned}
S_{h_n}(t) &= k_n S_{h_{n-1}}(t) + \hbar L^{-1}(\mathbf{B}_{1,n}[\bar{S}_{h_{n-1}}, \bar{I}_{h_{n-1}}, \bar{H}_{h_{n-1}}, \bar{R}_{h_{n-1}}, \bar{S}_{m_{n-1}}, \bar{I}_{m_{n-1}}]),\\
I_{h_n}(t) &= k_n I_{h_{n-1}}(t) + \hbar L^{-1}(\mathbf{B}_{2,n}[\bar{S}_{h_{n-1}}, \bar{I}_{h_{n-1}}, \bar{H}_{h_{n-1}}, \bar{R}_{h_{n-1}}, \bar{S}_{m_{n-1}}, \bar{I}_{m_{n-1}}]),\\
H_{h_n}(t) &= k_n H_{h_{n-1}}(t) + \hbar L^{-1}(\mathbf{B}_{3,n}[\bar{S}_{h_{n-1}}, \bar{I}_{h_{n-1}}, \bar{H}_{h_{n-1}}, \bar{R}_{h_{n-1}}, \bar{S}_{m_{n-1}}, \bar{I}_{m_{n-1}}]),\\
R_{h_n}(t) &= k_n R_{h_{n-1}}(t) + \hbar L^{-1}(\mathbf{B}_{4,n}[\bar{S}_{h_{n-1}}, \bar{I}_{h_{n-1}}, \bar{H}_{h_{n-1}}, \bar{R}_{h_{n-1}}, \bar{S}_{m_{n-1}}, \bar{I}_{m_{n-1}}]),\\
S_{m_n}(t) &= k_n S_{m_{n-1}}(t) + \hbar L^{-1}(\mathbf{B}_{5,n}[\bar{S}_{h_{n-1}}, \bar{I}_{h_{n-1}}, \bar{H}_{h_{n-1}}, \bar{R}_{h_{n-1}}, \bar{S}_{m_{n-1}}, \bar{I}_{m_{n-1}}]),\\
I_{m_n}(t) &= k_n I_{m_{n-1}}(t) + \hbar L^{-1}(\mathbf{B}_{6,n}[\bar{S}_{h_{n-1}}, \bar{I}_{h_{n-1}}, \bar{H}_{h_{n-1}}, \bar{R}_{h_{n-1}}, \bar{S}_{m_{n-1}}, \bar{I}_{m_{n-1}}]).
\end{aligned} \tag{5.6}
$$

Using the initial conditions and the above system, we have

$$
\begin{aligned}
S_{h_1}(t) &= [\Lambda_h^\alpha - \phi_m^\alpha\beta_m^\alpha I_{m_0}\frac{S_{h_0}}{N_h} - \gamma_h^\alpha S_{h_0}]\frac{\hbar t^\alpha}{\Gamma(\alpha+1)},\\
I_{h_1}(t) &= [\phi_m^\alpha\beta_m^\alpha I_{m_0}\frac{S_{h_0}}{N_h} - (\mu_h^\alpha + \tau_h^\alpha + \gamma_h^\alpha)I_{h_0}]\frac{\hbar t^\alpha}{\Gamma(\alpha+1)},\\
H_{h_1}(t) &= [\mu_h^\alpha I_{h_0} - (\epsilon_h^\alpha + \upsilon_h^\alpha + \gamma_h^\alpha)H_{h_0}]\frac{\hbar t^\alpha}{\Gamma(\alpha+1)},\\
R_{h_1}(t) &= [\tau_h^\alpha I_{h_0} + \upsilon_h^\alpha H_{h_0} - \gamma_h^\alpha R_{h_0}]\frac{\hbar t^\alpha}{\Gamma(\alpha+1)},\\
S_{m_1}(t) &= [\Lambda_m^\alpha - \phi_h^\alpha\beta_h^\alpha I_{h_0}\frac{S_{m_0}}{N_m} - \gamma_m^\alpha S_{m_0}]\frac{\hbar t^\alpha}{\Gamma(\alpha+1)},\\
I_{m_1}(t) &= [\phi_h^\alpha\beta_h^\alpha I_{h_0}\frac{S_{m_0}}{N_m} - \gamma_m^\alpha I_{m_0}]\frac{\hbar t^\alpha}{\Gamma(\alpha+1)},\\
S_{h_2}(t) =& \Lambda_h\frac{\hbar t^\alpha}{\Gamma(\alpha+1)} + (\gamma_h^\alpha\Lambda_h^\alpha - \gamma_h^\alpha\phi_m^\alpha\beta_m^\alpha I_{m_0}\frac{S_{h_0}}{N_h} - \gamma_h^{2\alpha}S_{h_0})\frac{\hbar^2 t^{2\alpha}}{\Gamma(2\alpha+1)}\\
&- (\phi_m^\alpha\beta_m^\alpha\phi_h^\alpha\beta_h^\alpha\Lambda_h^\alpha\frac{I_{h_0}S_{m_0}}{N_hN_m} - \phi_m^{2\alpha}\beta_m^{2\alpha}\phi_h^\alpha\beta_h^\alpha\frac{I_{h_0}I_{m_0}S_{h_0}S_{m_0}}{N_h^2N_m} - \phi_m^\alpha\phi_h^\alpha\beta_h^\alpha\beta_m^\alpha\gamma_h^\alpha\frac{S_{h_0}S_{m_0}I_{h_0}}{N_hN_m}\\
&- \phi_m^\alpha\beta_m^\alpha\gamma_h^\alpha\Lambda_h^\alpha - \phi_m^{2\alpha}\beta_m^{2\alpha}\gamma_h^\alpha\frac{I_{m_0}^2S_{h_0}}{N_h} + \gamma_h^\alpha\phi_m^\alpha\beta_m^\alpha S_{h_0}I_{m_0})\frac{\hbar^3 t^{3\alpha}}{\Gamma(3\alpha+1)},\\
I_{h_2}(t) =& \phi_m^\alpha\beta_m^\alpha(\phi_h^\alpha\beta_h^\alpha I_{h_0}\frac{S_{m_0}}{N_m} - \phi\gamma_m^\alpha I_{m_0})(\Lambda_h^\alpha - \phi_m^\alpha\beta_m^\alpha I_{m_0}\frac{S_{h_0}}{N_h} - \gamma_h^\alpha S_{h_0})\frac{\hbar^3 t^{3\alpha}}{\Gamma(3\alpha+1)}\\
&- (\mu_h^\alpha + \tau_h^\alpha + \gamma_h^\alpha)[(\phi_m^\alpha\beta_m^\alpha)I_{m_0}\frac{S_{h_0}}{N_h} - I_{h_0}(\mu_h^\alpha + \tau_h^\alpha + \gamma_h^\alpha)]\frac{\hbar^2 t^{2\alpha}}{\Gamma(2\alpha+1)},\\
H_{h_2}(t) =& [\phi_m^\alpha\beta_m^\alpha\mu_h^\alpha I_{m_0}\frac{S_{h_0}}{N_h} - \mu_h^\alpha(\mu_h^\alpha + \tau_h^\alpha + \gamma_h^\alpha)I_{h_0} - (\epsilon_h^\alpha + \upsilon_h^\alpha + \gamma_h^\alpha)\mu_h^\alpha I_{h_0}\\
&+ (\epsilon_h^\alpha + \upsilon_h^\alpha + \gamma_h^\alpha)^2 H_{h_0}]\frac{\hbar^2 t^{2\alpha}}{\Gamma(2\alpha+1)},\\
R_{h_2}(t) =& [\tau_m^\alpha\phi_m^\alpha\beta_m^\alpha I_{m_0}\frac{S_{h_0}}{N_h} - \tau_h^\alpha(\mu_h^\alpha + \tau_h^\alpha + \gamma_h^\alpha)I_{h_0} + \upsilon_h^\alpha\mu_h^\alpha I_{h_0} - \upsilon_h^\alpha(\epsilon_h^\alpha + \upsilon_h^\alpha + \gamma_h^\alpha)H_{h_0} - \gamma_h^\alpha\tau_h^\alpha I_{h_0}\\
&- \gamma_h^\alpha\upsilon_h^\alpha H_{h_0} + \gamma_h^{2\alpha}R_{h_0}]\frac{\hbar^2 t^{2\alpha}}{\Gamma(2\alpha+1)},\\
S_{m_2}(t) =& \Lambda_m^\alpha\frac{\hbar t^\alpha}{\Gamma(\alpha+1)} - \gamma_m^\alpha(\Lambda_m^\alpha - \phi_h^\alpha\beta_h^\alpha I_{h_0}\frac{S_{m_0}}{N_m} - \gamma_m^\alpha S_{m_0})\frac{\hbar^2 t^{2\alpha}}{\Gamma(2\alpha+1)} - [\phi_h^\alpha\beta_h^\alpha\phi_m^\alpha\beta_m^\alpha I_{m_o}\frac{S_{h_0}}{N_h}\\
&- (\mu_h^\alpha + \tau_h^\alpha + \gamma_h^\alpha)I_{h_0}\frac{(\Lambda_m^\alpha - \phi_h^\alpha\beta_h^\alpha I_{h_0}\frac{S_{m_0}}{N_m} - \gamma_m^\alpha S_{m_0})}{N_m}]\frac{\hbar^3 t^{3\alpha}}{\Gamma(3\alpha+1)},\\
I_{m_2}(t) =& -\gamma_m^\alpha(\Lambda_m^\alpha - \phi_h^\alpha\beta_h^\alpha I_{h_0}\frac{S_{m_0}}{N_m} - \gamma_m^\alpha S_{m_0})\frac{\hbar^2 t^{2\alpha}}{\Gamma(2\alpha+1)} + [(\phi_h^\alpha\beta_h^\alpha\phi_m^\alpha\beta_m^\alpha I_{m_0}\frac{S_{h_0}}{N_h}\\
&- \phi_h^\alpha\beta_h^\alpha(\mu_h^\alpha + \tau_h^\alpha + \gamma_h^\alpha)I_{h_0})(\Lambda_m^\alpha - \phi_h^\alpha\beta_h^\alpha I_{h_0}\frac{S_{m_0}}{N_m} - \gamma_m^\alpha S_{m_0})N_m]\frac{\hbar^3 t^{3\alpha}}{\Gamma(3\alpha+1)}.
\end{aligned}
$$

$$\vdots$$

We can get the rest of the term in a similar way. The q-HATM series solution for the FBM equation considered in Eq (5.1) is given by

$$\begin{aligned}
S_h(t) &= S_{h_0}(t) + \sum_{n=1}^{\infty} S_{h_n}(t)\left(\frac{1}{m}\right)^n, \\
I_h(t) &= I_{h_0}(t) + \sum_{n=1}^{\infty} I_{h_n}(t)\left(\frac{1}{m}\right)^n, \\
H_h(t) &= H_{h_0}(t) + \sum_{n=1}^{\infty} H_{h_n}(t)\left(\frac{1}{m}\right)^n, \\
R_h(t) &= R_{h_0}(t) + \sum_{n=1}^{\infty} R_{h_n}(t)\left(\frac{1}{m}\right)^n, \\
S_m(t) &= S_{m_0}(t) + \sum_{n=1}^{\infty} S_{m_n}(t)\left(\frac{1}{m}\right)^n, \\
I_m(t) &= I_{m_0}(t) + \sum_{n=1}^{\infty} I_{m_n}(t)\left(\frac{1}{m}\right)^n.
\end{aligned} \tag{5.7}$$

5.1. Boundedness of the solutions

In this section the total population is denoted as

$$N_h(t) = S_h(t) + I_h(t) + H_h(t) + R_h(t)$$

and the total mosquito population is denoted as

$$N_m(t) = S_m(t) + I_m(t).$$

The linearity of the Caputo operator in the above two different populations becomes,

$$\begin{aligned}
{}^{C}D_t^{\alpha} N_h(t) &= {}^{C}D_t^{\alpha} S_h(t) + {}^{C}D_t^{\alpha} I_h(t) + {}^{C}D_t^{\alpha} H_h(t) + {}^{C}D_t^{\alpha} R_h(t), \\
&= \Lambda_h^{\alpha} - \epsilon_h^{\alpha} H_h(t) - \gamma_h^{\alpha} N_h(t), \\
&\leq \Lambda_h^{\alpha} - \gamma_h^{\alpha} N_h(t).
\end{aligned} \tag{5.8}$$

and

$$\begin{aligned}
{}^{C}D_t^{\alpha} N_m(t) &= {}^{C}D_t^{\alpha} S_m(t) + {}^{C}D_t^{\alpha} I_m(t), \\
&\leq \Lambda_m^{\alpha} - \gamma_m^{\alpha} N_h(t).
\end{aligned} \tag{5.9}$$

We apply the Laplace transform method [27] to solve the Gronwall's inequality in (5.8) and (5.9) with initial condition $N(t_0) \geq 0$

$$L\{{}_0^{C}D_t^{\alpha} N_h(t) + \gamma_h^{\alpha} N_h(t)\} \leq L\{\Lambda_h^{\alpha}\},$$

and

$$L\{{}_0^C D_t^\alpha N_m(t) + \gamma_h^\alpha N_m(t)\} \leq L\{\Lambda_m^\alpha\}.$$

The linearity property of the Laplace transform gives

$$\begin{aligned} L\{{}_0^C D_t^\alpha N_h(t)\} + \gamma_h^\alpha L\{N_h(t)\} &\leq L\{\Lambda_h^\alpha\}, \\ S^\alpha L\{N_h(t)\} - \sum_{k=0}^{n-1} S^{\alpha-k-1} N_h^{(k)}(t_0) + \gamma_h^\alpha L\{N_h(t)\} &\leq \frac{\Lambda_h^\alpha}{S}, \\ L\{N_h(t)\} &\leq \frac{\Lambda_h^\alpha}{S(S^\alpha + \gamma_h^\alpha)} + \sum_{k=0}^{n-1} \frac{S^{\alpha-k-1}}{S^\alpha + \gamma_h^\alpha} N_h^{(k)}(t_0). \end{aligned} \tag{5.10}$$

and

$$\begin{aligned} L\{{}_0^C D_t^\alpha N_m(t)\} + \gamma_m^\alpha L\{N_m(t)\} &\leq L\{\Lambda_m^\alpha\}, \\ S^\alpha L\{N_m(t)\} - \sum_{k=0}^{n-1} S^{\alpha-k-1} N_m^{(k)}(t_0) + \gamma_m^\alpha L\{N_m(t)\} &\leq \frac{\Lambda_m^\alpha}{S}, \\ L\{N_m(t)\} &\leq \frac{\Lambda_m^\alpha}{S(S^\alpha + \gamma_m^\alpha)} + \sum_{k=0}^{n-1} \frac{S^{\alpha-k-1}}{S^\alpha + \gamma_m^\alpha} N_m^{(k)}(t_0). \end{aligned} \tag{5.11}$$

Splitting (5.10) and (5.11) to partial fraction gives the following:

$$\begin{aligned} L\{N_h(t)\} &\leq \Lambda_h^\alpha \left(\frac{1}{S} - \frac{S^{\alpha-1}}{S^\alpha + \gamma_h^\alpha}\right) + \sum_{k=0}^{n-1} \frac{S^{\alpha-k-1}}{S^\alpha + \gamma_h^\alpha} N_h^{(k)}(t_0), \\ &= \Lambda_h^\alpha \left(\frac{1}{S} - \frac{1}{S}\frac{1}{1 + \frac{\gamma_h^\alpha}{S^\alpha}}\right) + \sum_{k=0}^{n-1} \frac{1}{S^{k+1}} \frac{1}{1 + \frac{\gamma_h^\alpha}{S^\alpha}} N_h^{(k)}(t_0). \end{aligned}$$

and

$$\begin{aligned} L\{N_m(t)\} &\leq \Lambda_m^\alpha \left(\frac{1}{S} - \frac{S^{\alpha-1}}{S^\alpha + \gamma_m^\alpha}\right) + \sum_{k=0}^{n-1} \frac{S^{\alpha-k-1}}{S^\alpha + \gamma_m^\alpha} N_m^{(k)}(t_0), \\ &= \Lambda_m^\alpha \left(\frac{1}{S} - \frac{1}{S}\frac{1}{1 + \frac{\gamma_m^\alpha}{S^\alpha}}\right) + \sum_{k=0}^{n-1} \frac{1}{S^{k+1}} \frac{1}{1 + \frac{\gamma_m^\alpha}{S^\alpha}} N_m^{(k)}(t_0). \end{aligned}$$

According to Taylor series, we get

$$\frac{1}{1 + \frac{\gamma_h^\alpha}{S^\alpha}} = \sum_{n=0}^{\infty} \left(\frac{-\gamma_h^\alpha}{S^\alpha}\right)^n, \frac{1}{1 + \frac{\gamma_m^\alpha}{S^\alpha}} = \sum_{n=0}^{\infty} \left(\frac{-\gamma_m^\alpha}{S^\alpha}\right)^n.$$

Therefore

$$
\begin{aligned}
L\{N_h(t)\} &\leq \Lambda_h^\alpha \left(\frac{1}{S} - \frac{1}{S} \sum_{n=0}^{\infty} \left(\frac{-\gamma_h^\alpha}{S^\alpha} \right)^n \right) + \sum_{k=0}^{n-1} \frac{1}{S^{k+1}} N_h^{(k)}(t_0) \sum_{n=0}^{\infty} \left(\frac{-\gamma_h^\alpha}{S^\alpha} \right)^n, \\
&= \Lambda_h^\alpha \left(\frac{1}{S} - \sum_{n=0}^{\infty} \frac{\left(-\gamma_h^\alpha\right)^n}{S^{\alpha n+1}} \right) + \sum_{k=0}^{n-1} \sum_{n=0}^{\infty} \frac{1}{S^{k+1}} \frac{\left(-\gamma_h^\alpha\right)^n}{S^{\alpha n+k+1}} N_h^{(k)}(t_0).
\end{aligned} \tag{5.12}
$$

and

$$
\begin{aligned}
L\{N_m(t)\} &\leq \Lambda_m^\alpha \left(\frac{1}{S} - \frac{1}{S} \sum_{n=0}^{\infty} \left(\frac{-\gamma_m^\alpha}{S^\alpha} \right)^n \right) + \sum_{k=0}^{n-1} \frac{1}{S^{k+1}} N_m^{(k)}(t_0) \sum_{n=0}^{\infty} \left(\frac{-\gamma_m^\alpha}{S^\alpha} \right)^n, \\
&= \Lambda_m^\alpha \left(\frac{1}{S} - \sum_{n=0}^{\infty} \frac{\left(-\gamma_m^\alpha\right)^n}{S^{\alpha n+1}} \right) + \sum_{k=0}^{n-1} \sum_{n=0}^{\infty} \frac{1}{S^{k+1}} \frac{\left(-\gamma_m^\alpha\right)^n}{S^{\alpha n+k+1}} N_m^{(k)}(t_0).
\end{aligned} \tag{5.13}
$$

using the inverse Laplace transform of (5.12) and (5.13), we have

$$
N_h(t) \leq \Lambda_h^\alpha L^{-1}\left[\frac{1}{S}\right] - \Lambda_h^\alpha \sum_{n=0}^{\infty} (-\gamma_h^\alpha)^n L^{-1}\left[\frac{1}{S^{\alpha n+1}}\right] + \sum_{k=0}^{n-1} \sum_{n=0}^{\infty} (-\gamma_h^\alpha)^n N_h^{(k)}(t_0) L^{-1}\left[\frac{1}{S^{\alpha n+k+1}}\right],
$$

and

$$
N_m(t) \leq \Lambda_m^\alpha L^{-1}\left[\frac{1}{S}\right] - \Lambda_m^\alpha \sum_{n=0}^{\infty} (-\gamma_m^\alpha)^n L^{-1}\left[\frac{1}{S^{\alpha n+1}}\right] + \sum_{k=0}^{n-1} \sum_{n=0}^{\infty} (-\gamma_m^\alpha)^n N_m^{(k)}(t_0) L^{-1}\left[\frac{1}{S^{\alpha n+k+1}}\right].
$$

According to Laplace formula,

$$
L[t^m] = \frac{m!}{S^{m+1}} = \frac{\Gamma(m+1)}{S^{m+1}},
$$

or

$$
L^{-1}\left[\frac{1}{S^{m+1}}\right] = \frac{t^m}{\Gamma(m+1)}.
$$

Thus

$$
N_h(t) \leq \Lambda_h^\alpha - \Lambda_h^\alpha \sum_{n=0}^{\infty} (-\gamma_h^\alpha)^n \frac{t^{\alpha n}}{\Gamma(\alpha n+1)} + \sum_{k=0}^{n-1} \sum_{n=0}^{\infty} (-\gamma_h^\alpha)^n N_h^{(k)} \times (t_0) \frac{t^{\alpha n+k}}{\Gamma(\alpha n+k+1)},
$$

$$
N_h(t) \leq \Lambda_h^\alpha - \Lambda_h^\alpha \sum_{n=0}^{\infty} \frac{(-\gamma_h^\alpha t^\alpha)^n}{\Gamma(\alpha n+1)} + \sum_{k=0}^{n-1} \sum_{n=0}^{\infty} \frac{(-\gamma_h^\alpha t^\alpha)^n}{\Gamma(\alpha n+k+1)} t^k N_h^{(k)}(t_0).
$$

and

$$
N_m(t) \leq \Lambda_m^\alpha - \Lambda_m^\alpha \sum_{n=0}^{\infty} (-\gamma_m^\alpha)^n \frac{t^{\alpha n}}{\Gamma(\alpha n+1)} + \sum_{k=0}^{n-1} \sum_{n=0}^{\infty} (-\gamma_m^\alpha)^n N_m^{(k)} \times (t_0) \frac{t^{\alpha n+k}}{\Gamma(\alpha n+k+1)},
$$

$$
N_m(t) \leq \Lambda_m^\alpha - \Lambda_m^\alpha \sum_{n=0}^{\infty} \frac{(-\gamma_m^\alpha t^\alpha)^n}{\Gamma(\alpha n+1)} + \sum_{k=0}^{n-1} \sum_{n=0}^{\infty} \frac{(-\gamma_m^\alpha t^\alpha)^n}{\Gamma(\alpha n+k+1)} t^k N_m^{(k)}(t_0).
$$

Substituting the Mittag-Leffler function we get,

$$N_h(t) \leq \Lambda_h^\alpha[1 - E_1(-\gamma_h^\alpha t^\alpha)] + \sum_{k=0}^{n-1} E_{k+1}(-\gamma_h^\alpha t^\alpha) N_h^h(k)(t_0)t^k. \tag{5.14}$$

and

$$N_m(t) \leq \Lambda_m^\alpha[1 - E_1(-\gamma_m^\alpha t^\alpha)] + \sum_{k=0}^{n-1} E_{k+1}(-\gamma_m^\alpha t^\alpha) N_m^{(k)}(t_0)t^k. \tag{5.15}$$

where $E_1(-\gamma_h^\alpha t^\alpha)$,$E_{k+1}(-\gamma_h^\alpha t^\alpha)$, and $E_1(-\gamma_m^\alpha t^\alpha)$,$E_{k+1}(-\gamma_m^\alpha t^\alpha)$ are the series of Mittag-Leffler function (as in definition 4), so we say that the solution to the model is bounded. Thus,

$$\left\{(S_h(t), I_h(t), H_h(t), R_h(t)) \in \mathbb{R}_+^4 : S_h(t), I_h(t), H_h(t), R_h(t) \leq \Lambda_h^\alpha[1 - E_1(-\gamma_h^\alpha t^\alpha)] + \sum_{k=0}^{n-1} E_{k+1}(-\gamma_h^\alpha t^\alpha) N^{(k)}(t_0)t^k \right\}. \tag{5.16}$$

and

$$\left\{(S_m(t), I_m(t)) \in \mathbb{R}_+^2 : S_m(t), I_m(t) \leq \Lambda_m^\alpha\left[1 - E_1(-\gamma_m^\alpha t^\alpha)\right] + \sum_{k=0}^{n-1} E_{k+1}(-\gamma_m^\alpha t^\alpha) N^{(k)}(t_0)t^k \right\}. \tag{5.17}$$

5.2. Uniqueness of the solution

In this section we will prove the uniqueness of (3.1). Consider the system (3.1) written as

$$ {}_0^C D_t^\alpha x(t) = F(t, x),\ x(0) = x_0, \tag{5.18}$$

$$F(t, x) = Ax + g(x) + b,$$

$$x = x(t) = (S_h(t), I_h(t), H_h(t), R_h(t))^T, b = (\Lambda_h, o, o, o)^T,$$

$$\mathrm{A} = \begin{bmatrix} \gamma_h^\alpha & 0 & 0 & 0 \\ 0 & -\mu_h^\alpha - \tau_h^\alpha - \gamma_h^\alpha & 0 & 0 \\ 0 & \mu_h^\alpha & -\epsilon_h^\alpha - \upsilon_h^\alpha - \gamma_h^\alpha & 0 \\ 0 & \tau_h^\alpha & \upsilon_h^\alpha & -\gamma_h^\alpha \end{bmatrix},\ g(x(t)) = \begin{bmatrix} -\phi_m^\alpha \beta_m^\alpha I_m \frac{S_h}{N_h} \\ \phi_m^\alpha \beta_m^\alpha I_m \frac{S_h}{N_h} \\ 0 \\ 0 \end{bmatrix}. \tag{5.19}$$

Theorem 2. System (5.18) satisfies Lipschitz continuity

Proof. Since

$$\begin{aligned} |F(t, x) - F(t, x^*)| &= |A(x(t)) - A(x^*(t)) + g(x(t)) - g(x^*(t))| \\ &\leq (\|A\| + 1)\|x(t) - x^*(t)\|, \\ \|F(t, x(t)) - F(t, x^*(t))\| &\leq L\|x(t) - x^*(t)\|, L = \|A\| + 1 < \infty. \end{aligned} \tag{5.20}$$

It is clear that F is continuous and bounded function. Using Picard-Lindelof theorem [29] we establish the following theorem.

Theorem 3. Let $0 < \alpha < 1, I = [0, h^*] \subseteq \mathbb{R}$ and $J = |x(t) - x(0)| \leq k$ and let $f : I \times J \to \mathbb{R}$ be continuous bounded function, that is $\exists M > 0$ such that $|f(t, x)| \leq M$, Since f satisfies Lipschitz

conditions. If $Lk < M$, then there exists a unique $x \in C^{\alpha}[0, h^*]$ that holds for the initial value problem (5.18). Where $h^* = \min\{h, (\frac{k\Gamma(\alpha+1)}{M})^{\frac{1}{\alpha}}\}$.

Proof. Suppose $T = x \in C[0, h^*] : \|x(t) - x(0)\| \leq k$, Since $T \subseteq \mathbb{R}$ and its closed set, then T is complete metric space. The continuous system (5.18) can be transformed to equivalent equations as;

$$\begin{aligned} {}_0^C D_t^{-\alpha}[{}_0^C D_t^{\alpha} x(t)] &= {}_0^C D_t^{-\alpha} f(t, x)), \\ x(t) - x(0) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, x(\tau)) d\tau, \\ x(t) &= x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, x(\tau)) d\tau. \end{aligned} \tag{5.21}$$

Equation (5.21) is equivalent to Volterra integral equation that solves (5.18). Define an operator F in T

$$F[x](t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, x(\tau)) d\tau. \tag{5.22}$$

Now we need to proof that (5.22) satisfies the hypothesis of contradiction mapping principle. First to show $F : T \to T$,

$$\begin{aligned} |F[x(t)] - x(0)| &= \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, x(\tau)) d\tau \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} |f(\tau, x(\tau))| d\tau \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} M d\tau \\ &= \frac{M}{\Gamma(\alpha+1)} t^{\alpha} \\ &= \frac{M}{\Gamma(\alpha+1)} (h^*)^{\alpha} \\ &\leq \frac{M}{\Gamma(\alpha+1)} \frac{k\Gamma(\alpha+1)}{M}. \end{aligned} \tag{5.23}$$

Or, $x(0) - k \leq F[x](t) \leq x(0) + k, \forall t \in [0, h^*]$. Hence the operator F maps T onto itself. Secondly, to

show that T is a contradiction, we have

$$\begin{aligned}
|F[x](t) - F[x^*](t)| &= \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} [f(\tau, x(\tau)) - f(\tau, x^*(\tau))] d\tau \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} |f(\tau, x(\tau)) - f(\tau, x^*(\tau))| \, d\tau \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} L \|x - x^*\| d\tau \\
&= \frac{L}{\Gamma(\alpha)} \|x - x^*\| \int_0^t (t-\tau)^{\alpha-1} \tau^0 d\tau \\
&= \frac{L}{\Gamma(\alpha)} \|x - x^*\| \frac{\Gamma(\alpha)}{(\Gamma\alpha + 1)} t^\alpha \\
&= \frac{L}{\Gamma(\alpha)} \|x - x^*\| t^\alpha \\
&\leq \frac{L}{\Gamma(\alpha+1)} \|x - x^*\| (h^*)^\alpha \\
&\leq \frac{L}{\Gamma(\alpha+1)} \|x - x^*\| \frac{k\Gamma(\alpha+1)}{M}.
\end{aligned} \tag{5.24}$$

Since by hypothesis $\frac{Lk}{M} < 1$, then T is a contradiction and has a unique fixed point. Thus, system (5.18) has a unique solution in $\bar{\Omega}$. □

Theorem 4. The closed set $\Omega = \{(x_1, x_2, x_3, x_4) \in R_+^4 : 0 \leq x_1 + x_2 + x_3 + x_4 \leq M_1, 0 \leq x_5 + x_6 \leq M_2\}$ is a positive invariant set for the proposed fractional order system (3.1).

Proof. To prove that the system of Eq (3.1) has a non-negative solution, the system of Eq (3.1) implies

$$\begin{cases}
{}_0^C D_t^\alpha S_h|_{S_h=0} &= \Lambda_h^\alpha > 0, \\
{}_0^C D_t^\alpha I_h|_{I_h=0} &= \phi_m^\alpha \beta_m^\alpha I_m \frac{S_h}{N_h} \geq 0, \\
{}_0^C D_t^\alpha H_h|_{H_h=0} &= \mu_h^\alpha I_h \geq 0, \\
{}_0^C D_t^\alpha R_h|_{R_h=0} &= \tau_h^\alpha I_h + \upsilon_h^\alpha H_h \geq 0, \\
{}_0^C D_t^\alpha S_m|_{S_m=0} &= \Lambda_m^\alpha > 0, \\
{}_0^C D_t^\alpha I_m|_{I_m=0} &= \phi_h^\alpha \beta_h^\alpha I_h \frac{S_m}{N_m} \geq 0.
\end{cases} \tag{5.25}$$

Thus, the fractional system (3.1) has non-negative solutions. In the end, from the first four equations of the fractional system (3.1), we obtain ${}_0^C D_t^\alpha (x_1 + x_2 + x_3 + x_4) \leq \Lambda_h^\alpha - \gamma_h^\alpha (x_1 + x_2 + x_3 + x_4)$. Solving the above inequality, we obtain

$$(x_1(t) + x_2(t) + x_3(t) + x_4(t)) \leq \left(x_1(0) + x_2(0) + x_3(0) + x_4(0) - \frac{\Lambda_h^\alpha}{\gamma_h^\alpha} \right) E_\alpha(-\gamma_h^\alpha t^\alpha) + \frac{\Lambda_h^\alpha}{\gamma_h^\alpha}.$$

so by the asymptotic behavior of Mittag-Leffler function [25], we obtain $(x_1(t)+x_2(t)+x_3(t)+x_4(t)) \leq \frac{\Lambda_h^\alpha}{\gamma_h^\alpha} \cong M_1$. Taking the same steps for the last two equations of system (3.1), we get

$$x_5(t) + x_6(t) \leq M_2, M_2 = \frac{\Lambda_m^\alpha}{\gamma_m^\alpha}.$$

Hence, the closed set Ω is a positive invariant region for the fractional-order dengue model (3.1). □

6. Model equilibria and basic reproduction number

By getting the model equilibria from the fractional dengue model (3.1) we obtain and by setting

$$ {}^{C}D_t^{\alpha} S_h = {}^{C}D_t^{\alpha} I_h = {}^{C}D_t^{\alpha} H_h = {}^{C}D_t^{\alpha} R_h = {}^{C}D_t^{\alpha} S_m = {}^{C}D_t^{\alpha} I_m = 0. $$

So from some algebraic solution of (3.1) we get,

$$ \begin{cases} \Lambda_h^{\alpha} - \phi_m^{\alpha}\beta_m^{\alpha} I_m \frac{S_h}{N_h} - \gamma_h^{\alpha} S_h &= 0 \\ \phi_m^{\alpha}\beta_m^{\alpha} I_m \frac{S_h}{N_h} - (\mu_h^{\alpha} + \tau_h^{\alpha} + \gamma_h^{\alpha}) I_h &= 0 \\ \mu_h^{\alpha} I_h - (\epsilon_h^{\alpha} + \upsilon_h^{\alpha} + \gamma_h^{\alpha}) H_h &= 0 \\ \tau_h^{\alpha} I_h + \upsilon_h^{\alpha} H_h - \gamma_h^{\alpha} R_h &= 0 \\ \Lambda_m^{\alpha} - \phi_h^{\alpha}\beta_h^{\alpha} I_h \frac{S_m}{N_m} - \gamma_m^{\alpha} S_m &= 0 \\ \phi_h^{\alpha}\beta_h^{\alpha} I_h \frac{S_m}{N_m} - \gamma_m^{\alpha} I_m &= 0. \end{cases} \tag{6.1} $$

The fractional dengue model (3.1) obtain the following two equilibrium points: (1) The DFE for the model (3.1) is given by,

$$ E_0 = \left(\frac{\Lambda_h^{\alpha}}{\gamma_h^{\alpha}}, 0, 0, 0, \frac{\Lambda_m^{\alpha}}{\gamma_m^{\alpha}}, 0 \right). $$

(2) The EE for the model (3.1) is given by,

$$ E^* = (S_h^*, I_h^*, H_h^*, R_h^*, S_m^*, I_m^*). $$

where

$$ S_h^* = \frac{\Lambda_h^{\alpha}}{\tau_h^* + \gamma_h^{\alpha}}, I_h^* = \frac{\tau_h^* S_h^*}{\mu_h^{\alpha} + \tau_h^* + \gamma_h^{\alpha}}, H_h^* = \frac{\mu_h^{\alpha} I_h^*}{\epsilon_h^{\alpha} + \upsilon_h^{\alpha} + \gamma_h^{\alpha}}, R_h^* = \frac{\tau_h^* I_h^* + \upsilon_h^{\alpha} H_h^*}{\gamma_h^{\alpha}}, $$

$$ S_m^* = \frac{\Lambda_m^{\alpha}}{\tau_m^* + \gamma_m^{\alpha}}, I_m^* = \frac{\tau_m^* S_m^*}{\gamma_m^{\alpha}}. $$

where, $\tau_h^* = \frac{\phi_m^{\alpha}\beta_m^{\alpha} I_m^*}{N_h^*}$ and $\tau_m^* = \frac{\phi_h^{\alpha}\beta_h^{\alpha} I_h^*}{N_m^*}$. Observe that S_h^*, I_h^*, H_h^*, R_h^*, S_m^*, I_m^* are positive if and only if $\left[\phi_h^{\alpha}\phi_m^{\alpha}\beta_h^{\alpha}\beta_m^{\alpha} S_h S_m - N_h N_m \gamma_m^{\alpha}(\mu_h^{\alpha} + \tau_h^{\alpha} + \gamma_h^{\alpha})\right] > 0$. Calculate the reproduction number of the fractional model (3.1) by using the method of next generation matrix and the basic reproduction number present in [38]. Let us define a vector, $\mathrm{X} = [I_h, H_h, I_m]^T$, And

$$ \mathrm{f} = \begin{bmatrix} \phi_m^{\alpha}\beta_m^{\alpha} I_m \frac{S_h}{N_h} \\ 0 \\ \phi_h^{\alpha}\beta_h^{\alpha} I_h \frac{S_m}{N_m} \end{bmatrix}, \ \mathrm{v} = \begin{bmatrix} (\mu_h^{\alpha} + \tau_h^{\alpha} + \gamma_h^{\alpha}) I_h \\ (\epsilon_h^{\alpha} + \upsilon_h^{\alpha} + \gamma_h^{\alpha}) H_h - \mu_h^{\alpha} I_h \\ \gamma_m^{\alpha} I_m \end{bmatrix}. \tag{6.2} $$

$$ \mathcal{F} = \begin{bmatrix} 0 & 0 & \phi_m^{\alpha}\beta_m^{\alpha} \frac{S_h}{N_h} \\ 0 & 0 & 0 \\ \phi_h^{\alpha}\beta_h^{\alpha} \frac{S_m}{N_m} & 0 & 0 \end{bmatrix}, \ \mathcal{V} = \begin{bmatrix} (\mu_h^{\alpha} + \tau_h^{\alpha} + \gamma_h^{\alpha}) & 0 & 0 \\ -\mu_h^{\alpha} & (\epsilon_h^{\alpha} + \upsilon_h^{\alpha} + \gamma_h^{\alpha}) & 0 \\ 0 & 0 & \gamma_m^{\alpha} \end{bmatrix}. \tag{6.3} $$

Thus the basic reproduction number of the model (3.1) is

$$ R_0 = \rho(\mathcal{F}\mathcal{V}^{-1}) = \frac{\phi_h^{\alpha}\phi_m^{\alpha}\beta_h^{\alpha}\beta_m^{\alpha} S_h S_m}{N_h N_m \gamma_m^{\alpha}\left(\mu_h^{\alpha} + \tau_h^{\alpha} + \gamma_h^{\alpha}\right)}. $$

It is easy to prove that S_h^*, I_h^*, H_h^*, R_h^*, S_m^*, I_m^*, and $R^* > 0$ if and only if $R_0 > 1$.

7. Stability analysis

7.1. Analysis of DFE

Simplifying the stability of the DFE, Suppose DFE is $E_0 = (S_{h_0}, I_{h_0}, H_{h_0}, R_{h_0}, S_{m_0}, I_{m_0}) = \left(\frac{\Lambda_h}{\gamma_h}, 0, 0, 0, \frac{\Lambda_m}{\gamma_m}, 0\right)$ and the Jacobian matrix of the system in (3.1) can be written as

$$J(E_0) = \begin{bmatrix} -\gamma_h^\alpha & 0 & 0 & 0 & 0 & -\phi_m^\alpha \beta_m^\alpha \\ 0 & -(\mu_h^\alpha + \tau_h^\alpha + \gamma_h^\alpha) & 0 & 0 & 0 & \phi_m^\alpha \beta_m^\alpha \\ 0 & \mu_h^\alpha & -(\epsilon_h^\alpha + \upsilon_h^\alpha + \gamma_h^\alpha) & 0 & 0 & 0 \\ 0 & \tau_h^\alpha & \upsilon_h^\alpha & -\gamma_h^\alpha & 0 & 0 \\ 0 & -\phi_h^\alpha \beta_h^\alpha \frac{S_m}{N_m} & 0 & 0 & -\gamma_m^\alpha & 0 \\ 0 & \phi_h^\alpha \beta_h^\alpha \frac{S_m}{N_m} & 0 & 0 & \phi_h^\alpha \beta_h^\alpha \frac{I_h}{N_m} & -\gamma_m^\alpha \end{bmatrix} \tag{7.1}$$

Now calculating the Jacobian matrix J at DFE point E_0 and solved $\det(J - \lambda I)$, we obtain, $P_j(x) = (\lambda + \gamma_h^\alpha)^2(\lambda + \epsilon_h^\alpha + \upsilon_h^\alpha + \gamma_h^\alpha)(\lambda + \gamma_m^\alpha)(\lambda^2 + A\lambda + B)$, Where $A = \mu_h^\alpha + \tau_h^\alpha + \gamma_h^\alpha + \gamma_m^\alpha$ and

$$\begin{aligned} B &= \gamma_m^\alpha(\mu_h^\alpha + \tau_h^\alpha + \gamma_h^\alpha) - \phi_h^\alpha \phi_m^\alpha \beta_h^\alpha \beta_m^\alpha \frac{S_m}{N_m} \\ &= \gamma_m^\alpha(\mu_h^\alpha + \tau_h^\alpha + \gamma_h^\alpha)\left[1 - \frac{\phi_h^\alpha \phi_m^\alpha \beta_h^\alpha \beta_m^\alpha S_h}{N_m \gamma_m^\alpha N_h(\mu_h^\alpha + \tau_h^\alpha + \gamma_h^\alpha)}\right] \\ &= \gamma_m^\alpha\left(\mu_h^\alpha + \tau_h^\alpha + \gamma_h^\alpha\right)\left[1 - R_0\right]. \end{aligned}$$

It is easy to proof that if $\mathcal{R}_0 < 1$, then $A > 0$ and $B > 0$. This polynomial $\lambda^2 + A\lambda + B$ have two roots with negative real parts. Thats why, E_0 is locally stable because the real parts of six eigenvalues of the matrix $J(E_0)$ are all negative. Therefore, overall we can tell the DFE is stable when $B > 0$ and DFE is unstable when $B < 0$. The following theorem is presented for the global stability of the disease free equilibrium case E_0.

Theorem 5. The fractional dengue model given by (3.1) for the arbitrary fractional order $\alpha \in (0, 1]$, with $R_0 < 1$, is globally asymtotically stable.

Proof. The following Lyapunov function is considered for the proof of the global stability of the dengue fractional model (3.1):

$$G(t) = \eta_1\left(S_h - S_h^0 - S_h^0 log\frac{S_h}{S_h^0}\right) + \eta_2 I_h + \eta_3\left(H_h - H_h^0 - H_h^0 log\frac{H_h}{H_h^0}\right)H_h + \eta_4\left(S_m - S_m^0 - S_m^0 log\frac{S_m}{S_m^0}\right) + \eta_5 I_m,$$

where $\eta_i > 0$ for i= 1,2,3,4, are arbitrary constants to be determined later. We consider the result described in sect. 2 and taking the time derivative of G(t), we obtain

$${}^C D_t^\alpha G'(t) = \eta_1\left(1 - \frac{S_h^0}{S_h}\right){}^C D_t^\alpha S_h + \eta_2 {}^C D_t^\alpha I_h + \eta_3\left(1 - \frac{H_h^0}{H_h}\right){}^C D_t^\alpha H_h + \eta_4\left(1 - \frac{S_m^0}{S_m}\right){}^C D_t^\alpha S_m + \eta_5 {}^C D_t^\alpha I_m.$$

Considering the fractional system (3.1), we obtain

$$
\begin{aligned}
{}^{C}D_t^{\alpha}G(t) =& \eta_1\left(1-\frac{S_h^0}{S_h}\right)\left[\Lambda_h^{\alpha}-\phi_m^{\alpha}\beta_m^{\alpha}I_m\frac{S_h}{N_h}-\gamma_h^{\alpha}S_h\right]\\
&+\eta_2\left[\phi_m^{\alpha}\beta_m^{\alpha}I_m\frac{S_h}{N_h}-(\mu_h^{\alpha}+\tau_h^{\alpha}+\gamma_h^{\alpha})I_h\right]\\
&+\eta_3\left(1-\frac{H_h^0}{H_h}\right)[\mu_h^{\alpha}I_h-(\epsilon_h^{\alpha}+\upsilon_h^{\alpha}+\gamma_h^{\alpha})H_h]\\
&+\eta_4\left(1-\frac{S_m^0}{S_m}\right)\left[\Lambda_m^{\alpha}-\phi_h^{\alpha}\beta_h^{\alpha}I_h\frac{S_m}{N_m}-\gamma_m^{\alpha}S_m\right]\\
&+\eta_5\left[\phi_h^{\alpha}\beta_h^{\alpha}I_h\frac{S_m}{N_m}-\gamma_m^{\alpha}I_m\right].
\end{aligned}
\tag{7.2}
$$

Using the values of S_h^0 and S_m^0 at the DFE and simplifying we obtain

$$
{}^{C}D_t^{\alpha}G(t) = -\gamma_h^{\alpha}\gamma_m^{\alpha}\frac{(S_h-S_h^0)^2}{S_h}-\phi_m^{\alpha}\beta_m^{\alpha}\gamma_m^{\alpha}\frac{(S_m-S_m^0)^2}{S_m}-\gamma_m^{\alpha}(\mu_h^{\alpha}+\tau_h^{\alpha}+\gamma_h^{\alpha})I_h\left(1-R_0^2\right),
$$

where $\eta_1 = \eta_2 = \gamma_m^{\alpha}$, $\eta_4 = \eta_5 = \phi_h^{\alpha}\beta_h^{\alpha}$. Thus, ${}^{C}D_t^{\alpha}G(t)$ is negative for $R_0 \leq 1$. So it follows from the established results and those given by theorem 1 in [24,39] that the fractional dengue model is globally asymptotically stable at the DFE case E_0.

7.2. Analysis of EE

We prove the global stability results for the fractional model (3.1). Firstly we have the following results for the model (3.1) at the constant state:

$$
\Lambda_h^{\alpha} = \phi_m^{\alpha}\beta_m^{\alpha}I_m^*\frac{S_h^*}{N_h^*}+\gamma_h^{\alpha}S_h^*, \Lambda_m^{\alpha} = \frac{\phi_h^{\alpha}\beta_h^{\alpha}I_h^*}{N_m^*}+\gamma_m^{\alpha}S_m^*, \gamma_m^{\alpha} = \frac{\phi_h^{\alpha}\beta_h^{\alpha}S_m^*I_h^*}{N_m^*I_m^*},
$$

$$
\left(\mu_h^{\alpha}+\tau_h^{\alpha}+\gamma_h^{\alpha}\right)I_h^* = \phi_m^{\alpha}\beta_m^{\alpha}I_m^*\frac{S_h^*}{N_h^*}, \left(\epsilon_h^{\alpha}+\upsilon_h^{\alpha}+\gamma_h^{\alpha}\right)H_h^* = \mu_h^{\alpha}I_h^*.
$$

Now showing the global stability of the model (3.1) in the following theorem.

Theorem 6. If $R_0 > 1$, then the fractional dengue model (3.1) at E^* is globally asymptotically stable.

Proof. We suppose the following Lyapunov function:

$$
\begin{aligned}
L(t) =& \phi_h^{\alpha}\beta_h^{\alpha}I_h^*\frac{S_m^*}{N_m^*}\left[\left(S_h-S_h^*-S_H^*\log\frac{S_h}{S_h^*}\right)+\left(I_h-I_h^*-I_h^*\log\frac{I_h}{I_h^*}\right)+\left(H_h-H_h^*-H_h^*\log\frac{H_h}{H_h^*}\right)\right]\\
&+\phi_m^{\alpha}\beta_m^{\alpha}I_m^*\frac{S_h^*}{N_h^*}\left[\left(S_m-S_m^*-S_m^*\log\frac{S_m}{S_m^*}\right)+\left(I_m-I_m^*-I_m^*\log\frac{I_m}{I_m^*}\right)\right].
\end{aligned}
\tag{7.3}
$$

The derivative of L(t) with the application of the lemma 1 given in sec. 2 yields

$$
\begin{aligned}
{}^{C}D_t^{\alpha}L(t) =& \phi_h^{\alpha}\beta_h^{\alpha}I_h^{*}\frac{S_m^{*}}{N_m^{*}}\left[\left(1-\frac{S_h^{*}}{S_h}\right){}^{C}D_t^{\alpha}S_h+\left(1-\frac{I_h^{*}}{I_h}\right){}^{C}D_t^{\alpha}I_h+\left(1-\frac{H_h^{*}}{H_h}\right){}^{C}D_t^{\alpha}H_h\right]\\
&+\phi_m^{\alpha}\beta_m^{\alpha}I_m^{*}\frac{S_h^{*}}{N_h^{*}}\left[\left(1-\frac{S_m^{*}}{S_m}\right){}^{C}D_t^{\alpha}S_m+\left(1-\frac{I_m^{*}}{I_m}\right){}^{C}D_t^{\alpha}I_m\right].
\end{aligned}
\tag{7.4}
$$

From the fractional dengue model (3.1), we can consider the following way:

$$
\begin{aligned}
{}^{C}D_t^{\alpha}L(t) =& \phi_h^{\alpha}\beta_h^{\alpha}I_h^{*}\frac{S_m^{*}}{N_m^{*}}\left[\left(1-\frac{S_h^{*}}{S_h}\right)\left(\Lambda_h^{\alpha}-\phi_m^{\alpha}\beta_m^{\alpha}I_m\frac{S_h}{N_h}-\gamma_h^{\alpha}S_h\right)+\left(1-\frac{I_h^{*}}{I_h}\right)\left(\phi_m^{\alpha}\beta_m^{\alpha}I_m\frac{S_h}{N_h}-\left(\mu_h^{\alpha}+\tau_h^{\alpha}+\gamma_h^{\alpha}\right)I_h\right)\right.\\
&\left.+\left(1-\frac{H_h^{*}}{H_h}\right)\left(\mu_h^{\alpha}I_h-\left(\epsilon_h^{\alpha}+\upsilon_h^{\alpha}+\gamma_h^{\alpha}\right)H_h\right)\right]+\phi_m^{\alpha}\beta_m^{\alpha}I_m^{*}\frac{S_h^{*}}{N_h^{*}}\left[\left(1-\frac{S_m^{*}}{S_m}\right)\left(\Lambda_m^{\alpha}-\phi_h^{\alpha}\beta_h^{\alpha}I_h\frac{S_m}{N_m}-\gamma_m^{\alpha}I_m\right)\right.\\
&\left.+\left(1-\frac{I_m^{*}}{I_m}\right)\left(\phi_h^{\alpha}\beta_h^{\alpha}I_h\frac{S_m}{N_m}-\gamma_m^{\alpha}I_m\right)\right].
\end{aligned}
\tag{7.5}
$$

By direct calculation, we get the following:

$$
\begin{aligned}
\left(1-\frac{S_h^{*}}{S_h}\right){}^{C}D_t^{\alpha}S_h &= \left(1-\frac{S_h^{*}}{S_h}\right)\left(\Lambda_h^{\alpha}-\phi_m^{\alpha}\beta_m^{\alpha}I_m\frac{S_h}{N_h}-\gamma_h^{\alpha}S_h\right)\\
&= \left(1-\frac{S_h^{*}}{S_h}\right)\left(\phi_m^{\alpha}\beta_m^{\alpha}I_m^{*}\frac{S_h^{*}}{N_h^{*}}+\gamma_h^{\alpha}S_h^{*}-\phi_m^{\alpha}\beta_m^{\alpha}I_m\frac{S_h}{N_h}-\gamma_h^{\alpha}S_h\right)\\
&= -\gamma_h^{\alpha}\frac{\left(S_h-S_h^{*}\right)^2}{S_h}+\phi_m^{\alpha}\beta_m^{\alpha}I_m^{*}\frac{S_h^{*}}{N_h^{*}}\left(1-\frac{S_h^{*}}{S_h}-\frac{S_hI_mN_h^{*}}{S_h^{*}I_m^{*}}+\frac{I_mN_h^{*}}{I_m^{*}N_h}\right).\\
\left(1-\frac{I_h^{*}}{I_h}\right){}^{C}D_t^{\alpha}I_h &= \left(1-\frac{I_h^{*}}{I_h}\right)\left[\phi_m^{\alpha}\beta_m^{\alpha}I_m\frac{S_h}{N_h}-\left(\mu_h^{\alpha}+\tau_h^{\alpha}+\gamma_h^{\alpha}\right)I_h\right]\\
&= \left(1-\frac{I_h^{*}}{I_h}\right)\left(\phi_m^{\alpha}\beta_m^{\alpha}I_m\frac{S_h}{N_h}-\phi_m^{\alpha}\beta_m^{\alpha}I_m^{*}\frac{S_h^{*}}{N_h^{*}}\right)\\
&= \phi_m^{\alpha}\beta_m^{\alpha}I_m^{*}\frac{S_h^{*}}{N_h^{*}}\left(1-\frac{I_h}{I_h^{*}}-\frac{S_hI_mN_h^{*}I_h^{*}}{N_hS_h^{*}I_m^{*}I_h}+\frac{S_hI_mN_h^{*}}{S_h^{*}N_hI_m^{*}}\right).\\
\left(1-\frac{H_h^{*}}{H_h}\right){}^{C}D_t^{\alpha}H_h &= \left(1-\frac{H_h^{*}}{H_h}\right)\left[\mu_h^{\alpha}I_h-\left(\epsilon_h^{\alpha}+\upsilon_h^{\alpha}+\gamma_h^{\alpha}\right)H_h\right]\\
&= \left(1-\frac{H_h^{*}}{H_h}\right)\left(\mu_h^{\alpha}I_h-\mu_h^{\alpha}I_h^{*}\right)\\
&= \mu_h^{\alpha}I_h^{*}\left(1-\frac{H_h}{H_h^{*}}-\frac{I_hH_h^{*}}{I_h^{*}H_h}+\frac{I_h}{I_h^{*}}\right).
\end{aligned}
\tag{7.6}
$$

$$\begin{aligned}
\left(1-\frac{S_m^*}{S_m}\right){}^C D_t^\alpha S_m &= \left(1-\frac{S_m^*}{S_m}\right)\left(\Lambda_m^\alpha - \phi_h^\alpha\beta_h^\alpha I_h\frac{S_m}{N_m} - \gamma_m^\alpha S_m\right)\\
&= \left(1-\frac{S_m^*}{S_m}\right)\left(\phi_h^\alpha\beta_h^\alpha I_h^*\frac{S_m^*}{N_m^*} + \gamma_m^\alpha S_m^* - \phi_h^\alpha\beta_h^\alpha I_h\frac{S_m}{N_m} - \gamma_m^\alpha S_m\right)\\
&= -\gamma_m^\alpha\frac{(S_m-S_m^*)^2}{S_m} + \phi_h^\alpha\beta_h^\alpha I_h^*\frac{S_m^*}{N_m^*}\left(1-\frac{S_m^*}{S_m}-\frac{S_m I_h N_m^*}{N_m S_m^* I_h^*}+\frac{I_h N_m^*}{I_h^* N_m}\right).\\
\left(1-\frac{I_m^*}{I_m}\right){}^C D_t^\alpha I_m &= \left(1-\frac{I_m^*}{I_m}\right)\left(\phi_h^\alpha\beta_h^\alpha I_h\frac{S_m}{N_m} - \gamma_m^\alpha I_m\right)\\
&= \left(1-\frac{I_m^*}{I_m}\right)\left(\phi_h^\alpha\beta_h^\alpha I_h\frac{S_m}{N_m} - \phi_h^\alpha\beta_h^\alpha I_h^*\frac{S_m^*}{N_m^*}\right)\\
&= \phi_h^\alpha\beta_h^\alpha I_h^*\frac{S_m^*}{N_m^*}\left(1-\frac{I_m}{I_m^*}-\frac{S_m I_h N_m^* I_m^*}{N_m S_m^* I_h^* I_m}+\frac{S_m I_h N_m^*}{S_m^* N_m^* I_h}\right).
\end{aligned} \tag{7.7}$$

Using the above expressions (7.6) and (7.7) in Eq (7.5), we obtain

$$\begin{aligned}
{}^C D_t^\alpha L(t) = {}& \frac{\phi_m^\alpha\gamma_h^\alpha\beta_m^\alpha S_m^* I_h^* S_h^*}{N_h^*}\left(2-\frac{S_h^*}{S_h}-\frac{S_h}{S_h^*}\right)+\frac{\phi_h^\alpha\gamma_m^\alpha\beta_h^\alpha S_m^* I_m^* S_h^*}{N_h^*}\left(2-\frac{S_m^*}{S_m}-\frac{S_m}{S_m^*}\right)\\
&+\frac{\phi_h^\alpha\phi_m^\alpha\beta_h^\alpha\beta_m^\alpha S_h^* S_m^* I_h^* I_m^*}{(N_h^*)^2}\left(5-\frac{S_h^*}{S_h}-\frac{I_h^*}{I_h}-\frac{H_h^*}{H_h}-\frac{S_m^*}{S_m}-\frac{I_m^*}{I_m}-\frac{S_h I_m N_h^* I_h^*}{N_h S_h^* I_m^* I_m^* I_h}\right.\\
&\left.-\frac{I_h H_h^*}{I_h^* H_h}-\frac{S_m I_h N_h^* I_m^*}{N_h S_m^* I_h^* I_m}+\frac{I_m N_h^*}{N_h I_m^*}+\frac{I_h N_h^*}{N_h I_h^*}\right).
\end{aligned} \tag{7.8}$$

From Eq (7.8), we get the following result,

$$\left(2-\frac{S_h^*}{S_h}-\frac{S_h}{S_h^*}\right)\le 0, \left(2-\frac{S_m^*}{S_m}-\frac{S_m}{S_m^*}\right)\le 0$$

and, in the same way, if

$$\left(5-\frac{S_h^*}{S_h}-\frac{I_h^*}{I_h}-\frac{H_h^*}{H_h}-\frac{S_m^*}{S_m}-\frac{I_m^*}{I_m}-\frac{S_h I_m N_h^* I_h^*}{N_h S_h^* I_m^* I_m^* I_h}-\frac{I_h H_h^*}{I_h^* H_h}-\frac{S_m I_h N_h^* I_m^*}{N_h S_m^* I_h^* I_m}+\frac{I_m N_h^*}{N_h I_m^*}+\frac{I_h N_h^*}{N_h I_h^*}\right)\le 0$$

then ${}^C D_t^\alpha L(t) \le 0$, So it observes that, at the EE point E^*, the fractional dengue model is globally asymptotically stable at E^*, when $R_0 > 1$.

8. Sensitivity analysis

Sensitivity analysis helps us to identify parameters that have a big impact on the disease transmission. Such information is important not only for experimental design but also for data assimilation and reduction to complex nonlinear models [40]. This provides a good strategy to prevent and restrain the disease. The disease will be controlled and mitigated if we can change the value of parameters by the control strategies. Usually, in the epidemiological model, the analysis is used to discover parameters that have greatest influence on the basic reproduction number R_0 and should be targeted by the control strategies. The sensitivity indices of the R_0 are determined to allow us to measure which parameter has the greatest influence on the changes of R_0 and, hence, the greatest

effect in determining whether the disease can be eliminated in the population. The normalized forward sensitivity index of a variable (R_0) with respect to a parameter is the ratio of the relative change in the variable (R_0) to the relative change in the parameter. The systematic description of the sensitivity analysis of the different parameters in R_0 for the model is:

$$\delta_{\theta}^{R_0} = \frac{dRo}{d\theta}\frac{\theta}{Ro}.$$

So the basic reproduction number is,

$$R_0 = \rho(\mathcal{F}\mathcal{V}^{-1}) = \frac{\phi_h^{\alpha}\phi_m^{\alpha}\beta_h^{\alpha}\beta_m^{\alpha}S_hS_m}{N_hN_m\gamma_m^{\alpha}\left(\mu_h^{\alpha}+\tau_h^{\alpha}+\gamma_h^{\alpha}\right)}.$$

It is easy to verify that

$$\frac{dR_0}{d\phi_h}\frac{\phi_h}{R_0} = \frac{\phi_h\phi_m\beta_h\beta_m\gamma_m(\mu_h+\tau_h+\gamma_h)}{\gamma_m\phi_h\phi_m\beta_h\beta_m(\mu_h+\tau_h+\gamma_h)} = 1 > 0$$

$$\frac{dR_0}{d\phi_m}\frac{\phi_m}{R_0} = \frac{\phi_h\phi_m\beta_h\beta_m\gamma_m(\mu_h+\tau_h+\gamma_h)}{\gamma_m\phi_h\phi_m\beta_h\beta_m(\mu_h+\tau_h+\gamma_h)} = 1 > 0$$

$$\frac{dR_0}{d\beta_h}\frac{\beta_h}{R_0} = \frac{\phi_h\phi_m\beta_h\beta_m\gamma_m(\mu_h+\tau_h+\gamma_h)}{\phi_h\phi_m\beta_h\beta_m\gamma_m(\mu_h+\tau_h+\gamma_h)} = 1 > 0$$

$$\frac{dR_0}{d\beta_m}\frac{\beta_m}{R_0} = \frac{\phi_h\phi_m\beta_h\beta_m\gamma_m(\mu_h+\tau_h+\gamma_h)}{\phi_h\phi_m\beta_h\beta_m\gamma_m(\mu_h+\tau_h+\gamma_h)} = 1 > 0$$

$$\frac{dR_0}{d\gamma_m}\frac{\gamma_m}{R_0} = -\frac{\phi_h\phi_m\beta_h\beta_m\gamma_m^2(\mu_h+\tau_h+\gamma_h)}{\phi_h\phi_m\beta_h\beta_m\gamma_m^2(\mu_h+\tau_h+\gamma_h)} = -1 < 0$$

$$\frac{dR_0}{d\mu_h}\frac{\mu_h}{R_0} = -\frac{\mu_h}{(\mu_h+\tau_h+\gamma_h)} < 0$$

$$\frac{dR_0}{d\tau_h}\frac{\tau_h}{R_0} = -\frac{\tau_h}{(\mu_h+\tau_h+\gamma_h)} < 0$$

$$\frac{dR_0}{d\gamma_h}\frac{\gamma_h}{R_0} = -\frac{\gamma_h}{(\mu_h+\tau_h+\gamma_h)} < 0$$

Using the parameter values from Table 2, the sensitivity indices of R_0 with respect to the parameters are given in Table 3.

Table 3. Sensitivity indices of R_0 to the model parameters

Parameter	Sensitivity indices
ϕ_h	1
ϕ_m	1
β_h	1
β_m	1
γ_m	−1
γ_h	−0.11075182
τ_h	−0.08084074
μ_h	−0.80840744

We can conclude from Table 3 that the sensitivity indices are sign-related and, R_0 is more sensitive to the following parameters (ϕ_h,ϕ_m, β_h, β_m) increasing order and a corresponding decrease in R_0 as the following parameters (μ_h, γ_h, γ_m, τ_h) have a negative impact on R_0, that means an increase in these parameters will reduce R_0, while (ϕ_h, ϕ_m, β_h, β_m) has a positive impact, and reducing the value of these parameters will reduce R_0. After the above analytical results, we now perform an R_0 sensitivity analysis to find exact ways to choose the various parameters in R_0. The following can be inferred from the sensitive analysis:

(1). If can reduce the value of the transmission rates β_h, β_m, and biting rate ϕ_h, ϕ_m could be effective control measures to stop the spread of the dengue virus.

(2). If can increase the natural death rate of mosquitoes γ_m and natural recovery rate of infected humans τ_h so that it will not affect other susceptible individuals.

9. Numerical simulations

In this section, we carry out numerical simulations for system (3.1) by using Euler's algorithm method [41]. To have a numerical scheme, we write the model (3.1) in the following form:

$$^{C}D_t^{\alpha} g(t) = G\left(t, g(t)\right), \alpha \in (0,1], t \in [0,T], g(0) = g_0, 0 < T < \infty, \tag{9.1}$$

where $g = (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}_+^6$, $G(t, g(t))$ is used for a continuous real valued vector function, which additionally satisfies the Lipschitz condition and g_0 stands for initial state vector. Taking Caputo integral on both sides of (7.1) we get

$$g(t) = g_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\lambda)^{\alpha-1} G(\lambda, g(\lambda)) d\lambda. \tag{9.2}$$

To formulate an iterative scheme, we consider a uniform grid on [0,T] with $h = \frac{T-0}{m}$ is the step size and $m \in \mathbb{N}$. Thus, Eq (7.2) gets the structure as follows after making use of the Euler method

$$\begin{cases} g_{n+1} = g_0 + \frac{h^{\alpha}}{\Gamma(\alpha+1)} \Sigma_{j=0}^{n} ((n-j+1)^{\alpha} - (n-j)^{\alpha}) G(t_j, g(t_j)), \\ n = 0, 1, 2, ..., m. \end{cases} \tag{9.3}$$

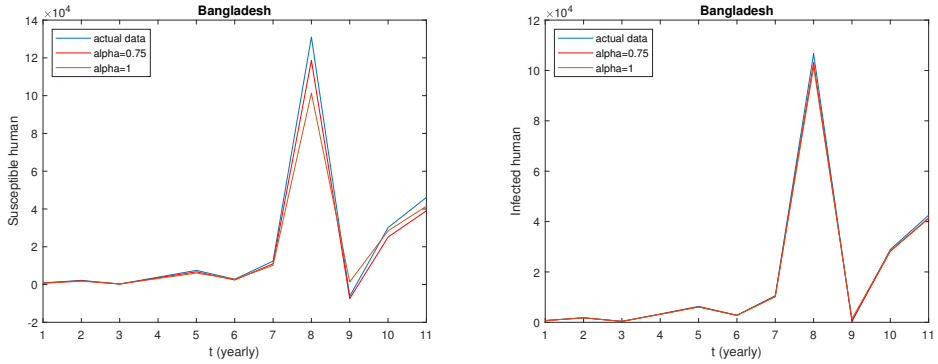
Thus, utilizing the above scheme (7.3), we deduced the following iterative formulae for the corresponding classes of the model (3.1).

We used the above approximation for the solution of our fractional system. In Figures (2) to (5), we demonstrate the dynamics of susceptible human (S_h), infected human (I_h), hospitalized human (H_h), recovered human (R_h), dead human (D_h), new cases of human, reported dengue cases of human, cumulative dengue cases of human, susceptible human with different biting rates, infected human with different biting rates, susceptible mosquitoes (S_m), infected mosquitoes (I_m), and infected mosquitoes (with different biting rates $b = 0.45, 0.50, 0.68$ with the variation of fractional order $\alpha = 0.75$, integer order and actual values. We noticed that the variation of fractional order has a great influence on the infection level of dengue in both populations. In other words, it can highly reduce the level of DF in the community. We demonstrated the effect of the biting rate ϕ_h of the mosquitoes on the dynamics of dengue and observed that the peak of infection can be greatly decreased by decreasing the biting rate ϕ_h. The mosquito biting and its generation further can be decreased by spraying or wasting the standing water around the home or inside the home, which has a great influence on the population of

mosquitoes that end up biting humans. Using bed nets, avoid to visit areas prone to mosquitoes, using mosquito repellent, and covering legs and arms by wearing long sleeves and long pants are useful to prevent biting of mosquitoes. These scenarios predict that the infection can be controlled and prevented by decreasing the index of memory and biting rate of vectors in the community. In this model, we use the real data of Bangladesh from (2012–2022) [42].

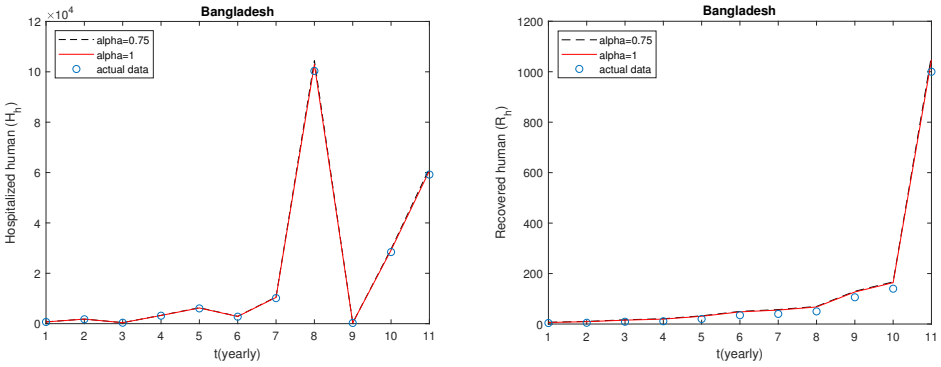
First, we simulate different values of fractional order α with fixed values of the model parameters. In this work, the dengue cases data of 11 years (2012–2022) are used with different parametric values for the numerical simulations based on a case study of Bangladesh cited from the literature; some are fitted, some are estimated and some are referred. We use the total population of Bangladesh, $N_h = 166303494$ [43]. The life expectancy in Bangladesh for the year 2022 is 72.87, so we estimate $\gamma_h = 1/72.87$ per year. The parameter Λ_h is estimated from $\Lambda_h/\gamma_h = 166303494$, and assumed that this is to be the limiting population in the disease absence, so $\Lambda_h = 2278130.05$ per year. For the initial values of the model variables, we use the total initial population $N_h(0) = 166303494$, so that $N_h(0) = S_h(0) + I_h(0) + H_h(0) + R_h(0)$ and $N_m(0) = S_m(0) + I_m(0)$, The initial conditions are assumed as $S_h(0) = 5000, I_h(0) = 1000, H_h(0) = 500, R_h(0) = 100, S_m(0) = 100000$, $I_m(0) = 80000$ and the parameter values are taken from the literature as given in Table 1. As illustrated in Figures 2 to 8, a smaller fractional order reduces the peak significantly and flattens the progression curve. The dengue cases reported in Bangladesh are shown in the following Figures and the solution obtained by the Euler algorithm method is presented in Figure 2 to 8 for different values of α. The number of infected human real data and simulated results will reach the highest level in eight months for $\alpha = 0.75, 1$. These scenarios predict that the infection can be controlled and prevented by decreasing the index of memory and biting rate of vectors in the community. Fig 6 and 8 describe the behavior of the solutions showing the dynamics of susceptible human, infected human and infected mosquito population for the biting rates 0.45 and 0.68 respectively.

$$
\begin{aligned}
{}^{c}S_{h_{p+1}} &= S_0 + \frac{h^\alpha}{\Gamma(\alpha+1)} \times \Sigma_{k=0}^{p} ((p-k+1)^\alpha - (p-k)^\alpha) \left(\Lambda_h^\alpha - \phi_m^\alpha \beta_m^\alpha I_m \frac{S_h}{N_h} - \gamma_h^\alpha S_h \right), \\
{}^{c}I_{h_{p+1}} &= I_0 + \frac{h^\alpha}{\Gamma(\alpha+1)} \times \Sigma_{k=0}^{p} ((p-k+1)^\alpha - (p-k)^\alpha) \left(\phi_m^\alpha \beta_m^\alpha I_m \frac{S_h}{N_h} - (\mu_h^\alpha + \tau_h^\alpha + \gamma_h^\alpha) I_h \right), \\
{}^{c}H_{h_{p+1}} &= H_0 + \frac{h^\alpha}{\Gamma(\alpha+1)} \times \Sigma_{k=0}^{p} ((p-k+1)^\alpha - (p-k)^\alpha) \left(\mu_h^\alpha I_h - (\epsilon_h^\alpha + \upsilon_h^\alpha + \gamma_h^\alpha) H_h \right), \\
{}^{c}R_{h_{p+1}} &= R_0 + \frac{h^\alpha}{\Gamma(\alpha+1)} \times \Sigma_{k=0}^{p} ((p-k+1)^\alpha - (p-k)^\alpha) \left(\tau_h^\alpha I_h + \upsilon_h^\alpha H_h - \gamma_h^\alpha R_h \right), \\
{}^{c}S_{m_{p+1}} &= X_0 + \frac{h^\alpha}{\Gamma(\alpha+1)} \times \Sigma_{k=0}^{p} ((p-k+1)^\alpha - (p-k)^\alpha) \left(\Lambda_m^\alpha - \phi_h^\alpha \beta_h^\alpha I_h \frac{S_m}{N_m} - \gamma_m^\alpha S_m \right), \\
{}^{c}I_{m_{p+1}} &= Y_0 + \frac{h^\alpha}{\Gamma(\alpha+1)} \times \Sigma_{k=0}^{p} ((p-k+1)^\alpha - (p-k)^\alpha) \left(\phi_h^\alpha \beta_h^\alpha I_h \frac{S_m}{N_m} - \gamma_m^\alpha I_m \right).
\end{aligned}
\tag{9.4}
$$



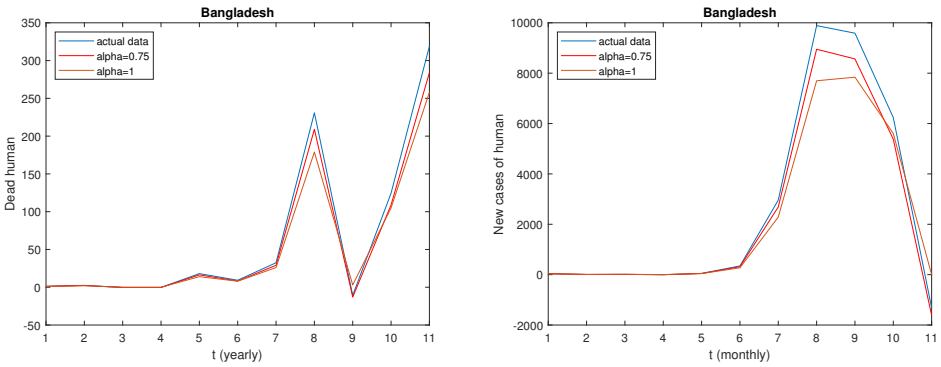
(a) (b)

Figure 2. Numerical simulation of (a) Susceptible humans $S_h(t)$ (b) Infected human $I_h(t)$ for different values of α and actual values with time (yearly).



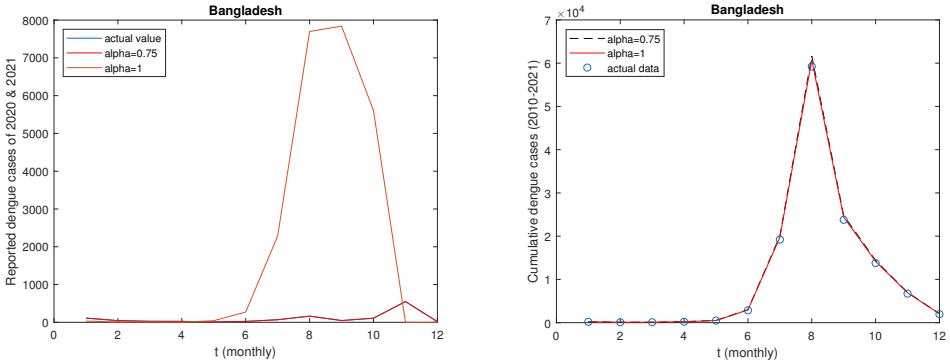
(a) (b)

Figure 3. Numerical simulation of (a) Hospitalized human $H_h(t)$ (b) Recovered human $R_h(t)$ for different values of α and actual values with time (yearly).



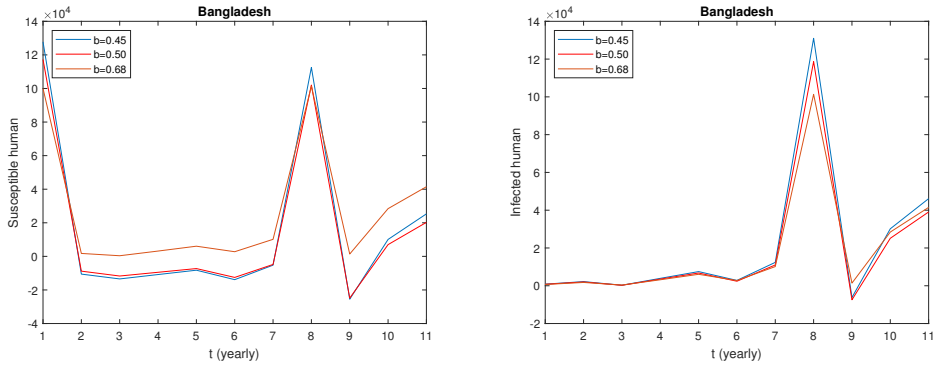
(a) (b)

Figure 4. Numerical simulation of (a) Death human $D_h(t)$ for integer and fractional values of α and actual values (yearly) (b) New cases of human for integer and fractional values of α and actual values (monthly).



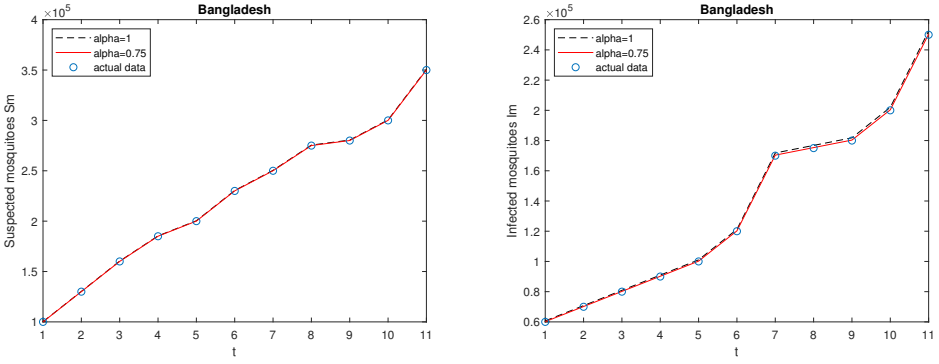
(a) (b)

Figure 5. Numerical simulation of (a) Reported dengue cases of human (b) Cumulative dengue cases of (2012–2022) with time (monthly) for integer and fractional values of α and actual values.



(a) (b)

Figure 6. Numerical simulation of (a) Suspected human $S_h(t)$ (b) Infected human $I_h(t)$ for different biting rates $b = 0.45, 0.50, 0.68$.



(a) (b)

Figure 7. Numerical simulation of (a) Suspected mosquitoes $S_m(t)$ (b) Infected mosquitoes $I_m(t)$ for different values of α and actual values (yearly).

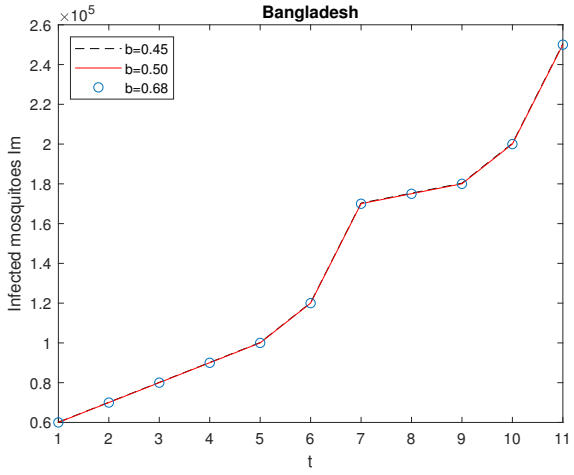


Figure 8. Numerical simulation of (a) Infected mosquitoes $I_m(t)$ for different values of biting rate $b = 0.45, 0.50, 0.68$.

10. Discussion and conclusions

The objective of this work is to understand, analyze and find the solution to the fractional epidemiological models. In this paper, we analyze a new fractional epidemic model for the transmission of dengue infection with a non-integer derivatives and are analysed using q-HATM. The existence of the solutions of the model is investigated by solving the fractional Grownwall's inequality using the Laplace transform approach. The positivity and boundedness of unique solutions are investigated. The basic reproduction number of the system is calculated by the next-generation method. We establish two equilibrium solutions, disease-free and endemic are obtained. Both local and global stability of the equilibria is investigated to depend on the magnitude of the basic reproduction ratio. Sensitivity analysis of R_0 is carried out to know the contribution of input factors in the results of R_0 and observed that ϕ_h, ϕ_m, β_h, β_m are the most critical parameters that highly contribute to the control and subsequent spread of dengue infection. We showed that the dengue infection is uniformly persistent in the system for $R_0 > 1$. Numerical simulations are carried out and dynamics of the populations are shown to vary for different values of α. We obtain feasible results for the dynamics of dengue infection with the variation of memory index α and suggest that the index of memory has a dominant influence on the system. We conclude that the fractional-order model can explore the dengue epidemic disease transmission model rather than the integer-order derivative models. We can suggest that fractional-order (index of memory) α and biting rate for humans and mosquitoes ϕ_h, ϕ_m can remarkably control and greatly decrease the level of disease in society.

DECLARATIONS

Ethics approval and consent to participate:
Not applicable.
Consent for publication:
Not applicable.

Availability of data and materials:
All data and materials used in this study are publicly available.

Competing Interest:
The authors declare that they have no competing interests.

Funding:
This work was supported by the Shanxi Science and Technology innovation team under grant no. (201805D131012-1), and the key projects of the Health Commission of Shanxi Province (2020XM18).

Authors' contributions:
The authors have read and approved the final manuscript.

Acknowledgments
The authors are thankful to the anonymous reviewers for the careful checking and suggestions that improved the presentation of the paper greatly. The authors thank the Chinese Government and the Complex Systems Research Centre, Shanxi University, for their support.

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