Research article

The effect of "fear" on two species competition

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Abstract: Non-consumptive effects such as fear of depredation, can strongly influence predator-prey dynamics. There are several ecological and social motivations for these effects in competitive systems as well. In this work we consider the classic two species ODE and PDE Lotka-Volterra competition models, where *one* of the competitors is "fearful" of the other. We find that the presence of fear can have several interesting dynamical effects on the classical competitive scenarios. Notably, for fear levels in certain regimes, we show novel bi-stability dynamics. Furthermore, in the spatially explicit setting, the effects of several spatially heterogeneous fear functions are investigated. In particular, we show that under certain integral restrictions on the fear function, a weak competition type situation can change to competitive exclusion. Applications of these results to ecological as well as sociopolitical settings are discussed, that connect to the "landscape of fear" (LOF) concept in ecology.

Keywords: reaction-diffusion system; Lotka-Volterra competition model; fear effect; bifurcation theory; ODE

1. Introduction

Fear, is defined as, "An unpleasant emotion caused by the belief that someone or something is dangerous" [1].

It is a complex emotion, that is critical as a safety measure, and can trigger the "fight or flight" response [2] - in particular it can change the way one acts, even when there is no threat present [3]. In predator-prey systems, this is most naturally observed among prey, due to their perceived threat of depredation [4]. This perception can lead to non-consumptive effects or trait-mediated interactions, which are behavioral, morphological or physiological changes in prey phenotype, due to this threat [4, 5]. Such effects are known to strongly influence predator-prey dynamics [6]. From a mathematical viewpoint, the effects of fear in predator-prey systems has been intensely investigated since the seminal work of Brown et al. [7], where optimal foraging theory is extended to consider a game theoretic setup,

played out by predator and prey, exhibiting stealth and fear, in which an animal follows a map or a "landscape of fear" (LOF), which describes its predation risk while it navigates the physical landscape. In recent work, Wang et al. [8], model fear of depredation, as a (predator) density dependent effect, that negatively effects the prey population. In essence, the prey's growth rate is modeled as a monotonically decreasing function of predator density. Dynamically, a key finding in [8] is that under the parametric restrictions of a Hopf bifurcation, an increase in the fear parameter (and prey's birth rate parameter) can alter the direction of a Hopf bifurcation from supercritical to subcritical. Thus, fear enables both supercritical and subcritical Hopf bifurcations, contrary to only the supercritical bifurcations found in classical predator-prey systems. In essence, the fear effect can change the fundamental cyclical patterns of predator-prey dynamics, leading to large scale ecological consequences [9]. Note, age/stage structure in the model can also have influence on the dynamics. For example, in [10] anti-predator sensitivity, as a consequence of fear, is introduced. Higher values of anti-predator sensitivity may cause predator extinction, thus adult prey persistence. However, lower values lead to co-existence, thus bi-stability is observed.

These results have since initiated a host of activities in diverse ecological scenarios such as when refuges are present [11, 12] or when the prey has tendencies to avoid predators [13]. Predator avoidance could cause its own cascade of dynamics, such as bats avoiding cave predators could fall prey to diurnal birds [14], or the green anolis lizard avoiding curly predatory tailed lizards could in turn introduce refuge competition [15]. A predator's response could also be influenced by interference pressures, for instance, via a Beddington-DeAngelis functional response [16]. Various works have considered the fear effect in case of group defense by the prey [10, 17]. It has been investigated in the context of cooperative and competitive systems within the larger predator-prey context. These include the fear effect when predators are cooperating, as earlier mentioned [14, 18] in the hunting process, or when they are hunting for competing prey [19]. These effects have been investigated in the three and multi-species settings as well [20, 21] where fear can damp population explosions [22]. Various authors have considered the fear effect in a stochastic setting [23] as well as a spatially explicit setting, in the context of taxis type movements, as well as pattern formation [24, 25]. It can also lead to chaotic dynamics [26]. However, the effect of fear has been far less investigated in classical monotone systems, such as purely cooperative or competitive two species systems - that are outside the predator-prey setting.

Competition among two species, typically modeled via the Lotka–Volterra competition model and its variants have been intensely investigated in the last few decades. These models take into account growth and inter/intraspecific competition [27], and predict well-observed states in biology of co-existence, competitive exclusion of one competitor, and bi-stability, and find diverse applications in ecology and invasion science [28–32]. There are several ecological motivations for competitors being fearful of each other. This is perhaps most naturally seen to occur with intraguild predation - a widespread phenomenon in many food webs, where competitors will kill and consume each other [33]. Recent evidence of non-consumptive effects exerted by intraguild predator mites (*Blattisocius dentriticus*) on their competitor (*Neoseiulus cucumeris*) show this can be an important factor in determining food web dynamics in biological control [34, 35]. However, there is strong evidence for fear in purely competitive two species systems *without* predatory effects. Barred owls (*Strix varia*) are a species of owl, native to eastern North America. They have expanded their range westward over the last century and are considered invasive in western North America. Currently, their range overlaps with the spotted owl (*Strix occidentalis*), which is native to the north west and western

North America. This has resulted in intense competition between the two species [36]. Barred owls exert a strong negative influence on spotted owls, threatening their possible competitive exclusion [37]. Field observations report frequent barred owl attacks on spotted owls, and even on surveyors imitating spotted owl calls [38]. There is also evidence of barred owls aggressively chasing spotted owls out of shared habitat - but *not* the opposite [39]. There is also other very recent empirical evidence to support such investigations. In [15] a series of 6 year long experiments are conducted in various Caribbean islands that aim to refute the theory of adaptive predation - which suggests that predators reduce dominant competitors, thus preventing competitive exclusion and enhancing coexistence in food webs. However, non-consumptive effects such as fear of depredation can have strongly influencing effects [15, 40–42]. [15] considers a series of experiments with two competing species of lizards, brown anolis (*Anolis sagrei*) that dwells on tree trunks, and green anolis (*Anolis smaragdinus*) that dwells on tree canopies. The experiments show that typically these species co-exist - due to a clear niche separation. However, the introduction of an intraguild predator, the curly tailed lizard (*Leiocephalus carinatus*) that dwells on the ground, causes (non-consumptive) fear driven effects. The brown anolis being fearful of possible depredation (as the lower half of tree trunks are within striking distance of the curly tailed lizard) moves upwards into the canopy, which is occupied by the green anolis. Herein interspecific competition intensifies leading to a loss of co-existence. However, what is most crucial in this study, is that the fecal analysis of the curly tail lizard shows that its diet included the brown anolis, in only 2 out of 51 samples examined. Thus the new dispersal pattern of the brown anolis and the "refuge competition" is driven strongly by a non-consumptive fear effect and not a consumptive one. Thus the brown anolis and the curly tail lizards are really competitors, as they have a strong overlap in dietary niche for several insects. However, the brown anolis is clearly fearful of the curly tailed lizard [15]. There is further evidence of non consumptive effects such as fear among competing aphid species, as well as competitors that feed on aphids [43, 44]. Such interplay between competition and predation has been investigated [45], where it is proposed that in many ecological processes, competition and predation are interlinked, and depending on niche overlap, one of them will dominate to drive the underlying dynamics. Such evidence clearly motivates considering fear dynamics into competitive two species model where one of the competitors is fearful of the other. Furthermore, one could look at more complex settings as well, consisting of several interacting predators and prey, across several trophic levels [46].

There are also several socio-economic-political settings, where pure competitors may be fearful of each other. Small/new businesses (as weaker competitors) may be fearful of large businesses (that are stronger/dominant competitors), due to their already large market share [47]. But large business may also be fearful of small local businesses, due to their familiarity with local nuances, that may yield competitive advantage at a small local scale [48] - Thus there could be strong spatial heterogeneity associated to the way in which competitors are fearful of each other. Fear is also conceivable among two competing political parties, where the weaker party on a national scale, may have a stronger voter bank at a regional scale [49] - This again is a clear example of spatial heterogeneity, in that a weaker competitor overall is dominant or exerts fear on a stronger competitor at a local/regional scale. Or perhaps two warring drug cartels, where the weaker cartel has certain local/territorial strongholds [50, 51] - within which they might be able to induce fear among the stronger cartel [50]. Such phenomenon becomes even more interesting in the spatially explicit case where this fear could be heterogeneous in the spatial domain of interest. This connects back to the LOF concept, where the

fear function is essentially the map that describes how the fear levels change as a species disperses over a physical landscape.

2. Model formulation

2.1. Recap of basic competition model

Consider the classical two species Lotka-Volterra ODE competition model,

$$
\begin{cases}
\dfrac{du}{dt} & = u(a_1 - b_1 u - c_1 v), \\[2mm]
\dfrac{dv}{dt} & = v(a_2 - b_2 v - c_2 u),
\end{cases}
\tag{2.1}
$$

where u and v are the population densities of two competing species, a_1 and a_2 are the intrinsic (per capita) growth rates, b_1 and b_2 are the intraspecific competition rates, c_1 and c_2 are the interspecific competition rates. All parameters considered are positive. The dynamics of this system are well studied [52]. We recap these briefly,

- $E_0 = (0, 0)$ is always unstable.

- $E_u = (\frac{a_1}{b_1}, 0)$ is globally asymptotically stable if $\dfrac{a_1}{a_2} > \max\left\{\dfrac{b_1}{c_2}, \dfrac{c_1}{b_2}\right\}$. Herein u is said to competitively exclude v.

- $E_v = (0, \frac{a_2}{b_2})$ is globally asymptotically stable if $\dfrac{a_1}{a_2} < \min\left\{\dfrac{b_1}{c_2}, \dfrac{c_1}{b_2}\right\}$. Herein v is said to competitively exclude u.

- $E^* = \left(\frac{a_1 b_2 - a_2 c_1}{b_1 b_2 - c_1 c_2}, \frac{a_2 b_1 - a_1 c_2}{b_1 b_2 - c_1 c_2}\right)$ exists when $b_1 b_2 - c_1 c_2 \neq 0$. The positivity of the equilibrium holds if $\frac{c_2}{b_1} < \frac{a_2}{a_1} < \frac{b_2}{c_1}$ and is globally asymptotically stable if $b_1 b_2 - c_1 c_2 > 0$. This is said to be the case of weak competition.

- If $b_1 b_2 - c_1 c_2 < 0$, then $E^* = \left(\frac{a_1 b_2 - a_2 c_1}{b_1 b_2 - c_1 c_2}, \frac{a_2 b_1 - a_1 c_2}{b_1 b_2 - c_1 c_2}\right)$ is unstable as a saddle. In this setting, one has initial condition dependent attraction to either $E_u(\frac{a_1}{b_1}, 0)$ or $E_v(0, \frac{a_2}{b_2})$. This is the case of strong competition.

We proceed by considering the effects of fear on the classical model (2.1), when *one* of the competitors is fearful of the other.

2.2. The fear model for two competitors

Species diffusion is ubiquitous in spatial ecology [53]. Species disperse to find mates, food and shelter [54]. Such movement is modeled often via reaction diffusion systems [53]. The spatially explicit Lotka-Volterra model, particularly in the case of heterogeneity in spatial resources has been intensely investigated, [27, 55–64]. Herein, we consider the spatially explicit version of the Lotka-Volterra competition model for two competing species u, v, where the competitor v is fearful of u. We model the fear effect as in [8], where the growth rate of the fearful competitor v, is not constant but rather density dependent. Essentially, the growth rate is decreased by a factor $\approx \frac{1}{1+ku}$, where $k \geq 0$ is

a fear coefficient. Thus a higher density of the competitor u increases the fear in v. When $k = 0$, the assumption is there is no fear and one recovers the classical spatially explicit model. If fear is present, we obtain the following spatially explicit model,

$$
\begin{aligned}
u_t &= d_1 \Delta u + a_1 u - b_1 u^2 - c_1 uv, \quad x \in \Omega, \\
v_t &= d_2 \Delta v + \frac{a_2 v}{1 + ku} - b_2 v^2 - c_2 uv, \quad x \in \Omega, \\
\frac{\partial u}{\partial v} &= \frac{\partial v}{\partial v} = 0, \quad \text{on} \quad \partial\Omega, \\
u(x, 0) &= u_0(x) > 0, \quad v(x, 0) = v_0(x) > 0.
\end{aligned}
\tag{2.2}
$$

Here $u(x, t), v(x, t)$ are the densities of two competing species, where v is also fearful of u. The species diffuse in a bounded domain $\Omega \subset \mathbb{R}^n$, with dispersal speeds d_1 and d_2, respectively. We prescribe Neumann boundary conditions, modeling the effect that the species do not immigrate or emigrate from the domain Ω. Also the initial data is chosen to be positive. We now proceed to study the dynamics of the above model, when various forms of fear are considered. We divide the dynamical analysis into two components, $d_1 = d_2 = 0$, the ODE case discussed next, after which we consider the PDE case $d_1 > 0, d_2 > 0$, where we allow a spatially heterogenous fear function as well, so $k = k(x)$.

3. The ODE case

3.1. The case of v fearing u

We consider the dynamics of (2.2), when $d_1 = d_2 = 0$, and we are in a bonafide ODE setting, where the competitor v is fearful of u,

$$
\begin{aligned}
\frac{du}{dt} &= a_1 u - b_1 u^2 - c_1 uv, \\
\frac{dv}{dt} &= \frac{a_2 v}{1 + ku} - b_2 v^2 - c_2 uv.
\end{aligned}
\tag{3.1}
$$

3.1.1. Existence

The nullclines associated with the system (3.1) are

$$
u(a_1 - b_1 u - c_1 v) = 0 \quad \text{and} \quad v\left(\frac{a_2}{1 + ku} - b_2 v - c_2 u\right).
$$

Hence, the boundary equilibrium points are obtained by substituting $u = 0$ and $v = 0$ in the above equations of the nullclines, respectively. Denote the boundary equilibrium points as $\widehat{E}_1 = (0, 0)$, $\widehat{E}_2 = (\frac{a_1}{b_1}, 0)$ and $\widehat{E}_3 = (0, \frac{a_2}{b_2})$.

For the interior equilibrium, substitute $u^* = \frac{a_1}{b_1} - \frac{c_1}{b_1} v^*$ in the second nullcline equation, i.e.,

$$
\frac{a_2}{1 + k\left(\frac{a_1}{b_1} - \frac{c_1}{b_1} v^*\right)} - b_2 v^* - c_2\left(\frac{a_1}{b_1} - \frac{c_1}{b_1} v^*\right) = 0.
$$

On simplification, we have that v^* solves a quadratic equation of the form $A_1(v^*)^2 + B_1 v^* + C_1 = 0$, where

$$
\begin{aligned}
A_1 &= c_1 k (b_2 b_1 - c_1 c_2), \\
B_1 &= b_1(c_1 c_2 - b_2 b_1) - a_1 k(b_2 b_1 - 2c_1 c_2), \\
C_1 &= b_1(a_2 b_1 - a_1 c_2) - a_1^2 c_2 k.
\end{aligned}
\tag{3.2}
$$

Let

$$
v_{1,2}^* = \frac{-B_1 \pm \sqrt{B_1^2 - 4A_1 C_1}}{2A_1}
\tag{3.3}
$$

be the two roots of above quadratic equation. Define discriminant as $\Delta_1 = B_1^2 - 4A_1 C_1$. WLOG assume $v_1^* < v_2^*$. Moreover, consider the following parametric restriction

$$
b_1(a_1 c_2 - a_2 b_1) + a_1^2 c_2 k < \left[(b_2 b_1 - c_1 c_2)(2a_1 k - b_1) - a_1 k c_1 c_2 \right] \frac{a_1}{c_1}.
\tag{3.4}
$$

We can prove the existence of a positive equilibrium point $\widehat{E}_4$ with the choice of specific parameters. Let us use Descartes's rule of sign to establish some sufficient conditions for the existence of one or two positive equilibrium points.

Two positive equilibrium points: Under the assumption $A_1 > 0, B_1 < 0, C_1 > 0$, i.e., $b_2 b_1 > 2c_1 c_2 > c_1 c_2$, $k < \frac{1}{a_1^2 c_2}\left(b_1^2 a_2 - a_1 c_2 b_1\right)$ and $\Delta_1 > 0$, we have two positive roots. In order to claim that these two roots correspond to two positive interior equilibria, we need some extra assumption given by:

$$
v_1^* < v_2^* := \frac{-B_1 \pm \sqrt{B_1^2 - 4A_1 C_1}}{2A_1} < \frac{a_1}{c_1} \implies -C_1 < (A_1 \frac{a_1}{c_1} + B_1)\frac{a_1}{c_1}.
$$

Hence, if $b_1(a_1 c_2 - a_2 b_1) + a_1^2 c_2 k < [(b_2 b_1 - c_1 c_2)(2a_1 k - b_1) - a_1 k c_1 c_2]\frac{a_1}{c_1}$, we have two positive interior equilibrium points $\widehat{E}_4 = (u_i^*, v_i^*)$ for $i = 1, 2$.

Remark 1. The existence of two positive internal equilibrium points can also be observed under the assumption $A_1 > 0, B_1 < 0, C_1 > 0$, and $\Delta_1 > 0$, which ensures that

$$
v_2^* = \frac{-B_1 + \sqrt{\Delta_1}}{2A_1} < -\frac{B_1}{A_1} < \frac{a_1}{c_1},
$$

and there must be two positive internal equilibria. Similarly, under the assumption $A_1 < 0, B_1 > 0, C_1 < 0$, and $\Delta_1 > 0$, we have the existence of two positive internal equilibria.

One positive equilibrium point: Under the assumption $A_1 > 0, B_1 < 0$, i.e., $b_2 b_1 > 2c_1 c_2 > c_1 c_2$, we have at least one positive root of the quadratic equation. If $C_1 < 0$, which is $k > \frac{1}{a_1^2 c_2}\left(b_1^2 a_2 - a_1 c_2 b_1\right)$ along with (3.4) gives existence of one positive equilibrium point $\widehat{E}_4 = (u^*, v^*)$. Moreover, if $C_1 > 0$, which is $k < \frac{1}{a_1^2 c_2}\left(b_1^2 a_2 - a_1 c_2 b_1\right)$, $\Delta_1 > 0$, and $v_1^* < \frac{a_1}{c_1}$ and $v_2^* > \frac{a_1}{c_1}$, then we have existence of one positive equilibrium point $\widehat{E}_4 = (u^*, v^*)$.

We formulate all these restrictions as an existence theorem:

Theorem 3.1. *For the given ODE system* (3.1)*, we always have three boundary equilibrium points, namely* $\widehat{E}_1 := (0,0), \widehat{E}_2 := (\frac{a_1}{b_1}, 0)$ *and* $\widehat{E}_3 := (0, \frac{a_2}{b_2})$. *For the case of two positive interior equilibrium points* $\widehat{E}_4 = \left(u_i^*, v_i^*\right)_{i=1,2}$, *we have the following parametric restrictions:*

$$b_2 b_1 > 2c_1 c_2 > c_1 c_2, \quad k < \frac{1}{a_1^2 c_2}\left[b_1^2 a_2 - a_1 c_2 b_1\right], \quad \Delta_1 > 0, \quad \& \quad (3.4) \ holds \ true.$$

Lastly, for the case of one positive interior equilibrium point $\widehat{E}_4 = (u^*, v^*)$*, we have either one of the following choices of parametric restrictions:*

1) $b_2 b_1 > 2c_1 c_2 > c_1 c_2, \quad k > \frac{1}{a_1^2 c_2}\left(b_1^2 a_2 - a_1 c_2 b_1\right)$ *and* (3.4) *holds true.*

2) $b_2 b_1 > 2c_1 c_2 > c_1 c_2, \quad k < \frac{1}{a_1^2 c_2}\left(b_1^2 a_2 - a_1 c_2 b_1\right), \Delta_1 > 0$ *and* $v_1^* < \frac{a_1}{c_1}$ *and* $v_2^* > \frac{a_1}{c_1}$.

3) $b_2 b_1 < c_1 c_2 < 2c_1 c_2, \quad k > \frac{1}{a_1^2 c_2}\left(b_1^2 a_2 - a_1 c_2 b_1\right), \Delta_1 > 0$ *and* $v_1^* < \frac{a_1}{c_1}$ *and* $v_2^* > \frac{a_1}{c_1}$,

where v_i^* *are the roots of the quadratic equation defined as* (3.3)*.*

We now provide several lemmas, so that we can compare the effect of fear to the classical competition case.

Lemma 3.2. *Consider the given ODE system* (3.1)*, with* $k = 0$*, such that the system is in the weak competition setting with* $b_2 b_1 > 2c_1 c_2$*. Then for a fear coefficient* k *such that* $k > k_c = \frac{1}{a_1^2 c_2}\left(b_1^2 a_2 - a_1 c_2 b_1\right)$, $(\frac{a_1}{b_1}, 0)$ *is globally asymptotically stable. That is,* u *will competitively exclude* v.

Remark 2. For the existence of one positive interior equilibrium point for the given system (3.1), we have either one of the following choices of parametric restrictions: For weak competition,

$$b_2 b_1 > 2c_1 c_2 > c_1 c_2 \quad \& \quad \frac{c_2}{b_1} < \frac{a_2}{a_1} < \frac{b_2}{c_1}, \tag{3.5}$$

and for strong competition,

$$b_2 b_1 < c_1 c_2 < 2c_1 c_2 \quad \& \quad \frac{c_2}{b_1} > \frac{a_2}{a_1} > \frac{b_2}{c_1}. \tag{3.6}$$

3.1.2. Linear stability analysis

We next perform stability analysis on the equilibrium points of system (3.1). The Jacobian matrix of system (3.1) is given by

$$J(u^*, v^*) = \begin{pmatrix} a_1 - 2b_1 u^* - c_1 v^* & -c_1 u^* \\ -\dfrac{a_2 k v^*}{(ku^* + 1)^2} - c_2 v^* & \dfrac{a_2}{ku^* + 1} - 2b_2 v^* - c_2 u^* \end{pmatrix}. \tag{3.7}$$

We state the following lemmas:

Lemma 3.3. $\widehat{E}_1$ *is unstable.*

Proof. On evaluating (3.7) at $\widehat{E}_1$, we have

$$J(\widehat{E}_1) = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}.$$

Being a triangular matrix, we know that the above matrix has two positive eigenvalues a_1 and a_2. Hence, the equilibrium point $\widehat{E}_1$ is unstable. □

Lemma 3.4. $\widehat{E}_2$ *is locally stable iff* $k > \dfrac{a_2 b_1^2 - c_2 a_1 b_1}{a_1^2 c_2}$.

Proof. We again evaluate (3.7) at $\widehat{E}_2$ and obtain

$$J(\widehat{E}_2) = \begin{pmatrix} -a_1 & -\dfrac{c_1 a_1}{b_1} \\ 0 & \dfrac{a_2 b_1}{b_1 + k a_1} - \dfrac{c_2 a_1}{b_1} \end{pmatrix}.$$

Being a triangular matrix, the above matrix has two eigenvalues, $\lambda_1 = -a_1$ and $\lambda_2 = \frac{a_2 b_1}{b_1 + k a_1} - \frac{c_2 a_1}{b_1}$. As λ_1 is always negative, if we can show that λ_2 is negative, we are done. We make the assumption that,

$$k > \frac{a_2 b_1^2 - c_2 a_1 b_1}{a_1^2 c_2} \iff \lambda_2 = \frac{a_2 b_1}{b_1 + k a_1} - \frac{c_2 a_1}{b_1} < 0.$$

Therefore, the boundary equilibrium point $\widehat{E}_2$ is locally stable. □

Local stability of $\widehat{E}_2$ actually implies global stability, we can see this via a simple geometric argument.

Lemma 3.5. $\widehat{E}_2$ *is globally stable if* $k > k_c = \dfrac{a_2 b_1^2 - c_2 a_1 b_1}{a_1^2 c_2}$.

Proof. Consider the nullclines of u and v, where,

$$v = \frac{1}{c_1}(a_1 - b_1 u) \quad \& \quad v = \frac{1}{b_2}\left(\frac{a_2}{1 + ku} - c_2 u\right).$$

In order to establish the global stability of $\widehat{E}_2$, via the geometry of the nullclines, it suffices to show that

$$\frac{a_1 - b_1 u}{c_1} > \frac{1}{b_2}\left(\frac{a_2}{1 + ku} - c_2 u\right)$$

when $u = \frac{a_1}{b_1}$, i.e., when $k > \frac{a_2 b_1^2 - c_2 a_1 b_1}{a_1^2 c_2}$. Herein, the v-nullcline lies completely below the u-nullcline, and via the convexity of the v-nullcline, it lies completely below the straight line connecting its v and u intercepts - which lies completely below the u-nullcline. Now, via the standard theory of competition and a comparison argument, where v is compared to the $\tilde{v}$ that is a solution to the straight line nullcline connecting the v and u intercepts of the v-nullcline, we have the global stability of $\widehat{E}_2$. □
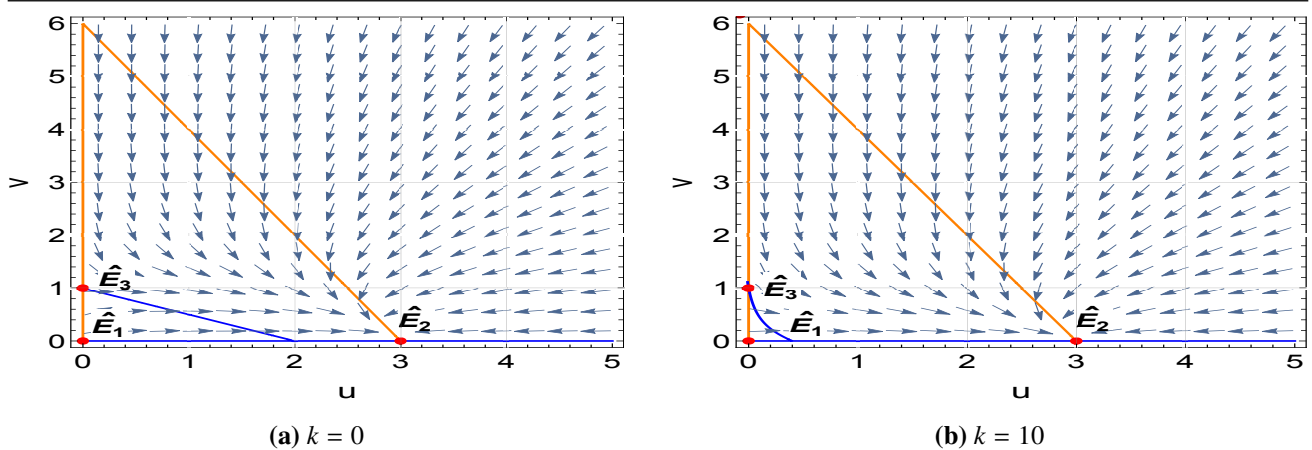
Figure 1. Phase plots showing dynamics under competition exclusion parametric restriction. The orange curve is the u-nullcline and blue curve is the v-nullcline. Here, $(u^*, 0)$ always wins. Parameters used are $a_1 = 3, a_2 = 1, b_1 = 1, b_2 = 1, c_1 = 0.5, c_2 = 0.5$ and this choice meets the parametric constraint of the Lemma 3.4.

Lemma 3.6. *The equilibrium point $\widehat{E}_3$ is locally stable iff $a_1 b_2 < c_1 a_2$.*

Proof. Similar evaluation of (3.7) at $\widehat{E}_3$ yields

$$J(\widehat{E}_3) = \begin{pmatrix} a_1 - \dfrac{c_1 a_2}{b_2} & 0 \\ -\dfrac{k a_2^2}{b_2} - \dfrac{c_2 a_2}{b_2} & -a_1 \end{pmatrix}.$$

Being a triangular matrix, the above matrix has two eigenvalues, $\lambda_1 = -a_1$ and $\lambda_2 = a_1 - \dfrac{c_1 a_2}{b_2}$. Under the assumed parametric restriction,

$$a_1 b_2 < c_1 a_2 \iff \lambda_2 < 0.$$

Hence, the equilibrium point $\widehat{E}_3$ is locally stable.

$\square$

Remark 3. This is essentially the parametric restriction for the case when v excludes u, see Figure 4. Thus under this parametric restriction, $\widehat{E}_3$ is locally stable for all k, but globally stable only for certain k values.

Lemma 3.7. *The interior equilibrium $\widehat{E}_4$ exists and is locally stable if $0 < k < \frac{1}{a_2}\left(\frac{b_1 b_2}{c_1} - c_2\right)$.*

Proof. On evaluating (3.7) again at $\widehat{E}_4$, we have

$$J(\widehat{E}_4) = \begin{pmatrix} -b_1 u^* & -c_1 u^* \\ -\dfrac{k a_2 v^*}{(1 + k u^*)^2} - c_2 v^* & -b_2 v^* \end{pmatrix}.$$

For the local stability of $\widehat{E}_4$, it is enough to show that $\mathrm{Tr}(J(\widehat{E}_4)) < 0$ and $\det(J(\widehat{E}_4)) > 0$. Simple computations yield

$$\mathrm{Tr}(J(\widehat{E}_4)) = -b_1 u^* - b_2 v^* < 0,$$

and

$$\det(J(\widehat{E}_4)) = b_1 b_2 u^* v^* - c_1 u^* \left(\frac{k a_2 v^*}{(1 + k u^*)^2} + c_2 v^* \right) = u^* v^* \left\{ b_1 b_2 - c_1 \left(\frac{k a_2}{(1 + k u^*)^2} + c_2 \right) \right\}.$$

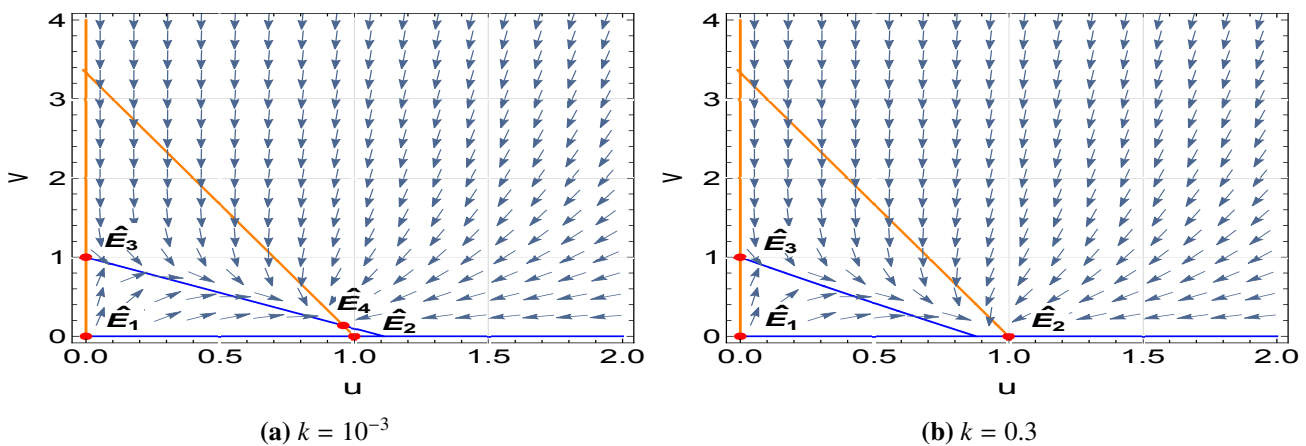Note that

$$k a_2 > \frac{k a_2}{(1 + k u^*)^2}.$$

Therefore, if k is chosen such that, $0 < k < \frac{1}{a_2} \left(\frac{b_1 b_2}{c_1} - c_2 \right)$, then,

$$b_1 b_2 > c_1 (k a_2 + c_2) > c_1 \left[\frac{k a_2}{(1 + k u^*)^2} + c_2 \right] \implies \det(J(\widehat{E}_4)) > 0,$$

and the result follows.

$\square$

(a) $k = 10^{-3}$ **(b)** $k = 0.3$

Figure 2. Phase plots showing various dynamics under weak competition parametric restriction. Here (u^*, v^*) is a sink and both $(0, v^*)$ and $(u^*, 0)$ are saddles in (A). In (B) there is competition exclusion and $(u^*, 0)$ wins. The orange curve is the u-nullcline and blue curve is the v-nullcline. Parameters used are $a_1 = 1, a_2 = 2, b_1 = 1, b_2 = 2, c_1 = 0.3, c_2 = 1.8$ and this choice meets the parametric constraint of the Lemma 3.7.

Lemma 3.8. *The interior equilibrium $\widehat{E}_4$ exists and is a saddle if*

$$\frac{b_2 b_1}{c_1} - c_2 < \frac{k a_2 b_1^2}{(b_1 + k a_1)^2}.$$

Proof. In order to claim that the interior equilibrium $\widehat{E}_4$ is a saddle, it is enough to show that $\text{Tr}(J(\widehat{E}_4)) < 0$ and $\det(J(\widehat{E}_4)) < 0$. We have that

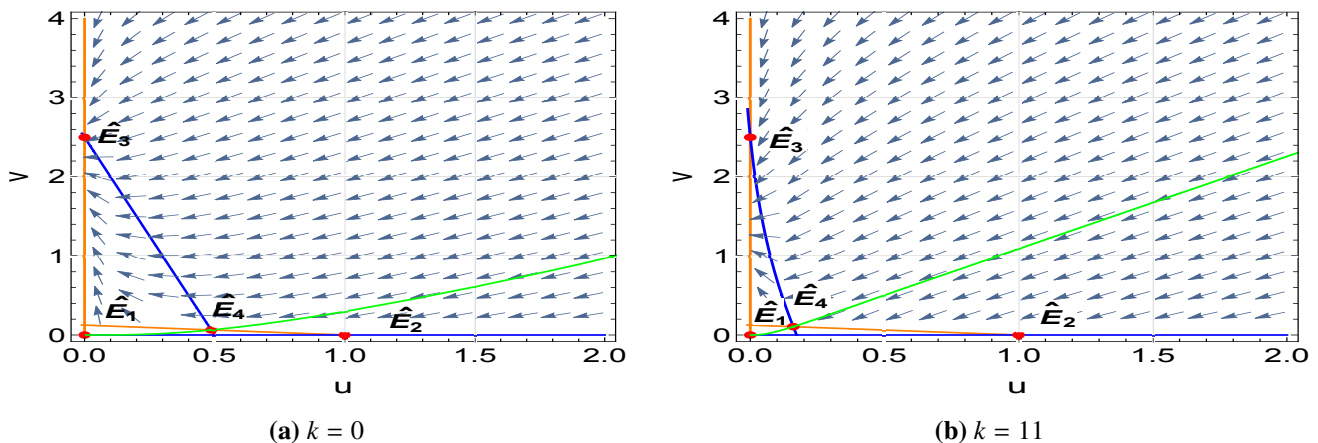$$\text{Tr}(J(\widehat{E}_4)) = -b_1 u^* - b_2 v^* < 0,$$

and

$$\det(J(\widehat{E}_4)) = b_1 b_2 u^* v^* - c_1 u^* \left(\frac{k a_2 v^*}{(1 + k u^*)^2} + c_2 v^* \right)$$

$$= u^* v^* \left\{ b_1 b_2 - c_1 \left(\frac{ka_2}{(1 + ku^*)^2} + c_2 \right) \right\}.$$

Under the assumption and density of reals, we have

$$\frac{b_2 b_1}{c_1} - c_2 < \frac{ka_2}{(1 + \frac{ka_1}{b_1})^2} < \frac{ka_2}{(1 + ku^*)^2} \implies \det(J(\widehat{E}_4)) < 0.$$

Hence, $\widehat{E}_4$ is a saddle. □

(a) $k = 0$ **(b)** $k = 11$

Figure 3. Phase diagrams showing dynamics under strong competition parametric restriction. The orange curve is the u-nullcline, blue curve is the v-nullcline and separatrix/stable manifold is in green. Here, (u^*, v^*) is saddle. Parameters used are $a_1 = 0.5, a_2 = 2, b_1 = 0.5, b_2 = 0.8, c_1 = 4, c_2 = 4$ and this choice meets the parametric constraint of the Lemma 3.8.

Lemma 3.9. *Consider the system* (3.1). *For any given $k > 0$ such that, there exist two positive interior equilibria, a sink and a saddle.*

Proof. Theorem 3.1 provides conditions under which two positive interior equilibria exists. Since stability has to alternate by standard theory [65] for planar systems, one of the equilibrium is stable while the other is unstable. Instability as a source is impossible due to the lack of periodic dynamics in the system via Lemma 3.10. Thus the unstable equilibrium must be a saddle. This proves the lemma. □

Lemma 3.10. *Consider the ODE system* (3.1). *There do not exist any periodic orbits for the system, for any values of the fear parameter k.*

Proof. Consider the function $\phi(u, v) = \frac{1}{uv}$ where u and v are both non-zero. Let,

$$F_1(u, v) = a_1 u - b_1 u^2 - c_1 uv,$$
$$F_2(u, v) = \frac{a_2 v}{1 + ku} - b_2 v^2 - c_2 uv.$$

Then we have

$$
\begin{aligned}
\frac{\partial(F_1\phi)}{\partial u} + \frac{\partial(F_2\phi)}{\partial v} &= \frac{\partial}{\partial u}\left(\frac{1}{uv}(a_1 u - b_1 u^2 - c_1 uv)\right) + \\
&\quad \frac{\partial}{\partial v}\left(\frac{1}{uv}\left(\frac{a_2 v}{1+ku} - b_2 v^2 - c_2 uv\right)\right) \\
&= -\frac{b_1}{v} - \frac{b_2}{u} < 0.
\end{aligned}
$$

The result follows by application of the Dulac criterion [65]. □

Corollary 1. *Consider the ODE system* (3.1)*, with the fear term modeled as* $\dfrac{a_2 v}{1+kF(u)}$*, where* $F(u) \in C^{0,1}$*. There do not exist any periodic orbits for the system, for any values of the fear parameter* k*.*

Proof. The result follows via direct application of the Dulac function from Lemma 3.10. □

(a) $k = 0$ **(b)** $k = 15$
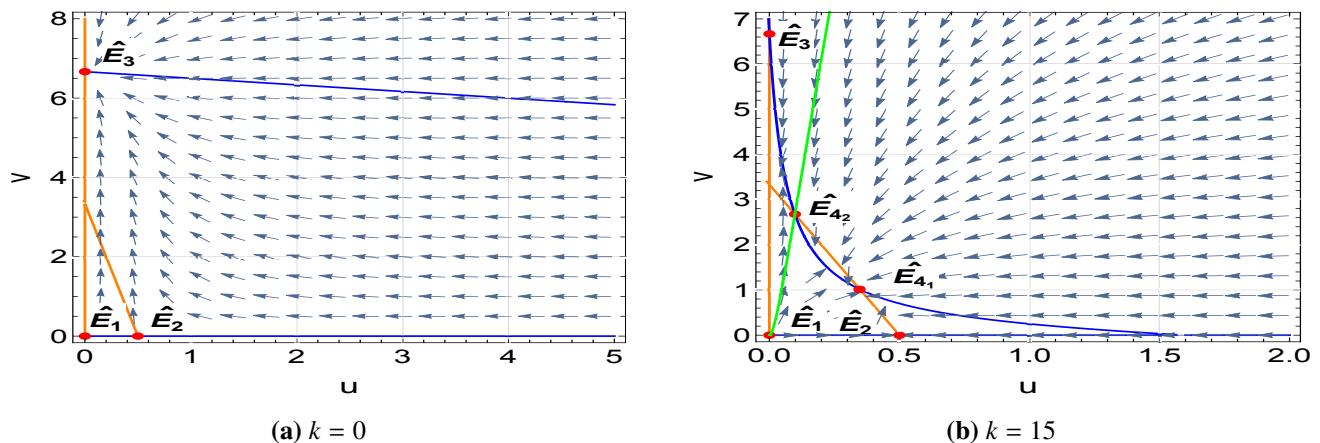
Figure 4. Phase plot showing competition exclusion in (*A*) when fear is absent in both competing species where $(0, v^*)$ wins. In (*B*), we observe the occurrence of two positive interior equilibria when $k = 15$. The orange curve is the u-nullcline, blue curve is the v-nullcline and separatrix/stable manifold is in green. Parameters used are $a_1 = 1, a_2 = 2, b_1 = 2, b_2 = 0.3, c_1 = 0.3, c_2 = 0.05$ and this choice meets the parametric constraint of the Lemma 3.9.

Remark 4. Some observations:

1) From Figure 4, we observe that when there is no fear, then $(0, v^*)$ is globally stable. For a sufficiently large level of fear in species v, a bi-stability situation is created. That is, for a certain initial data, species u is completely excluded by v and initial data is attracted to the co-existence state (See Figure 4).

2) If $(0, v^*)$ is globally attracting, a much higher level of fear in species v (≈ 200) can change the dynamics to a strong competition type case. However most data in this setting is attracted to (u^*, v^*). For initial data (u_0, v_0) to be attracted to $(0, v^*)$, we would need $v_0 \gg 1, u_0 \ll 1$.

3) Lemma 3.10 shows the lack of periodic orbits in (3.1), which could be interpreted as the inability of the fear factor to destabilize a steady state through a Hopf bifurcation to cyclical dynamics. This result could be extended to a wider class of fear models if fear in the second competitor was modeled as $\frac{a_2 v}{1 + k F(u)}$, where $F(u)$ was a general density dependent fear effect, not necessarily linear. See Corollary 1 which shows non existence of periodic orbits. Thus the result can be extended to a more general class of models than (3.1).

4) One could consider the case of u fearing v, but since the fear model is symmetric, most of the results will be similar. Thus we do not consider this case in detail, rather relegate a few technicalities to the appendix Section A.1, that rigorously show that the case of u fearing v and v fearing u, are indeed models that are symmetric.

3.2. Bifurcation Analysis

A bifurcation is said to occur in a dynamical system when the behavior of solutions changes when a parameter is varied. Bifurcation analysis is useful in understanding and measuring these qualitative changes as the system switches from stable to unstable and vice-versa.

3.2.1. Saddle-node bifurcation

The following theorem is connected to the existence of a saddle-node bifurcation for the growth rate a_1 when there is no fear effect in competitor u.

Theorem 3.11. *The ODE system* (3.1) *undergoes a saddle-node bifurcation around* $\widehat{E_4}$ *at* $a_1 = a_1^* = \frac{2(ku^*+1)^2 \left(b_1 u^* (2b_2 v^* + c_2 u^*) + b_2 c_1 v^{*2} \right) - a_2 (2b_1 u^* (ku^*+1) + c_1 v^* (2ku^*+1))}{(ku^*+1)((ku^*+1)(2b_2 v^* + c_2 u^*) - a_2)}$ *and when* $c_1 > \frac{a_2 (2b_2 v^* + c_2 u^*)}{u^* v^* \left(ka_2 + (1+ku^*)^2 c_2 \right)}$.

Proof. We shall use the Sotomayor's Theorem [65] to show the occurrence of a saddle-node bifurcation at $a_1 = a_1^*$. At $a_1 = a_1^*$, we can have $\det(J) = 0$ and for $c_1 > \frac{a_2 (2b_2 v^* + c_2 u^*)}{u^* v^* \left(ka_2 + (1+ku^*)^2 c_2 \right)}$, $\text{Tr}(J) < 0$. This implies that, $\det(J)$ admits a zero eigenvalue. Now let $P = (p_1, p_2)^T$ and $Q = (q_1, q_2)^T$ be the eigenvectors of J and J^T corresponding to the zero eigenvalue respectively.

We have that, $P = \left(\frac{-B}{A}, 1 \right)^T$ and $Q = \left(\frac{-A}{F}, 1 \right)^T$

where $A = a_1 - 2b_1 u^* - c_1 v^*$, $B = -c_1 u^*$, $F = -\frac{a_2 k v^*}{(1 + ku^*)^2} - c_2 v^*$.

Now, let $X = (X_1, X_2)^T$ where

$$X_1 = a_1 u - b_1 u^2 - c_1 uv,$$
$$X_2 = \frac{a_2 v}{1 + ku} - b_2 v^2 - c_2 uv.$$

Furthermore,

$$Q^T X_{a_1}(\widehat{E_4}, a_1^*) = \left(\frac{-A}{F}, 1 \right) (u^*, 0)^T = -\frac{Au^*}{F} \neq 0$$

and

$$Q^T [D^2 X(\widehat{E_4}, a_1^*)(P, P)] =$$
$$\left(\frac{-A}{F}, 1 \right) \left(\frac{2B}{A} (b_1 dfrac{B}{A} + c_1), \frac{2a_2 kB}{A(1 + ku^*)^2} \left[1 - \frac{kv^*}{1 + ku^*} \right] - \frac{BC_2}{A} - b_2 \right)^T \neq 0.$$

Therefore by the Sotomayor's theorem system (3.1) undergoes a saddle-node bifurcation at $a_1 = a_1^*$ around $\widehat{E}_4$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

Remark 5. Consider the case of two interior equilibria, such as in Figure 4. Then decreasing the fear coefficient k results in a saddle-node bifurcation, where the interior equilibrium (the saddle $\widehat{E}_4$ and the node $\widehat{E}_5$) collide and disappear, resulting in the boundary equilibrium $\widehat{E}_2 = (u^*, 0)$ becoming globally asymptotically stable. Thus in this setting, a certain critical level of "fear" can maintain a co-existence state, but fear less than this level takes the system back to a competitive exclusion type scenario. This transition occurs via a saddle-node bifurcation. This can be rigorously proven by adopting the methods of Theorem 3.11, to the parameter k instead of using a_1.

3.2.2. Transcritical bifurcation

Theorem 3.12. *The model* (3.1) *undergoes a transcritical bifurcation around* $\widehat{E}_3$ *when* $a_1 = a_1^* = \dfrac{c_1 a_2}{b_2}$ *and* $c_1 \neq \dfrac{b_1 b_2}{ka_2 + c_2}$.

Proof. The Jacobian matrix for system (3.1) evaluated at $\widehat{E}_3$ with $a_1^* = \dfrac{c_1 a_2}{b_2}$ is given as

$$J = \begin{pmatrix} 0 & 0 \\ -\dfrac{a_2}{b_2}\left(a_2 k + c_2\right) & -a_2 \end{pmatrix}. \tag{3.8}$$

The corresponding eigenvalues to the Jacobian of (3.1) in Eq (3.8) are $\lambda_1 = 0$ and $\lambda_2 = -a_2$. Clearly, there is a zero eigenvalue at $a_1 = a_1^* = \dfrac{c_1 a_2}{b_2}$. Next, we let $W = (w_1, w_2)^T$ and $Z = (z_1, z_2)^T$ represent the eigenvectors related to the zero eigenvalue of the matrices J and J^T respectively.

We obtain $W = \left(-\dfrac{b_2}{ka_2 + c_2}, 1\right)^T$ and $Z = (1, 0)^T$. Now, let $X = (X_1, X_2)^T$ as already defined. Presently, we validate the transversality conditions using the Sotomayor's theorem [65]. Now,

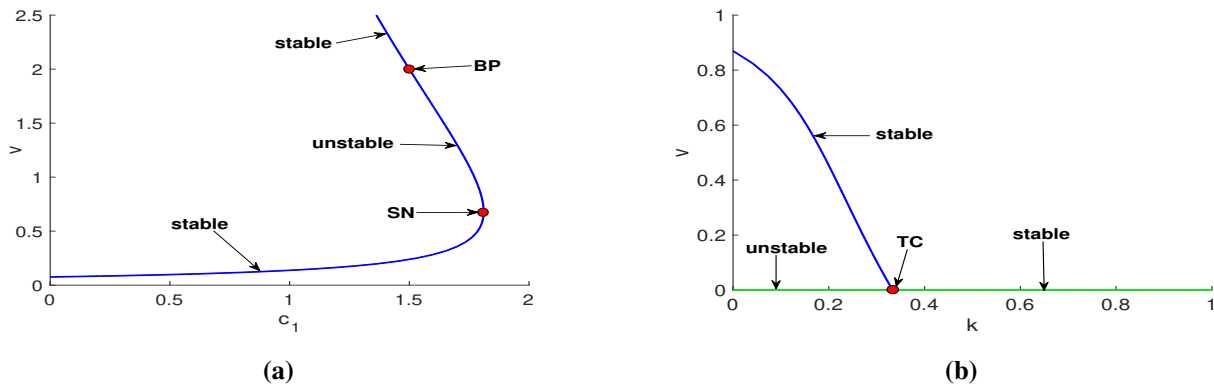$$Z^T X_{a_1}(\widehat{E}_3, a_1^*) = (1, 0)\,(0, 0)^T = 0.$$

Also,

$$Z^T \left[DX_{a_1}\left(\widehat{E}_3, a_1^*\right) W\right] = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

$$= -\dfrac{b_2}{ka_2 + c_2} \neq 0$$

and

$$Z^T \left[D^2 X\left(\widehat{E}_3, a_1^*\right)(W, W)\right] = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \dfrac{2b_2}{ka_2 + c_2}\left(c_1 - \dfrac{b_1 b_2}{ka_2 + c_2}\right) \\ \dfrac{2a_2^2 k^2 b_2}{(ka_2 + c_2)^2} \end{pmatrix}$$

$$= \dfrac{2b_2}{ka_2 + c_2}\left(c_1 - \dfrac{b_1 b_2}{ka_2 + c_2}\right) \neq 0.$$

Therefore by the Sotomayor's theorem system (3.1) experiences a transcritical bifurcation at $a_1 = a_1^* = \dfrac{c_1 a_2}{b_2}$ around $\widehat{E}_3$. $\square$


Figure 5. Bifurcation diagrams showing the impacts of the interspecific competition rate c_1 and the fear effect parameter k. In (A) we observe the occurrence of a saddle-node bifurcation at $c_1 = c_1^* = 1.8107$. The parameters used are $a_2 = 3.6, b_1 = 1, b_2 = 1.8, a_1 = 3, c_2 = 0.5, k = 0.4$. A transcritical bifurcation is also observed in (B) at $k = k^* = 0.3333$. The parameters are chosen as $a_2 = 1, a_1 = 2, b_1 = 1, b_2 = 1$ and $c_2 = 0.3, c_1 = 1.8$. (Note: TC=Transcritical point, SN = Saddle-Node point, BP = Branch Point.)
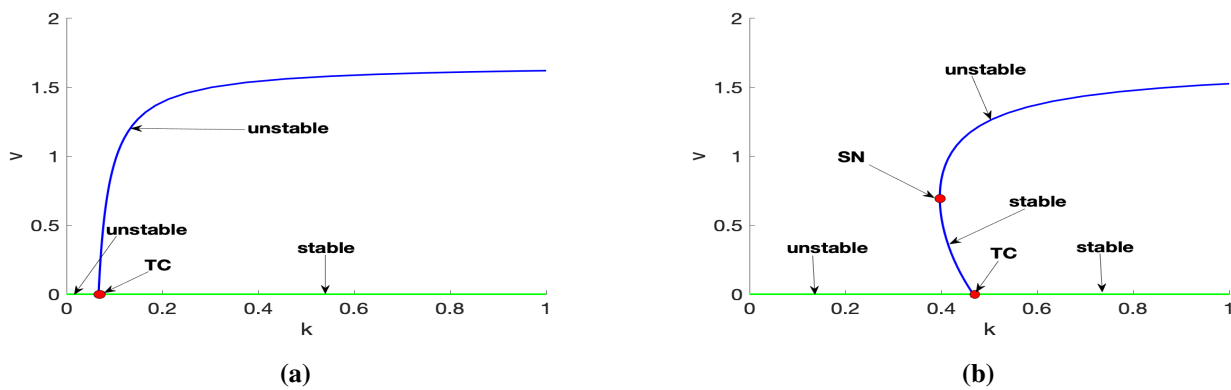
Figure 6. Bifurcation diagrams showing the impact of the fear parameter k with different parameter sets. The parameters are chosen as $a_1 = 3, b_1 = 1, c_2 = 0.5, c_1 = 1.8$. In (A), $a_2 = 1.8, b_2 = 1$ and a transcritical bifurcation is observed at $k = k^* = 0.0666$. In (B), $a_2 = 3.6, b_2 = 1.8$ and we observe the occurrence of a saddle-node at $k = k^* = 0.3966$ and a transcritical bifurcation at $k = k^* = 0.4666$. (Note: TC = Transcritical point, SN = Saddle-Node point.)

Remark 6. Consider the case of one interior equilibrium, such as in Figure 2 - that is, we are in the weak competition case, when there is no fear or $k = 0$. Now increasing the fear coefficient k, results in a transcritical bifurcation, where the interior equilibrium (the node $\widehat{E}_4$) and the boundary saddle equilibrium $\widehat{E}_2 = (u^*, 0)$ collide, exchange stability, after which $\widehat{E}_4$ now moves to the 4th quadrant,

while $\widehat{E}_2$ becomes globally asymptotically stable. Thus in this setting, a certain critical level of "fear" can move the system from a weak competition setting of coexistence to a competitive exclusion type scenario. This transition occurs via a transcritical bifurcation. This is rigorously proven in Theorem 3.12.
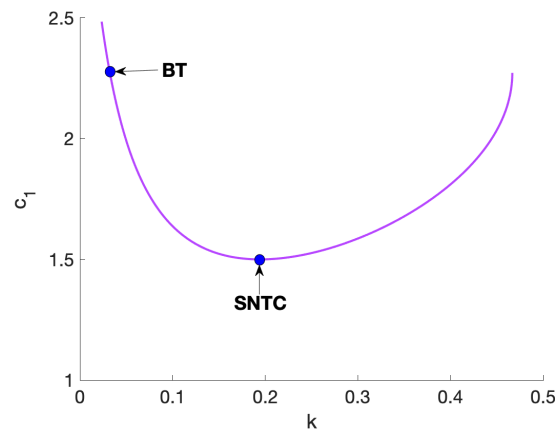
Figure 7. Two parameter bifurcation diagram for parameters k and c_1. We observe the occurrence of a Saddle-Node Transcritical (SNTC) bifurcation and a Bogdanov-Takens (BT) bifurcation. SNTC bifurcation occurs around $k = 0.1945$ and $c_1 = 1.5$. BT bifurcation occurs around $k = 0.0323$ and $c_1 = 2.2738$. Parameters used are chosen from Figure 6(B).

Table 1. Effect of fear on classical competition dynamics.

	Classical case	$k > 0$
(i)	CE $(u^*, 0)$	Species v is competitively excluded for both small and large k.
(ii)	CE $(0, v^*)$	1) If $\frac{a_2 b_1^2 k}{(b_1 + k a_1)^2} < \frac{b_1 b_2 - c_1 c_2}{c_1}$, then species u is competitively excluded.
		2) Possibility of one interior saddle equilibrium, with a large level of k.
		3) Possibility of two positive interior equilibria, one sink and one saddle, with an intermediate level of k.
(iii)	weak competition	1) If $k < k_c = \frac{b_2 b_1 - c_2 c_1}{a_2 c_1}$ via Thm 3.1 yields co-existence.
		2) If $k > k_c$, then species v is competitively excluded.
(iv)	strong competition	Interior equilibrium always exists and is a saddle.

4. The PDE Case

In this section we consider the dynamics of (2.2), when $d_1, d_2 > 0$, and we are in a bonafide PDE setting, where the competitor v is fearful of u.

4.1. Notations and preliminary observations

To prove global existence of solutions to (2.2), it suffices to derive uniform estimate on the $\mathbb{L}^p$ norms of the R.H.S. of (2.2), for some $p > \frac{n}{2}$. Classical theory will then yield global existence [66]. The usual norms in spaces $\mathbb{L}^p(\Omega)$, $\mathbb{L}^\infty(\Omega)$ and $\mathbb{C}\left(\overline{\Omega}\right)$ are respectively denoted by

$$\|u\|_p^p = \int_\Omega |u(x)|^p \, dx, \ \|u\|_\infty = ess \ sup_{x\in\Omega} |u(x)|. \tag{4.1}$$

To this end, we use standard techniques [67]. We first recall classical results guaranteeing non-negativity of solutions, local and global existence [67, 68]:

Lemma 4.1. *Let us consider the following $m \times m$ - reaction diffusion system: for all $i = 1, ..., m$,*

$$\partial_t u_i - d_i \Delta u_i = f_i(u_1, ..., u_m) \ in \ \mathbb{R}_+ \times \Omega, \ \partial_\nu u_i = 0 \ on \ \partial\Omega, u_i(0) = u_{i0}, \tag{4.2}$$

where $d_i \in (0, +\infty)$, $f = (f_1, ..., f_m) : \mathbb{R}^m \to \mathbb{R}^m$ is $C^1(\Omega)$ and $u_{i0} \in L^\infty(\Omega)$. Then there exists a $T > 0$ and a unique classical solution of (4.2) on $[0, T)$. If T^* denotes the greatest of these $T's$, then

$$\left[\sup_{t\in[0,T^*), 1\leq i\leq m} \|u_i(t)\|_{L^\infty(\Omega)} < +\infty \right] \implies [T^* = +\infty].$$

If the nonlinearity $(f_i)_{1\leq i\leq m}$ is moreover quasi-positive, which means

$$\forall i = 1, ..., m, \ \ \forall u_1, ..., u_m \geq 0, \ \ f_i(u_1, ..., u_{i-1}, 0, u_{i+1}, ..., u_m) \geq 0,$$

then

$$[\forall i = 1, ..., m, u_{i0} \geq 0] \implies [\forall i = 1, ..., m, \ \forall t \in [0, T^*), u_i(t) \geq 0].$$

Lemma 4.2. *Using the same notations and hypotheses as in Lemma 4.1, suppose moreover that f has at most polynomial growth and that there exists $\mathbf{b} \in \mathbb{R}^m$ and a lower triangular invertible matrix P with nonnegative entries such that*

$$\forall r \in [0, +\infty)^m, \quad Pf(r) \leq \left[1 + \sum_{i=1}^m r_i\right]\mathbf{b}.$$

Then, for $u_0 \in L^\infty(\Omega, \mathbb{R}_+^m)$, the system (4.2) has a strong global solution.

Under these assumptions, the following local existence result is well known, see D. Henry [66].

Theorem 4.3. *The system (4.4) admits a unique, classical solution (u, v) on $[0, T_{max}] \times \Omega$. If $T_{max} < \infty$ then*

$$\lim_{t \nearrow T_{max}} \left\{ \|u(t, .)\|_\infty + \|v(t, .)\|_\infty \right\} = \infty, \tag{4.3}$$

where T_{max} denotes the eventual blow-up time in $\mathbb{L}^\infty(\Omega)$.

The next result follows from the application of standard theory [69].

Theorem 4.4. *Consider the reaction diffusion system (2.2). For spatially homogenous initial data $u_0 \equiv c, v_0 \equiv d$, with $c, d > 0$, then the dynamics of (4.4) and its resulting kinetic (ODE) system, when $d_1 = d_2 = 0$ in (2.2), are equivalent.*

4.2. Spatially heterogeneous fear

Our objective now is to consider the case of a fear function that may be heterogeneous in space. A motivation for this comes from several ecological and sociological settings. For example it is very common for prey to be highly fearful closer to a predators lair, but less fearful in a region of refuge [12], or in regions of high density due to group defense [70]. Furthermore, a conceivably weaker drug cartel, could have certain localized strongholds, within which they would be more feared by stronger groups. To these ends, it is conceivable that the fear coefficient k is not a constant, but actually varies in the spatial domain Ω, so $k = k(x)$, which could take different forms depending on the application at hand. This is also in line with the LOF concept [7]. Thus we consider the following spatially explicit version of (3.1), with heterogeneous fear function $k(x)$, resulting in the following reaction diffusion system,

$$
\begin{aligned}
u_t &= d_1(u)_{xx} + a_1 u - b_1 u^2 - c_1 uv, \quad x \in \Omega, \\
v_t &= d_2(v)_{xx} + \frac{a_2 v}{1 + k(x)u} - b_2 v^2 - c_2 uv, \quad x \in \Omega, \\
\frac{\partial u}{\partial v} &= \frac{\partial v}{\partial v} = 0, \quad \text{on} \quad \partial \Omega. \\
u(x, 0) &= u_0(x) \equiv c > 0, \quad v(x, 0) = v_0(x) \equiv d > 0,
\end{aligned}
\tag{4.4}
$$

where $\Omega \subset \mathbb{R}^n$. We assume Neumann boundary conditions. Also, we prescribe spatially homogeneous (flat) initial conditions $u(x, 0) = u_0(x) \equiv c > 0$, $v(x, 0) = v_0(x) \equiv d > 0$. Furthermore, we impose the following restrictions on the fear function $k(x)$,

$$
\begin{aligned}
&(i) \quad k(x) \in C^1(\Omega), \\
&(ii) \quad k(x) \geq 0, \\
&(iii) \quad \text{If } k(x) \equiv 0 \text{ on } \Omega_1 \subset \Omega, \text{ then } |\Omega_1| = 0. \\
&(iv) \quad \text{If } k(x) \equiv 0 \text{ on } \cup_{i=1}^n \Omega_i \subset \Omega, \text{ then } \Sigma_{i=1}^n |\Omega_i| = 0.
\end{aligned}
\tag{4.5}
$$

Remark 7. If $k(x) \equiv 0$ on $\Omega_1 \subset \Omega$, with $|\Omega_1| > \delta > 0$, or $k(x) \equiv 0$ on $\cup_{i=1}^n \Omega_i \subset \Omega$, with $\Sigma_{i=1}^n |\Omega_i| > \delta > 0$, that is, on non-trivial parts of the domain, the analysis is notoriously difficult, as one now is dealing with a *degenerate* problem. See [71, 72] for results on this problem. This case is not in the scope of the current manuscript.

Since the nonlinear right hand side of (4.4) is continuously differentiable on $\mathbb{R}^+ \times \mathbb{R}^+$, then for any initial data in $\mathbb{C}(\overline{\Omega})$ or $\mathbb{L}^p(\Omega)$, $p \in (1, +\infty)$, it is standard to estimate the $\mathbb{L}^p$−norms of the solutions and thus deduce global existence. Standard theory will apply even in the case of a bonafide fear function $k(x)$, because due to our assumptions on the form of k, standard comparison arguments will apply [73]. Thus applying the classical methods above, via Theorem 4.3, and Lemmas 4.1 and 4.2, we can state the following lemmas:

Lemma 4.5. *Consider the reaction diffusion system* (4.4), *for $k(x)$ such that the assumtions via* (4.5) *hold. Then solutions to* (4.4) *are non-negative, as long as they initiate from positive initial conditions.*

Lemma 4.6. *Consider the reaction diffusion system* (4.4). *For $k(x)$ such that the assumtions via* (4.5) *hold, the solutions to* (4.4) *are classical, that is for $(u_0, v_0) \in \mathbb{L}^\infty(\Omega)$, $(u, v) \in C^1(0, T; C^2(\Omega))$, $\forall T$.*

Our goal in this section is to investigate the dynamics of (4.4). Herein we will use the comparison technique, and compare to the ODE cases of classical competition, or the constant fear function case, where the dynamics are well known.

Remark 8. The analysis in this section are primarily focused on the choice of spatially homogenous (flat) initial data.

We begin by defining the following systems of PDEs,

$$\overline{u}_t = d_1(\overline{u})_{xx} + a_1\overline{u} - b_1\overline{u}^2 - c_1\overline{u}\overline{v},$$
$$\overline{v}_t = d_2(\overline{v})_{xx} + a_2\overline{v} - b_2\overline{v}^2 - c_2\overline{u}\overline{v}, \tag{4.6}$$

$$\widehat{u}_t = d_1(\widehat{u})_{xx} + a_1\widehat{u} - b_1\widehat{u}^2 - c_1\widehat{u}\widehat{v},$$
$$\widehat{v}_t = d_2(\widehat{v})_{xx} + \frac{a_2\widehat{v}}{1 + \widehat{\mathbf{k}}\widehat{u}} - b_2\widehat{v}^2 - c_2\widehat{u}\widehat{v}, \tag{4.7}$$

$$\widetilde{u}_t = d_1(\widetilde{u})_{xx} + a_1\widetilde{u} - b_1\widetilde{u}^2 - c_1\widetilde{u}\widetilde{v},$$
$$\widetilde{v}_t = d_2(\widetilde{v})_{xx} + \frac{a_2\widetilde{v}}{1 + \widetilde{\mathbf{k}}\widetilde{u}} - b_2\widetilde{v}^2 - c_2\widetilde{u}\widetilde{v}, \tag{4.8}$$

$$\tilde{u}_t = d_1(\tilde{u})_{xx} + a_1\tilde{u} - b_1\tilde{u}^2 - c_1\tilde{u}\tilde{v},$$
$$\tilde{v}_t = d_2(\tilde{v})_{xx} + \frac{a_2\tilde{v}}{1 + \widetilde{\mathbf{k}}\frac{a_1}{b_1}} - b_2\tilde{v}^2 - c_2\tilde{u}\tilde{v}, \tag{4.9}$$

where

$$\widehat{\mathbf{k}} = \min_{x\in\Omega} k(x), \qquad \widetilde{\mathbf{k}} = \max_{x\in\Omega} k(x). \tag{4.10}$$

We assume Neumann boundary conditions for all of the reaction diffusion systems (4.6)–(4.9). Also in each of the systems we prescribe spatially homogenous (flat) initial conditions $u(x,0) = u_0(x) \equiv c > 0$, $v(x,0) = v_0(x) \equiv d > 0$.

We now state the following lemma:

Lemma 4.7. *Consider the reaction diffusion system* (4.4)*, as well as the reaction diffusion systems* (4.6)–(4.9)*. Then the following point wise comparison holds,*

$$\tilde{v} \le \widetilde{v} \le v \le \widehat{v} \le \overline{v}.$$

Proof. Note via positivity of solutions to (4.4), (4.6)–(4.9), the definitions via (4.10), and the upper bound on u, which is a solution to (4.4) of $\frac{a_1}{b_1}$, (derived via comparison to the logistic equation), we have

$$\frac{a_2}{1 + \widetilde{\mathbf{k}}\frac{a_1}{b_1}} \le \frac{a_2}{1 + \widetilde{\mathbf{k}}u(x)} \le \frac{a_2}{1 + k(x)u(x)} \le \frac{a_2}{1 + \widehat{\mathbf{k}}u(x)} \le a_2, \quad \forall x \in \Omega. \tag{4.11}$$

Thus the result follows via standard comparison theory [73]. □

4.3. The competitive exclusion case

Theorem 4.8. *Consider the reaction diffusion system* (4.4), *for a fear function* $k(x)$, *such that the assumptions via* (4.5) *are met, and*

$$\widetilde{\mathbf{k}} > \frac{a_2 b_1^2 - c_2 a_1 b_1}{a_1^2 c_2} \quad \text{and} \quad \frac{a_1}{a_2} > \max\left\{\frac{b_1}{c_2}, \frac{c_1}{b_2}\right\}.$$

Then the solution (u, v) *to* (4.4) *converges uniformly to the spatially homogenous state* $(\frac{a_1}{b_1}, 0)$ *as* $t \to \infty$.

Proof. From the classical theory of competition [52], we know the dynamics for (4.6), that is in the competitive exclusion case, when

$$\frac{a_1}{a_2} > \max\left\{\frac{b_1}{c_2}, \frac{c_1}{b_2}\right\},$$

we have

$$(\overline{u}, \overline{v}) \to \left(\frac{a_1}{b_1}, 0\right).$$

Moreover, under the assumption

$$\widetilde{\mathbf{k}} > \frac{a_2 b_1^2 - c_2 a_1 b_1}{a_1^2 c_2},$$

and making use of Lemma 3.5, along with the use of Theorem 4.4, we have $(\widetilde{u}, \widetilde{v}) \to \left(\frac{a_1}{b_1}, 0\right)$. Now using Lemma 4.7 we have,

$$\widetilde{v} \leq v \leq \overline{v},$$

which entails,

$$\lim_{t \to \infty}(\widetilde{u}, \widetilde{v}) \leq \lim_{t \to \infty}(u, v) \leq \lim_{t \to \infty}(\overline{u}, \overline{v}),$$

subsequently,

$$\left(\frac{a_1}{b_1}, 0\right) \leq \lim_{t \to \infty}(u, v) \leq \left(\frac{a_1}{b_1}, 0\right).$$

Now using a squeezing argument, in the limit that $t \to \infty$, we have uniform convergence of solutions of (4.4), i.e.,

$$(u, v) \to \left(\frac{a_1}{b_1}, 0\right)$$

as $t \to \infty$. $\square$

Using the positivity of solutions the requirement on $k(x)$ and so in turn on $\widetilde{\mathbf{k}}$, can be weakened to derive a stronger result.

Theorem 4.9. *Consider the reaction diffusion system* (4.4), *for a fear function* $k(x)$, *such that the assumptions via* (4.5) *are met, and* $\frac{a_1}{a_2} > \max\left\{\frac{b_1}{c_2}, \frac{c_1}{b_2}\right\}$. *Then the solution* (u, v) *to* (4.4) *converges uniformly to the spatially homogeneous state* $(\frac{a_1}{b_1}, 0)$ *as* $t \to \infty$.

Proof. From the classical theory of competition [52], we know the dynamics for (4.6), that is in the competitive exclusion case, when

$$\frac{a_1}{a_2} > \max\Big\{\frac{b_1}{c_2}, \frac{c_1}{b_2}\Big\},$$

we have

$$(\bar{u}, \bar{v}) \to \Big(\frac{a_1}{b_1}, 0\Big).$$

Now using Lemma 4.7, we have

$$v \leq \bar{v}.$$

Using the non negativity of solutions to (4.4) via Lemma 4.5, entails,

$$0 \leq \lim_{t \to \infty} v \leq \lim_{t \to \infty}(\bar{v}) = 0,$$

subsequently,

$$\lim_{t \to \infty}(u, v) \to \Big(\frac{a_1}{b_1}, 0\Big).$$

$\square$

Theorem 4.10. *Consider the reaction diffusion system* (4.4), *for a fear function k(x), such that the assumptions via* (4.5) *are met, and*

$$\frac{a_2 b_1^2 \widetilde{\mathbf{k}}}{(b_1 + \widetilde{\mathbf{k}} a_1)^2} < \frac{b_1 b_2 - c_1 c_2}{c_1} \quad and \quad \frac{a_1}{a_2} < \min\Big\{\frac{b_1}{c_2}, \frac{c_1}{b_2}\Big\}.$$

Then the solution (u, v) *to* (4.4) *converges uniformly to the spatially homogeneous state* $(0, \frac{a_2}{b_2})$ *as* $t \to \infty$.

Proof. From the classical theory of competition [52], we know the dynamics for (4.6), that is in the competitive exclusion case, when

$$\frac{a_1}{a_2} < \min\Big\{\frac{b_1}{c_2}, \frac{c_1}{b_2}\Big\},$$

we have

$$(\bar{u}, \bar{v}) \to \Big(0, \frac{a_2}{b_2}\Big).$$

Consider the nullclines of (4.8). If

$$\frac{d}{du}\Big[\frac{a_1 - b_1 u}{c_1}\Big] < \frac{d}{du}\Big[\frac{1}{b_2}\Big(\frac{a_2}{1 + \widetilde{\mathbf{k}} u} - c_2 u\Big)\Big],$$

it follows via the geometry of the nullclines, that the *v*-nullcline remains above the *u*-nullcline, and thus $(\bar{u}, \bar{v}) \to \Big(0, \frac{a_2}{b_2}\Big)$. On simplification, and using the upper bound estimate for *u*, we have

$$\frac{a_2 b_1^2 \widetilde{\mathbf{k}}}{(b_1 + \widetilde{\mathbf{k}} a_1)^2} < \frac{a_2 \widetilde{\mathbf{k}}}{(1 + \widetilde{\mathbf{k}} u)^2} < \Big(\frac{b_1 b_2 - c_1 c_2}{c_1}\Big).$$

Now using Lemma 4.7 we have,

$$\widetilde{v} \leq v \leq \overline{v},$$

which entails,

$$\lim_{t \to \infty}(\widetilde{u}, \widetilde{v}) \leq \lim_{t \to \infty}(u, v) \leq \lim_{t \to \infty}(\overline{u}, \overline{v}),$$
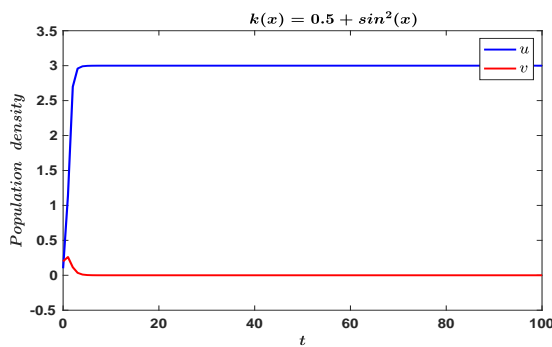
subsequently,

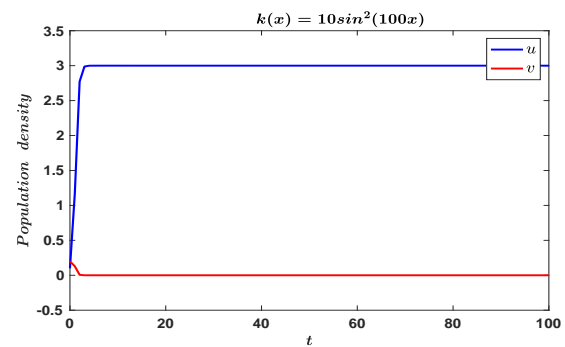$$\left(0, \frac{a_2}{b_2}\right) \leq \lim_{t \to \infty}(u, v) \leq \left(0, \frac{a_2}{b_2}\right).$$

Now using a squeezing argument, in the limit that $t \to \infty$, we have uniform convergence of solutions of (4.4), i.e.,

$$(u, v) \to \left(0, \frac{a_2}{b_2}\right)$$

as $t \to \infty$. □

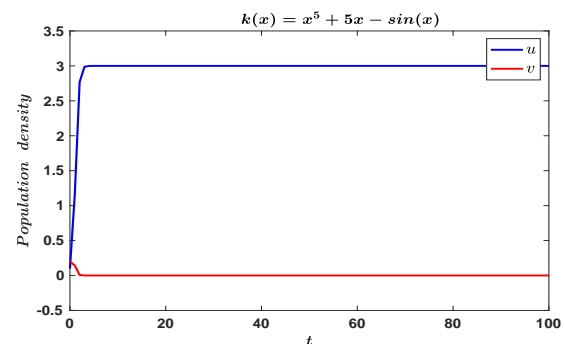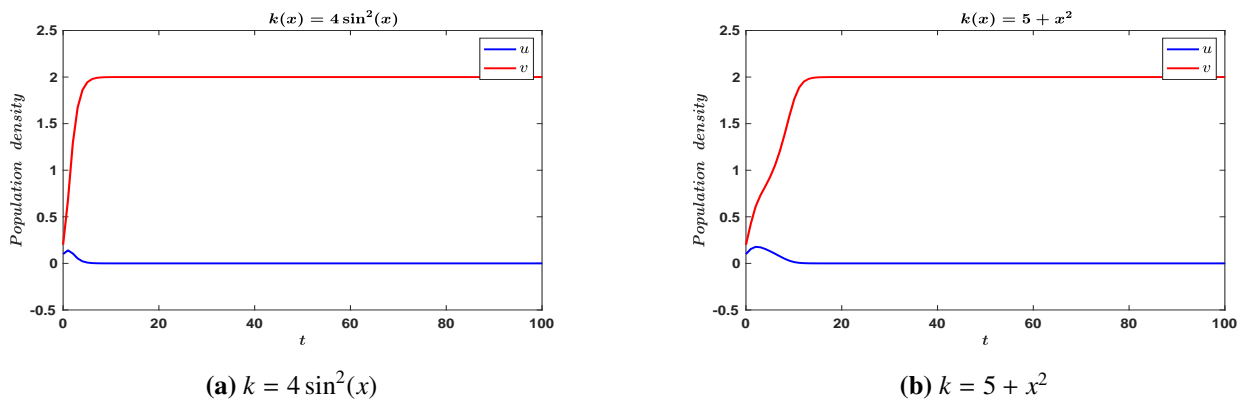**(a)** $k = 0.5 + \sin^2(x)$

(b) $k = 10\sin^2(100x)$
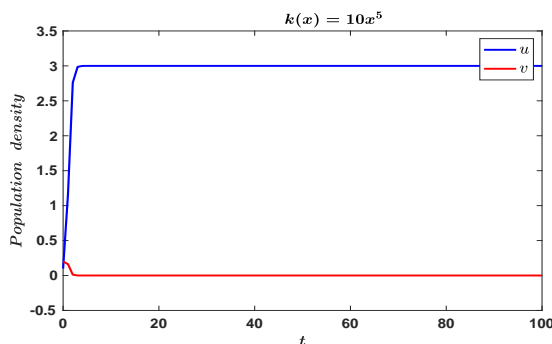
(c) $k = 10x^5$

(d) $k = x^5 + 5x - \sin(x)$

Figure 8. Numerical simulation of (4.4) for the case of competition exclusion in $\Omega = [0, 1]$. The parameters are chosen as $[u_0, v_0] = [0.1, 0.2], d_1 = 1, d_2 = 1, a_1 = 3, a_2 = 1, b_1 = b_2 = 1$ and $c_1 = c_2 = 0.5$ and this choice meets the parametric constraint of the Theorem 4.8.

(a) $k = 4\sin^2(x)$ **(b)** $k = 5 + x^2$

Figure 9. Numerical simulation of (4.4) for the case of competition exclusion in $\Omega = [0, 1]$. The parameters are chosen as $[u_0, v_0] = [0.1, 0.2], d_1 = 1, d_2 = 1, a_1 = 1, a_2 = 2, b_1 = 2, b_2 = 1, c_1 = 1$ and $c_2 = 1$ and this choice meets the parametric constraint of the Theorem 4.10.

4.4. The weak competition case

We state the following result:

Lemma 4.11. *Consider the reaction diffusion system* (4.4)*, for a fear function $k(x)$, such that the assumptions via* (4.5) *are met, with*

$$\widetilde{\mathbf{k}} > \frac{a_2 b_1^2 - c_2 a_1 b_1}{a_1^2 c_2} \quad and \quad b_1 b_2 > 2c_1 c_2.$$

Then the solution (u, v) to (4.4) *converges uniformly to the spatially homogeneous state $(u^*, 0)$ as $t \to \infty$.*

Proof. Via Lemma 3.2, and the parametric restrictions assumed we have,

$$\widetilde{v} \leq v \leq \widehat{v},$$

and the result follows via similar analysis as in Theorem 4.8. $\qquad\square$

Lemma 4.12. *Consider the reaction diffusion system* (4.4)*, such that system is in the weak competition case when $k(x) \equiv 0$. Then given $0 < \epsilon \ll 1$, there exists a fear function $k_\epsilon(x)$, for which the assumptions via* (4.5) *are met, such that the solution (u, v) to* (4.4) *with the fear function $k_\epsilon(x)$, converges uniformly to a spatially homogeneous state (u^*, v^*) as $t \to \infty$.*

Proof. Given $0 < \epsilon \ll 1$, we can always construct a k_ϵ, s.t $k_c - \epsilon \leq \widehat{k_\epsilon}$, whereas $\widetilde{k_\epsilon} \leq k_c + \epsilon$. Thus via Lemma 4.7 we have,
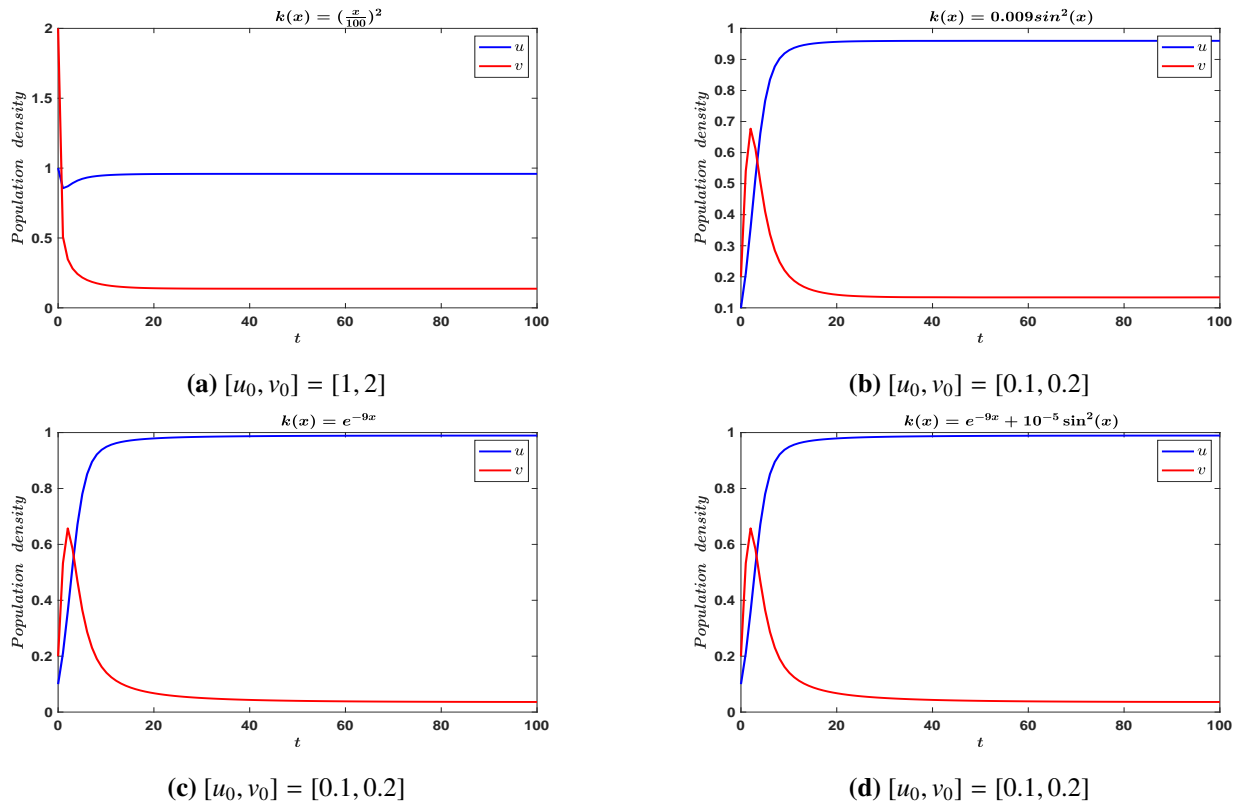
$$\widetilde{v} \leq v_\epsilon \leq \widehat{v}.$$

Lemma 3.7 ensures we have $(\widetilde{u}, \widetilde{v}) \to (u^*, v^*)$ and $(\widehat{u}, \widehat{v}) \to (u^{**}, v^{**})$, where the spatially homogeneous solutions may be different. Hence, via squeezing argument, we can take $\epsilon \to 0$, to yield

the uniform convergence of solutions, i.e.,

$$\lim_{\epsilon \to 0} \lim_{t \to \infty} (u_\epsilon, v_\epsilon) \to (u^*, v^*).$$

This proves the lemma. □

(a) $[u_0, v_0] = [1, 2]$

(b) $[u_0, v_0] = [0.1, 0.2]$

(c) $[u_0, v_0] = [0.1, 0.2]$

(d) $[u_0, v_0] = [0.1, 0.2]$

Figure 10. Numerical simulation of (4.4) for the case of weak-competition in $\Omega = [0, 1]$. The parameters are chosen as $d_1 = 1, d_2 = 1, a_1 = 1, a_2 = 2, b_1 = 1, b_2 = 2, c_1 = 0.3$ and $c_2 = 1.8$ and this choice meets the parametric constraint of the Conjecture 1.

The numerical simulations above motivate the following conjecture:

Conjecture 1. *Consider the reaction diffusion system* (4.4)*, for a fear function k(x), such that the assumptions via* (4.5) *are met, with*

$$\widetilde{\mathbf{k}} < \frac{1}{a_2} \left(\frac{b_1 b_2}{c_1} - c_2 \right),$$

and the parameters follow (3.5) *and Theorem* 3.1*, then the solution* (u, v) *to* (4.4) *converges uniformly to the spatially homogenous state* (u^*, v^*) *as* $t \to \infty$.

Remark 9. We see via Lemma 3.5, and Lemma 3.7, that in the ODE case, if we are in the weak competition setting without fear, then a critical amount of fear k_c is *both* sufficient and necessarily required to change the system's dynamics to a competitive exclusion type scenario. In the PDE case, where the fear function $k(x)$ can be spatially heterogeneous, this requirement is certainly sufficient, as seen via Theorem 4.13, but *not* necessary, in a point wise sense.

This result is stated next:

Theorem 4.13. *Consider the reaction diffusion system (4.4), with initial data $u_0(x) \geq 1$, s.t $u \nearrow u^*(u$ increases to u^*), such that system is in the weak competition setting when $k(x) \equiv 0$. For a fear function $k(x)$, s.t the assumptions via (4.5) are met, the solution (u, v) converges uniformly to the spatially homogeneous state $(\frac{a_1}{b_1}, 0)$ as $t \to \infty$, if the following condition holds,*

$$\frac{C_1}{|\Omega|} \int_\Omega \frac{1}{1 + k(x)} dx < \left(\frac{1}{1 + k_c \left(\frac{a_1}{b_1} \right)} \right), \tag{4.12}$$

where k_c is as defined in Lemma 3.5.

Proof. Consider (4.4), integrating the equation over Ω yields,

$$
\begin{aligned}
&\frac{d}{dt} \int_\Omega v dx \\
&= \int_\Omega \left(\frac{a_2 v}{1 + k(x)u} - b_2 v^2 - c_2 uv \right) dx \\
&\leq \int_\Omega \left(\frac{a_2 v}{1 + k(x)u_0} - b_2 v^2 - c_2 uv \right) dx \\
&= \frac{1}{1 + k(x^*)u_0} \int_\Omega a_2 v dx - \int_\Omega \left(b_2 v^2 + c_2 uv \right) dx \\
&\leq \frac{1}{1 + k(x^*)} \int_\Omega a_2 v dx - \int_\Omega \left(b_2 v^2 + c_2 uv \right) dx \\
&\leq \left(C_1 \int_\Omega \frac{1}{1 + k(x)} dx \right) \int_\Omega a_2 v dx - \int_\Omega \left(b_2 v^2 + c_2 uv \right) dx \\
&\leq \left(\frac{1}{1 + k_c \frac{a_1}{b_1}} \right) \int_\Omega a_2 v dx - \int_\Omega \left(b_2 v^2 + c_2 uv \right) dx \\
&\leq \int_\Omega \left(\frac{a_2}{1 + k_c \frac{a_1}{b_1}} v - b_2 v^2 - c_2 uv \right) dx.
\end{aligned}
\tag{4.13}
$$

This follows via the mean value theorem for integrals. We can now compare,

$$\int_\Omega v dx < \int_\Omega \tilde{v} dx, \tag{4.14}$$

where $\tilde{v}$ solves $\frac{d\tilde{v}}{dt} = \frac{a_2}{1 + k_c \frac{a_1}{b_1}} \tilde{v} - b_2 \tilde{v}^2 - c_2 u \tilde{v}$. Thus using Theorem 4.4, Lemma 3.2 and the positivity of solutions, this entails

$$0 \leq \lim_{t \to \infty} \int_\Omega v dx \leq \lim_{t \to \infty} \int_\Omega \tilde{v} dx = 0. \tag{4.15}$$

The bounds on v, standard Lebesgue convergence theorems, and squeezing argument entail,

$$\int_\Omega (\lim_{t\to\infty} v)dx \to 0, \tag{4.16}$$

which implies the uniform convergence,

$$\lim_{t\to\infty}(u, v) \to \left(\frac{a_1}{b_1}, 0\right).$$

$\square$

Remark 10. The C_1 in Theorem 4.13 is a pure constant, that will depend on the size of the domain Ω, and the other problem parameters, but not on the initial data or the spatial variable x.

Remark 11. Clearly $k(x)$ could be chosen such that it lies below k_c for a portion of the domain, and above k_c on some portion of the domain - thus $\min_{x\in\Omega} k(x)$ does not lie uniformly above k_c. Yet via Theorem 4.13, we see that, one can change the system's dynamics and bring it to a competitive exclusion type scenario, from a coexistence situation.

The next result gives a lower estimate for the fear function k.

Lemma 4.14. *Consider the fear function $k(x)$ in Theorem 4.13, then we have,*

$$\int_\Omega k(x)dx \geq |\Omega|\left(1 - \frac{1}{C_1}\left(\frac{1}{1 + k_c\left(\frac{a_1}{b_1}\right)}\right)\right).$$

Proof. We have that for any fear function $k(x)$ satisfying (4.5),

$$\frac{1}{1 + k(x)} > 1 - k(x) \implies \int_\Omega \frac{1}{1 + k(x)}dx > |\Omega| - \int_\Omega k(x)dx. \tag{4.17}$$

Now using Theorem 4.13 the result follows. $\square$

4.5. The strong competition case

Theorem 4.15. *Consider the reaction diffusion system (4.4), for a fear function $k(x)$, such that the assumptions via (4.5) are met, with $b_1b_2 < c_1c_2$. Then there exists sufficiently large positive initial data $[u_0, v_0]$ for which the solution (u, v) to (4.4) converges uniformly to the spatially homogenous state $(0, \frac{a_2}{b_2})$ as $t \to \infty$, while there also exists sufficiently small positive initial data $[u_1, v_1]$ for which solution (u, v) to (4.4) converges uniformly to the spatially homogenous state $(\frac{a_1}{b_1}, 0)$ as $t \to \infty$.*

Proof. Consider the system (4.6). From the classical strong competition parametric restrictions, $b_1b_2 < c_1c_2$ that are assumed, as well as Lemma 4.7, we can make use of standard competition theory and use the stable manifold theorem, i.e., $\exists W_s(E_4) \in C^1$ separatrix, such that for initial data $(\bar{u}_0, \bar{v}_0)$ chosen above $W_s(E_4)$ the solution $(\bar{u}, \bar{v}) \to (0, v^*)$ and for initial data chosen below $W_s(E_4)$, $(\bar{u}, \bar{v}) \to (u^*, 0)$. Here E_4 is the interior saddle equilibrium to the kinetic (ODE) system for (4.6). Moreover, since

$\dfrac{a_2}{1 + \widetilde{\mathbf{k}}\frac{a_1}{b_1}} \leq a_2$, and $b_1 b_2 < c_1 c_2$, we have that for (4.10) we still remain in the strong competition case, and via standard theory again, $W_s^1(E_4^*) \in C^1$ separatrix, such that for initial data $(\widehat{u_0}, \widehat{v_0})$ chosen above $W_s^1(E_4^*)$, the solution $(\tilde{u}, \tilde{v}) \to (0, v^*)$ and for initial data chosen below $W_s^1(E_4)$, $(\tilde{u}, \tilde{v}) \to (u^*, 0)$. Here E_4^* is the interior saddle equilibrium to the kinetic (ODE) system for (4.9). Now since $\dfrac{a_2}{1 + \widetilde{\mathbf{k}}\frac{a_1}{b_1}} \leq a_2$, the v component of E_4^* is higher than the v component of E_4. Now using the C^1 property of the separatricies $W_s^1(E_4^*), W_s(E_4)$, we have the existence of a wedge V emanating from E_4, s.t within V we have $W_s^1(E_4^*) \geq W_s(E_4)$. Note via Lemma 4.7, we have $\tilde{v} \leq v \leq \bar{v}$. Let us consider positive initial data (u_0, v_0) chosen large enough, within V s.t. $(u_0, v_0) > W_s^1(E_4^*) > W_s(E_4)$, we will have

$$\left\{(0, v^*)\right\} \leq \left\{(u, v)\right\} \leq \left\{(0, v^*)\right\}.$$

On the other hand, for positive initial data (u_1, v_1) chosen small enough via an analogous construction we will have

$$\left\{(u^*, 0)\right\} \leq \left\{(u, v)\right\} \leq \left\{(u^*, 0)\right\}.$$

This proves the theorem.

$\square$
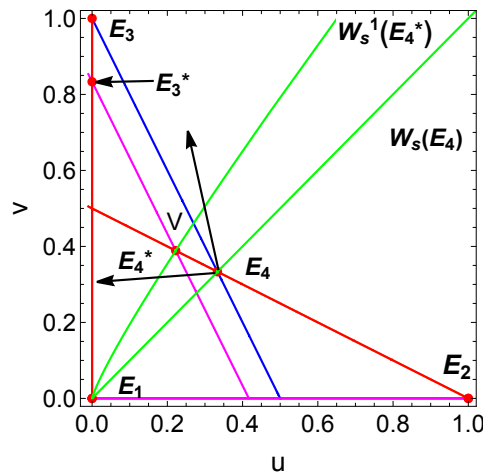
Figure 11. Numerical simulation for the strong-competition PDE case when species u is fearing species v in $\Omega = [0, 1]$. The parameters are chosen as $a_1 = 1, a_2 = 1, b_1 = 1, b_2 = 1, c_1 = 2$ and $c_2 = 2$. Equilibria: $E_1 = E_1^* = (0, 0), E_2 = E_2^* = (1, 0), E_3 = (0, 1), E_3^* = (0, 0.833333), E_4 = \left(\frac{1}{3}, \frac{1}{3}\right)$ and $E_4^* = (0.222222, 0.388889)$. $W_s(E_4)$ $(k = 0)$ and $W_s^1(E_4^*)$ $(k = 0.2)$ are two sepratrices passing through E_4 and E_4^* respectively. The C^1 property of the separatrices $W_s^1(E_4^*), W_s(E_4)$, shows a wedge V emanating from E_4, s.t within V we have $W_s^1(E_4^*) \geq W_s(E_4)$. The u-nullcline is in red for $k = 0$ and $k = 0.2$. For $k = 0$, v-nullcline is in blue. For $k = 0.2$, v-nullcline is in magenta.
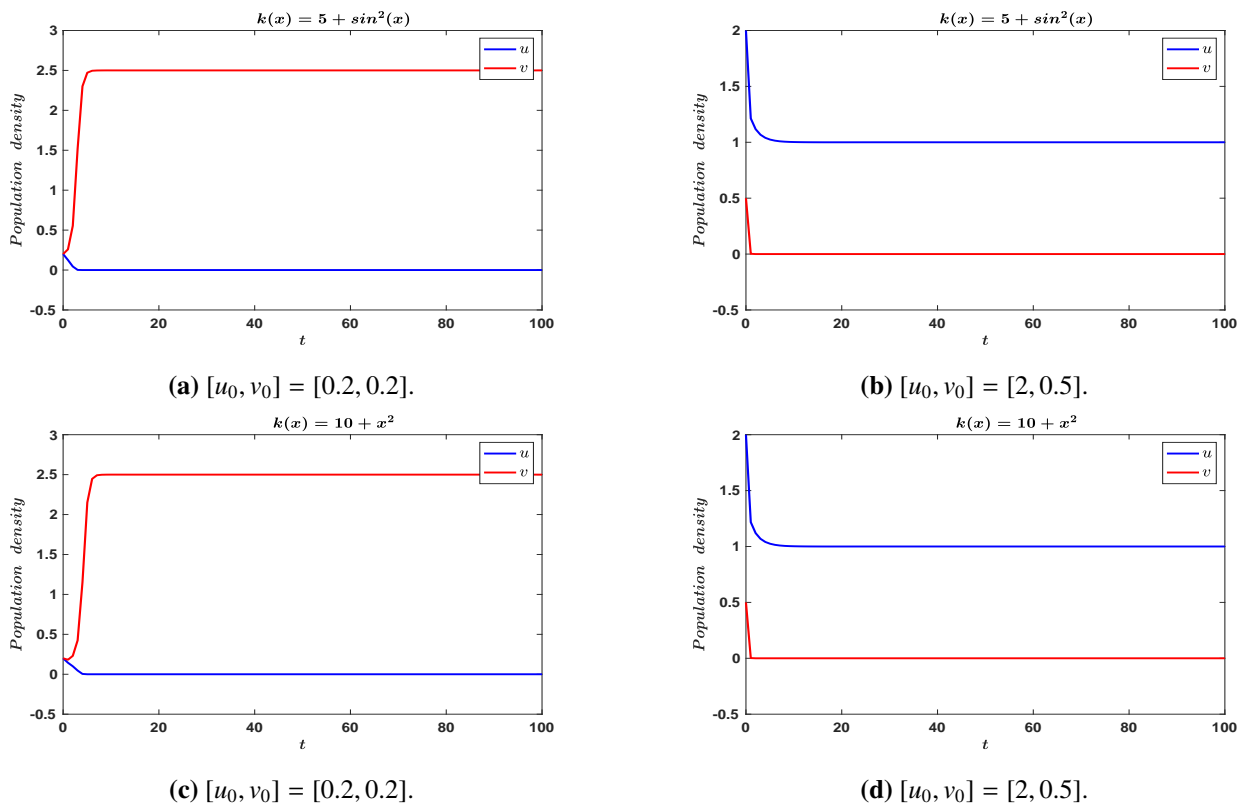
Figure 12. Numerical simulation of (4.4) for the case of strong-competition in $\Omega = [0, 1]$. The parameters are chosen as $d_1 = 1, d_2 = 1, a_1 = 0.5, a_2 = 2, b_1 = 0.5, b_2 = 0.8$ and $c_1 = c_2 = 4$ and this choice meets the parametric constraint of the Theorem 4.15.
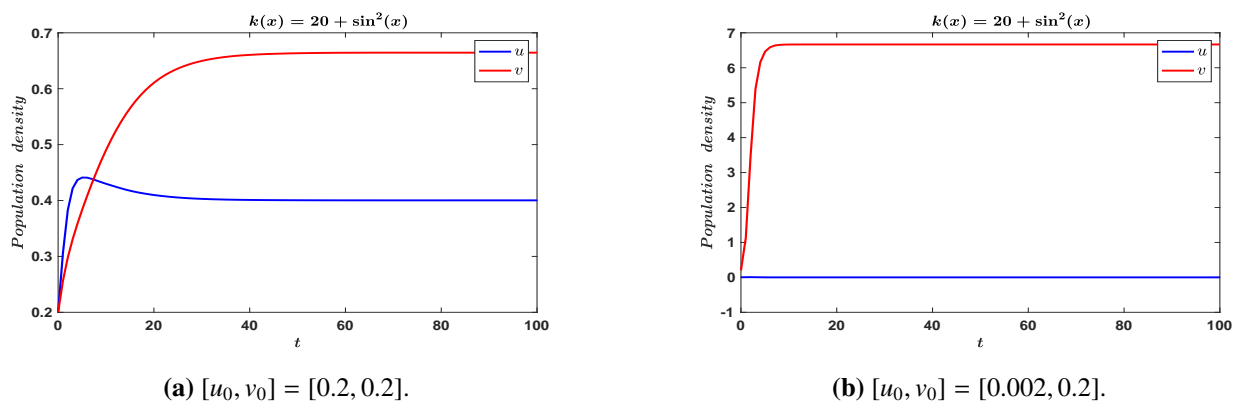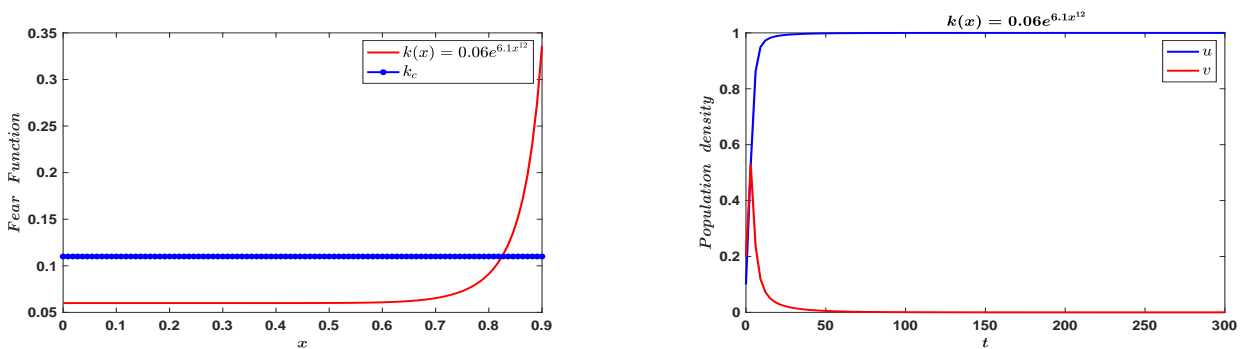

Figure 13. Numerical simulation of (4.4) for the case of two-positive interior equilibria in $\Omega = [0, 1]$. The parameters are chosen as $d_1 = 1, d_2 = 1, a_1 = 1, a_2 = 2, b_1 = 2, b_2 = 0.3$ and $c_1 = 0.3, c_2 = 0.05$ and this choice meets the parametric constraint of the Theorem 3.9.

From the observations in the above numerical simulations, we can state a conjecture concerning positive interior equilibrium.

Conjecture 2. *Consider the system (4.4). For any non-negative fear function $k(x)$ such that*

$$\frac{b_2 b_1}{c_1} - c_2 < \frac{k a_2 b_1^2}{(b_1 + k a_1)^2} \tag{4.18}$$

and parametric restrictions given by (4.5) and Theorem 3.1 hold true, for $\mathbf{k} = \widetilde{\mathbf{k}}, \hat{\mathbf{k}}$. There exists some data $[u_0, v_0]$, for which the solution $(u, v) \to (u^*, v^*)$, and for some choice of data $[u_1, v_1]$, the solution converges to the boundary equilibrium $(0, v^*)$.
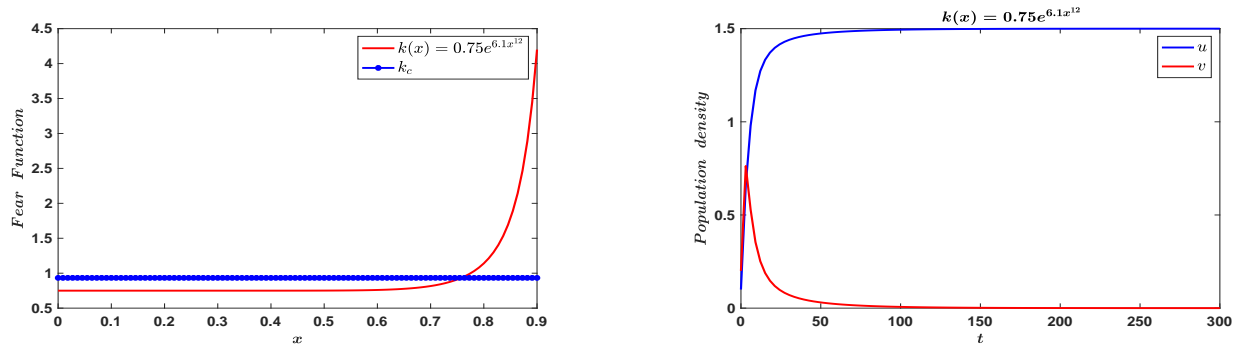

Figure 14. Numerical simulation for the case ($v^* < u^*$). The parameters are chosen as $[u_0, v_0] = [0.1, 0.2], d_1 = 1, d_2 = 1, a_1 = 1, a_2 = 2, b_1 = 1, b_2 = 2, c_1 = 0.3$ and $c_2 = 1.8$. The interior equilibrium $[u^*, v^*] = [0.958904, 0.136986]$ and the fear threshold $k_c = \frac{a_2 b_1^2 - c_2 a_1 b_1}{a_1^2 c_2} = 0.11$ and $C_1 = 1$ and this choice meets the parametric constraint of the Theorem 4.13 and Lemma 4.14.

5. Numerical simulation

All of our PDE simulations for the case of a spatially heterogeneous fear function were performed in `MATLAB` R2021b, using the `MATLAB` inbuilt function pdepe, which is used in solving $1-D$ parabolic and elliptic PDEs. The simulations were run on an $8-$core CPU, Apple $M1$ pro-chip-based workstation. In this configuration, a typical simulation takes between 5–7 seconds when the spatial domain is taken to be $[0, 1]$, and is partitioned into 1000 sub-intervals. The simulations' parameters are chosen to meet the parametric constraints of the theorems. In the spatially explicit setting, we have used the comparison theory to determine point-wise restrictions on the fear functions such that competitive exclusion or strong competition-type dynamics abound. These are shown via Theorems 4.8, 4.10 and 4.15, whereas Figures 8, 9 and 12 can infer the numerical validation of this result. We also have numerically validated that in the spatially explicit setting, fear can change a situation of weak competition to a competitive exclusion type scenario for fear functions with certain $\mathbb{L}^1$ restrictions. This is proved in Theorem 4.13 and Lemma 4.14. Moreover, we have performed a time series analysis over a long period to numerically validate the result, see Figures 14–16. Various heterogeneous fear functions are constructed to demonstrate these results numerically; see Figure 17b. All the details about the parameters and their choice are explicitly mentioned in the captions of the figures of numerical simulation. Essentially in all of the PDE and ODE simulation, parametric values are chosen that meet the parametric restrictions of the Lemmas and Theorems that are being numerically demonstrated.

Also we assume fairly competitive populations, and do not want one population exceeding more than three times the other. This (under scaling) restricts the choice of the parameters to lie in the range $(0, 3]$. This also assists in easier visualization of the dynamics.

Figure 15. Numerical simulation for the case $(u^* < v^*)$. The parameters are chosen as $[u_0, v_0] = [0.1, 0.2], d_1 = 1, d_2 = 1, a_1 = 1.5, a_2 = 1.8, b_1 = 1, b_2 = 1, c_1 = 0.8$ and $c_2 = 0.5$. The interior equilibrium $[u^*, v^*] = [0.1, 1.75]$ and the fear threshold $k_c = \frac{a_2 b_1^2 - c_2 a_1 b_1}{a_1^2 c_2} = 0.933$ and $C_1 = 0.8$ and this choice meets the parametric constraint of the Theorem 4.13 and Lemma 4.14.
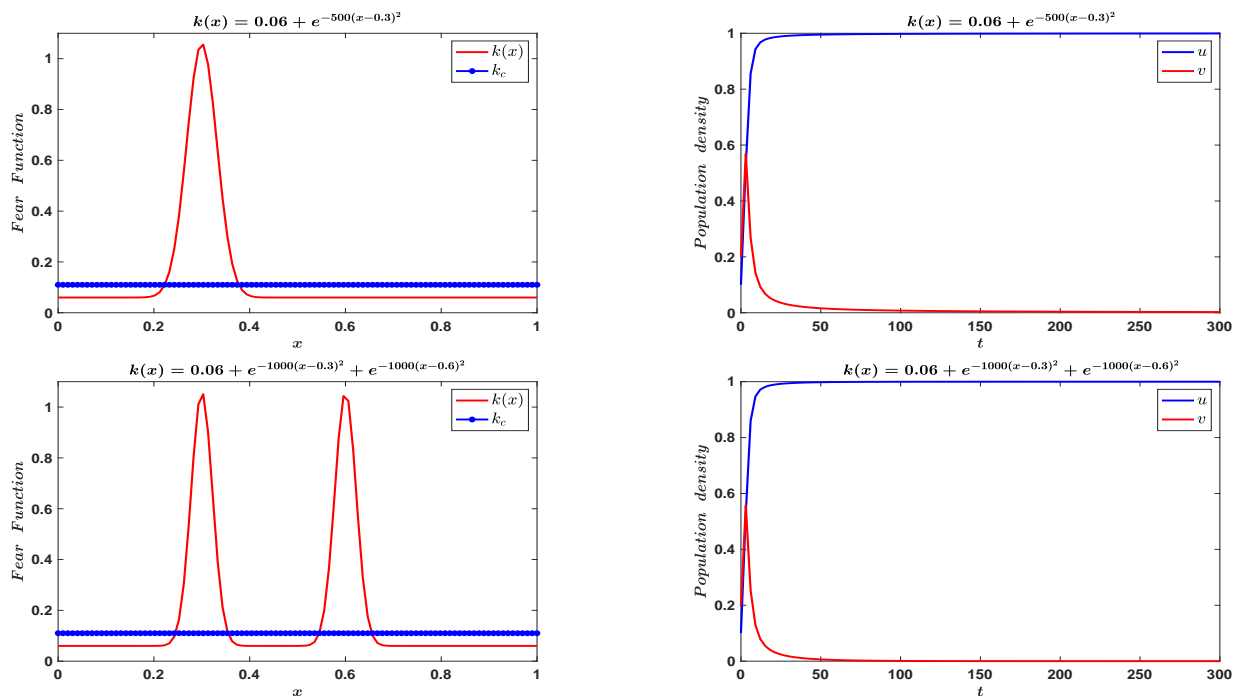
Figure 16. The parameters are chosen as $[u_0, v_0] = [0.1, 0.2], d_1 = 1, d_2 = 1, a_1 = 1, a_2 = 2, b_1 = 1, b_2 = 2, c_1 = 0.3$ and $c_2 = 1.8$. The interior equilibrium $[u^*, v^*] = [0.958904, 0.136986]$ and the fear threshold $k_c = \frac{a_2 b_1^2 - c_2 a_1 b_1}{a_1^2 c_2} = 0.11$ and $C_1 = 1$ and this choice meets the parametric constraint of the Theorem 4.13 and Lemma 4.14.

(a) $f = 0.5$

(b) Various heterogeneous fear functions
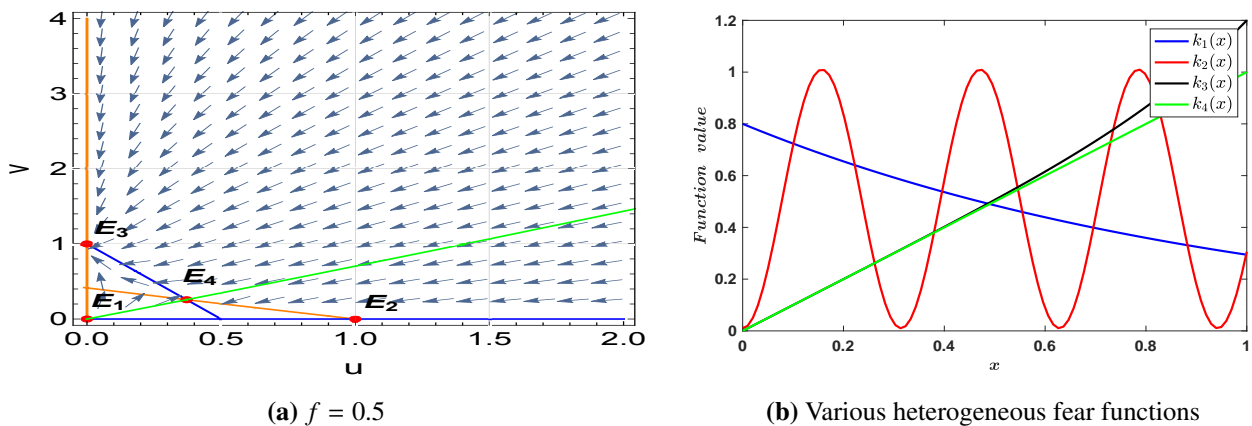
Figure 17. Various heterogeneous fear functions (A) Phase plots showing dynamics under strong competition parametric restriction of system (A.1). Parameters used are $a_1 = 1, a_2 = 1, b_1 = 1, b_2 = 1, c_1 = 2, c_2 = 2$ and $f = 0.5$ (B) Illustration of various heterogeneous fear functions. $k_1(x) = 0.8e^{-x}, k_2(x) = 0.01 + \sin^2(10x), k_3(x) = x + 0.2x^5$ and $k_4(x) = x$.

6. Discussion

In this section we discuss several aspects of the lemmas and theorems described in the current manuscript. We first briefly recap our key results.

- Sufficiently large fear can change a situation of competitive exclusion, to a strong competition type scenario, where there is bi-stability between boundary equilibrium. Dynamically, this occurs via a transcritical bifurcation. This is shown via Lemma 3.12, see Figure 6.

- Fear in a certain parametric regime can change a situation of competitive exclusion to bi-stability between boundary equilibrium and interior equilibrium, see Figure 4. Dynamically, this occurs via a saddle-node bifurcation. This is shown via Lemma 3.11, see Figure 5. This is in sharp contrast with classical competition theory, where bi-stability occurs only between boundary equilibria.

- Sufficiently large fear can change a situation of weak competition to a competitive exclusion type scenario. This is shown via Lemma 3.5, see Figure 2.

- Fear cannot qualitatively change a strong competition type scenario. This is shown via Lemma 3.8, see Figure 3. Also, fear cannot produce periodic orbits. This is demonstrated via Lemma 3.10 and Corollary 1.

- In the spatially explicit setting, comparison theory is used to determine point-wise restrictions on the fear functions such that competitive exclusion or strong competition type dynamics abounds. These are shown via Theorems 4.8, 4.10 and 4.15, see Figures 8, 9 and 12.

- In the spatially explicit setting, fear can change a situation of weak competition to a competitive exclusion type scenario, for fear functions with certain $\mathbb{L}^1$ restrictions. This is shown via Theorem 4.13 and Lemma 4.14, see Figures [14–16]. In particular, the fear functions need not lie uniformly above the critical fear levels derived in the ODE case via Lemma 3.5.

- Various heterogeneous fear functions are constructed to demonstrate these results numerically, see Figure 17b. Applications of these to ecological as well as socio-political settings are discussed in Section 6.

We see that in the competitive exclusion case, where $(u^*, 0)$ is globally stable, a small amount of fear in u, can bring about a strong competition type situation - via a transcritical bifurcation. Where after bifurcation there appears an interior equilibrium, which is a saddle, and one has initial condition dependent attraction to $(u^*, 0)$ or $(0, v^*)$. See Figures 4 and 6]. However, a slightly larger level of fear, can bring down the separatrix drastically see Figure 17a. Furthermore, here (depending on parametric restrictions) only a very small quantity of fear can create two interior equilibriums, see Figure 6, where there are both weak and strong competition dynamics at play, where most initial conditions are attracted to an interior equilibrium, but certain initial conditions (u_0, v_0) (where $u_0 \ll 1, v_0 \gg 1$), are attracted to $(0, v^*)$. It is interesting to think about this in an invasion setting. If u were an invasive species, and v a resident species, then in the absence of fear in u, it can invade and thus exclude v. However, if the resident v can instill just a small amount of fear in the invader u, coexistence for most initial conditions is possible. If one plays devil's advocate, and switches the role of u and v, (u resident, v invader), then we see in the absence of fear in the resident, it will exclude the invader, but if it can instill some fear, then coexistence is possible, as earlier alluded to. However, having too much fear in either case, yields only one interior equilibrium, which is a saddle and we are in a strong competition type setting, with initial condition dependent attraction to either $(u^*, 0)$ or $(0, v^*)$. Thus if coexistence (for most initial conditions) is sought after, it is advantageous to induce a little fear - but not too much fear. Dynamically, this is seen because in the former case we have a saddle-node bifurcation occurring first, followed by a transcritical bifurcation, see Figure 6b, as opposed to the later case, where only a transcritical bifurcation occurs, see Figure 6a. Also, a 2 parameter bifurcation in the $k - c_1$ parameter space reveals a rich higher co-dimensional bifurcation structure. A Bogdanov Takens (BT) bifurcation, as well as a saddle node transcritical (SNTC) bifurcation is observed. Future work could entail further theoretical investigations of the SNTC [74], herein numerical investigations are also apt [75,76]. In this work we assume fear comes from the physical presence of a competitor. However even past presence such as by scent or other chemical cues can cause fear [77]. It is worthwhile considering modeling such a scenario in future work.

This phenomenon becomes even more interesting in sociological or political settings, particularly when we are in the spatially explicit scenario. Consider the case of law enforcement trying to control crime in high crime areas, where traditionally control efforts have failed, indicative of the competitive exclusion scenario, or a coexistence scenario, where the criminal groups have not been managed so coexist at high levels, and have not been eradicated. The question becomes, what is the optimal design of the fear function $k(x)$, that could now yield a co-existence scenario—or one that can yield a competitive exclusion scenario. We see via Theorem 4.13, that this function could be very small in some areas of the spatial domain, and large enough in others, so that it would change dynamics - this is also seen numerically, where "hotspot" type fear functions have been utilized to generate a competitive exclusion type scenario, see Figures 14–16. Similar ideas for control and policing activities have been explored in [78,79]. Essentially, the fear function can be modeled through various functional types, that also preserve spatial features of the underlying domain. For instance, consider competition between two warring drug cartels, where the weaker cartel has certain territorial strongholds. This situation can be modeled through oscillatory functions like $\sin^2(nx)$, where $n \in \mathbb{N}$.

Moreover, for complex ecological models, the result of these theorems also holds, where we need to model the fear responses by a sequence of functions or in the form of piece-wise functions. We display some of these functions in Figure 17b. Many of the relevant theorems have been tested by choosing such functions (See Figures 8, 10, 12 and 13). However, we have not incorporated non-smooth fear function or non-smooth density dependence herein [31, 80–82], and this remains an apt direction for future work. In essence the fear model considered can be placed in a much broader context than ecological. Corollary 1 enables us to provide a mechanistic description of the fear function in much broader contexts. Also, in sociopolitical domains, the fear effect does not have to be thought of as consumptive vs non-consumptive. For example a new small business that is fearful of a larger established business might not invest in advertising or expanding clientele in established market, rather search for niche markets - which might hinder its growth. Interesting future work could consider competition models with a fear effect where the outcome of the competition whether contest or scramble was explicit.

In the weak competition case without fear, once there is a critical level of fear, in either u or v, a competitive exclusion type phenomenon will occur, see Lemma 3.5. This occurs again via a transcritical bifurcation, see Theorem 3.12 and Figure 6. This again is interesting in the spatially explicit setting, as via Theorem 4.8, where the fear function need not be above k_c so as to induce competitive exclusion. We have explored certain types of fear functions herein, see Figures 8,14–16, but the effect of functions such as in Figure 17b, remain to be explored. Here one could look at the effect of fear in fragmented domains vs domains that are not, via deriving or enforcing conditions relating the fear function $k(x)$ to the resource function $m(x)$, see similar ideas explored in [83]. The results are symmetric even if u is assumed to be fearful of v, and one considers the spatially explicit case, i.e., when a competitive exclusion situation, can change to a bi-stability type situation, where the bi-stability is between the $(u^*, 0)$ and a coexistence equilibrium. A further increase of the fear function causes a strong stability situation to occur. Similar dynamics are seen in changing a weak competition case to a competitive exclusion case. We are also motivated to studying the fear effect in competitive systems modeled differently - such as bringing in this effect in the competitive coefficients, rather than the birth rate. In addition, for competitive systems, one might intuitively think that the "dominant" population causes fear in the inferior population, which further weakens the inferior population. However, simulations show this is incorrect. Particularly if one considers fear in both competitors. The interplay of the heterogenous fear functions $k(x), f(x)$, can actually lead to a co-existence scenario, even when one competitor is dominant with and without the fear effect. This counterintuitive situation warrants much further investigation.

An interesting question is the dynamical consequence of Allee effects on the fear models considered. Note, Allee effects in competition systems have been considered [84, 85], and results show the possibility of alternate stable states, and even tri-stability situations possible, where the occurrence of two interior saddle equilibrium, cause the phase to be partitioned such that initial data can be attracted to $(u^*, 0)$, $(0, v^*)$ or $(0,0)$. However, whether a weak or strong Allee effect is considered, periodic orbits have not been shown to occur [84–86]. The possibility of two interior saddles does not preclude the existence of heteroclinic orbits, say via a saddle-saddle collision–although such dynamics have not been rigorously proved to the best of our knowledge. The *interplay* of the fear effect with an Allee effect could well change this situation. For example with a simple strong Allee effect the Dulac function used in Lemma 3.10 would not be sufficient to claim

non-existence of limit cycle dynamics. Thus such investigations would make for interesting future studies. It would be interesting to explore applications of these results to the case of competing political parties, where a nationally weaker party with smaller voter bank, could have several local strongholds. What levels of fear need to be instilled in those strongholds so as to enforce competitive advantage, is an apt question. Herein estimates on the $\mathbb{L}^1(\Omega)$ norm of $k(x)$, would be interesting to derive. Currently Lemma 4.14, only gives lower estimates - but sharp upper estimates or even just upper estimates are unknown, and would make for interesting future work. The effect of fear on the strong competition setting is perhaps the least interesting dynamically. Herein, having fear in u or v, only shifts the interior saddle equilibrium - but qualitatively the dynamics remain the same, as in a strong competition type scenario persists, no matter what level of fear (f or k) is chosen. Also it would be of interest to rigorously prove Conjectures 1 and 2. Our current numerical evidence clearly motivates their validity. It would also be of interest to consider fractional derivative extensions of the fear model (3.1). Such problems have been much investigated recently [87–90].

Also, in [8], it is found that various monotonically decreasing fear functions essentially yield the same dynamics. This has not been tested in the current work, in the case of competitive systems. Neither have we investigated rigorously the case of both competitors being fearful of each other with possible different fear functions, indicative of different LOF for each competitor. Here again, upper estimates on the fear functions would be useful and we could allude to methods and techniques explored in [91] and in [92]. Here, one could consider a spatially dependent growth function as well, such as in [92], and attempt to derive conditions relating the function describing the resources to the fear function. Furthermore, it would make for interesting future work if certain choices of (density) dependence or fear functions which lead to degenerate dynamics, can cause periodic orbits. In the current scenario Lemma 3.10 and Corollary 1, do not allow limit cycle dynamics. All in all, we hope these questions lead to future investigations of the fear effect in competitive systems, as a host of rich applications exist.

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Conflict of interest

The author declare that they have no conflict of interest.

References

1. M. W. Dictionary, Merriam-webster, Available from: https://www.merriam-webster.com/.

2. H. S. Bracha, Freeze, flight, fight, fright, faint: Adaptationist perspectives on the acute stress response spectrum, *CNS Spectrums*, **9** (2004), 679–685. https://doi.org/10.1017/S1092852900001954

3. W. Cresswell, Predation in bird populations, *J. Ornithol.*, **152** (2011), 251–263. https://doi.org/10.1007/s10336-010-0638-1

4. M. J. Sheriff, S. D. Peacor, D. Hawlena, M. Thaker, Non-consumptive predator effects on prey population size: a dearth of evidence, *J. Animal Ecol.*, **89** (2020), 1302–1316. https://doi.org/10.1111/1365-2656.13213

5. S. D. Peacor, B. L. Peckarsky, G. C. Trussell, J. R. Vonesh, Costs of predator-induced phenotypic plasticity: a graphical model for predicting the contribution of nonconsumptive and consumptive effects of predators on prey, *Oecologia*, **171** (2013), 1–10. https://doi.org/10.1007/s00442-012-2394-9

6. B. L. Peckarsky, P. A. Abrams, D. I. Bolnick, L. M. Dill, J. H. Grabowski, B. Luttbeg, et al., Revisiting the classics: considering nonconsumptive effects in textbook examples of predator–prey interactions, *Ecology*, **89** (2008), 2416–2425. https://doi.org/10.1890/07-1131.1

7. J. S. Brown, J. W. Laundré, M. Gurung, The ecology of fear: optimal foraging, game theory, and trophic interactions, *J. Mammal.*, **80** (1999), 385–399. https://doi.org/10.2307/1383287

8. X. Wang, L. Zanette, X. Zou, Modelling the fear effect in predator–prey interactions, *J. Math. Biol.*, **73** (2016), 1179–1204. https://doi.org/10.1007/s00285-016-0989-1

9. P. A. Abrams, The evolution of predator-prey interactions: theory and evidence, *Ann. Rev. Ecol. Syst.*, **31** (2000), 79–105. https://doi.org/10.1146/annurev.ecolsys.31.1.79

10. P. Panday, N. Pal, S. Samanta, P. Tryjanowski, J. Chattopadhyay, Dynamics of a stage-structured predator-prey model: cost and benefit of fear-induced group defense, *J. Theor. Biol.*, **528** (2021), 110846. https://doi.org/10.1016/j.jtbi.2021.110846

11. J. Wang, Y. Cai, S. Fu, W. Wang, The effect of the fear factor on the dynamics of a predator-prey model incorporating the prey refuge, *Chaos Interdiscip. J. Nonlinear Sci.*, **29** (2019), 083109. https://doi.org/10.1063/1.5111121

12. H. Zhang, Y. Cai, S. Fu, W. Wang, Impact of the fear effect in a prey-predator model incorporating a prey refuge, *Appl. Math. Comput.*, **356** (2019), 328–337. https://doi.org/10.1016/j.amc.2019.03.034

13. X. Wang, X. Zou, Modeling the fear effect in predator–prey interactions with adaptive avoidance of predators, *Bull. Math. Biol.*, **79** (2017), 1325–1359. https://doi.org/10.1007/s11538-017-0287-0

14. P. Mikula, F. Morelli, R. K. Lučan, D. N. Jones, P. Tryjanowski, Bats as prey of diurnal birds: a global perspective, *Mammal Rev.*, **46** (2016), 160–174. https://doi.org/10.1111/mam.12060

15. R. M. Pringle, T. R. Kartzinel, T. M. Palmer, T. J. Thurman, K. Fox-Dobbs, C. C. Xu, et al., Predator-induced collapse of niche structure and species coexistence, *Nature*, **570** (2019), 58–64. https://doi.org/10.1038/s41586-019-1264-6

16. S. Pal, S. Majhi, S. Mandal, N. Pal, Role of fear in a predator–prey model with Beddington–Deangelis functional response, *Z. Naturforsch. A*, **74** (2019), 581–595. https://doi.org/10.1515/zna-2018-0449

17. M. Das, G. Samanta, A prey-predator fractional order model with fear effect and group defense, *Int. J. Dyn. Control*, **9** (2021), 334–349. https://doi.org/10.1080/21642583.2021.1907259

18. S. Pal, N. Pal, S. Samanta, J. Chattopadhyay, Effect of hunting cooperation and fear in a predator-prey model, *Ecol. Complexity*, **39** (2019), 100770. https://doi.org/10.1016/j.ecocom.2019.100770

19. A. Yousef, A. A. Thirthar, A. L. Alaoui, P. Panja, T. Abdeljawad, The hunting cooperation of a predator under two prey's competition and fear-effect in the prey-predator fractional-order model, *AIMS Math.*, **7** (2022), 5463–5479.

20. P. Panday, N. Pal, S. Samanta, J. Chattopadhyay, Stability and bifurcation analysis of a three-species food chain model with fear, *Int. J. Bifurcation Chaos*, **28** (2018), 1850009. https://doi.org/10.1142/S0218127418500098

21. P. Panday, N. Pal, S. Samanta, J. Chattopadhyay, A three species food chain model with fear induced trophic cascade, *Int. J. Appl. Comput. Math.*, **5** (2019), 1–26. https://doi.org/10.1007/s40819-018-0585-8

22. H. Verma, K. Antwi-Fordjour, M. Hossain, N. Pal, R. D. Parshad, P. Mathur, A "double" fear effect in a tri-trophic food chain model, *Eur. Phys. J. Plus*, **136** (2021), 1–17. https://doi.org/10.1140/epjp/s13360-020-01001-7

23. A. Das, G. Samanta, Modeling the fear effect on a stochastic prey–predator system with additional food for the predator, *J. Phys. A Math. Theor.*, **51** (2018), 465601. https://doi.org/10.1088/1751-8121/aae4c6

24. S. Wu, J. Wang, J. Shi, Dynamics and pattern formation of a diffusive predator–prey model with predator-taxis, *Math. Models Methods Appl. Sci.*, **28** (2018), 2275–2312. https://doi.org/10.1142/S0218202518400158

25. X. Wang, X. Zou, Pattern formation of a predator-prey model with the cost of anti-predator behaviors, *Math. Biosci. Eng.*, **15** (2018), 775. https://doi.org/10.1088/1742-2140/aaa076

26. D. Duan, B. Niu, J. Wei, Hopf-Hopf bifurcation and chaotic attractors in a delayed diffusive predator-prey model with fear effect, *Chaos Solitons Fractals*, **123** (2019), 206–216. https://doi.org/10.1016/j.chaos.2019.04.012

27. Y. Lou, Some challenging mathematical problems in evolution of dispersal and population dynamics, in *Tutorials in Mathematical Biosciences IV*, (2008), 171–205.

28. P. J. den Boer, The present status of the competitive exclusion principle, *Trends Ecol. Evol.*, **1** (1986), 25–28. https://doi.org/10.1016/0169-5347(86)90064-9

29. P. Chesson, Mechanisms of maintenance of species diversity, *Ann. Rev. Ecol. Syst.*, **31** (2000), 343–366. https://doi.org/10.1146/annurev.ecolsys.31.1.343

30. A. Okubo, P. K. Maini, M. H. Williamson, J. D. Murray, On the spatial spread of the grey squirrel in Britain, *Proc. R. Soc. London. B. Biol. Sci.*, **238** (1989), 113–125. https://doi.org/10.1098/rspb.1989.0070

31. J. Sugie, R. Kohno, R. Miyazaki, On a predator-prey system of Holling type, *Proc. Am. Math. Soc.*, **125** (1997), 2041–2050. https://doi.org/10.1090/S0002-9939-97-03901-4

32. M. Arim, S. R. Abades, P. E. Neill, M. Lima, P. A. Marquet, Spread dynamics of invasive species, *Proc. Nat. Acad. Sci.*, **103** (2006), 374–378. https://doi.org/10.1073/pnas.0504272102

33. G. A. Polis, C. A. Myers, R. D. Holt, The ecology and evolution of intraguild predation: potential competitors that eat each other, *Ann. Rev. Ecol. Syst.*, **20** (1989), 297–330. https://doi.org/10.1146/annurev.es.20.110189.001501

34. X. Gu, K. Zhang, Z. Q. Zhang, Non-consumptive effects of intraguild predator Blattisocius dentriticus (Berlese) on the development and prey consumption of Neoseiulus cucumeris (Oudemans), *Syst. Appl. Acarol.*, **27** (2022), 1475–1482.

35. R. Van Driesche, T. S. Bellows, *Biological Control*, Springer Science & Business Media, 2012.

36. L. L. Long, J. D. Wolfe, Review of the effects of barred owls on spotted owls, *J. Wildlife Manage.*, **83** (2019), 1281–1296.

37. N. J. Van Lanen, A. B. Franklin, K. P. Huyvaert, R. F. Reiser II, P. C. Carlson, Who hits and hoots at whom? Potential for interference competition between barred and northern spotted owls, *Biol. Conserv.*, **144** (2011), 2194–2201. https://doi.org/10.1016/j.biocon.2011.05.011

38. R. Gutiérrez, M. Cody, S. Courtney, D. Kennedy, Assessment of the potential threat of the northern barred owl, *Sci. Eval. Status Northern Spotted Owl*, **2004** (2004), 1–51.

39. J. D. Wiens, R. G. Anthony, E. D. Forsman, Competitive interactions and resource partitioning between northern spotted owls and barred owls in western Oregon, *Wildlife Monogr.*, **185** (2014), 1–50. https://doi.org/10.1002/wmon.1009

40. O. J. Schmitz, Fearful effects on ecological competitors, *Nature*, **570** (2019), 43–44. https://doi.org/10.1038/d41586-019-01712-7

41. B. L. Peckarsky, P. A. Abrams, D. I. Bolnick, L. M. Dill, J. H. Grabowski, B. Luttbeg, et al., Revisiting the classics: considering nonconsumptive effects in textbook examples of predator–prey interactions, *Ecology*, **89** (2008), 2416–2425. https://doi.org/10.1890/07-1131.1

42. O. Schmitz, Predator and prey functional traits: understanding the adaptive machinery driving predator–prey interactions, *F1000Res.*, **6** (2017).

43. J. Michaud, P. R. Barbosa, C. L. Bain, J. B. Torres, Extending the "ecology of fear" beyond prey: reciprocal nonconsumptive effects among competing aphid predators, *Environ. Entomol.*, **45** (2016), 1398–1403.

44. M. H. Bayoumy, H. S. Awadalla, D. M. Fathy, T. M. Majerus, Beyond killing: intra-and interspecific nonconsumptive effects among aphidophagous competitors, *Ecol. Entomol.*, **43** (2018), 794–803.

45. P. Chesson, J. J. Kuang, The interaction between predation and competition, *Nature*, **456** (2008), 235–238. https://doi.org/10.1038/nature07248

46. M. Arim, P. A. Marquet, Intraguild predation: a widespread interaction related to species biology, *Ecol. Lett.*, **7** (2004), 557–564.

47. S.-E. Byun, S. Han, H. Kim, C. Centrallo, Us small retail businesses' perception of competition: Looking through a lens of fear, confidence, or cooperation, *J. Retailing Consumer Serv.*, **52** (2020), 101925.

48. Ø. Moen, The relationship between firm size, competitive advantages and export performance revisited, *Int. Small Bus. J.*, **18** (1999), 53–72. https://doi.org/10.1177/0266242699181003

49. H. Löfgren, The communist party of India (marxist) and the left government in west Bengal, 1977–2011: Strains of governance and socialist imagination, *Stud. Indian Polit.*, **4** (2016), 102–115. https://doi.org/10.1177/2321023016634947

50. T. Wainwright, *Narconomics: How to Run a Drug Cartel*, PublicAffairs, 2016.

51. J. P. Sullivan, R. J. Bunker, Drug cartels, street gangs, and warlords, *Small Wars Insurgencies*, **13** (2002), 40–53. https://doi.org/10.1080/09592310208559180

52. J. D. Murray, *Mathematical Biology*, Springer, 1993. https://doi.org/10.1007/978-3-662-08542-4

53. R. S. Cantrell, C. Cosner, *Spatial Ecology via Reaction-Diffusion Equations*, John Wiley & Sons, 2004.

54. A. Okubo, S. A. Levin, *Diffusion and Ecological Problems: Modern Perspectives*, Springer, 2001. https://doi.org/10.1007/978-1-4757-4978-6

55. J. Dockery, V. Hutson, K. Mischaikow, M. Pernarowski, The evolution of slow dispersal rates: a reaction diffusion model, *J. Math. Biol.*, **37** (1998), 61–83. https://doi.org/10.1007/s002850050120

56. X. He, W. M. Ni, The effects of diffusion and spatial variation in Lotka–Volterra competition–diffusion system I: heterogeneity vs. homogeneity, *J. Differ. Equations*, **254** (2013), 528–546. https://doi.org/10.1016/j.jde.2012.08.032

57. A. Hastings, Can spatial variation alone lead to selection for dispersal?, *Theor. Popul. Biol.*, **24** (1983), 244–251. https://doi.org/10.1016/0040-5809(83)90027-8

58. X. He, W. M. Ni, The effects of diffusion and spatial variation in Lotka–Volterra competition–diffusion system II: the general case, *J. Differ. Equations*, **254** (2013), 4088–4108. https://doi.org/10.1016/j.jde.2013.02.009

59. X. He, W. M. Ni, Global dynamics of the Lotka–Volterra competition–diffusion system with equal amount of total resources, II, *Cal. Var. Partial Differ. Equations*, **55** (2016), 1–20. https://doi.org/10.1007/s00526-015-0942-y

60. H. Ninomiya, Separatrices of competition-diffusion equations, in *Reaction-diffusion Equations And Their Applications And Computational Aspects-Proceedings Of The China-Japan Symposium*, 1997, 118.

61. K. Y. Lam, W. M. Ni, Uniqueness and complete dynamics in heterogeneous competition-diffusion systems, *SIAM J. Appl. Math.*, **72** (2012), 1695–1712. https://doi.org/10.1137/120869481

62. Y. Lou, On the effects of migration and spatial heterogeneity on single and multiple species, *J. Differ. Equations*, **223** (2006), 400–426. https://doi.org/10.1016/j.jde.2005.05.010

63. Y. Lou, S. Martínez, P. Poláčik, Loops and branches of coexistence states in a Lotka–Volterra ompetition model, *J. Differ. Equations*, **230** (2006), 720–742. https://doi.org/10.1016/j.jde.2006.04.005

64. W. M. Ni, Complete dynamics in a heterogeneous competition-diffusion system, *East China Normal Univ. Univ. Minnesota*, **2012** (2012).

65. L. Perko, *Differential Equations and Dynamical Systems*, Springer Science & Business Media, 2013.

66. D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Springer, 2006.

67. J. Morgan, Global existence for semilinear parabolic systems, *SIAM J. Math. Anal.*, **20** (1989), 1128–1144.

68. M. Pierre, Global existence in reaction-diffusion systems with control of mass: a survey, *Milan J. Math.*, **78** (2010), 417–455. https://doi.org/10.1007/s00032-010-0133-4

69. K. Kishimoto, H. F. Weinberger, The spatial homogeneity of stable equilibria of some reaction-diffusion systems on convex domains, *J. Differ. Equations*, **58** (1985), 15–21. https://doi.org/10.1016/0022-0396(85)90020-8

70. S. K. Sasmal, Y. Takeuchi, Dynamics of a predator-prey system with fear and group defense, *J. Math. Anal. Appl.*, **481** (2020), 123471.

71. Y. Du, Effects of a degeneracy in the competition model: part I. classical and generalized steady-state solutions, *J. Differ. Equations*, **181** (2002), 92–132. https://doi.org/10.1006/jdeq.2001.4074

72. Y. Du, Effects of a degeneracy in the competition model: Part II. perturbation and dynamical behaviour, *J. Differ. Equations*, **181** (2002), 133–164. https://doi.org/10.1006/jdeq.2001.4075

73. D. Gilbarg, N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, 1977.

74. M. Krupa, M. Wechselberger, Local analysis near a folded saddle-node singularity, *J. Differ. Equations*, **248** (2010), 2841–2888. https://doi.org/10.1016/j.jde.2010.02.006

75. V. Hajnová, L. Přibylová, Bifurcation manifolds in predator–prey models computed by Gröbner basis method, *Math. Biosci.*, **312** (2019), 1–7. https://doi.org/10.1016/j.mbs.2019.03.008

76. L. van Veen, M. Hoti, Automatic detection of saddle-node–transcritical interactions, *Int. J. Bifurcation Chaos*, **29** (2019), 1950104. https://doi.org/10.1142/S0218127419501049

77. A. Ekner, P. Tryjanowski, Do small hole nesting passerines detect cues left by a predator? A test on winter roosting sites, *Acta Ornithol.*, **43** (2008), 107–111. https://doi.org/10.3161/000164508X345392

78. N. Rodríguez, Q. Wang, L. Zhang, Understanding the effects of on-and off-hotspot policing: Evidence of hotspot, oscillating, and chaotic activities, *SIAM J. Appl. Dyn. Syst.*, **20** (2021), 1882–1916.

79. H. Berestycki, N. Rodriguez, L. Ryzhik, Traveling wave solutions in a reaction-diffusion model for criminal activity, *Multiscale Model. Simul.*, **11** (2013), 1097–1126. https://doi.org/10.1137/12089884X

80. R. D. Parshad, K. Antwi-Fordjour, E. M. Takyi, Some novel results in two species competition, *SIAM J. Appl. Math.*, **81** (2021), 1847–1869. https://doi.org/10.1137/20M1387274

81. R. D. Parshad, S. Wickramsooriya, S. Bailey, A remark on "biological control through provision of additional food to predators: A theoretical study", *Theor. Popul. Biol.*, **132** (2020), 60–68. https://doi.org/10.1016/j.tpb.2007.03.011

82. A. P. Farrell, J. P. Collins, A. L. Greer, H. R. Thieme, Do fatal infectious diseases eradicate host species?, *J. Math. Biol.*, **77** (2018), 2103–2164. https://doi.org/10.1007/s00285-018-1249-3

83. D. L. DeAngelis, W. M. Ni, B. Zhang, Dispersal and spatial heterogeneity: single species, *J. Math. Biol.*, **72** (2016), 239–254. https://doi.org/10.1007/s00285-015-0879-y

84. G. Wang, X. G. Liang, F. Z. Wang, The competitive dynamics of populations subject to an Allee effect, *Ecol. Model.*, **124** (1999), 183–192. https://doi.org/10.1016/S0304-3800(99)00160-X

85. M. De Silva, S. R. J. Jang, Competitive exclusion and coexistence in a Lotka–Volterra competition model with Allee effects and stocking, *Lett. Biomath.*, **2** (2015), 56–66. https://doi.org/10.30707/LiB2.1DeSilva

86. P. A. Naik, Z. Eskandari, M. Yavuz, J. Zu, Complex dynamics of a discrete-time Bazykin–Berezovskaya prey-predator model with a strong Allee effect, *J. Comput. Appl. Math.*, **413** (2022), 114401. https://doi.org/10.1016/j.cam.2022.114401

87. M. Yavuz, N. Sene, Stability analysis and numerical computation of the fractional predator–prey model with the harvesting rate, *Fractal Fract.*, **4** (2020), 35. https://doi.org/10.3390/fractalfract4030035

88. Z. Hammouch, M. Yavuz, N. Özdemir, Numerical solutions and synchronization of a variable-order fractional chaotic system, *Math. Model. Numer. Simul. Appl.*, **1** (2021), 11–23. https://doi.org/10.53391/mmnsa.2021.01.002

89. M. Gholami, R. K. Ghaziani, Z. Eskandari, Three-dimensional fractional system with the stability condition and chaos control, *Math. Model. Numer. Simul. Appl.*, **2** (2022), 41–47. https://doi.org/10.53391/mmnsa.2022.01.004

90. H. Joshi, B. K. Jha, Chaos of calcium diffusion in Parkinson's infectious disease model and treatment mechanism via Hilfer fractional derivative, *Math. Model. Numer. Simul. Appl.*, **1** (2021), 84–94.

91. I. Mazari, G. Nadin, Y. Privat, Optimal location of resources maximizing the total population size in logistic models, *J. Math. Pures Appl.*, **134** (2020), 1–35. https://doi.org/10.1016/j.matpur.2019.10.008

92. D. DeAngelis, B. Zhang, W. M. Ni, Y. Wang, Carrying capacity of a population diffusing in a heterogeneous environment, *Mathematics*, **8** (2020), 49. https://doi.org/10.3390/math8010049

Appendix

A.1. The case of u fearing v

Here we shall consider the case when the competitor u is being fearful of v. Modeling as earlier yields,

$$\frac{du}{dt} = \frac{a_1 u}{1 + fv} - b_1 u^2 - c_1 uv,$$
$$\frac{dv}{dt} = a_2 v - b_2 v^2 - c_2 uv. \tag{A.1}$$

A.2. Existence

The system (A.1) can have one or two positive interior equilibria, but not three. This is established via the following lemma:

Lemma A.2.1. *Consider the ODE system* (A.1), *with* $f = 0$. *If we are in the competitive exclusion, weak-competition, or strong-competition setting, then for any* $f > 0$, *under the parametric restrictions in* (A.2), *there do not exist three positive interior equilibria.*

Proof. On substituting $u^* = \frac{1}{b_1}(\frac{a_1}{1+fv^*} - c_1 v^*)$ in the second nullcline equation, we get the following third order polynomial in v^*:

$$B_2(v^*)^3 + C_2(v^*)^2 + D_2(v^*) + E_2 = 0,$$

where

$$\begin{aligned}
B_2 &= b_1 f^2 (c_1 c_2 - b_1 b_2), \\
C_2 &= a_2 b_1^2 f^2 + 2 b_1 f (c_1 c_2 - b_2 b_1), \\
D_2 &= b_1 f (-a_1 c_2 + 2 a_2 b_1) + b_1 (c_1 c_2 - b_2 b_1), \\
E_2 &= b_1 (-a_1 c_2 + a_2 b_1).
\end{aligned} \tag{A.1}$$

We will require the following conditions to satisfy Descartes's rule of signs, so as to obtain three positive roots to the cubic equation. These are:

$$\begin{aligned}
c_1 c_2 - b_1 b_2 &< 0, \\
a_2 b_1 f + 2(c_1 c_2 - b_2 b_1) &> 0, \\
f(-a_1 c_2 + 2 a_2 b_1) &< -(c_1 c_2 - b_2 b_1), \\
-a_1 c_2 + a_2 b_1 &> 0.
\end{aligned} \tag{A.2}$$

Let us prove this is an impossible claim by contradiction: First, assume the parameter set satisfies both competitive exclusion-state parametric restriction and (A.2). We know that competitive exclusion-state is asymptotically stable if $\frac{a_1}{a_2} > \max\left\{\frac{b_1}{c_2}, \frac{c_1}{b_2}\right\}$. On using these parametric restriction, $-a_1 c_2 + a_2 b_1 < 0$, which is a contradiction to the last inequality in (A.2).

If the parameter set satisfies both the strong competition state parametric restriction and (A.2), then we have a contradiction because of the first inequality in (A.2), as under the strong competition state parametric restriction that inequality should be positive.

For the weak competition, recall the parametric restrictions:

$$\frac{c_2}{b_1} < \frac{a_2}{a_1} < \frac{b_2}{c_1}.$$

Let us re-write the third inequality in (A.2),

$$-(c_1 c_2 - b_2 b_1) - f(-a_1 c_2 + 2 a_2 b_1) > 0.$$

On adding the second and third inequality in (A.2), we have

$$(c_1 c_2 - b_1 b_2) + f(a_1 c_2 - a_2 b_1) > 0,$$

which is a contradiction as the added inequality should be negative by the parametric restrictions of weak-competition. This proves the lemma.

$\square$

Also note:

Remark 12. The cubic equation obtained in (A.1) can be transformed to a quadratic polynomial by reducing $(b_1 f v^* + b_1)$ on both sides, i.e.,

$$\widetilde{B_2}(v^*)^2 + \widetilde{C_2}v^* + \widetilde{D_2} = 0,$$

where

$$\widetilde{B_2} = f(c_1c_2 - b_1b_2), \quad \widetilde{C_2} = (c_1c_2 - b_1b_2) + a_2b_1f, \quad \widetilde{D_2} = a_2b_1 - a_1c_2.$$

Thus in essence models (3.1) and (A.1) are symmetric, and it suffices to consider (3.1).