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Research article

A discrete mixed distribution: Statistical and reliability properties with applications to model COVID-19 data in various countries

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Abstract: The aim of this paper is to introduce a discrete mixture model from the point of view of reliability and ordered statistics theoretically and practically for modeling extreme and outliers' observations. The base distribution can be expressed as a mixture of gamma and Lindley models. A wide range of the reported model structural properties are investigated. This includes the shape of the probability mass function, hazard rate function, reversed hazard rate function, min-max models, mean residual life, mean past life, moments, order statistics and L-moment statistics. These properties can be formulated as closed forms. It is found that the proposed model can be used effectively to evaluate over- and under-dispersed phenomena. Moreover, it can be applied to analyze asymmetric data under extreme and outliers' notes. To get the competent estimators for modeling observations, the maximum likelihood approach is utilized under conditions of the Newton-Raphson numerical technique. A simulation study is carried out to examine the bias and mean squared error of the estimators. Finally, the flexibility of the discrete mixture model is explained by discussing three COVID-19 data sets.

Keywords: mixed distributions; discretization technique; failure analysis; simulation; Covid-19; statistics and numerical data

1. Introduction

The data generated by the daily work environment are more complex in nature at present, and therefore many lifetime models have been listed in the literature to analyze and evaluate these data. Determining which probability distribution should be adopted to make inferences from the data under study is a very important problem in statistics. For these reasons, great efforts have been spent over the years in developing large categories of distributions along with relevant statistical methodologies.

See, for instance, El-Gohary et al. [1], Saboor et al. [2], Jia et al. [3], Fernandez [4], Alizadeh et al. [5], Kumar et al. [6], and references cited therein. Nedjar and Zeghdoudi [7] proposed a mixture of gamma$(2, \tau)$ and Lindley(ε) (MGL) distributions. The probability density function (PDF) of the MGL distribution can be expressed as

$$g(x; \varepsilon, \tau) = \frac{\tau^2}{\varepsilon(1+\tau)} \left([\tau\varepsilon + \varepsilon - \tau]\, x + 1\right) e^{-\tau x};\ x > 0, \varepsilon > 0, \tau > 0. \tag{1.1}$$

Unfortunately, Eq (1.1) is not a proper PDF. Messaadia and Zeghdoudi [8] corrected the parameter space to be $\varepsilon \geq \frac{\tau}{1+\tau}$ and $\tau > 0$, and consequently the modified PDF of the MGL model can be written as

$$f(x; \varepsilon, \tau) = \frac{\tau^2}{\varepsilon(1+\tau)} \left([\tau\varepsilon + \varepsilon - \tau]\, x + 1\right) e^{-\tau x};\ x > 0, \varepsilon \geq \frac{\tau}{1+\tau}, \tau > 0. \tag{1.2}$$

The survival function (SF) corresponding to Eq (1.2) can be formulated as

$$S(x; \varepsilon, \tau) = \frac{(\tau x + 1)(\tau\varepsilon + \varepsilon - \tau) + \tau}{\varepsilon(1+\tau)} e^{-\tau x};\ x > 0, \varepsilon \geq \frac{\tau}{1+\tau}, \tau > 0. \tag{1.3}$$

The quantile function (QF) is

$$Q_X(u) = \begin{cases} -\frac{\varepsilon(1+\tau)}{\tau[\varepsilon(1+\tau)-\tau]} - \frac{1}{\tau} W_{-1}\left(\frac{\varepsilon(1+\tau)(u-1)}{\varepsilon(1+\tau)-\tau} e^{-\frac{\varepsilon(1+\tau)}{\varepsilon(1+\tau)-\tau}}\right); & \varepsilon > \frac{\tau}{1+\tau} \\ \frac{-\ln(1-u)}{\tau} & ;\ \ \varepsilon = \frac{\tau}{1+\tau}, \end{cases} \tag{1.4}$$

where W_{-1} denotes the negative branch of the Lambert W function, and $\frac{-\ln(1-u)}{\tau}$ is the QF of the exponential model. Sometimes, survival trials produce data that are discrete in nature either because of the limitations of the measuring instruments or their inherent characteristics. The study and analysis of counting data plays an important role in many fields of applied sciences, such as economics, engineering, marketing, medicine, and insurance. Counting datasets are often modeled utilizing the Poisson model. However, the Poisson model cannot handle hyper-scattered datasets. Therefore, it is reasonable to model such cases via an appropriate discrete distribution. Discretization of a continuous distribution can be created by using several methods. The most widely utilized technique is the survival discretization approach. For a given continuous random variable X with SF $S(x; \xi) = \Pr(X > x)$, we can obtain the discretized version as

$$\Pr(X = x) = S(x; \xi) - S(x+1; \xi);\ x = 0, 1, 2, 3, ... \tag{1.5}$$

For more details, Roy [9]. This technique has received a lot of attention in recent years. See, for instance, Gómez-Déniz and Calderín-Ojeda [10], Bebbington et al. [11], Nekoukhou et al. [12], Alamatsaz et al. [13], El-Morshedy et al. [14], Gillariose et al. [15], Singh et al. [16,17], Eliwa and El-Morshedy [18], Altun et al. [19] and references cited therein. In this paper, a discrete distribution MGL (DsMGL) will be discussed from the point of view of reliability and ordered statistics theoretically and practically to analyze extreme and outliers' notes. This is because in El-Morshedy et al. [20], simple statistical characteristics and a regression model were presented only in a small whole sample (outliers were not included). Further, the previous paper ignored the reliability, order statistics, and L-moments measures which can be applied in the fields of biomedicine and engineering. Given the importance of reliability and structured statistical measures, the authors sought to discuss neglected characteristics as well as model extreme and outliers' observations. Thus, the motives for this study can be summarized as follows: to formulate statistical characteristics as closed forms; to model dispersed-positively-skewed real data under extreme and outliers' observations; to provide a consistently better fit than other discrete models known in the current statistical literature, especially over-dispersed models; and to prove that the proposed model can be applied to discuss zero-inflated observations.

The article is organized as follows. In Section 2, the DsMGL distribution is proposed. Various properties are derived in Sections 3 and 4. In Section 5, the DsMGL parameters are estimated by utilizing the maximum likelihood approach. Simulation study is discussed in Section 6. In Section 7, three real data sets are analyzed. Finally, some conclusions and future work are listed in Section 8.

2. Discrete analogue of MGL distribution

Recall Eq (1.3), and the SF of the DsMGL distribution can be expressed as

$$S(x;\varepsilon,\delta)=\frac{(1-\ln\delta^{x+1})(\varepsilon-\varepsilon\ln\delta+\ln\delta)-\ln\delta}{\varepsilon(1-\ln\delta)}\delta^{x+1};\ x\in\mathbb{N}_0, \tag{2.1}$$

where $\varepsilon\geq\frac{-\ln\delta}{1-\ln\delta}$, $0<e^{-\tau}=\delta<1$, and $\mathbb{N}_0=\{0,1,2,3,...\}$. The corresponding PMF to Eq (2.1) can be introduced as

$$P_x(x;\varepsilon,\delta)=\frac{\delta^x}{1-\ln\delta}\left\{1-\delta-\ln\delta\,[1+x-\delta(x+2)]+(1-\frac{1}{\varepsilon})(\ln\delta)^2\,[x-\delta(x+1)]\right\};\ x\in\mathbb{N}_0. \tag{2.2}$$

The CDF can be reported as

$$F(x;\varepsilon,\delta)=1-\frac{(1-\ln\delta^{x+1})(\varepsilon-\varepsilon\ln\delta+\ln\delta)-\ln\delta}{\varepsilon(1-\ln\delta)}\delta^{x+1};\ x\in\mathbb{N}_0. \tag{2.3}$$

Figure 1 shows the PMF of the DsMGL model based on various values of the parameters ε and δ.

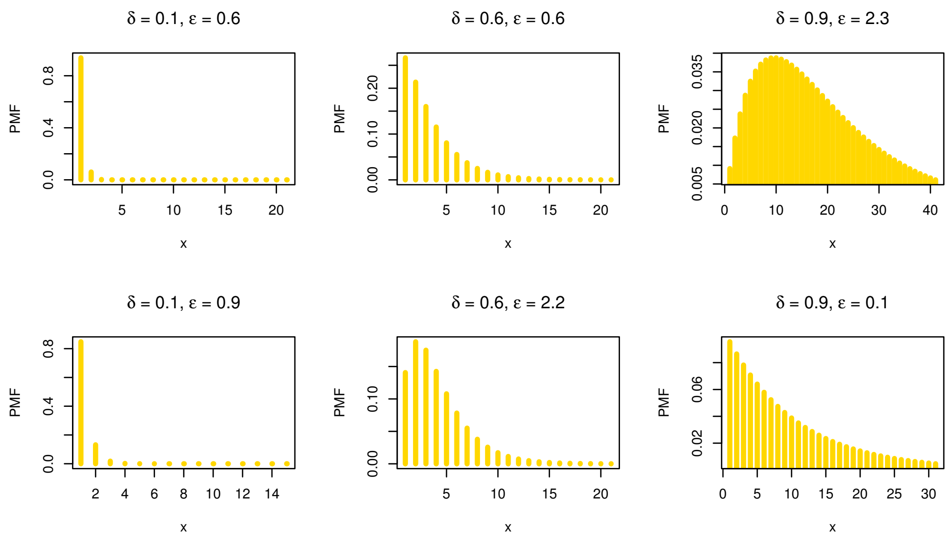

Figure 1. The PMF for the DsMGL distribution.

As can be noted, the PMF can take unimodal or decreasing form. Moreover, it can be used as a statistical approach to model zero-inflated observations under positive skew.

3. Reliability analytics

In reliability theory, the forward iteration "remaining" time and the past time are two very important measurements in the theory of renewal processes. Consider, for example, the lifetime of a wireless link when a new packet arrives in wireless networks. Reliability studies model the remaining life of a component. If something has survived that far, how long can it be expected to survive? This is the question answered by mean residual life (MRL). In the discrete setting, the MRL, say Θ_i, is defined as

$$\Theta_i = \mathbf{E}\left(T - i|T \geq i\right) = \frac{1}{1 - F(i-1;\varepsilon,\delta)} \sum_{j=i+1}^{q} \left[1 - F(j-1;\varepsilon,\delta)\right]; \;\; i \in \mathbb{N}_0, \tag{3.1}$$

where $\mathbb{N}_0 = \{0, 1, 2, 3, ..., q\}$ for $0 < q < \infty$. Thus, for the DsMGL model, the MRL is given as

$$\Theta_i = -\delta \frac{2(\varepsilon-1)(\delta i - i - 1)\ln\delta - \varepsilon(\delta(i+1) - i - 2)\ln\delta + \varepsilon(\delta-1)}{(\delta-1)^2\left\{(1 - i\ln\delta)(\varepsilon - \varepsilon\ln\delta + \ln\delta) - \ln\delta\right\}}; \;\; i \in \mathbb{N}_0. \tag{3.2}$$

Furthermore, in the discrete setting, the mean past life (MPL), say Θ_i^*, is defined as

$$\Theta_i^* = \mathbf{E}\left(i - T|T < i\right) = \frac{1}{F(i-1;\varepsilon,\delta)} \sum_{m=1}^{i} F(m-1;\varepsilon,\delta); \;\; i \in \mathbb{N}_0 - \{0\}. \tag{3.3}$$

So, the MPL for the DsMGL model can be represented as

$$\begin{aligned}\Theta_i^* = &\frac{i\varepsilon\delta^{i+2}\ln\delta - i\varepsilon\delta^{i+1}\ln\delta + i\varepsilon\delta^2(\ln\delta - 1) - 2i\delta^{i+2}\ln\delta + 2(i+1)\delta^{i+1}\ln\delta - \varepsilon\delta^{i+2}(\ln\delta - 1)}{(2i\varepsilon\delta^i\ln\delta - 2i\delta^i\ln\delta - \varepsilon\delta^i\ln\delta + \varepsilon\delta^i + \varepsilon\ln\delta - \varepsilon)(\delta-1)^2} \\ &+ \frac{\varepsilon\delta^2(\ln\delta - 1) - 2\delta\ln\delta + (1-2\delta)i\varepsilon\ln\delta + 2i\varepsilon\delta - \varepsilon\delta^{i+1} + \varepsilon\delta - i\varepsilon}{(2i\varepsilon\delta^i\ln\delta - 2i\delta^i\ln\delta - \varepsilon\delta^i\ln\delta + \varepsilon\delta^i + \varepsilon\ln\delta - \varepsilon)(\delta-1)^2}.\end{aligned} \tag{3.4}$$

For $i \in \mathbb{N}_0$, we get $\Theta_i^* \leq i$. The CDF of the DsMGL model can be recovered by the MPL as

$$F(k;\varepsilon,\delta) = F(0;\varepsilon,\delta)\prod_{i=1}^{k}\left[\frac{\Theta_i^*}{\Theta_{i+1}^* - 1}\right]; \; k \in \mathbb{N}_0 - \{0\}, \tag{3.5}$$

where $F(0;\varepsilon,\delta) = \left(\prod_{i=1}^{q}\left[\frac{\Theta_i^*}{\Theta_{i+1}^*-1}\right]\right)^{-1}$ and $0 < q < \infty$. Thus, the mean of the DsMGL model can be expressed as

$$\text{Mean} = i - \Theta_i^* F(i-1;\varepsilon,\delta) + \Theta_i\left[1 - F(i-1;\varepsilon,\delta)\right]; \; i \in \mathbb{N}_0 - \{0\}. \tag{3.6}$$

The reversed hazard rate function (RHRF) can be expressed as a function in MPL as

$$r(i;\varepsilon,\delta) = \frac{1 - \Theta_{i+1}^* + \Theta_i^*}{\Theta_i^*}; \; i \in \mathbb{N}_0 - \{0\}. \tag{3.7}$$

Further, the RHRF can be proposed as

$$r(x;\varepsilon,\delta)=\frac{1-\delta-\ln\delta\,[1+x-\delta(x+2)]+(1-\frac{1}{\varepsilon})(\ln\delta)^2\,[x-\delta(x+1)]}{\varepsilon(1-\ln\delta)-[(1-\ln\delta^{x+1})(\varepsilon-\varepsilon\ln\delta+\ln\delta)-\ln\delta]\,\delta^{x+1}}\varepsilon\delta^x;\quad x\in\mathbb{N}_0. \tag{3.8}$$

Figure 2 shows the RHRF plots for different values of the DsMGL parameters.

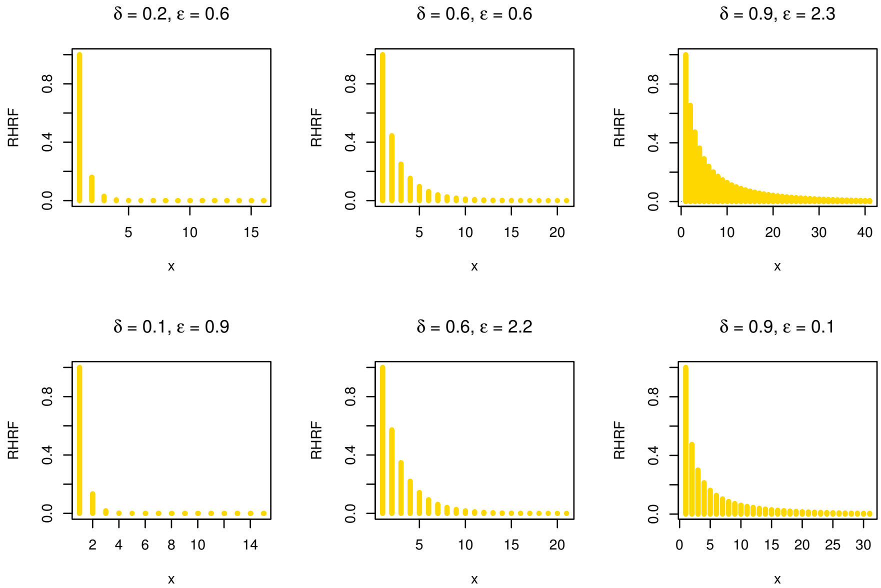

Figure 2. The RHRF for the DsMGL model

Suppose W and H are two independent DsMGL random variables (RVs) with parameters (ε_1,δ_1) and (ε_2,δ_2), respectively. Then, the RHRF of $T=\min(W,H)$ and $L=\max(W,H)$ can be formulated as

$$r_T(x;\mathbf{\Lambda})=1-\frac{1-\left\{\frac{(1-\ln\delta_1^x)(\varepsilon_1-\varepsilon_1\ln\delta_1+\ln\delta_1)-\ln\delta_1}{\varepsilon_1(1-\ln\delta_1)}\delta_1^x\right\}\left\{\frac{(1-\ln\delta_2^x)(\varepsilon_2-\varepsilon_2\ln\delta_2+\ln\delta_2)-\ln\delta_2}{\varepsilon_2(1-\ln\delta_2)}\delta_2^x\right\}}{1-\left\{\frac{(1-\ln\delta_1^{x+1})(\varepsilon_1-\varepsilon_1\ln\delta_1+\ln\delta_1)-\ln\delta_1}{\varepsilon_1(1-\ln\delta_1)}\delta_1^{x+1}\right\}\left\{\frac{(1-\ln\delta_2^{x+1})(\varepsilon_2-\varepsilon_2\ln\delta_2+\ln\delta_2)-\ln\delta_2}{\varepsilon_2(1-\ln\delta_2)}\delta_2^{x+1}\right\}}, \tag{3.9}$$

and

$$\begin{aligned} r_L(x;\mathbf{\Lambda})=&\frac{1-\delta_1-\ln\delta_1\,[1+x-\delta_1(x+2)]+(1-\frac{1}{\varepsilon_1})(\ln\delta_1)^2\,[x-\delta_1(x+1)]}{\varepsilon_1(1-\ln\delta_1)-\left[(1-\ln\delta_1^{x+1})(\varepsilon_1-\varepsilon_1\ln\delta_1+\ln\delta_1)-\ln\delta_1\right]\delta_1^{x+1}}\varepsilon_1\delta_1^x\\ &+\frac{1-\delta_2-\ln\delta_2\,[1+x-\delta_2(x+2)]+(1-\frac{1}{\varepsilon_2})(\ln\delta_2)^2\,[x-\delta_2(x+1)]}{\varepsilon_2(1-\ln\delta_2)-\left[(1-\ln\delta_2^{x+1})(\varepsilon_2-\varepsilon_2\ln\delta_2+\ln\delta_2)-\ln\delta_2\right]\delta_2^{x+1}}\varepsilon_2\delta_2^x\\ &-\frac{1-\delta_1-\ln\delta_1\,[1+x-\delta_1(x+2)]+(1-\frac{1}{\varepsilon_1})(\ln\delta_1)^2\,[x-\delta_1(x+1)]}{\varepsilon_1(1-\ln\delta_1)-\left[(1-\ln\delta_1^{x+1})(\varepsilon_1-\varepsilon_1\ln\delta_1+\ln\delta_1)-\ln\delta_1\right]\delta_1^{x+1}}\varepsilon_1\delta_1^x\\ &\times\frac{1-\delta_2-\ln\delta_2\,[1+x-\delta_2(x+2)]+(1-\frac{1}{\varepsilon_2})(\ln\delta_2)^2\,[x-\delta_2(x+1)]}{\varepsilon_2(1-\ln\delta_2)-\left[(1-\ln\delta_2^{x+1})(\varepsilon_2-\varepsilon_2\ln\delta_2+\ln\delta_2)-\ln\delta_2\right]\delta_2^{x+1}}\varepsilon_2\delta_2^x. \end{aligned} \tag{3.10}$$

Since the RHRFs of the two RVs W and H are decreasing, then the RHRFs of $T = \min(W, H)$ and $L = \max(W, H)$ are also decreasing. Another important measure in survival analysis theory is called the hazard rate function (HRF). If X is a DsMGL random variable, then the HRF can be expressed as

$$h(x; \varepsilon, \delta) = 1 - \frac{\left[(1 - \ln \delta^{x+1})(\varepsilon - \varepsilon \ln \delta + \ln \delta) - \ln \delta\right] \delta}{(1 - \ln \delta^{x})(\varepsilon - \varepsilon \ln \delta + \ln \delta) - \ln \delta}; \quad x \in \mathbb{N}_0, \tag{3.11}$$

where $h(x; \varepsilon, \delta) = \frac{P_x(x;\varepsilon,\delta)}{S(x-1;\varepsilon,\delta)}$. Figure 3 shows the HRF plots for different values of the DsMGL parameters. It should be noted that the new paradigm can be used to discuss any phenomena of an increasing unilateral form.

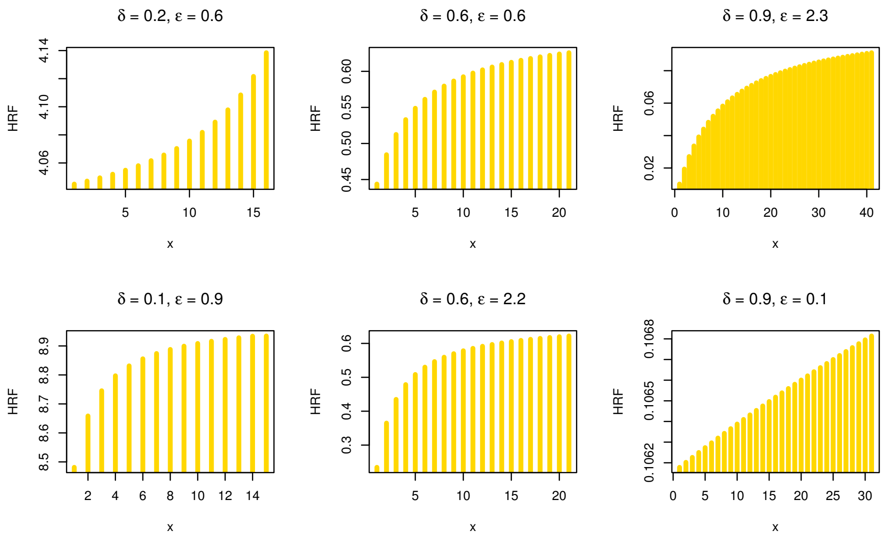

Figure 3. The HRF for the DsMGL distribution.

Suppose W and H are two independent RVs with parameters DsMGL$(\varepsilon_1, \delta_2)$ and DsMGL$(\varepsilon_2, \delta_2)$, respectively. Then, the HRF of $T = \min(W, H)$ can be formulated as

$$\begin{aligned} h_T(x; \mathbf{\Lambda}) &= \frac{\Pr(\min(W, H) = x)}{\Pr(\min(W, H) \geq x)} \\ &= \frac{\Pr(\min(W, H) \geq x) - \Pr(\min(W, H) \geq x + 1)}{\Pr(\min(W, H) \geq x)} \\ &= \frac{\Pr(W \geq x)\Pr(H \geq x) - \Pr(W \geq x + 1)\Pr(H \geq x + 1)}{\Pr(W \geq x)\Pr(V \geq x)} \\ &= \frac{\Pr(W \geq x)\Pr(H = x) + \Pr(W = x)\Pr(H \geq x) - \Pr(W = x)\Pr(H = x)}{\Pr(W \geq x)\Pr(H \geq x)}, \end{aligned}$$

where $\mathbf{\Lambda} = (\varepsilon_1, \varepsilon_2, \delta_1, \delta_2)$. Then,

$$h_T(x; \mathbf{\Lambda}) = \left\{1 - \frac{\left[(1 - \ln \delta_1^{x+1})(\varepsilon_1 - \varepsilon_1 \ln \delta_1 + \ln \delta_1) - \ln \delta_1\right] \delta_1}{(1 - \ln \delta_1^{x})(\varepsilon_1 - \varepsilon_1 \ln \delta_1 + \ln \delta_1) - \ln \delta_1}\right\}$$

$$+\left\{1-\frac{\left[(1-\ln\delta_2^{x+1})(\varepsilon_2-\varepsilon_2\ln\delta_2+\ln\delta_2)-\ln\delta_2\right]\delta_2}{(1-\ln\delta_2^{x})(\varepsilon_2-\varepsilon_2\ln\delta_2+\ln\delta_2)-\ln\delta_2}\right\}$$
$$-\left\{1-\frac{\left[(1-\ln\delta_1^{x+1})(\varepsilon_1-\varepsilon_1\ln\delta_1+\ln\delta_1)-\ln\delta_1\right]\delta_1}{(1-\ln\delta_1^{x})(\varepsilon_1-\varepsilon_1\ln\delta_1+\ln\delta_1)-\ln\delta_1}\right\}$$
$$\times\left\{1-\frac{\left[(1-\ln\delta_2^{x+1})(\varepsilon_2-\varepsilon_2\ln\delta_2+\ln\delta_2)-\ln\delta_2\right]\delta_2}{(1-\ln\delta_2^{x})(\varepsilon_2-\varepsilon_2\ln\delta_2+\ln\delta_2)-\ln\delta_2}\right\}. \tag{3.12}$$

Since the HRFs of the two RVs W and H are increasing, the HRF of $T = \min(W, H)$ is also increasing. Similarly, the HRF of $L = \max(W, H)$ can be expressed as

$$h_L(x;\mathbf{\Lambda}) = 1-\frac{1-\left\{1-\frac{(1-\ln\delta_1^{x+1})(\varepsilon_1-\varepsilon_1\ln\delta_1+\ln\delta_1)-\ln\delta_1}{\varepsilon_1(1-\ln\delta_1)}\delta_1^{x+1}\right\}\left\{1-\frac{(1-\ln\delta_2^{x+1})(\varepsilon_2-\varepsilon_2\ln\delta_2+\ln\delta_2)-\ln\delta_2}{\varepsilon_2(1-\ln\delta_2)}\delta_2^{x+1}\right\}}{1-\left\{1-\frac{(1-\ln\delta_1^{x})(\varepsilon_1-\varepsilon_1\ln\delta_1+\ln\delta_1)-\ln\delta_1}{\varepsilon_1(1-\ln\delta_1)}\delta_1^{x}\right\}\left\{1-\frac{(1-\ln\delta_2^{x})(\varepsilon_2-\varepsilon_2\ln\delta_2+\ln\delta_2)-\ln\delta_2}{\varepsilon_2(1-\ln\delta_2)}\delta_2^{x}\right\}}. \tag{3.13}$$

4. Some distributive properties

4.1. The DsMGL distribution for order statistics: PMF and moments

Assume $X_{1:l}, X_{2:l}, ..., X_{l:l}$ are the corresponding order statistics (OS) of the random sample (RS) $X_1, X_2, ..., X_l$ from the DsMGL model. Then, the CDF of the ith OS is given as

$$F_{i:l}(x;\varepsilon,\delta) = \sum_{b=i}^{l}\binom{l}{b}[F_i(x;\varepsilon,\delta]^b\,[1-F_i(x;\varepsilon,\delta)]^{l-b}$$
$$= \sum_{b=i}^{l}\sum_{j=0}^{l-b}\vartheta_{(m)}^{(l,b)}F_i(x;\varepsilon,\delta,b+j), \tag{4.1}$$

where $\vartheta_{(m)}^{(l,b)} = (-1)^j\binom{l}{b}\binom{l-b}{j}$. The corresponding PMF to Eq (4.1) can be listed as

$$f_{i:l}(x;\varepsilon,\delta) = F_{i:l}(x;\varepsilon,\delta) - F_{i:l}(x-1;\varepsilon,\delta)$$
$$= \sum_{b=i}^{l}\sum_{j=0}^{l-b}\vartheta_{(m)}^{(l,b)}f_i(x;\varepsilon,\delta,b+j), \tag{4.2}$$

where $f_i(x;\varepsilon,\delta,b+j)$ represents the PMF of the exponentiated DsMGL distribution with power parameter $b + j$. So, the vth moments of $X_{i:l}$ can be written as

$$E(X_{i:l}^v) = \sum_{x=0}^{\infty}\sum_{b=i}^{l}\sum_{j=0}^{l-b}\vartheta_{(m)}^{(l,b)}x^v f_i(x;\varepsilon,\delta,b+j). \tag{4.3}$$

Hosking [21] has defined the L-moment (L-MT) to show the descriptive statistics for the probability model. The L-MT of the DsMGL can be formulated as

$$\Upsilon o = \frac{1}{o}\sum_{c=0}^{o-1}(-1)^c\binom{o-1}{c}E\left(X_{o-c:o}\right). \tag{4.4}$$

Using Eq (4.4), L-MT of mean = Υ_1, L-MT coefficient of variation = $\frac{\Upsilon_2}{\Upsilon_1}$, L-MT coefficient of skewness = $\frac{\Upsilon_3}{\Upsilon_2}$, and L-MT coefficient of kurtosis = $\frac{\Upsilon_4}{\Upsilon_2}$.

4.2. *The original moments for data behavior scan*

The shape of any probability model can be described by its different moments. Based on the first four moments, the mean "$E(X)$", variance "$Var(X)$", index of dispersion "$D(X)$", skewness "$Sk(X)$", and kurtosis "$Ku(X)$" can be derived. Let X be a DsMGL random variable. Then, the probability generating function (PGF), say $\Upsilon(z)$, can be formulated as

$$\begin{aligned}\Upsilon(z) &= \sum_{x=0}^{\infty} z^x P_x(x;\varepsilon,\delta)\\ &= \frac{1}{1-\ln\delta}\sum_{x=0}^{\infty}\left\{1-\delta-\ln\delta\,[1+x-\delta(x+2)]+(1-\frac{1}{\varepsilon})(\ln\delta)^2\,[x-\delta(x+1)]\right\}(z\delta)^x\\ &= \frac{-2\delta(\varepsilon-1)(z-1)\ln\delta+\varepsilon\left(\delta^2 z-2\delta+1\right)\ln\delta-\varepsilon(\delta-1)(\delta z-1)}{\varepsilon(\ln\delta-1)(\delta z-1)^2},\end{aligned} \tag{4.5}$$

where the power series converges absolutely at least for all complex numbers z with $|z| \leq 1$. Equation (4.5) can be derived utilizing the Maple software program. Thus, the first four moments of the DsMGL model can be listed as

$$E(X) = -\delta\frac{(\varepsilon-1)\ln\delta^2+\varepsilon(\delta-2)\ln\delta-\varepsilon(\delta-1)}{\varepsilon(\ln\delta-1)(\delta-1)^2}, \tag{4.6}$$

$$E(X^2) = \delta\frac{(3\varepsilon\delta+\varepsilon-3\delta-1)\ln\delta^2+\varepsilon(\delta^2-3\delta-2)\ln\delta-\varepsilon\delta^2+\varepsilon}{\varepsilon(\ln\delta-1)(\delta-1)^3}, \tag{4.7}$$

$$E(X^3) = -\delta\frac{14\left(\delta^2+\frac{10}{7}\delta+\frac{1}{7}\right)(\varepsilon-1)\ln\delta+\varepsilon(\delta+2)(\delta^2-6\delta-1)\ln\delta-\varepsilon(\delta-1)(\delta^2+4\delta+1)}{\varepsilon(\ln\delta-1)(\delta-1)^4}, \tag{4.8}$$

and

$$\begin{aligned}E(X^4) = \delta&\frac{30\left(\delta^3+\frac{11}{3}\delta^2+\frac{5}{3}\delta+\frac{1}{15}\right)(\varepsilon-1)\ln\delta+\varepsilon(\delta^4-5\delta^3-55\delta^2-35\delta-2)\ln\delta}{\varepsilon(\ln\delta-1)(\delta-1)^5}\\ &-\frac{\delta^4+10\delta^3-10\delta-1}{(\ln\delta-1)(\delta-1)^5}.\end{aligned} \tag{4.9}$$

According to the previous moments "$E(X^r);\ r = 1,2,3,4$", the $E(X)$, $Var(X)$, $Sk(X)$, and $Ku(X)$ can be derived in closed forms. Table 1 reports some numerical results for the DsMGL model under different values of the distribution parameters.

Table 1. Some descriptive statistics under $\delta = 0.05$ and various values of ε.

	ε					
Measure	1.1	1.4	1.7	2.1	4.2	8.4
$E(X)$	0.10548	0.12972	0.14540	0.15934	0.18897	0.20378
$Var(X)$	0.11102	0.13466	0.14933	0.16196	0.18750	0.19962
$D(X)$	1.05252	1.03809	1.02701	1.01641	0.99223	0.97955
$Sk(X)$	3.31652	2.93747	2.73625	2.57885	2.29485	2.17274
$Ku(X)$	14.9956	12.2687	10.94861	9.97976	8.37673	7.74605

The DsMGL is capable of modeling positively skewed and leptokurtic datasets. Further, it is appropriate for modeling under- (over-) dispersed phenomena where $\frac{Var(X)}{|E(X)|} < (>)1$ for some parameter values.

5. Estimation of the DsMGL parameters

In this segment, the maximum likelihood estimates (MLEs) of the model parameters are determined. Let $X_1, X_2, ..., X_n$ be a RS of size n from the DsMGL distribution. The log-likelihood function (LL) can be written as

$$LL(x;\varepsilon,\delta) = \ln\delta\sum_{i=1}^{n} x_i - n\ln(1-\ln\delta) + \sum_{i=1}^{n}\ln\left(1-\delta-\ln\delta\,[1+x_i-\delta(x_i+2)] + (1-\frac{1}{\varepsilon})(\ln\delta)^2\,[x_i-\delta(x_i+1)]\right). \tag{5.1}$$

The MLEs of the parameters ε and δ, say $\widehat{\varepsilon}$ and $\widehat{\delta}$, are derived by $(\widehat{\varepsilon},\widehat{\delta}) = \text{argmax}_{(\varepsilon,\delta)}\,(L)$ or, in an equivalent approach in our case, $(\widehat{\varepsilon},\widehat{\delta}) = \text{argmax}_{(\varepsilon,\delta)}\,(LL)$. To provide more practicalities, the normal equations can be formulated as

$$\frac{\partial LL(x;\varepsilon,\delta)}{\partial\varepsilon} = \sum_{i=1}^{n}\frac{\left(\frac{\ln\delta}{\varepsilon}\right)^2[x_i-\delta(x_i+1)]}{1-\delta-\ln\delta\,[1+x_i-\delta(x_i+2)] + (1-\frac{1}{\varepsilon})(\ln\delta)^2\,[x_i-\delta(x_i+1)]} \tag{5.2}$$

and

$$\begin{aligned}\frac{\partial LL(x;\varepsilon,\delta)}{\partial\delta} &= \frac{1}{\delta}\sum_{i=1}^{n}x_i + \frac{n}{\delta\,(1-\ln\delta)} \\ &+ \sum_{i=1}^{n}\frac{(x_i+2)\,(\ln\delta+1) - \frac{1+x_i}{\delta} - 1 + \ln\delta\left(1-\frac{1}{\varepsilon}\right)\left(\frac{2x_i-2\delta(x_i+1)}{\delta} - (x_i+1)\ln\delta\right)}{1-\delta-\ln\delta\,[1+x_i-\delta(x_i+2)] + (1-\frac{1}{\varepsilon})(\ln\delta)^2\,[x_i-\delta(x_i+1)]}.\end{aligned} \tag{5.3}$$

The two previous equations cannot be solved analytically; therefore, a mathematical package such as the R program based on an iterative procedure such as the Newton-Raphson (numerical optimization approach) can be used to obtain the estimators.

6. Simulation: Appreciative behavior of estimators

In this segment, Monte Carlo simulation was performed to prove the efficiency of the DsMGL model utilizing the maximum likelihood approach. The performance of the MLE with respect to sample size

n is tested. The evaluation is based on a simulation study:

1) Generate 10000 samples of size $n = 10, 11, 12, ..., 60$ from the DsMGL model based on four cases as follows: case I: ($\delta = 0.1$ and $\varepsilon = 0.9$), case II: ($\delta = 0.3$ and $\varepsilon = 0.9$), case III: ($\delta = 0.5$ and $\varepsilon = 0.9$), and case IV: ($\delta = 0.7$ and $\varepsilon = 0.9$).
2) Generate 10000 samples of size $n = 30, 70, 140, 200, 300, 400, 600$ from the DsMGL model based on four cases as follows: case V: ($\delta = 0.5$ and $\varepsilon = 0.4$), case VI: ($\delta = 0.5$ and $\varepsilon = 0.5$), case VII: ($\delta = 0.5$ and $\varepsilon = 1.5$), and case VIII: ($\delta = 0.5$ and $\varepsilon = 2.0$).
3) Compute the MLEs for the 10000 samples, say $\widehat{\varepsilon}_j$ and $\widehat{\delta}_j$ for $j = 1, 2, 3, ..., 1000$.
4) Compute the biases, mean-squared errors (MSE), and mean relative errors (MRE).
5) The empirical results are shown in Figures 4–7 and Tables 2 and 3.

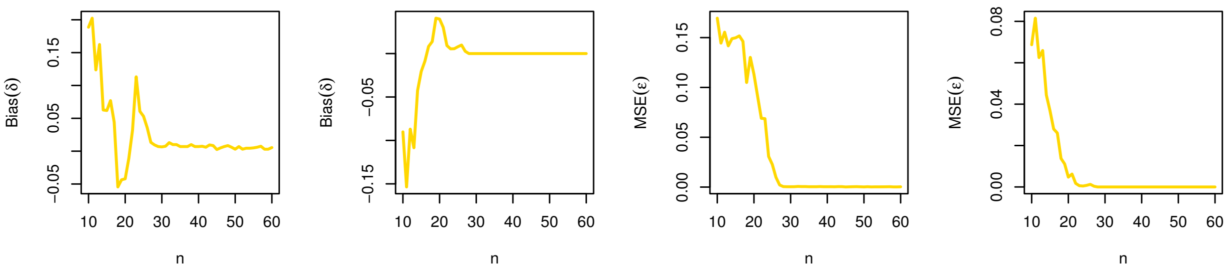

Figure 4. The empirical results based on case I.

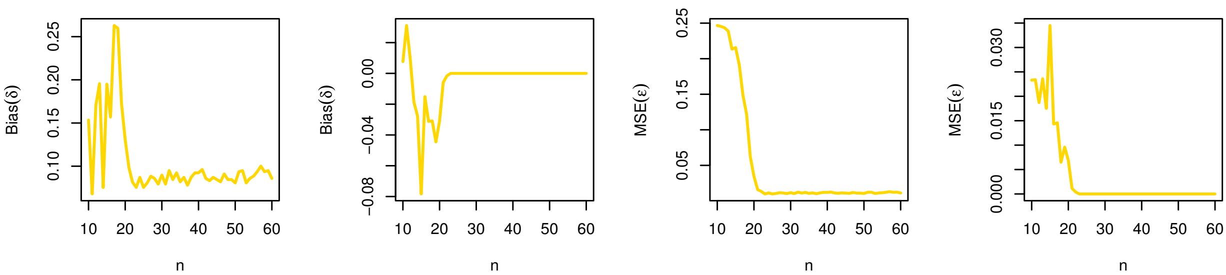

Figure 5. The empirical results based on case II.

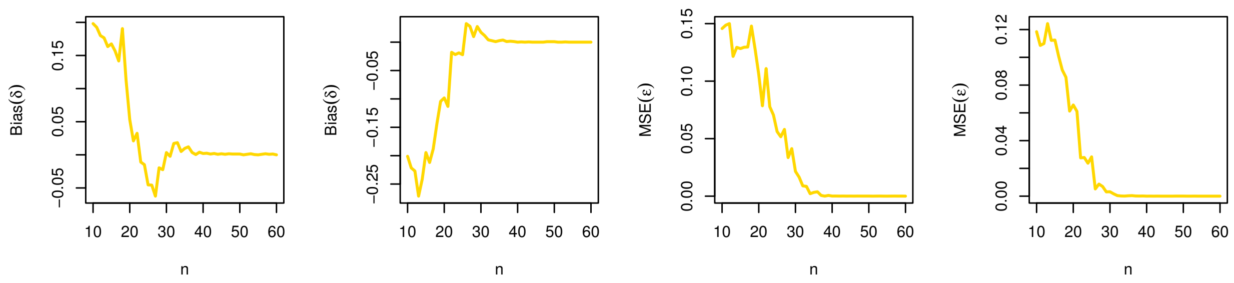

Figure 6. The empirical results based on case III.

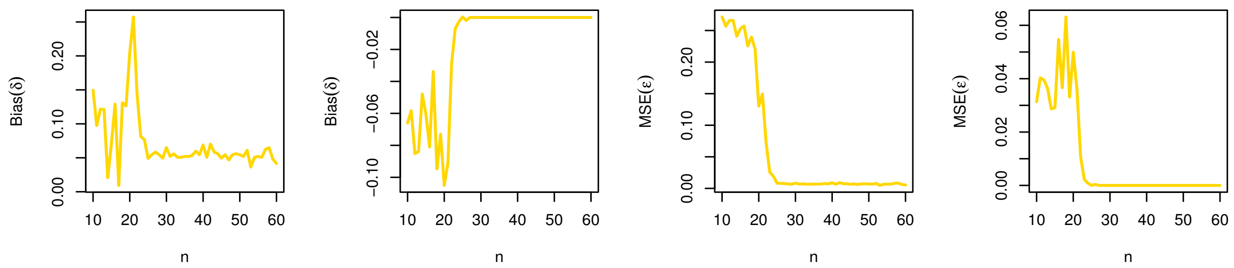

Figure 7. The empirical results based on case IV.

Table 2. The empirical simulation results for schemes V and VI.

		Scheme V			Scheme VI		
Parameter	n	Bias	MSE	MRE	Bias	MSE	MRE
δ	30	0.10056591	0.09006876	0.06336994	0.11632194	0.08963175	0.08223158
	70	0.09865782	0.08830367	0.04163261	0.10223666	0.06202190	0.08063179
	140	0.04369310	0.08200239	0.04001453	0.09220169	0.06101429	0.07312016
	200	0.02299658	0.07703927	0.03192237	0.08139327	0.04002236	0.06350036
	300	0.02126971	0.06005479	0.02101206	0.08023636	0.03237014	0.05519738
	400	0.01733369	0.04079031	0.00700269	0.05193079	0.01112539	0.04980373
	600	0.00234104	0.00201034	0.00023694	0.00292733	0.00026456	0.00015654
ε	30	0.09369110	0.07793171	0.06131304	0.05969637	0.04320158	0.05336947
	70	0.08567341	0.07122367	0.05510375	0.05223691	0.03201392	0.05102125
	140	0.07011036	0.06379035	0.05022693	0.04112566	0.03036549	0.04020024
	200	0.04105604	0.05380086	0.04500102	0.03233697	0.02002137	0.03102236
	300	0.03367101	0.04522564	0.04103979	0.03102973	0.01122024	0.01909439
	400	0.02098200	0.03474369	0.03139064	0.01018674	0.00889001	0.00306970
	600	0.00523689	0.00236844	0.00053671	0.00235659	0.00059458	0.00002518

Table 3. The empirical simulation results for schemes VII and VIII.

Parameter	n	Scheme VII			Scheme VIII		
		Bias	MSE	MRE	Bias	MSE	MRE
δ	30	0.025687193	0.01466875	0.01399737	0.06696575	0.08763274	0.06353187
	70	0.022658763	0.01136219	0.10145190	0.06132640	0.06923671	0.06123671
	140	0.019365811	0.00855157	0.00500215	0.05426598	0.05110123	0.05922014
	200	0.015330240	0.00431502	0.00362307	0.05120134	0.04320139	0.04623665
	300	0.013695875	0.00400236	0.00306981	0.05026971	0.03915618	0.04010656
	400	0.008532179	0.00100153	0.00168892	0.04902790	0.03101569	0.03410973
	600	0.001036942	0.00010265	0.00023296	0.00265235	0.00021691	0.00139237
ε	30	0.108067988	0.01123177	0.08307104	0.07437669	0.07782736	0.09069587
	70	0.103365789	0.00998796	0.08223676	0.06722364	0.05220104	0.08133658
	140	0.089445810	0.00912348	0.07911671	0.04749836	0.03002336	0.06310263
	200	0.071002158	0.00722369	0.06600163	0.03229659	0.01024787	0.05154031
	300	0.053210199	0.00426971	0.03697274	0.03193261	0.01010915	0.03974904
	400	0.009571377	0.00379104	0.00409265	0.02410124	0.00580048	0.02122360
	600	0.000531041	0.00022165	0.00023291	0.00124304	0.00083190	0.00212642

According to Figures 4–7 and Tables 2 and 3, the magnitude of bias, MSE, and MRE always decrease to zero as n grows (consistency property). Thus, the MLE approach can be utilized effectively for the parameter's estimation.

7. Data modeling

In this section, we demonstrate the resilience of the DsMGL distribution in modeling COVID-19 data. Fitted models are compared using some criteria, namely, the $-LL$, Akaike-information-criterion (AIC), modified-AIC (MAIC), Hannan-Quinn-information-criterion (HQIC), Bayesian-information-criterion (BIC), and Kolmogorov-Smirnov (K-S) test with its corresponding P-value. We will compare the flexibility of the DsMGL model with some of the competitive models such as discrete Lindley (DsL), discrete Burr-Hatke (DsBH), new discrete distribution with one parameter (NDsIP) (see, Eliwa and El-Morshedy, 2022), discrete Rayleigh (DsR), discrete Pareto (DsPa), discrete Burr type XII (DsB-XII), and modified negative binomial (NeBi) models.

7.1. Data set I: COVID-19 in Hong Kong

The data is listed in (*https://www.worldometers.info/coronavirus/country/china-hong-kong-sar/*) and represents the daily new cases in Hong Kong for COVID-19 from Feb. 15, 2020, to Oct. 25, 2020. The initial shape/form of this data is reported in Figure 8 using non-parametric (N-P) methods like strip, box, violin and QQ plots. It is noted that there are extreme and outliers' observations.

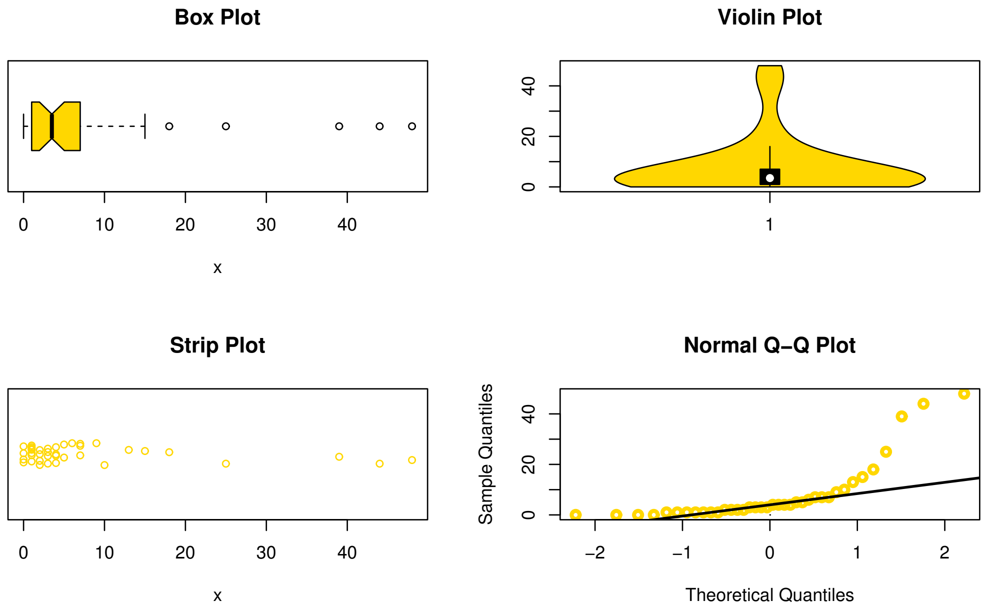

Figure 8. The N-P plots for COVID-19 data in Hong Kong.

The MLEs with their corresponding standard error (SE), confidence interval (C. I) for the parameter(s) and goodness-of-fit test (G-O-F-T) for COVID-19 data in Hong Kong, are listed in Tables 4 and 5.

Table 4. The MLEs with their corresponding SE and C. I for COVID-19 data in Hong Kong.

	ε			δ		
Model	MLE	SE	C. I	MLE	SE	C. I
DsMGL	0.8887	0.0242	[0.8390, 0.9361]	0.1062	0.0373	[0.0355, 0.1786]
DsL	0.8052	0.0200	[0.7660, 0.8451]	–	–	–
NDsIP	0.9110	0.0151	[0.8820, 0.9390]	–	–	–
DsR	0.9951	0.1361	[0.7280, 1.000]	–	–	–
DsB-XII	0.8173	0.0740	[0.6722, 0.9622]	3.2981	1.4272	[0.5013, 6.0962]
DsPa	0.5592	0.0531	[0.4515, 0.6603]	–	–	–
DsBH	0.9954	0.0132	[0.9695, 1.000]	–	–	–
NeBi	0.9203	0.0221	[0.8772, 0.9634]	0.6859	0.1023	[0.4854, 0.8864]

Table 5. The G-O-F-T for COVID-19 data in Hong Kong.

	Model							
Statistic	DsMGL	DsL	NDsIP	DsR	DsB-XII	DsPa	DsBH	NeBi
$-LL$	118.951	127.120	121.353	161.193	118.992	124.333	130.953	119.237
AIC	241.892	256.234	244.717	324.384	241.980	250.662	263.902	242.473
MAIC	242.233	256.343	244.824	324.493	242.325	250.777	264.014	242.816
BIC	245.178	257.877	246.352	326.024	245.263	252.302	265.543	245.748
HQIC	243.065	256.825	245.295	324.963	243.157	251.254	264.484	243.638
K-S	0.157	0.246	0.186	0.761	0.277	0.367	0.565	0.167
P-value	0.305	0.020	0.146	≤ 0.001	0.006	≤ 0.001	≤ 0.001	0.285

The DsMGL model is the best among all the discussed models, because it has the lowest value among $-LL$, AIC, MAIC, BIC, HQIC and K-S. Moreover, the DsMGL model has the highest P-value among all tested distributions. Figures 9 and 10 show the estimated CDF and P-P plots for all reported models from which the distribution adequacy of the DsMGL model can be noted clearly. Thus, the COVID-19 data in Hong Kong plausibly came from the DsMGL model.

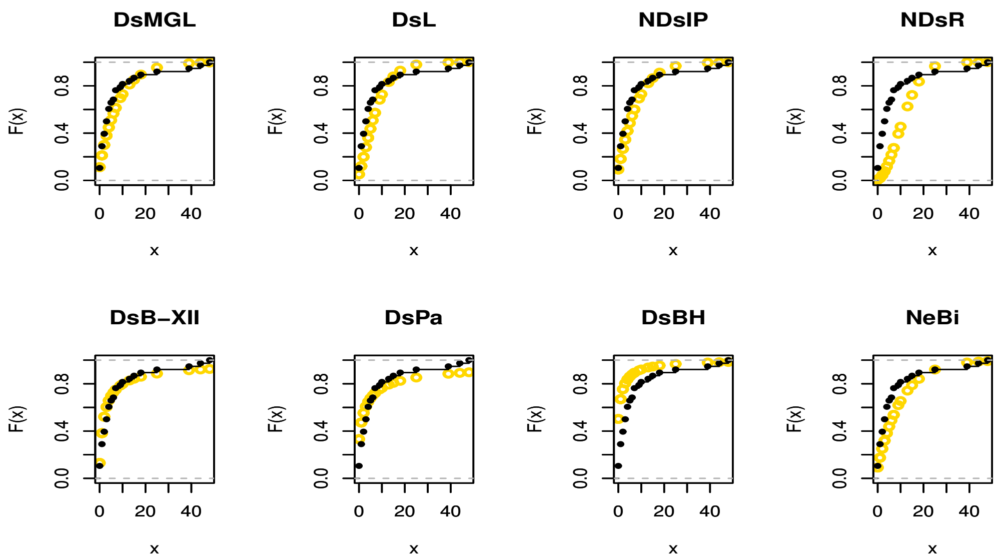

Figure 9. The estimated CDFs for COVID-19 data in Hong Kong.

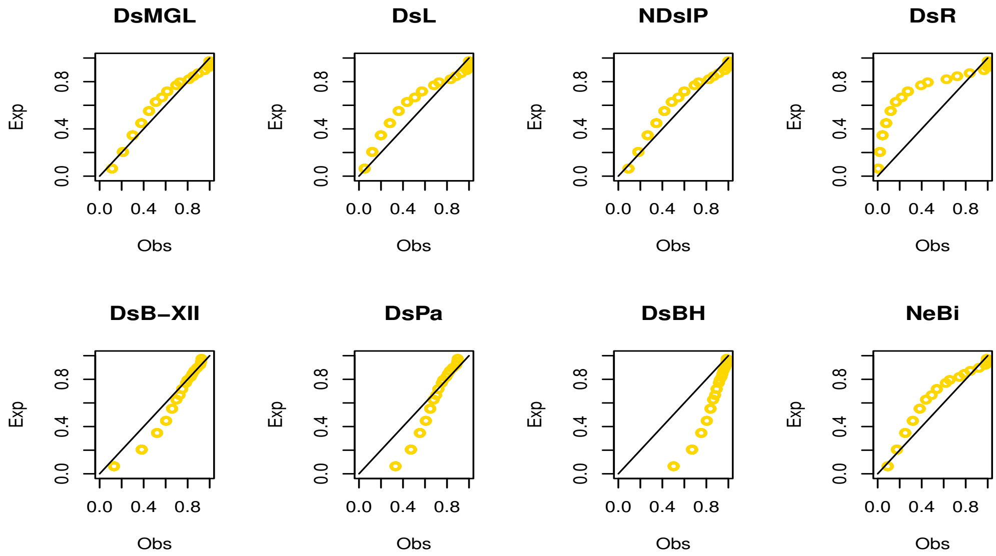

Figure 10. The P-P plots for COVID-19 data in Hong Kong.

Table 6 lists some descriptive statistics (DEST) for COVID-19 data in Hong Kong utilizing the DsMGL model.

Table 6. Some DEST for COVID-19 data in Hong Kong.

$E(X)$	$Var(X)$	$D(X)$	$Sk(X)$	$Ku(X)$
0.1843	0.2096	1.1374	2.7779	12.1489

The data reported here suffer from excessive dispersion "$D(X) > 1$". Moreover, it is moderately right skewed "$Sk(X) > 0$" and leptokurtic "$Ku(X) > 3$".

7.2. *Data set II: COVID-19 in Iraq*

The data is reported in (https://www.worldometers.info/coronavirus/country/iraq/) and represents the daily new cases in Iraq for COVID-19 from Feb. 15, 2020, to 25 Oct. 25, 2020. In Figure 11, the N-P plots are sketched. It is noted that there is an extreme observation.

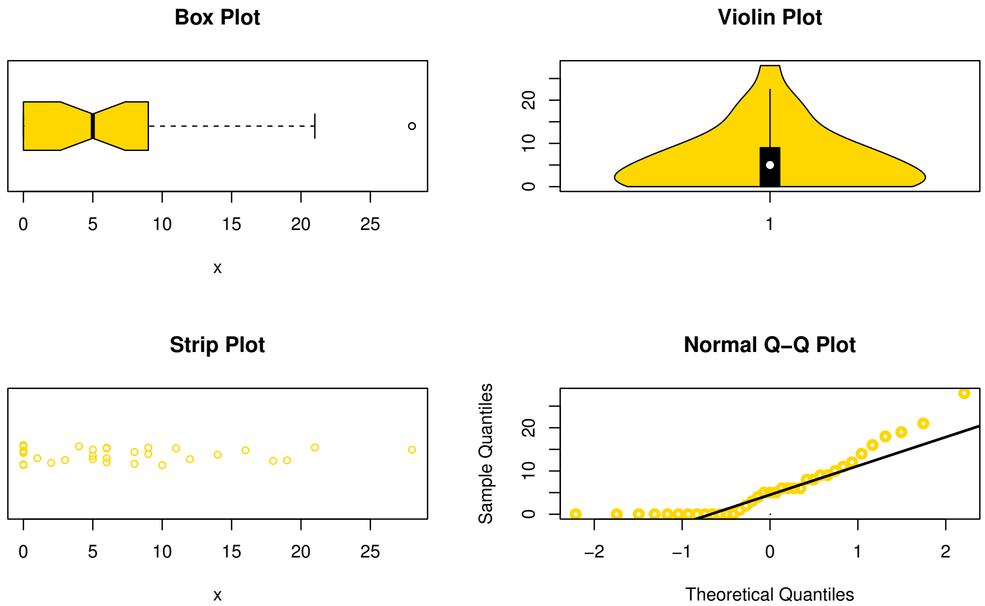

Figure 11. The N-P plots for COVID-19 data in Iraq.

The MLEs with their corresponding SE, C. I for the parameter(s) and G-O-F-T for COVID-19 data in Iraq, are listed in Tables 7 and 8.

Table 7. The MLEs with their corresponding SE and C. I for COVID-19 data in Iraq.

	ε			δ		
Model	MLE	SE	C. I	MLE	SE	C. I
DsMGL	0.863	0.078	[0.709, 1.015]	0.129	0.144	[0, 0.411]
DsL	0.768	0.024	[0.721, 0.815]	–	–	–
NDsIP	0.895	0.015	[0.865, 0.926]	–	–	–
DsR	0.989	0.0142	[0.710, 1]	–	–	–
DsB-XII	0.531	0.087	[0.361, 0.701]	1.009	0.253	[0.513, 1.506]
DsPa	0.528	0.056	[0.419, 0.637]	–	–	–
DsBH	0.989	0.019	[0.953, 1]	–	–	–
NeBi	0.928	0.053	[0.824, 1.032]	0.486	0.141	[0.209, 0.762]

Table 8. The G-O-F-T for COVID-19 data in Iraq.

	Model							
Statistic	DsMGL	DsL	NDsIP	DsR	DsB-XII	DsPa	DsBH	NeBi
$-LL$	107.731	114.313	109.230	138.243	112.394	112.393	116.527	109.426
AIC	219.464	230.635	220.452	278.495	228.771	226.775	235.054	222.852
MAIC	219.812	230.747	220.573	278.603	229.123	226.893	235.163	223.205
BIC	222.683	232.244	222.707	280.092	231.993	228.387	236.664	226.074
HQIC	220.595	231.195	221.025	279.051	229.914	227.343	235.623	223.988
K-S	0.214	0.279	0.2423	0.736	0.355	0.357	0.505	0.251
P-value	0.068	0.006	0.026	≤ 0.0001	0.0002	0.0002	≤ 0.0001	0.029

The DsMGL model is the best among all the studied models. Figures 12 and 13 show the estimated CDFs and P-P plots for COVID-19 data in Iraq.

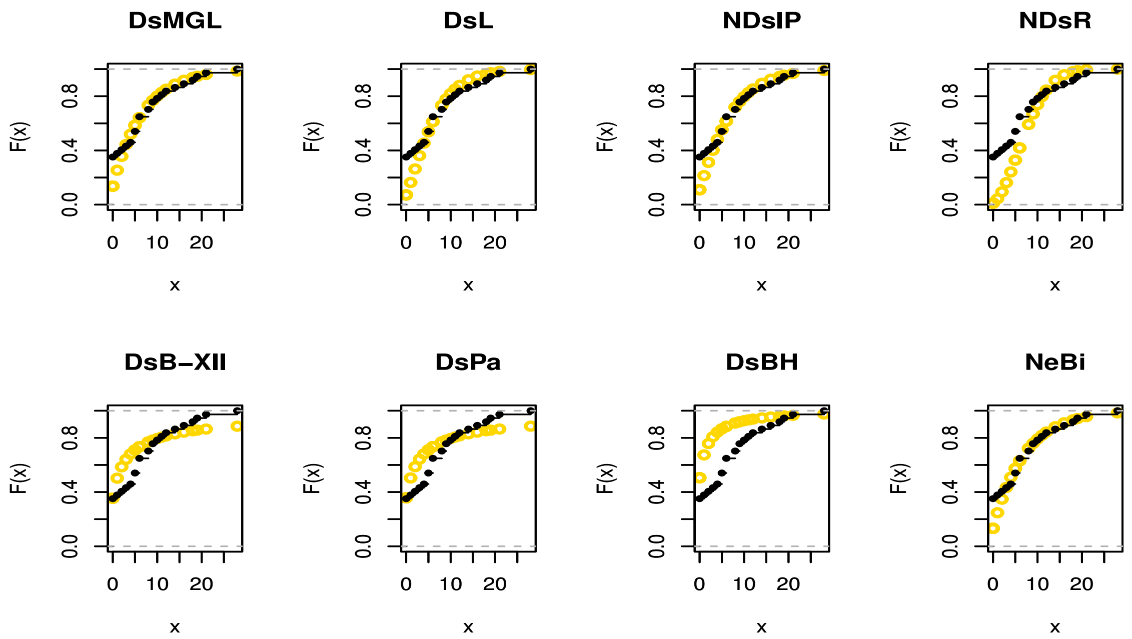

Figure 12. The estimated CDFs for COVID-19 data in Iraq.

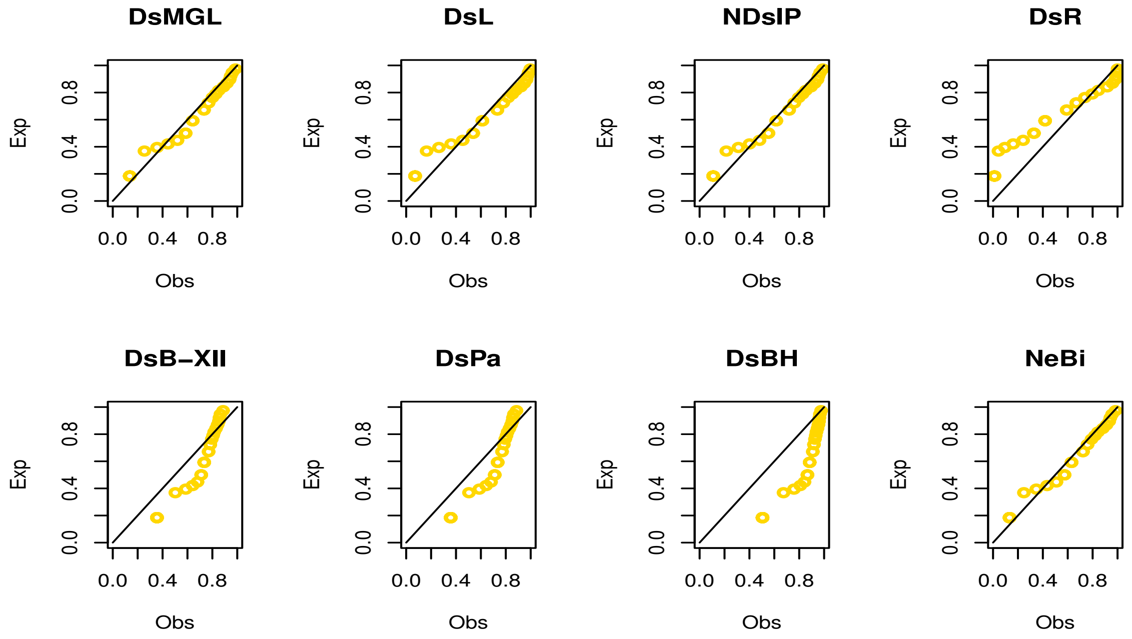

Figure 13. The P-P plots for COVID-19 data in Iraq.

From Figures 13 and 14, the data set plausibly came from the DsMGL model. Table 9 lists some information for COVID-19 data in Iraq based on the DsMGL model.

Table 9. Some DEST for COVID-19 data in Iraq.

$E(X)$	$Var(X)$	$D(X)$	$Sk(X)$	$Ku(X)$
0.2252	0.2640	1.1724	2.6089	11.2663

The data listed here suffer from excessive dispersion. Furthermore, it is moderately right skewed and leptokurtic.

7.3. Data set III: COVID-19 in Saudi Arabia

The data is reported in (*https://en.wikipedia.org/wiki/2020_coronavirus_pandemic_in_Saudi_Arabia*) and represents the daily new cases in Saudi Arabia for COVID-19 from March 1, 2020, to Sep. 13, 2021. In Figure 14, the N-P plots are reported. It is found that there are extreme observations.

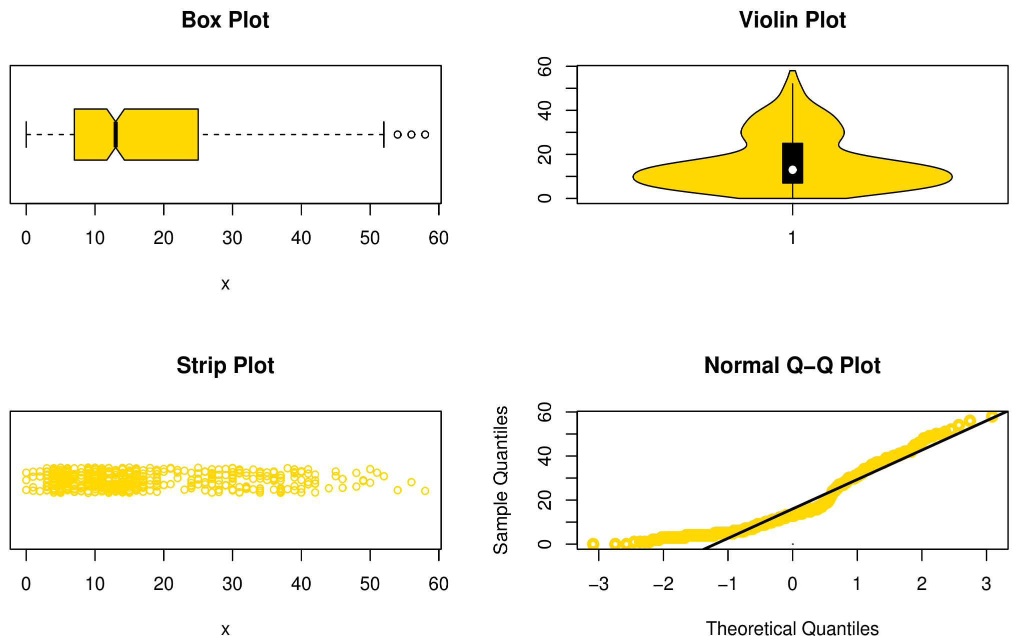

Figure 14. The N-P plots for COVID-19 data in Saudi Arabia.

The MLEs with their corresponding SE, C. I for the parameter(s) and G-O-F-T for COVID-19 data in Saudi Arabia are reported in Tables 10 and 11.

Table 10. The MLEs with their corresponding SE and C. I for COVID-19 data in Saudi Arabia.

	ε			δ		
Model	MLE	SE	C. I	MLE	SE	C. I
DsMGL	0.891	0.0003	0.884, 0.897]	0.302	0.023	[0.300, 0.305]
DsL	0.896	0.003	[0.889, 0.902]	–	–	–
NDsIP	0.956	0.002	[0.953, 0.959]	–	–	–
DsR	0.998	0.001	[0.995, 1]	–	–	–
DsB-XII	0.909	0.028	[0.854, 0.964]	4.067	1.300	[1.519, 6.616]
DsPa	0.688	0.012	[0.665, 0.711]	–	–	–
DsBH	0.998	0.002	[0.994, 1.002]	–	–	–
NeBi	0.521	0.125	[0.276, 0.766]	0.745	0.045	[0.657, 0.833]

Table 11. The G-O-F-T for COVID-19 data in Saudi Arabia.

	Model							
Statistic	DsMGL	DsL	NDsIP	DsR	DsB-XII	DsPa	DsBH	NeBi
$-LL$	1844.941	1849.332	1879.599	1894.439	2244.195	2301.366	2641.779	1847.236
AIC	3693.882	3693.881	3761.199	3792.878	4492.393	4604.732	5285.559	3698.472
MAIC	3693.904	3693.904	3761.207	3792.902	4492.415	4604.741	5285.567	3698.496
BIC	3702.285	3702.285	3765.401	3801.283	4500.796	4608.934	5289.761	3706.877
HQIC	3697.183	3697.180	3762.849	3796.178	4495.691	4606.382	5287.209	3701.772
K-S	0.071	0.079	0.169	0.241	0.415	0.439	0.787	0.075
P-value	0.139	0.129	0.023	0.004	0	0	0	0.134

According to Table 10, the DsMGL model is the best among all the tested models. Figures 15 and 16 show the estimated CDFs and P-P plots for COVID-19 data in Saudi Arabia. It is found that the data set plausibly came from DsMGL model.

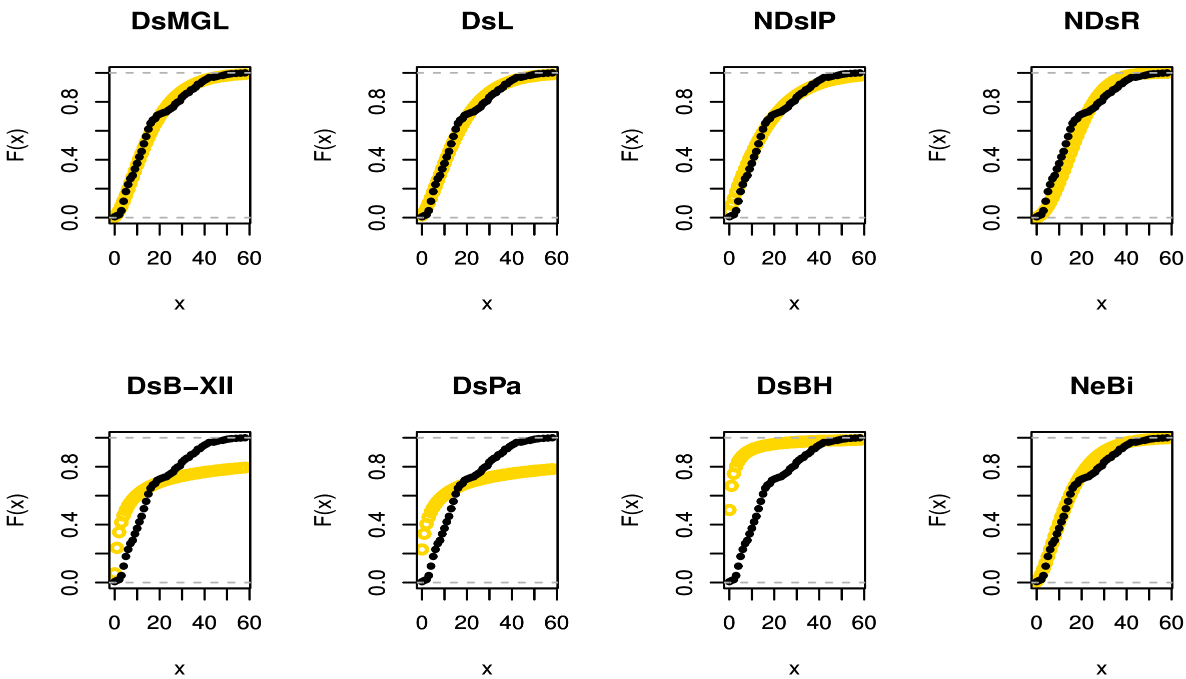

Figure 15. The estimated CDFs for COVID-19 data in Saudi Arabia.

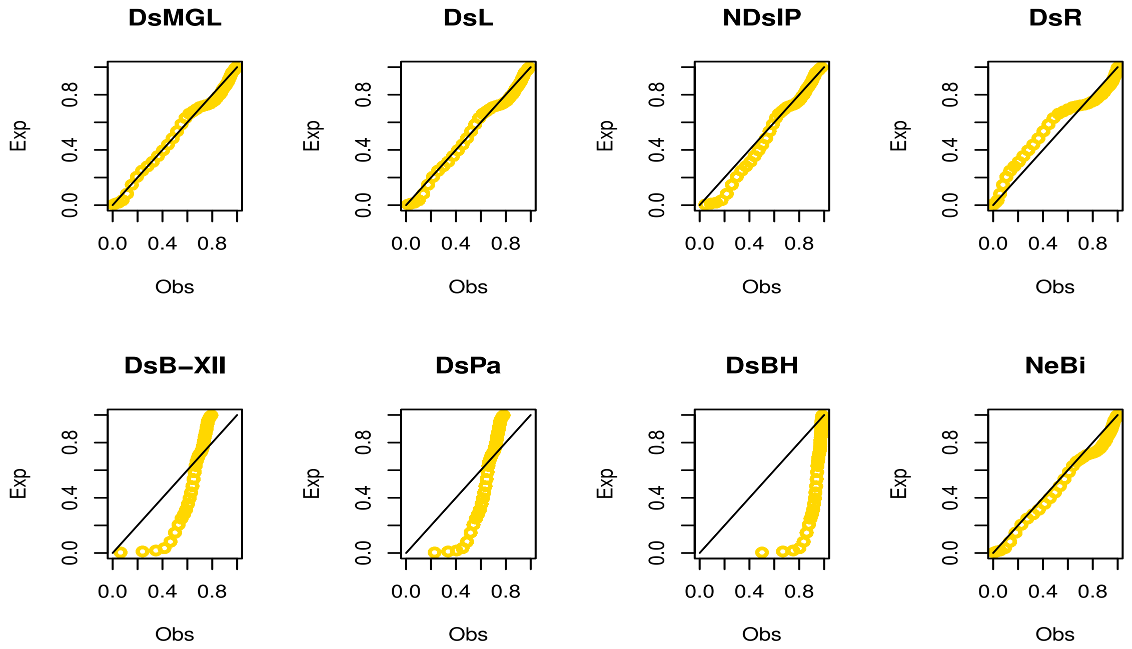

Figure 16. The P-P plots for COVID-19 data in Saudi Arabia.

Table 12 lists some DEST around COVID-19 data in Saudi Arabia by utilizing the DsMGL model.

Table 12. Some DEST for COVID-19 data in Saudi Arabia.

$E(X)$	$Var(X)$	$D(X)$	$Sk(X)$	$Ku(X)$
16.777	150.421	8.966	1.006	3.212

The data presented here suffer from excessive dispersion. Moreover, it is moderately right skewed and leptokurtic.

8. Conclusions remarks and future works

In this article, we have developed a new two parameter discrete model, named as discrete mixture gamma-Lindley (DsMGL) distribution. Various important distributional characteristics of the DsMGL distribution have been discussed. One of the important virtues of this newly developed model is that it can not only discuss over-dispersed, under-dispersed, positively skewed, and leptokurtic data sets, but it can also be applied for modeling increasing failure time data (due to its increasing HRF). The unknown parameters of the DsMGL model have been discussed under a method of maximum likelihood (ML) estimation. A detailed MCMC evaluation has been conducted to measure the behavior of the estimators. This numerical study shows that the ML estimation measures work satisfactorily. Finally, the modeling flexibility of the DsMGL distribution has been explored via three real data sets on COVID-19. For future work, the authors will utilize the DsMGL model to generate a bivariate extension distribution based on a shock models approach for modeling bivariate data.

Moreover, the first-order integer-valued regression model and autoregressive process will be studied in detail.

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