Research article

Propagation dynamics in an SIRS model with general incidence functions

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Abstract: This paper studies the initial value problems and traveling wave solutions in an SIRS model with general incidence functions. Linearizing the infected equation at the disease free steady state, we can define a threshold if the corresponding basic reproduction ratio in kinetic system is larger than the unit. When the initial condition for the infected is compactly supported, we prove that the threshold is the spreading speed for three unknown functions. At the same time, this threshold is the minimal wave speed for traveling wave solutions modeling the disease spreading process. If the corresponding basic reproduction ratio in kinetic system is smaller than the unit, then we confirm the extinction of the infected and the nonexistence of nonconstant traveling waves.

Keywords: comparison principle; upper and lower solutions; asymptotic spreading; minimal wave speed; spreading speed

1. Introduction

Since the pioneering work by Kermack and McKendrick [1], many mathematical models dividing the total population into several different classes have been established to model the disease spreading. Such models often involve the transmission among different classes. SIRS (susceptible/infected/recovered/susceptible) models are established by dividing the total population into three classes, which also assume that the recovered individuals have temporary immunity and the recovered may become the susceptible. When the new recruits are susceptible and the treatment behavior is considered, one typical SIRS model has the following form

$$\begin{cases} \frac{dS}{dt} = b - dS - f(I)S + \gamma R, \\ \frac{dI}{dt} = f(I)S - (d + \mu)I - T(I), \\ \frac{dR}{dt} = -(d + \gamma)R + \mu I + T(I), \end{cases} \tag{1.1}$$

in which $S(t), I(t), R(t)$ respectively denote the density of the susceptible, infected, recovered population at time t, f and T are incidence and treatment functions respectively, b is the recruitment rate of the population, d reflects the per capita natural death rate of the population, μ is the per capita natural recovery rate of the infective individuals, and γ denotes the per capita rate of recovered individuals who lose immunity and return to the susceptible class. In the past decades, much attention has been paid to the rich dynamics of (1.1) by taking different parameters and functions, see a very recent work by Pan et al. [2] and references cited therein. In particular, with different incidence functions f, (1.1) may admit different dynamics properties including oscillation and bifurcation conclusions. As mentioned in Pan et al. [2], different treatment functions T have been proposed by many researchers for different epidemic scenarios. One traditional form for T is the linear type, which is reasonable when the medical resource is sufficient or the treated receives adequate health care and the treated is timely.

Besides the total number for each class, the spatial distribution of individuals is also important to understand the disease spreading and controlling [3–5]. For SIRS models with spatial factor, we see a very recent work [6] and some references cited therein. Basing on model (1.1) and taking the linear treatment function, we consider the spatial factor in the following SIRS model

$$\begin{cases} \frac{\partial S}{\partial t} = D\frac{\partial^2 S}{\partial x^2} + b - dS - f(I)S + \gamma R, \\ \frac{\partial I}{\partial t} = D\frac{\partial^2 I}{\partial x^2} + f(I)S - (d + \mu + \beta)I, \\ \frac{\partial R}{\partial t} = D\frac{\partial^2 R}{\partial x^2} - (d + \gamma)R + (\mu + \beta)I, \end{cases} \tag{1.2}$$

in which $S(x, t), I(x, t), R(x, t)$ respectively denote the density of the susceptible, infected, recovered population at time $t > 0$ and at location $x \in \mathbb{R}$, $D > 0$ is the diffusion parameter and $\beta > 0$ describes the treatment ratio, other parameters are similar to those in (1.1). By considering the dynamics of (1.2), we may theoretically understand the spatial and temporal evolution features of the disease.

To formulate the disease spreading, much attention has been paid to the dynamics of different epidemic reaction-diffusion models. In particular, the spreading speed for initial value problem [7] and minimal wave speed for traveling wave solutions [8, 9] of epidemic models have been widely studied. From these speeds, it is possible to explore some factors that affect the disease spreading, we may see some earlier works by Brown and Carr [10], Diekmann [11, 12], Thieme [13, 14]. We first define the spreading speed for a nonnegative function [7]. A positive constant c is called the *spreading speed* of a nonnegative function $u(x, t)$, $x \in \mathbb{R}, t > 0$ if

$$\limsup\nolimits_{t\to\infty} \sup\nolimits_{|x|>(c+\epsilon)t} u(x, t) = 0, \qquad \liminf\nolimits_{t\to\infty} \inf\nolimits_{|x|<(c-\epsilon)t} u(x, t) > 0$$

for any given $\epsilon \in (0, c)$. Therefore, the speed for I shows the ability for disease spreading, which is from the viewpoint of the observer moving rightward or leftward at constant speeds $c \pm \epsilon$ (see Weinberger et al. [15]). Moreover, a traveling wave solution is a special entire solution having the form

$$(S(x, t), I(x, t), R(x, t)) = (s(z), i(z), r(z)), \quad z = x + ct \in \mathbb{R},$$

in which $c \in \mathbb{R}$ is the wave speed while $s, i, r \in C^2(\mathbb{R}, \mathbb{R})$ are the wave profile functions (here, $C^2 : \mathbb{R} \to \mathbb{R}$ means a function that is twice differentiable). Hence, they must satisfy

$$\begin{cases} Ds'' - cs' + b - ds - f(i)s + \gamma r = 0, \\ Di'' - ci' + f(i)s - (d + \mu + \beta)i = 0, \\ Dr'' - cr' - (d + \gamma)r + (\mu + \beta)i = 0, \end{cases} \tag{1.3}$$

where $\cdot' = \frac{d\cdot}{dz}, \cdot'' = \frac{d^2\cdot}{dz^2}, z \in \mathbb{R}$. To model the invasion of the infected or the disease spreading, we require that these functions are nonnegative and

$$\lim_{z\to-\infty}(s(z), i(z), r(z)) = (b/d, 0, 0), \qquad \liminf_{z\to\infty}[s(z)i(z)r(z)] > 0. \tag{1.4}$$

Note that $z = x + ct$, then when $c > 0$, at any given location $x \in \mathbb{R}$, $z \to \pm\infty$ if and only if $t \to \pm\infty$. Therefore, such a traveling wave solution models that the disease spreads at a constant speed c, and the minimal wave speed shows the ability that the disease spreads in the wave form.

In this paper, we shall explore the disease spreading ability by giving the spreading speed for $I, R, \max\{0, b/d - S\}$ when the infected initially appears in a finite interval and the minimal wave speed for traveling wave solutions. For the purpose, we need the following assumptions on incidence function f.

(F1) $f : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous such that $f(0) = 0, f(x) > 0, x > 0$; $\lim_{x\to0^+} \frac{f(x)}{x} =: f'(0) > 0$ exists such that $f'(0)x - Lx^{1+\alpha} \le f(x) \le f'(0)x, x > 0$ for some $L > 0, \alpha > 0$.

(F2) $f'(0)b/d - (d + \mu + \beta) > 0$.

(F3) There exists $I^* > 0$ such that

$$f(I^*)(b/d - I^* - (\mu + \beta)I^*/(d + \gamma)) - (d + \mu + \beta)I^* = 0.$$

Assume that $0 < \underline{I} \le \overline{I} < b/d$ such that

$$\begin{cases} f(\overline{I})(b/d - \overline{I} - (\mu + \beta)\underline{I}/(d + \gamma)) - (d + \mu + \beta)\overline{I} \le 0, \\ f(\underline{I})(b/d - \underline{I} - (\mu + \beta)\overline{I}/(d + \gamma)) - (d + \mu + \beta)\underline{I} \ge 0. \end{cases} \tag{1.5}$$

Then $\underline{I} = \overline{I} = I^*$.

(F4) $f'(0)b/d - (d + \mu + \beta) < 0$.

With (F1), we see that the corresponding basic reproduction ratio $\mathcal{R}_0$ of (1.2) is $f'(0)b/[d(d + \mu + \beta)]$. When (F1) and (F2) hold such that the ratio $\mathcal{R}_0$ is larger than 1, we define

$$c^* = 2\sqrt{D[f'(0)b/d - (d + \mu + \beta)]}.$$

We shall mainly prove that c^* is the spreading speed of $I, R, \max\{0, b/d - S\}$ as well as the minimal wave speed of traveling wave solutions when the disease spreads successfully. From this definition, it is possible to find some features on the disease spreading. For example, the ratio γ does not affect the spreading speed, but it may change the final state for an epidemic disease by increasing the infected. Moreover, when the ratio $\mathcal{R}_0$ is smaller than 1 or (F4) holds, we shall show the extinction of the infected as well as the nonexistence of nonconstant traveling waves.

The rest of this article is organized as follows. In Section 2, we investigate the spreading speed in initial value problems. The minimal wave speed of traveling wave solutions is studied in Section 3. Further discussion and numerical examples are presented in Section 4.

2. Spreading speed

In this part, we study the initial value problem

$$
\begin{cases}
\frac{\partial S}{\partial t} = D\frac{\partial^2 S}{\partial x^2} + b - dS - f(I)S + \gamma R, \\
\frac{\partial I}{\partial t} = D\frac{\partial^2 I}{\partial x^2} + f(I)S - (d + \mu + \beta)I, \\
\frac{\partial R}{\partial t} = D\frac{\partial^2 R}{\partial x^2} - (d + \gamma)R + (\mu + \beta)I, \\
S(x, 0) = S_0(x), I(x, 0) = I_0(x), R(x, 0) = R_0(x),
\end{cases}
\tag{2.1}
$$

in which $x \in \mathbb{R}, t > 0$, the initial condition satisfies the following assumption that will be imposed through this section

(I) $S_0(x), I_0(x), R_0(x) : \mathbb{R} \to \mathbb{R}^+$ are bounded and continuous, $I_0(x)$ admits nonempty compact support.

With the initial condition, we have the following conclusion.

Lemma 2.1. (2.1) *has a global classical bounded positive solution such that the following items hold.*

1) $S(x, t) > 0, I(x, t) > 0, R(x, t) > 0, x \in \mathbb{R}, t > 0.$

2) $\lim_{t \to \infty}(S(x, t) + I(x, t) + R(x, t)) = b/d$ *uniformly in* $x \in \mathbb{R}$, *and there exist* $M > 0, \delta > 0$ *such that*

$$
|S(x, t) + I(x, t) + R(x, t) - b/d| \le Me^{-\delta t}, \quad x \in \mathbb{R}, t \ge 0.
$$

In particular, if $S(x, T) + I(x, T) + R(x, T) \le b/d$ *for all* $x \in \mathbb{R}$ *and some* $T > 0$, *then*

$$
S(x, t) + I(x, t) + R(x, t) \le b/d, \quad x \in \mathbb{R}, t \ge T.
$$

3) The following functions are uniformly continuous and bounded

$$
u_t(x, t), u_x(x, t), u_{tx}(x, t), u_{tt}(x, t), u_{xx}(x, t), u_{xxt}(x, t), u_{xxx}(x, t), \quad u \in \{S, I, R\}
$$

for all $x \in \mathbb{R}, t \ge 1.$

4) There exists $\underline{S} > 0$ *such that* $S(x, t) \ge \underline{S}, x \in \mathbb{R}, t \ge 1.$

Proof. The local existence of bounded classical solution holds by the basic theory of reaction-diffusion equations [16]. We just show the bounds by comparison principle. Firstly, the nonlinearity in the second equation is quasipositive [17], we see that $I(x, t)$ is nonnegative once the solution exists, which further leads to the nonnegative of R, S in the existence interval. Adding $N = S + I + R$, we have

$$
\frac{\partial N}{\partial t} = D\frac{\partial^2 N}{\partial x^2} + b - dN, \quad x \in \mathbb{R}, t > 0,
$$

with initial condition

$$
N(x, 0) = S(x, 0) + I(x, 0) + R(x, 0), \quad x \in \mathbb{R}.
$$

Then the existence of M, δ follows by the ODE $u'(t) = b - du(t), u(0) = \sup_{x \in \mathbb{R}} N(x, 0)$ or $u(0) = \inf_{x \in \mathbb{R}} N(x, 0)$, which indicates the global existence. Applying the uniform boundedness, we obtain the regularity by the basic theory of reaction-diffusion equations [16].

Finally, the nonnegative implies that

$$\frac{\partial S}{\partial t} \geq D\frac{\partial^2 S}{\partial x^2} + b - dS - f(I)S, x \in \mathbb{R}, t > 0.$$

Let $K = \max_{x \in [0, b/d+M]} f(x)$. Then

$$\frac{\partial S}{\partial t} \geq D\frac{\partial^2 S}{\partial x^2} + b - dS - KS, x \in \mathbb{R}, t > 0.$$

Define $\underline{S} = s(1) > 0$, where $s(t)$ is defined by

$$s'(t) = b - ds(t) - Ks(t), t > 0, \quad s(0) = 0.$$

Due to the comparison principle, we have $S(x, t) \geq \underline{S}, x \in \mathbb{R}, t \geq 1$. The proof is complete.

Basing on the existence and uniqueness of classical solutions, we have the following spreading property of (2.1).

Theorem 2.2. *Assume that (F1) and (F2) hold. Then c^* is the spreading speed of three functions $I, R, \max\{0, b/d - S\}$ defined by (2.1). Further suppose that (F3) holds, then*

$$\limsup_{t \to \infty} \sup_{|x| < (c^*-\epsilon)t} \left[|I(x, t) - I^*| + \left| R(x, t) - \frac{(\mu + \beta)I^*}{d + \gamma} \right| \right.$$

$$\left. + \left| S(x, t) + \frac{(\mu + \beta + d + \gamma)I^*}{d + \gamma} - \frac{b}{d} \right| \right] = 0 \qquad (2.2)$$

for any given small $\epsilon > 0$. When (F1) and (F4) are true, we have

$$\lim_{t \to \infty} [I(x, t) + R(x, t) + |S(x, t) - b/d|] = 0 \text{ uniformly in } x \in \mathbb{R}. \qquad (2.3)$$

We now prove this theorem by the following Lemmas 2.3–2.6.

Lemma 2.3. *Assume that (F1) and (F2) hold. Then for any given $\epsilon > 0$, we have*

$$\limsup_{t \to \infty} \sup_{|x| > (c^*+\epsilon)t} [|S(x, t) - b/d| + I(x, t) + R(x, t)] = 0.$$

Proof. Fix $\epsilon > 0$. We first prove the conclusion on I. By Lemma 2.1, we have

$$S(x, t) + I(x, t) + R(x, t) \leq b/d + Me^{-\delta t}, \quad x \in \mathbb{R}, t > 0.$$

Therefore, I satisfies

$$\begin{aligned}
\frac{\partial I}{\partial t} &= D\frac{\partial^2 I}{\partial x^2} + f(I)S - (d + \mu + \beta)I \\
&= D\frac{\partial^2 I}{\partial x^2} + f(I)(S + I + R - I - R) - (d + \mu + \beta)I \\
&\leq D\frac{\partial^2 I}{\partial x^2} + f(I)(b/d + Me^{-\delta t} - I - R) - (d + \mu + \beta)I
\end{aligned}$$

$$\leq D\frac{\partial^2 I}{\partial x^2} + f(I)(b/d + Me^{-\delta t} - I) - (d + \mu + \beta)I$$

$$\leq D\frac{\partial^2 I}{\partial x^2} + f'(0)I(b/d + Me^{-\delta t} - I) - (d + \mu + \beta)I$$

$$= D\frac{\partial^2 I}{\partial x^2} + I\left[f'(0)\left(b/d + Me^{-\delta t}\right) - (d + \mu + \beta) - f'(0)I\right]$$

$$\leq D\frac{\partial^2 I}{\partial x^2} + I\left[f'(0)\left(b/d + Me^{-\delta t}\right) - (d + \mu + \beta)\right](1 - \varepsilon I)$$

for all $x \in \mathbb{R}, t > 0$, where $\varepsilon > 0$ such that $\varepsilon[f'(0)(b/d + M) - (d + \mu + \beta)] < f'(0)$.

Consider the auxiliary equation

$$\frac{\partial \bar{I}}{\partial t} = D\frac{\partial^2 \bar{I}}{\partial x^2} + \bar{I}\left[f'(0)\left(b/d + Me^{-\delta t}\right) - (d + \mu + \beta)\right](1 - \varepsilon\bar{I}), \quad x \in \mathbb{R}, t > 0$$

with initial condition $\bar{I}(x, 0) = I(x, 0)$. Note that $\lim_{t\to\infty} Me^{-\delta t} = 0$ such that

$$\liminf_{t\to\infty} \frac{1}{T} \int_t^{t+T} \left[f'(0)\left(b/d + Me^{-\delta s}\right) - (d + \mu + \beta)\right] ds$$

$$= \limsup_{t\to\infty} \frac{1}{T} \int_t^{t+T} \left[f'(0)\left(b/d + Me^{-\delta s}\right) - (d + \mu + \beta)\right] ds$$

$$= f'(0)b/d - (d + \mu + \beta) \text{ uniformly in } T > 0.$$

Applying Nadin and Rossi [18, Proposition 2.6] and Rossi [19, Proposition 1.6], $\bar{I}(x, t)$ satisfies

$$\limsup_{t\to\infty} \sup_{|x|>(c^*+\epsilon)t} \bar{I}(x, t) = 0$$

when the initial value $\bar{I}(x, 0)$ admits nonempty compact support. Due to the comparison principle, we obtain a similar result for $I(x, t)$. From the special structure

$$R(x, t) = \frac{e^{-(d+\gamma)t}}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4Dt}} R(y, 0)dy + (\mu + \beta) \int_0^t \frac{e^{-(d+\gamma)(t-s)}}{\sqrt{4\pi D(t-s)}} ds \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4D(t-s)}} I(y, s)dy,$$

R also satisfies the desired conclusion.

Using Lemma 2.1, we can obtain the conclusion on $S(x, t)$. The proof is complete.

Lemma 2.4. *Assume that (F1) and (F2) hold. Then for any given $\epsilon > 0$, we have*

$$\liminf_{t\to\infty} \inf_{|x|<(c^*-\epsilon)t}[I(x, t)R(x, t)] > 0, \qquad \liminf_{t\to\infty} \inf_{|x|<(c^*-\epsilon)t}[b/d - S(x, t)] > 0.$$

Proof. When the result on I, R holds, then Lemma 2.1 implies that

$$\limsup_{t\to\infty} \sup_{|x|<(c^*-\epsilon)t} S(x, t) < b/d,$$

which leads to the conclusion on S. From the linear structure of R, we can obtain the desirable result on R once $\liminf_{t\to\infty} \inf_{|x|<(c^*-\epsilon)t} I(x, t) > 0$.

Now, we verify the conclusion $\liminf_{t\to\infty}\inf_{|x|<(c^*-\epsilon)t} I(x,t) > 0$ for any given constant $\epsilon > 0$. Firstly, we select small constant $\varepsilon > 0$ such that

$$Dx^2 - (c^* - \epsilon)x + f'(0)[b/d - 4\varepsilon] - (d + \mu + \beta) > 0, \qquad x \in \mathbb{R},$$

which holds by the property that $Dx^2 - c^*x + f'(0)b/d - (d + \mu + \beta) = 0$ has a unique positive root. Since we study the long time behavior, we select a constant $T > 1$ such that

$$\begin{aligned}
\frac{\partial I}{\partial t} &= D\frac{\partial^2 I}{\partial x^2} + f(I)S - (d + \mu + \beta)I \\
&= D\frac{\partial^2 I}{\partial x^2} + f(I)(S + I + R - I - R) - (d + \mu + \beta)I \\
&\geq D\frac{\partial^2 I}{\partial x^2} + f(I)(b/d - Me^{-\delta t} - I - R) - (d + \mu + \beta)I \\
&\geq D\frac{\partial^2 I}{\partial x^2} + f(I)(b/d - \varepsilon - I - R) - (d + \mu + \beta)I
\end{aligned}$$

when $t > T$. Due to the special structure of R, we select a constant $T_1 > T$ such that

$$\begin{aligned}
R(x,t) &= \frac{e^{-(d+\gamma)t}}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4Dt}} R(y,0)dy + (\mu + \beta) \int_0^t \frac{e^{-(d+\gamma)(t-s)}}{\sqrt{4\pi D(t-s)}}ds \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4D(t-s)}} I(y,s)dy \\
&\leq \varepsilon + (\mu + \beta) \int_{1/T_1}^{T_1} \frac{e^{-(d+\gamma)s}}{\sqrt{4\pi Ds}}ds \int_{-T_1}^{T_1} e^{-\frac{(x-y)^2}{4Ds}} I(y,t-s)dy
\end{aligned}$$

for all $t > 2T_1, x \in \mathbb{R}$. Here, the selection of T_1 is independent of x, t once $t > 0$ is large. If

$$(\mu + \beta) \int_{1/T_1}^{T_1} \frac{e^{-(d+\gamma)s}}{\sqrt{4\pi Ds}}ds \int_{-T_1}^{T_1} e^{-\frac{(x-y)^2}{4Ds}} I(y,t-s)dy < \varepsilon,$$

then

$$\frac{\partial I}{\partial t} \geq D\frac{\partial^2 I}{\partial x^2} + f(I)(b/d - 3\varepsilon - I) - (d + \mu + \beta)I.$$

If

$$(\mu + \beta) \int_{1/T_1}^{T_1} \frac{e^{-(d+\gamma)s}}{\sqrt{4\pi Ds}}ds \int_{-T_1}^{T_1} e^{-\frac{(x-y)^2}{4Ds}} I(y,t-s)dy \geq \varepsilon, \tag{2.4}$$

then the uniform continuity in Lemma 2.1 implies that there exist

$$\delta > 0, \quad y' \in [x - T_1, x + T_1], \quad s' \in [t - T_1, t - 1/T_1]$$

such that $I(y', s') > \delta$. Evidently, δ is also independent of x, t and only depends on ε. Again by the uniform continuity, there exists $\sigma \in (0,1)$ such that $I(y, s') > \delta/2$ if $|y - y'| \leq \sigma$. Here, σ is also independent of x, t. Since

$$I(x,t) \geq \frac{e^{-(d+\mu+\beta)s}}{\sqrt{4\pi Ds}} \int_{-(T_1+1)}^{T_1+1} e^{-\frac{(x-y)^2}{4Ds}} I(y,t-s)dy, \quad s \in [1/T_1, T_1],$$

we get a positive constant υ

$$0 < \upsilon \leq \inf_{y\in[-2T_1-1,2T_1+1]} I(x-y,t),$$

which is independent of x, t once (2.4) holds. In fact, we can define v by

$$v = \inf_{x \in [-T_1-1, T_1+1], t \in [1/T_1, T_1]} \underline{I}(x, t),$$

where $\underline{I}(x, t)$ is defined by

$$\frac{\partial \underline{I}}{\partial t} = D \frac{\partial^2 \underline{I}}{\partial x^2} - (d + \mu + \beta)\underline{I}, \quad x \in \mathbb{R}, t > 0$$

with initial condition

$$\underline{I}(x, 0) = \delta/2, |x| < \sigma; \quad \underline{I}(x, 0) = 0, |x| \geq \sigma.$$

Then the boundedness of $I + R$ implies that we can select a constant $m > 0$ such that

$$-I(x, t) - R(x, t) \geq -mv \geq -mI(x, t)$$

once (2.4) is true. That is, when $t > 0$ is large, we find that

$$\frac{\partial I}{\partial t} \geq D \frac{\partial^2 I}{\partial x^2} + f(I)(b/d - 3\varepsilon - mI) - (d + \mu + \beta)I.$$

Using the comparison principle and the positivity of $I(x, t)$, we have

$$\liminf_{t \to \infty} \inf_{|x| < 2 \sqrt{D[f'(0)(b/d - 4\varepsilon) - (d+\mu+\beta)]t}} I(x, t) > 0.$$

The proof is complete.

When the spreading occurs, we also have the following conclusion.

Lemma 2.5. *Assume that (F1)–(F3) hold. Then (2.2) is true.*

Proof. We shall prove this by a contradiction discussion basing on the following Claim, which is motivated by Ducrot [20].

Claim. *Assume that $(S_\infty, I_\infty, R_\infty)$ is a bounded nonnegative entire solution to (1.2) such that $\inf_{x, t \in \mathbb{R}}[S_\infty(x, t)I_\infty(x, t)R_\infty(x, t)] > 0$. Then*

$$(S_\infty(x, t), I_\infty(x, t), R_\infty(x, t)) = \left(\frac{b}{d} - \frac{(\mu + \beta + d + \gamma)I^*}{d + \gamma}, I^*, \frac{(\mu + \beta)I^*}{d + \gamma} \right), x, t \in \mathbb{R}.$$

The proof of the Claim will be given later. Were the statement of this lemma false, then we can select small constants $\epsilon' > 0, \sigma > 0$ and a strictly increasing sequence $\{t_n\}$, a sequence $\{x_n\}$ with

$$\lim_{n \to \infty} t_n = \infty, \quad |x_n| \leq (c^* - \epsilon')t_n, \quad n \in \mathbb{N}$$

and

$$\left[|I(x_n, t_n) - I^*| + \left| R(x_n, t_n) - \frac{(\mu + \beta)I^*}{d + \gamma} \right| \right.$$
$$\left. + \left| S(x_n, t_n) + \frac{(\mu + \beta + d + \gamma)I^*}{d + \gamma} - \frac{b}{d} \right| \right] > \sigma. \tag{2.5}$$

Consider the sequence

$$(S_n(x,t), I_n(x,t), R_n(x,t)) = (S(x + x_n, t + t_n), I(x + x_n, t + t_n), R(x + x_n, t + t_n)), n \in \mathbb{N}.$$

Due to the smoothness in Lemma 2.1, after selecting a proper subsequence and still using the same notation, $(S_n(x,t), I_n(x,t), R_n(x,t))$ locally uniformly converges to an entire solution $(S_\infty, I_\infty, R_\infty)$ to (2.1). The spreading properties in Lemma 2.4 imply that $\inf_{x,t\in\mathbb{R}}[S_\infty(x,t)I_\infty(x,t)R_\infty(x,t)] > 0$. From the selection of $\{x_n\}, \{t_n\}$, we have

$$\left[|I_\infty(0,0) - I^*| + \left| R_\infty(0,0) - \frac{(\mu + \beta)I^*}{d + \gamma} \right| \right.$$
$$\left. + \left| S_\infty(0,0) + \frac{(\mu + \beta + d + \gamma)I^*}{d + \gamma} - \frac{b}{d} \right| \right] \geq \sigma,$$

which contradicts to the Claim. Therefore, (2.2) is true.

We now verify the Claim. Firstly, let $N_\infty = S_\infty + I_\infty + R_\infty$, then

$$\frac{\partial N_\infty(x,t)}{\partial t} = D\frac{\partial^2 N_\infty(x,t)}{\partial x^2} + b - dN_\infty(x,t), x,t \in \mathbb{R}.$$

Then the stability and boundedness imply that $N_\infty(x,t) = b/d, x,t \in \mathbb{R}$. Therefore,

$$\begin{cases} \frac{\partial I_\infty}{\partial t} = D\frac{\partial^2 I_\infty}{\partial x^2} + f(I_\infty)(b/d - I_\infty - R_\infty) - (d + \mu + \beta)I_\infty, \\ \frac{\partial R_\infty}{\partial t} = D\frac{\partial^2 R_\infty}{\partial x^2} - (d + \gamma)R_\infty + (\mu + \beta)I_\infty. \end{cases} \tag{2.6}$$

From the definition of entire solution, we see that

$$R_\infty(x,t) = (\mu + \beta)\int_0^\infty \frac{e^{-(d+\gamma)s}}{\sqrt{4\pi Ds}}ds \int_{-\infty}^\infty e^{-\frac{y^2}{4Ds}}I_\infty(x - y, t - s)dy =: (K * I_\infty)(x,t) \tag{2.7}$$

for all $x, t \in \mathbb{R}$. That is, (2.6) becomes

$$\frac{\partial I_\infty}{\partial t} = D\frac{\partial^2 I_\infty}{\partial x^2} + f(I_\infty)(b/d - I_\infty - (K * I_\infty)(x,t)) - (d + \mu + \beta)I_\infty \tag{2.8}$$

for all $x, t \in \mathbb{R}$. Let $H > 0$ be a constant such that

$$f(x)(b/d - x - (\mu + \beta)b/(d(d + \gamma))) - (d + \mu + \beta)x + Hx, \qquad x \in [0, b/d]$$

is increasing. The corresponding integral equation of (2.8) is

$$I_\infty(x,t) = \int_0^\infty \frac{e^{-Hs}}{\sqrt{4\pi Ds}}ds \int_{-\infty}^\infty e^{-\frac{y^2}{4Ds}} \times \{[H - (d + \mu + \beta)]I_\infty(x - y, t - s) \tag{2.9}$$
$$+ f(I_\infty(x - y, t - s))(b/d - I_\infty(x - y, t - s) - (K * I_\infty)(x - y, t - s))\} dy.$$

Define two positive constants

$$\sup_{x,t\in\mathbb{R}} I_\infty(x,t) = \overline{I}, \quad \inf_{x,t\in\mathbb{R}} I_\infty(x,t) = \underline{I}.$$

Then the monotonicity in (2.9) implies that (1.5) holds. Applying (F3), we see that $I_\infty = I^*$, which further leads to the conclusions for S_∞, R_∞ by (2.7). The proof is complete.

Lemma 2.6. *Assume that (F1) and (F4) hold. Then* (2.3) *is true.*

Proof. By the positivity of R, we see that

$$
\begin{aligned}
\frac{\partial I}{\partial t} &= D\frac{\partial^2 I}{\partial x^2} + f(I)S - (d + \mu + \beta)I \\
&\leq D\frac{\partial^2 I}{\partial x^2} + f(I)(b/d + Me^{-\delta t} - I) - (d + \mu + \beta)I.
\end{aligned}
$$

Due to the comparison principle, we have $I(x, t) \leq \bar{I}(t)$, $x \in \mathbb{R}$, $t > 0$, where $\bar{I}(t)$ is defined by

$$
\bar{I}'(t) = f(\bar{I}(t))(b/d + Me^{-\delta t} - \bar{I}(t)) - (d + \mu + \beta)\bar{I}(t), t > 0, \quad \bar{I}(0) = \sup_{x \in \mathbb{R}} I(x, 0),
$$

and is the spatially homogeneous solution to the following equation

$$
\begin{cases}
\frac{\partial u(x,t)}{\partial t} = D\frac{\partial^2 u(x,t)}{\partial x^2} + f(u(x,t))(b/d + Me^{-\delta t} - u(x,t)) - (d + \mu + \beta)u(x,t), & x \in \mathbb{R}, t > 0, \\
u(x, 0) = \sup_{x \in \mathbb{R}} I(x, 0), x \in \mathbb{R}.
\end{cases}
$$

Since (F4) implies that $\lim_{t \to \infty} \bar{I}(t) = 0$, we have

$$
\lim_{t \to \infty} I(x, t) = 0 \text{ uniformly in } x \in \mathbb{R},
$$

which further indicates the conclusions of S, R. The proof is complete.

Before ending this section, we make the following remarks.

Remark 2.7. Even if $S(x, 0) + I(x, 0) + R(x, 0) = b/d$, $x \in \mathbb{R}$, we can not obtain a cooperative autonomous subsystem of I, R, which is different from that in some epidemic models and predator-prey systems including [21]. For asymptotic spreading in noncooperative systems of epidemic models and predator-prey systems, we may refer to recent works in [20, 22–27].

Remark 2.8. From our proof, we find that (F3) leads to a convergence conclusion, which ensures the uniqueness of positive entire solutions. In fact, the convergence (2.2) holds once the positive entire solution with positive infimum of (1.2) is unique or the positive constant steady state of (1.2) is asymptotic stable in (2.1). With given f, it is possible to obtain further sufficient conditions such that (F3) holds. For example, if $f(I) = f'(0)I$ holds and $(\mu + \beta)/(d + \gamma)$ is small, then (F3) holds.

Remark 2.9. Lemmas 2.4–2.6 remain true once $I_0(x)$ has nonempty support.

3. Traveling wave solutions

In this part, we study the existence or nonexistence of (1.3) and (1.4). Letting $n(z) = s(z) + i(z) + r(z)$, $z \in \mathbb{R}$, we find that

$$
\begin{cases}
Dn'' - cn' + b - dn = 0, \\
\lim_{z \to -\infty} n(z) = b/d, \quad \liminf_{z \to \infty} n(z) > 0.
\end{cases}
\tag{3.1}
$$

Recall the fact that every bounded solution to

$$\frac{\partial u(x,t)}{\partial t} = D\frac{\partial^2 u(x,t)}{\partial x^2} + b - du(x,t)$$

uniformly converges to b/d. Then the bounded solution to (3.1) must be $n(z) = b/d, z \in \mathbb{R}$. Therefore, we need to investigate the existence or nonexistence of

$$\begin{cases} Di'' - ci' + f(i)(b/d - i - r) - (d + \mu + \beta)i = 0, \\ Dr'' - cr' - (d + \gamma)r + (\mu + \beta)i = 0, \\ \lim_{z \to -\infty}(i(z), r(z)) = (0, 0), \liminf_{z \to \infty} i(z)r(z) > 0. \end{cases} \quad (3.2)$$

Again by the fact that a traveling wave solution is a special entire solution, from the viewpoint of initial value problem as well as the theory of semigroup theory, we see that

$$r(z) = (\mu + \beta)\int_0^\infty \frac{e^{-(d+\gamma)s}}{\sqrt{4\pi Ds}}ds \int_{-\infty}^\infty e^{-\frac{y^2}{4Ds}}i(z - y - cs)dy =: (J * i)(z)$$

for all $z \in \mathbb{R}$. Therefore, it suffices to consider

$$\begin{cases} Di''(z) - ci'(z) + f(i(z))(b/d - i(z) - (J * i)(z)) - (d + \mu + \beta)i(z) = 0, \\ \lim_{z \to -\infty} i(z) = 0, \liminf_{z \to \infty} i(z) > 0, \end{cases} \quad (3.3)$$

and we first investigate the positive solution to

$$Di''(z) - ci'(z) + f(i(z))(b/d - i(z) - (J * i)(z)) - (d + \mu + \beta)i(z) = 0, z \in \mathbb{R}. \quad (3.4)$$

Remark 3.1. With the appearance of $(J * i)(z)$, this equation does not satisfy the monotone conditions in [15, 28–30].

Lemma 3.2. *Assume that $\bar{i}, \underline{i}$ are two nonnegative continuous functions satisfying the following items.*

1) $0 \le \underline{i}(z) \le \bar{i}(z) \le b/d, z \in \mathbb{R}$.

2) They are twice differentiable except in a set $\mathcal{Z}$ with finite elements such that

$$\underline{i}'(z+) \ge \underline{i}'(z-), \quad \bar{i}'(z-) \ge \bar{i}'(z+), \quad z \in \mathcal{Z}.$$

3) If $z \in \mathbb{R} \setminus \mathcal{Z}$, then

$$D\bar{i}''(z) - c\bar{i}'(z) + f(\bar{i}(z))(b/d - \bar{i}(z) - (J * \underline{i})(z)) - (d + \mu + \beta)\bar{i}(z) \le 0, \quad (3.5)$$

$$D\underline{i}''(z) - c\underline{i}'(z) + f(\underline{i}(z))(b/d - \underline{i}(z) - (J * \bar{i})(z)) - (d + \mu + \beta)\underline{i}(z) \ge 0. \quad (3.6)$$

Then (3.4) has a solution $i(z)$ such that $\underline{i}(z) \le i(z) \le \bar{i}(z), z \in \mathbb{R}$.

The proof of Lemma 3.2 is similar to that of Lin and Ruan [31, Section 4], and we omit it here. In the above lemma, $\bar{i}$ is an upper solution while $\underline{i}$ is a lower solution to (3.4). In particular, to study the existence of traveling waves, the recipe of upper and lower solutions has been proved to be useful, see some results in [32–38], which include some recent applications in different epidemic models [39–47]. Using Lemma 3.2, we can obtain the following existence result.

Lemma 3.3. *Assume that (F1) and (F2) hold. Then for any given $c \geq c^*$, (3.4) has a bounded positive nonconstant solution.*

Proof. We now construct upper and lower solutions. For each given $c > c^*$, we define two positive constants

$$\rho_1 = \frac{c - \sqrt{c^2 - 4D[f'(0)b/d - (d + \mu + \beta)]}}{2D},$$

$$\rho_2 = \frac{c + \sqrt{c^2 - 4D[f'(0)b/d - (d + \mu + \beta)]}}{2D},$$

and continuous functions

$$\bar{i}(z) = \min\{e^{\rho_1 z}, b/d\}, \underline{i}(z) = \max\{0, e^{\rho_1 z} - Qe^{\eta \rho_1 z}\},$$

in which $\eta - 1 > 0$ is small such that $\eta \rho_1 < \rho_2$ and $Q > 1$ is large. From Appendix A, we obtain a pair of upper and lower solutions of (3.4).

When $c = c^*$, we define $\rho_1 = \frac{c}{2D}$. Further select constant $P > b/d$ such that $(-z + P)e^{\rho_1 z}$ is nondecreasing in $z < 0$. Let $z_1 < 0$ be the unique negative root such that $(-z + P)e^{\rho_1 z} = b/d$. We define

$$\bar{i}(z) = \begin{cases} (-z + P)e^{\rho_1 z}, z \leq z_1, \\ b/d, z \geq z_1, \end{cases} \quad \underline{i}(z) = \begin{cases} (-z - Q\sqrt{-z})e^{\rho_1 z}, z < -Q^2, \\ 0, z \geq -Q^2, \end{cases}$$

in which $Q > 1$ is a large constant. From Appendix B, we obtain a pair of upper and lower solutions of (3.4) with $c = c^*$. The proof is complete.

Lemma 3.4. *Assume that (F1) and (F2) hold. If $c \geq c^*$, then (3.3) has a positive solution.*

Proof. When (3.4) has a positive solution, (1.3) also has a positive solution, in which the positivity of $i(z) > 0$ for small $z < 0$ is clear from the lower solution. By the upper and lower solutions, $\lim_{z \to -\infty} i(z) = 0$ holds. Since a traveling wave solution is a special entire solution, we can regard it as a special initial value problem when the initial condition is given by the positive wave profile. That is $S(x, t) = s(z), I(x, t) = i(z), R(x, t) = r(z)$ satisfy

$$\begin{cases} \frac{\partial S}{\partial t} = D\frac{\partial^2 S}{\partial x^2} + b - dS - f(I)S + \gamma R, \\ \frac{\partial I}{\partial t} = D\frac{\partial^2 I}{\partial x^2} + f(I)S - (d + \mu + \beta)I, \\ \frac{\partial R}{\partial t} = D\frac{\partial^2 R}{\partial x^2} - (d + \gamma)R + (\mu + \beta)I, \\ S(x, t) = s(x), I(x, t) = i(x), R(x, t) = r(x), x \in \mathbb{R}, t > 0. \end{cases} \tag{3.7}$$

By Lemma 2.4 and Remark 2.9, we have $\liminf_{t \to \infty} I(0, t) > 0$. By the fact $z = 0 + ct \to \infty$ as $t \to \infty$, the limit behavior as $z \to \infty$ also holds from the invariant form of traveling wave solutions. The proof is complete.

Lemma 3.5. *Assume that (F1)–(F3) hold. If $c \geq c^*$, then (3.3) has a solution satisfying $\lim_{z \to \infty} i(z) = I^*$.*

Proof. Similar to Lemma 3.4, if (F1)–(F3) hold, then a solution to (3.3) also satisfies $\lim_{z \to \infty} i(z) = I^*$. The proof is complete.

Lemma 3.6. *Assume that (F1) and (F2) hold. If $c < c^*$, then (3.3) does not have a positive solution.*

Proof. Were the statement false, then for some $c_1 \in (0, c^*)$, (3.3) with $c = c_1$ has a positive solution. We first prove the strictly positivity of traveling waves in the sense that $i(z) > 0, z \in \mathbb{R}$. Let $H > 0$ be defined in (2.9), and

$$h_1 = \frac{c_1 - \sqrt{c_1^2 + 4DH}}{2D}, \quad h_2 = \frac{c_1 + \sqrt{c_1^2 + 4DH}}{2D}.$$

Then

$$i(z) = \frac{1}{D(h_2 - h_1)} \left[\int_{-\infty}^{\infty} \min \left\{ e^{h_1(z-s)}, e^{h_2(z-s)} \right\} \right] \times$$
$$\left[Hi(s) + f(i(s))(b/d - i(s) - (J * i)(s)) - (d + \mu + \beta)i(s) \right] ds$$

for all $z \in \mathbb{R}$, which implies the strict positivity of $i(z)$ once $i(z_0) > 0$ for some $z_0 > 0$.

Let $\epsilon > 0$ such that

$$Dx^2 - c_1 x + f'(0)(b/d - 2\epsilon) - (d + \mu + \beta) > 0, \quad x \in \mathbb{R}.$$

Then c_1 is smaller than the spreading speed of the following Fisher-KPP equation

$$\frac{\partial I(x, t)}{\partial t} = D \frac{\partial^2 I(x, t)}{\partial x^2} + f(I(x, t))(b/d - 2\epsilon - MI(x, t)) - (d + \mu + \beta)I(x, t) \tag{3.8}$$

with nonempty compact supported initial condition, in which M is a given nonnegative constant. Select $z_1 < 0$ such that $(J * i)(z) < \epsilon$ if $z < z_1$, which is admissible from the limit behavior when $z \to -\infty$. When $z \geq z_1$, then the strict positivity and limit behavior as $z \to \infty$ implies that $\inf_{z \geq z_1} i(z) > 0, \sup_{z \in \mathbb{R}} i(z) < \infty$, and so $(J * i)(z) < (M - 1)i(z)$ for all $z \geq z_1$ and some $M > 1$. That is $i(z)$ satisfies

$$Di''(z) - c_1 i'(z) + f(i(z))(b/d - 2\epsilon - Mi(z)) - (d + \mu + \beta)i(z) \leq 0, \quad z \in \mathbb{R},$$

which implies that $i(z) = I(x + c_1 t)$ is an upper solution of (3.8). Therefore, the spreading speed of (3.8) is not larger than c_1. A contradiction occurs. The proof is complete.

Lemma 3.7. *Assume that (F1) and (F4) hold. Then (3.3) does not have a positive solution for all $c \in \mathbb{R}$.*

Proof. We argue it by contradiction. Assume that $I(x, t) = i(x + ct)$ is a solution to (3.3) for some given constant c. Then it is the unique solution of

$$\begin{cases} \frac{\partial I(x,t)}{\partial t} = D \frac{\partial^2 I(x,t)}{\partial x^2} + f(I(x, t))(b/d - I(x, t) - (J * I)(x, t)) - (d + \mu + \beta)I(x, t), & x \in \mathbb{R}, t > 0, \\ I(x, s) = i(x + cs), & x \in \mathbb{R}, s \leq 0. \end{cases}$$

Then the positivity implies that

$$\frac{\partial I(x, t)}{\partial t} \leq D \frac{\partial^2 I(x, t)}{\partial x^2} + f(I(x, t))(b/d - I(x, t)) - (d + \mu + \beta)I(x, t)$$

for all $x \in \mathbb{R}, t > 0$. Therefore, $\lim_{t \to \infty} I(x, t) = 0$ uniformly in $x \in \mathbb{R}$. From the invariant form of traveling wave solutions, we see that $i(z) \equiv 0, z \in \mathbb{R}$. A contradiction occurs. The proof is complete.

Summarizing what we have done, we give the main result of this section.

Theorem 3.8. *Assume that (F1) and (F2) hold. Then c^* is the minimal wave speed of (1.3) and (1.4). In addition, if (F3) is true, then c^* is the minimal wave speed of (1.3) with*

$$\begin{cases} \lim_{z\to-\infty}(s(z), i(z), r(z)) = (b/d, 0, 0), \\ \lim_{z\to\infty}(s(z), i(z), r(z)) = \left(\frac{b}{d} - \frac{(\mu+\beta+d+\gamma)I^*}{d+\gamma}, I^*, \frac{(\mu+\beta)I^*}{d+\gamma}\right). \end{cases}$$

Moreover, any positive solution to (1.3) and (1.4) is strictly positive. When (F1) and (F4) are true, then (1.3) and (1.4) does not have a positive solution for all $c \in \mathbb{R}$.

Remark 3.9. From the proof of Lemma 3.7, when the traveling wave solution is concerned, (F4) can be replaced by the following condition (C).

(C) $\lim_{t\to\infty} \mathcal{I}(t) = 0$, in which $\mathcal{I}$ is defined by

$$\frac{d\mathcal{I}(t)}{dt} = f(\mathcal{I}(t))(b/d - \mathcal{I}(t)) - (d + \mu + \beta)\mathcal{I}(t), t > 0; \quad \mathcal{I}(0) > 0.$$

When f is strictly subhomogeneous (e.g., $f(x) = x/(1 + x)$), the condition (C) contains $f'(0)b/d = d + \mu + \beta$. Similarly, if $S(x, T) + I(x, T) + R(x, T) \le b/d$ for $x \in \mathbb{R}$ and some $T \ge 0$, then Lemma 2.6 holds once (C) is true.

4. Conclusions and discussion

In this article, we studied the propagation dynamics of the coupled system (1.2). A speed that reflects the spreading ability of the disease was given. This speed implies some factors that affect the disease spreading. To establish the spreading speed, some auxiliary equations were constructed, which have nice spreading properties. To show the minimal wave speed, we utilized the upper and lower solutions and the theory of asymptotic spreading.

The first shortcoming of the model (1.2) is that three classes have the same diffusion ratio. When different diffusive parameters are concerned, we can not directly obtain the conclusion for $S + I + R$. Therefore, our discussion does not work directly. Consider

$$\begin{cases} \frac{\partial S}{\partial t} = D_1 \frac{\partial^2 S}{\partial x^2} + b - dS - f(I)S + \gamma R, \\ \frac{\partial I}{\partial t} = D \frac{\partial^2 I}{\partial x^2} + f(I)S - (d + \mu + \beta)I, \\ \frac{\partial R}{\partial t} = D_3 \frac{\partial^2 R}{\partial x^2} - (d + \gamma)R + (\mu + \beta)I, \end{cases} \quad (4.1)$$

in which $x \in \mathbb{R}, t > 0$, and the initial condition satisfies (I). We may find some features that are different from our study. For example, the boundedness of I and $\limsup_{t\to\infty} \sup_{x\in\mathbb{R}} S(x, t) \le b/d$ are not clear although we believe they are true. Because of the occurrence of γR, some important recipes for $\gamma = 0$ such as those in Ducrot [20] do not work directly to obtain the uniform boundedness. These questions need further study. Note that the appearance of R strongly depends on the successful invasion of I and the equation of R is linear, we further conjecture that the spreading speed of I in (4.1) is independent of γ. Moreover, at $(b/d, 0, 0)$, the corresponding linearized equation of I only depends on D, then very likely the spreading speed depends on D only. That is, the spreading speed

of $I(x,t), R(x,t), \max\{b/d - S(x,t), 0\}$ is c^* for any nonnegative $D_1 \geq 0, D_3 \geq 0$ once $\mathcal{R}_0 > 1$ holds and $I(x,0)$ has nonempty compact support. Once the conjecture is proved, we may find that only the diffusion of the infected affects the disease spreading ability.

We now numerically verify the conjecture that different diffusion parameters in the first and third equations do not change the spreading speed. For this purpose, we first show four examples by considering the initial value problem

$$
\begin{cases}
\frac{\partial S}{\partial t} = D_1 \frac{\partial^2 S}{\partial x^2} + 0.05 - 0.05S - 2IS + 0.05R, \, x \in \mathbb{R}, t > 0, \\
\frac{\partial I}{\partial t} = 0.24 \frac{\partial^2 I}{\partial x^2} + 2IS - 0.5I, \, x \in \mathbb{R}, t > 0, \\
\frac{\partial R}{\partial t} = D_3 \frac{\partial^2 R}{\partial x^2} - 0.1R + 0.45I, \, x \in \mathbb{R}, t > 0
\end{cases}
\tag{4.2}
$$

with initial condition

$$1 - S(x,0) = R(x,0) = 0, x \in \mathbb{R}; \quad I(x,0) = \cos x, |x| \leq \pi/2, I(x,0) = 0, |x| \geq \pi/2.$$

Then $c^* = 1.2$. We explore this case by four examples to show the role of different diffusion parameters.
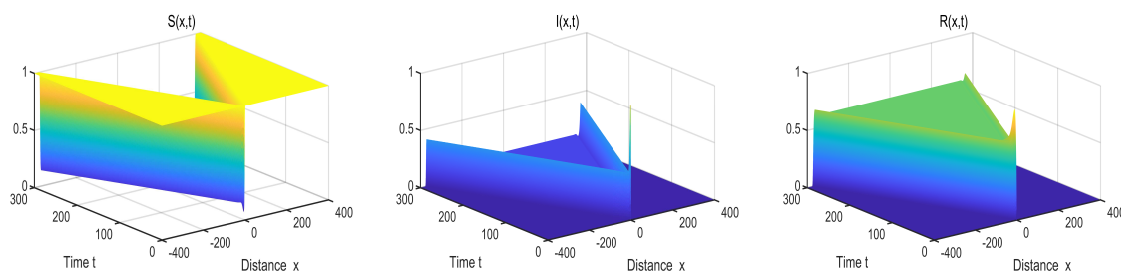
Example 4.1. Take $D_1 = D_3 = 0.24$ in (4.2).

Due to the fact that for any given $t > 0$,

$$\lim_{|x|\to\infty} S(x,t) = 1, \lim_{|x|\to\infty} I(x,t) = 0, \lim_{|x|\to\infty} R(x,t) = 0,$$

we add the following Dirichlet boundary condition to simulate (4.2)

$$S(\pm\mathcal{L}, t) = 1, I(\pm\mathcal{L}, t) = R(\pm\mathcal{L}, t) = 0$$

for $\mathcal{L} = 520, t \in [0, 300]$, and the method is applied to all examples in this part. This case is covered by our conclusion. The evolution using the Dirichlet condition is simulated by Figures 1 and 2. Figure 1 shows a property that the spreading speed exists. By the spatial movement of level sets in Figure 2, the approximate invasion speed is 1.25. Here, we use a Dirichlet problem to approximate the Cauchy problem on whole $\mathbb{R}$. There are some other methods that can well simulate this kind of problem, e.g., spectral methods [48–50].
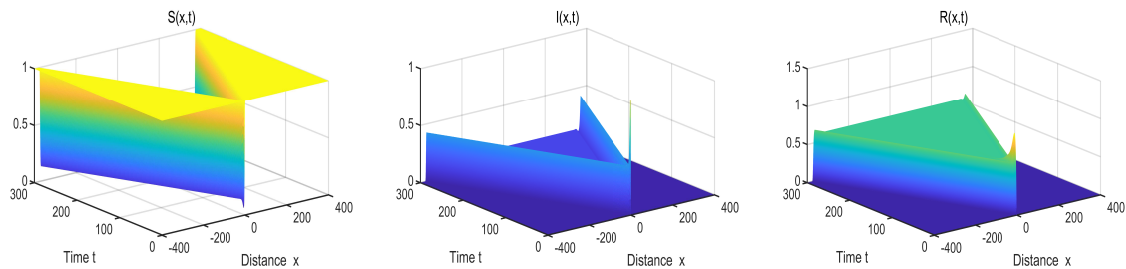

**Figure 1.** The spatial plots of $(S(x,t), I(x,t), R(x,t)), t \in [0, 300]$ defined by Example 4.1.
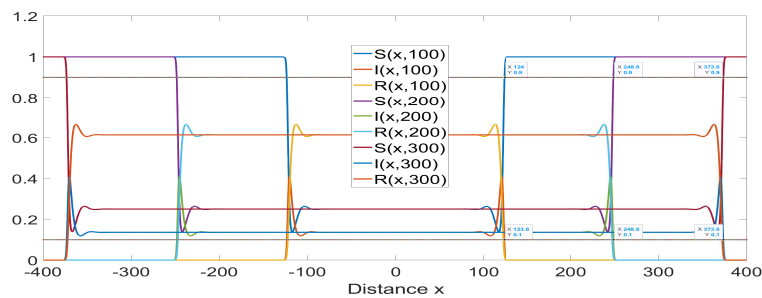
Figure 2. The snapshoots of $(S, I, R)(x, t), t = 100, 200, 300$ defined by Example 4.1.

Example 4.2. Take $D_1 = D_3 = 0.05$ in (4.2).

For this case, the numerical result is presented in Figures 3 and 4, from which we find that the spreading properties are similar to those in Example 4.1. In particular, from the movement of level sets in Figure 4, the spreading speed is close to 1.25.

**Figure 3.** The spatial plots of $(S(x, t), I(x, t), R(x, t)), t \in [0, 300]$ defined by Example 4.2.
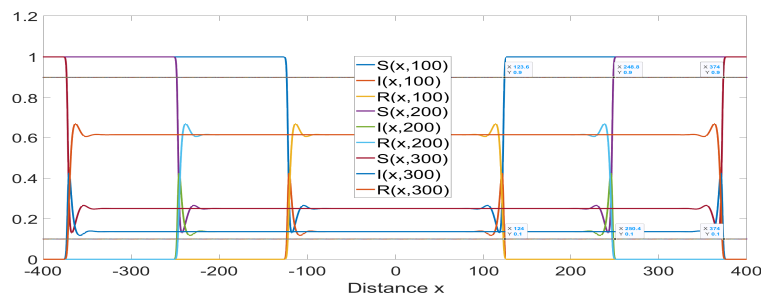
Figure 4. The snapshoots of $(S, I, R)(x, t), t = 100, 200, 300$ defined by Example 4.2.

Example 4.3. Take $D_1 = 0, D_3 = 1$ in (4.2).

For this case, we give several snapshoots in Figure 5. Although the diffusion parameter $D_1 = 0$, we obtain a rough speed that is close to the speeds in Examples 4.1 and 4.2. Very likely, the degenerate diffusion of the susceptible does not change the invasion of I.
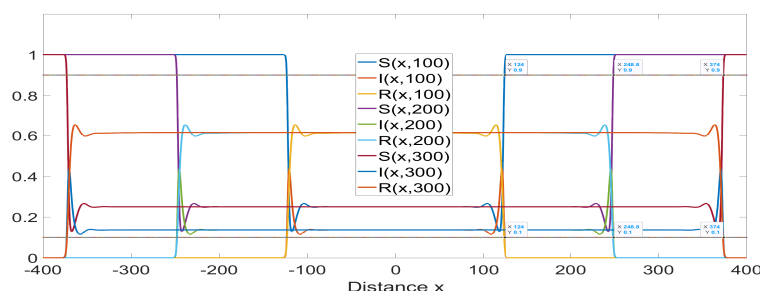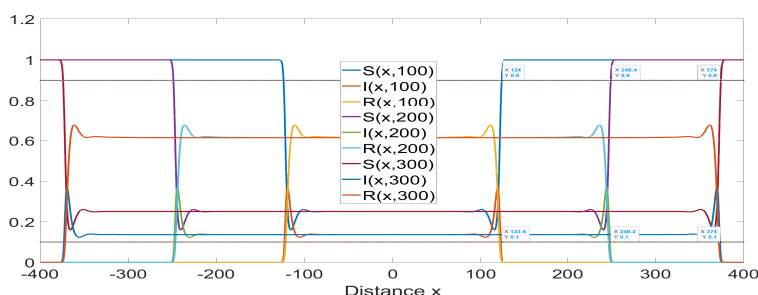
Figure 5. The snapshoots of $(S, I, R)(x, t), t = 100, 200, 300$ defined by Example 4.3.

Example 4.4. Take $D_1 = 1, D_3 = 0$ in (4.2).

For this case, the diffusion parameter $D_3 = 0$ is degenerate. We give the simulation in Figure 6, from which the rough speed for Example 4.4 is close to the speeds in Examples 4.1–4.3. By Examples 4.2–4.4, we may find that D_1, D_3 do not change the spreading speed. Moreover, in Examples 4.1–4.4, the final size of the infected is significantly smaller than 0.2.
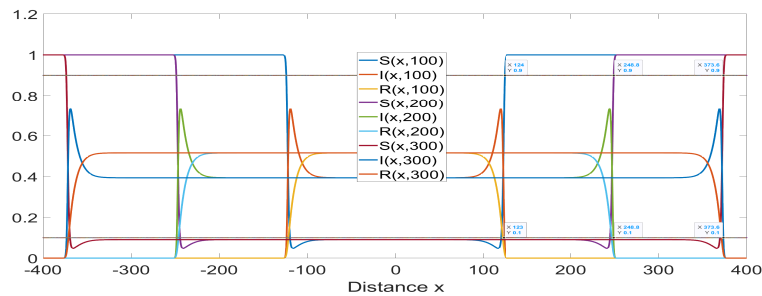

Figure 6. The snapshoots of $(S, I, R)(x, t), t = 100, 200, 300$ defined by Example 4.4.

We also consider the role of treatment. In this article, we take a linear treatment function. When the resource is limited or the treated is delayed, there are several possible cases. For example, the resource is dispersive such that the infected can not receive sufficient treatment care, or there is a maximum number for adequate medical treatment. For this case, we need a nonlinear function to model the medical treatment. Due to the limited or dispersive medical resource or the delayed treatment, at any given location, the eventual infected number may be significantly larger than the case of sufficient medical resource. We illustrate the phenomenon by the following example.

Example 4.5. We consider the system

$$\begin{cases} \frac{\partial S}{\partial t} = 0.5\frac{\partial^2 S}{\partial x^2} + 0.05 - 0.05S - 2IS + 0.05R, x \in \mathbb{R}, t > 0, \\ \frac{\partial I}{\partial t} = 0.24\frac{\partial^2 I}{\partial x^2} + 2IS - 0.1I - \frac{0.4I}{1+10I}, x \in \mathbb{R}, t > 0, \\ \frac{\partial R}{\partial t} = 0.6\frac{\partial^2 R}{\partial x^2} - 0.1R + 0.05I + \frac{0.4I}{1+10I}, x \in \mathbb{R}, t > 0, \\ S(x,0) - 1 = R(x,0) = 0, x \in \mathbb{R}, I(x,0) = \cos x, |x| \le \pi/2, I(x,0) = 0, |x| \ge \pi/2. \end{cases} \quad (4.3)$$

In this case, the treatment function shows the effect of the infected being delayed for treatment and the reverse effect of the infected being delayed for treatment [51]. For the second equation in (4.3),

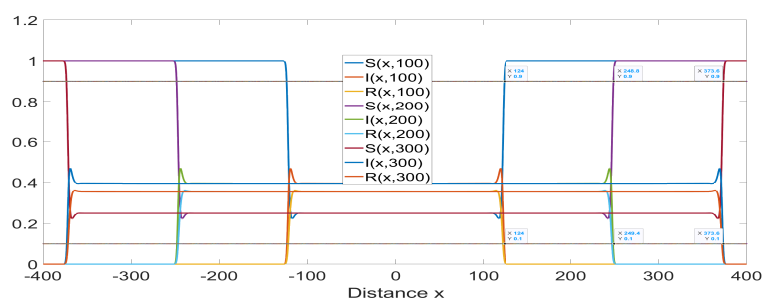**Figure 7.** The snapshoots of $(S, I, R)(x, t), t = 100, 200, 300$ defined by Example 4.5.

linearizing it at $(1, 0, 0)$, we may obtain the same equation as those in (4.2). From Figure 7, we see that the spreading speed is close to that in Examples 4.1–4.4. However, the ratio for final infected is significantly larger than those in Examples 4.1–4.4. That is, the delayed treatment may be harm to the disease controlling.

Finally, comparing with the SIR model, SIRS involves the recovered individuals that partly lose the immunity. At least when the solution locally uniformly converges to the constant positive steady state, it is possible that at any given spatial position, larger ratio γ may lead to larger ratio of the eventual infected since the superinfection is possible. We give a numerical example to illustrate the role of γ.

Example 4.6. We consider the system

$$
\begin{cases}
\frac{\partial S}{\partial t} = 0.05 \frac{\partial^2 S}{\partial x^2} + 0.05 - 0.05S - 2IS + 0.45R, x \in \mathbb{R}, t > 0, \\
\frac{\partial I}{\partial t} = 0.24 \frac{\partial^2 I}{\partial x^2} + 2IS - 0.5I, x \in \mathbb{R}, t > 0, \\
\frac{\partial R}{\partial t} = 0.05 \frac{\partial^2 R}{\partial x^2} - 0.5R + 0.45I, x \in \mathbb{R}, t > 0, \\
S(x, 0) - 1 = R(x, 0) = 0, x \in \mathbb{R}, I(x, 0) = \cos x, |x| \le \pi/2, I(x, 0) = 0, |x| \ge \pi/2.
\end{cases}
\tag{4.4}
$$

From Figure 8, we see that the spreading speed is close to that in the previous examples. But the final state of the infected is larger than that in the first example. At the same time, the susceptible at any given location will eventually have a relative smaller size than that in Examples 4.1–4.4. The conclusion is also clear since high ratio that loses immunity is harm to the disease controlling.

Figure 8. The snapshoots of $(S, I, R)(x, t), t = 100, 200, 300$ defined by Example 4.6.

Comparing Examples 4.1–4.4 with 4.5 and 4.6, we introduce different treatment functions and ratios that the recovered partly lose the immunity, which leads to different final size of the infected. In

application, it is absolutely important to show the different phenomena caused by them such as the role of vaccine, see several very recent works [52–54]. Here, we only gave some theoretical conclusions for a special model. To do sensitive analysis with possible real data in these models deserves further investigation.

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Conflict of interest

The authors declare there is no conflict of interest.

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Appendix

In this part, we show the detail verification in the proof of Lemma 3.3.

A. The case $c > c^*$

For each fixed $c > c^*$, it suffices to verify (3.5) and (3.6) when $\bar{i}(z), \underline{i}(z)$ are twice differentiable. If

$\bar{i}(z) = e^{\rho_1 z} < b/d$, then

$$
\begin{aligned}
& D\bar{i}''(z) - c\bar{i}'(z) + f(\bar{i}(z))(b/d - \bar{i}(z) - (J * \underline{i})(z)) - (d + \mu + \beta)\bar{i}(z) \\
\le\ & D\bar{i}''(z) - c\bar{i}'(z) + bf(\bar{i}(z))/d - (d + \mu + \beta)\bar{i}(z) \\
\le\ & D\bar{i}''(z) - c\bar{i}'(z) + bf'(0)\bar{i}(z)/d - (d + \mu + \beta)\bar{i}(z) \\
=\ & D\rho_1^2 e^{\rho_1 z} - c\rho_1 e^{\rho_1 z} + [bf'(0)/d - (d + \mu + \beta)] e^{\rho_1 z} \\
=\ & e^{\rho_1 z}[D\rho_1^2 - c\rho_1 + bf'(0)/d - (d + \mu + \beta)] \\
=\ & 0.
\end{aligned}
$$

If $\bar{i}(z) = b/d < e^{\rho_1 z}$, then

$$
\begin{aligned}
& D\bar{i}''(z) - c\bar{i}'(z) + f(\bar{i}(z))(b/d - \bar{i}(z) - (J * \underline{i})(z)) - (d + \mu + \beta)\bar{i}(z) \\
<\ & D\bar{i}''(z) - c\bar{i}'(z) + f(\bar{i}(z))(b/d - \bar{i}(z)) \\
=\ & 0.
\end{aligned}
$$

and this completes the proof of (3.5).

We now consider (3.6), which is clear if $\underline{i}(z) = 0 > e^{\rho_1 z} - Qe^{\eta\rho_1 z}$. Note that we can select a large constant $M > 1/\rho_1$ such that

$$
\bar{i}(z) \le Me^{z/M}, \quad (J * \bar{i})(z) \le Me^{z/M}, \quad z \in \mathbb{R}. \tag{A.1}
$$

After fixing M, we select $\eta - 1 > 0$ small enough such that

$$
\eta\rho_1 < \min\{\rho_2, \rho_1 + 1/M, 2\rho_1, (1 + \alpha)\rho_1\},
$$

which implies that

$$
D\eta^2\rho_1^2 - c\eta\rho_1 - (d + \mu + \beta) + bf'(0)/d < 0.
$$

When $\underline{i}(z) = e^{\rho_1 z} - Qe^{\eta\rho_1 z} > 0$, then $Q > 1$ leads to $z < 0$ and

$$
\max\left\{e^{(1+\alpha)\rho_1 z}, e^{2\rho_1 z}, e^{(\rho_1 + 1/M)z}\right\} < e^{\eta\rho_1 z}, \quad z < 0.
$$

Hence, we obtain

$$
\begin{aligned}
& D\underline{i}''(z) - c\underline{i}'(z) + f(\underline{i}(z))(b/d - \underline{i}(z) - (J * \bar{i})(z)) - (d + \mu + \beta)\underline{i}(z) \\
\ge\ & D\underline{i}''(z) - c\underline{i}'(z) - (d + \mu + \beta)\underline{i}(z) + bf'(0)\underline{i}(z)/d - L\underline{i}^{1+\alpha}(z)b/d \\
& -f(\underline{i}(z))\underline{i}(z) - f(\underline{i}(z))(J * \bar{i})(z) \\
\ge\ & D\underline{i}''(z) - c\underline{i}'(z) - (d + \mu + \beta)\underline{i}(z) + bf'(0)\underline{i}(z)/d \\
& -L\underline{i}^{1+\alpha}(z)b/d - f'(0)\underline{i}(z)\underline{i}(z) - f'(0)\underline{i}(z)(J * \bar{i})(z) \\
\ge\ & D\underline{i}''(z) - c\underline{i}'(z) - (d + \mu + \beta)\underline{i}(z) + bf'(0)\underline{i}(z)/d \\
& -L\bar{i}^{1+\alpha}(z)b/d - f'(0)\bar{i}^2(z) - f'(0)\bar{i}(z)(J * \bar{i})(z) \\
\ge\ & D\underline{i}''(z) - c\underline{i}'(z) - (d + \mu + \beta)\underline{i}(z) + bf'(0)\underline{i}(z)/d \\
& -L\bar{i}^{1+\alpha}(z)b/d - f'(0)\bar{i}^2(z) - Mf'(0)\bar{i}(z)e^{\frac{z}{M}}
\end{aligned}
$$

$$
\begin{aligned}
&= e^{\rho_1 z}\left[D\rho_1^2 - c\rho_1 - (d + \mu + \beta) + bf'(0)/d\right] \\
&\quad -Qe^{\eta\rho_1 z}\left[D\eta^2\rho_1^2 - c\eta\rho_1 - (d + \mu + \beta) + bf'(0)/d\right] \\
&\quad -L\bar{i}^{-1+\alpha}(z)b/d - f'(0)\bar{i}^2(z) - Mf'(0)\bar{i}(z)e^{\frac{z}{M}} \\
&= -Qe^{\eta\rho_1 z}\left[D\eta^2\rho_1^2 - c\eta\rho_1 - (d + \mu + \beta) + bf'(0)/d\right] \\
&\quad -L\bar{i}^{-1+\alpha}(z)b/d - f'(0)\bar{i}^2(z) - Mf'(0)\bar{i}(z)e^{\frac{z}{M}} \\
&= -Qe^{\eta\rho_1 z}\left[D\eta^2\rho_1^2 - c\eta\rho_1 - (d + \mu + \beta) + bf'(0)/d\right] \\
&\quad -Lbe^{(1+\alpha)\rho_1 z}/d - f'(0)e^{2\rho_1 z} - Mf'(0)e^{(\rho_1 + 1/M)z} \\
&\geq 0,
\end{aligned}
$$

which holds provided that

$$
Q > 1 - \frac{Lb/d + f'(0) + Mf'(0)}{D\eta^2\rho_1^2 - c\eta\rho_1 - (d + \mu + \beta) + bf'(0)/d} > 1.
$$

Then (3.6) is true and we complete the proof.

B. The case $c = c^*$

In this part, we verify (3.5) and (3.6) when $\bar{i}(z), \underline{i}(z)$ are twice differentiable. If $\bar{i}(z) = (-z + P)e^{\rho_1 z}$, then

$$
\begin{aligned}
&D\bar{i}''(z) - c\bar{i}'(z) + f(\bar{i}(z))(b/d - \bar{i}(z) - (J * \underline{i})(z)) - (d + \mu + \beta)\bar{i}(z) \\
\leq\ & D\bar{i}''(z) - c\bar{i}'(z) + f(\bar{i}(z))b/d - (d + \mu + \beta)\bar{i}(z) \\
\leq\ & D\bar{i}''(z) - c\bar{i}'(z) + [f'(0)b/d - (d + \mu + \beta)]\bar{i}(z) \\
=\ & -2D\rho_1 e^{\rho_1 z} + D(-z + P)\rho_1^2 e^{\rho_1 z} + ce^{\rho_1 z} - c(-z + P)\rho_1 e^{\rho_1 z} \\
&\quad + [f'(0)b/d - (d + \mu + \beta)](-z + P)e^{\rho_1 z} \\
=\ & \left[D\rho_1^2 - c\rho_1 + f'(0)b/d - (d + \mu + \beta)\right](-z + P)e^{\rho_1 z} + (-2D\rho_1 + c)e^{\rho_1 z} \\
=\ & 0.
\end{aligned}
$$

If $\bar{i}(z) = b/d$, then the proof is the same as that in Appendix A, which completes the proof of (3.5).

We now consider (3.6), which is clear if $\underline{i}(z) = 0$. For this case, (A.1) still holds with a large constant $M > 0$. Moreover, we select a constant $m > 0$ such that

$$
Lb\bar{i}^{-1+\alpha}(z)/d \leq me^{(1+\alpha/2)\rho_1 z}, \quad Mf'(0)\bar{i}(z)e^{\frac{z}{M}} \leq m(1 + \alpha/2)e^{\rho_1 z + z/(2M)},
$$
$$
f'(0)\bar{i}^2(z) \leq me^{3\rho_1 z/2}, \quad z \leq -1.
$$

Using the property, we now verify (3.6) if $\underline{i}(z) = -ze^{\rho_1 z} - Q\sqrt{-z}e^{\rho_1 z} > 0$. By direct calculation, we have

$$
\begin{aligned}
\underline{i}'(z) &= \rho_1(-z - Q\sqrt{-z})e^{\rho_1 z} - e^{\rho_1 z} + \frac{Q}{2\sqrt{-z}}e^{\rho_1 z}, \\
\underline{i}''(z) &= \rho_1^2(-z - Q\sqrt{-z})e^{\rho_1 z} - 2\rho_1 e^{\rho_1 z} + \frac{Q\rho_1}{\sqrt{-z}}e^{\rho_1 z} + \frac{Q}{4\sqrt{(-z)^3}}e^{\rho_1 z}.
\end{aligned}
$$

Letting $Q > 1$ be large, we see that $z < -Q^2$ such that

$$
\begin{aligned}
& D\underline{i}''(z) - c\underline{i}'(z) + f(\underline{i}(z))(b/d - \underline{i}(z) - (J * \bar{i})(z)) - (d + \mu + \beta)\underline{i}(z) \\
\geq\ & D\underline{i}''(z) - c\underline{i}'(z) - (d + \mu + \beta)\underline{i}(z) + bf'(0)\underline{i}(z)/d - L\underline{i}^{1+\alpha}(z)b/d \\
& -f(\underline{i}(z))\bar{i}(z) - f(\underline{i}(z))(J * \bar{i})(z) \\
\geq\ & D\underline{i}''(z) - c\underline{i}'(z) - (d + \mu + \beta)\underline{i}(z) + bf'(0)\underline{i}(z)/d \\
& -L\underline{i}^{1+\alpha}(z)b/d - f'(0)\underline{i}(z)\bar{i}(z) - f'(0)\underline{i}(z)(J * \bar{i})(z) \\
\geq\ & D\underline{i}''(z) - c\underline{i}'(z) - (d + \mu + \beta)\underline{i}(z) + bf'(0)\underline{i}(z)/d \\
& -L\bar{i}^{1+\alpha}(z)b/d - f'(0)\bar{i}^2(z) - f'(0)\bar{i}(z)(J * \bar{i})(z) \\
=\ & \left[e^{\rho_1 z} - \frac{Q e^{\rho_1 z}}{2\sqrt{-z}} \right][-2D\rho_1 + c] + \frac{DQ}{4\sqrt{(-z)^3}} e^{\rho_1 z} \\
& -(z + Q\sqrt{-z})e^{\rho_1 z}\left[D\rho_1^2 - c\rho_1 - (d + \mu + \beta) + bf'(0)/d \right] \\
& -L\bar{i}^{1+\alpha}(z)b/d - f'(0)\bar{i}^2(z) - f'(0)\bar{i}(z)(J * \bar{i})(z) \\
\geq\ & \frac{DQ}{4\sqrt{(-z)^3}} e^{\rho_1 z} - L\bar{i}^{1+\alpha}(z)b/d - f'(0)\bar{i}^2(z) - Mf'(0)\bar{i}(z)e^{\frac{z}{M}} \\
\geq\ & 0
\end{aligned}
$$

since

$$
\lim_{z \to -\infty} \left[e^{Az} \sqrt{(-z)^3} \right] = 0 \text{ uniformly for all } A \geq A_0
$$

with any given $A_0 > 0$. We complete the proof of (3.6) for $c = c^*$.

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