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Research article

Sharp conditions for a class of nonlinear Schrödinger equations

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Abstract: This paper studies a class of nonlinear Schrödinger equations in two space dimensions. By constructing a variational problem and the so-called invariant manifolds of the evolution flow, we get a sharp condition for global existence and blow-up of solutions.

Keywords: nonlinear Schrödinger equations; global existence; blow-up; sharp conditions; invariant manifolds

1. Introduction

In this paper, we study the Cauchy problem for the nonlinear Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta u + u g_1(|u|^2) = 0, \\ u(0) = u_0 \in H^1(\mathbb{R}^2), \end{cases} \tag{1.1}$$

where $u(t,x) : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{C}$, $g_1 \in C^1(\mathbb{R}, \mathbb{R}_+)$ is a positive real function satisfying $g_1(0) = 0$, and

$$g(z) := z g_1(|z|^2), \quad G(z) := \int_0^{|z|} g(s)\mathrm{d}s.$$

We assume $g(u)$ satisfies the following conditions:

$$\text{(H)} \qquad \begin{aligned} &\text{(i) } g \in C^1 \text{ and } g(0) = g'(0) = 0. \\ &\text{(ii) } g(u) \text{ is monotone, and is convex for } u > 0, \text{ concave for } u < 0. \\ &\text{(iii) } (p+1)G(u) \le ug(u), |ug(u)| \le \gamma |G(u)|, \text{ where } 2 < p+1 \le \gamma < \infty. \end{aligned}$$

Recently, the qualitative research on the nonlinear fourth-order Schrödinger equations has been widely performed, and the corresponding results have greatly developed the mathematical theory of

Schrödinger equations (see for instance [1–3] and the references therein). Here we restrict our attention to the nonlinear second-order Schrödinger equations. Monomial semilinear Schrödinger equation

$$i\partial_t u + \Delta u + \mu |u|^{p-1} u = 0,\ p > 1,\ u : (-T^*, T^*) \times \mathbb{R}^d \to \mathbb{C} \tag{1.2}$$

is called defocusing if $\mu = -1$ and focusing if $\mu = 1$. The solutions of (1.2) satisfy conservation of mass

$$M(u(t)) = \frac{1}{2}\|u(t)\|^2_{L^2(\mathbb{R}^d)}$$

and Hamiltonian

$$H_p(u(t)) := \|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} - \frac{2\mu}{p+1}\int_{\mathbb{R}^d} |u(t,x)|^{p+1}\mathrm{d}x.$$

For $d = 2$, when $p > 1$ the Cauchy problem for nonlinear Schrödinger equation (1.2) is energy subcritical [4]. As is well known, the problems with exponential nonlinear terms have lots of applications, for instance the self-trapped laser beams in plasma [5]. Cazenave [6] considered the Cauchy problem for the Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta u + F(u) = 0, & t \in \mathbb{R}, \quad x \in \mathbb{R}^2, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^2, \end{cases}$$

with the decreasing exponential nonlinear term $F(u)$, and showed global well-posedness and scattering. Generally, the problems with increasing exponential nonlinear terms are more complicated because there are no a priori L^∞-estimates on the nonlinear terms. Furthermore, in view of its relationship with the critical Moser-Trudinger inequality, the two-dimensional case is interesting (see [7, 8]). For the higher-dimensional case, we refer the readers to [3, 9–13] and the references therein.

Later on, Colliander et al. [14] considered the Cauchy problem for the Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta u = u(e^{4\pi|u|^2} - 1), \quad t \in \mathbb{R},\ x \in \mathbb{R}^2, \\ u(0) = u_0 \in H^1(\mathbb{R}^2). \end{cases}$$

They obtained global well-posedness under the situation that the initial data u_0 satisfies

$$H(u_0) = \|\nabla u_0\|^2_{L^2(\mathbb{R}^2)} + \frac{1}{4\pi}\|e^{4\pi|u_0|^2} - 1 - 4\pi|u_0|^2\|_{L^1(\mathbb{R}^2)} \leq 1$$

and an instability when $H(u_0) > 1$. Saanouni [4] used the Strichartz estimate and some embedding inequalities to get the global existence result of the Cauchy problem for semilinear Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta u + ug_1(|u|^2) = 0, \quad t \in \mathbb{R},\ x \in \mathbb{R}^2, \\ u(0) = u_0 \in H^1(\mathbb{R}^2), \end{cases}$$

in the subcritical case

$$\begin{cases} g_1(0) = g_1'(0) = 0, \\ \forall \alpha > 0,\ \exists C_\alpha > 0 \text{ s.t. } |g(s)| \leq C_\alpha e^{\alpha s^2}, \quad s \in \mathbb{R}, \\ (D-2)G(r) > 0 \ \text{ and } \ (D-2)^2 G(r) \geq 0, \quad \forall r > 0. \end{cases}$$

In critical case

$$\begin{cases} g_1(0) = g_1'(0) = 0, \\ \lim\limits_{|u|\to\infty} G_L(u)/uG_L'(u) = 0, \\ \exists k_0 > 0 \text{ s.t } \lim\limits_{|u|\to\infty} G_L''(u)e^{-k|u|^2} = \{0 \text{ if } k > k_0,\ \infty \text{ if } k < k_0\}, \\ \exists \varepsilon > 0 \text{ s.t } (D-4-\varepsilon)G(r) \geq 0 \ \text{ and } \ (D-2)(D-4-\varepsilon)G(r) \geq 0, \quad \forall r > 0, \end{cases}$$

he got the blow-up result for the above equation under some assumptions. However, the sharp conditions for global existence and blow-up of solutions of the problem is still unsolved. In the present paper, we aim to consider this by the concavity arguments and the potential well theory (see for instance [3, 9–13, 15–23] and the references therein).

The outline of our paper is as follows. In Section 2, we show a few propositions and lemmas. Moreover, we introduce some functionals and invariant manifolds. In Section 3, we provide a sharp condition for global existence and blow-up of solutions of problem (1.1).

In this paper, we use $\|\cdot\|_{H^1}$ to stand for the norm of $H^1(\mathbb{R}^2)$ and $\|\cdot\|$ of $L^2(\mathbb{R}^2)$. For simplicity, hereafter, we will denote $\int_{\mathbb{R}^2} \cdot \mathrm{d}x$ by $\int \cdot$.

2. Preliminaries

Regarding problem (1.1), we define the energy space in the course of nature by

$$H := \left\{ u \in H^1(\mathbb{R}^2) \middle| \int |x|^2|u|^2 < \infty \right\}$$

with the inner product

$$(u, v) = \int (\nabla u \nabla \bar{v} + u\bar{v}),$$

where $\bar{v}$ denotes the conjugate function of v.

Proposition 2.1 ([24]). *Let $\varphi_0 \in H$. Then the Cauchy problem (1.1) has a unique solution $u \in C([0, T); H)$, where $T \leq \infty$ is the maximal existence time of the solution. Moreover, we have alternative: $T = \infty$, or $T < \infty$ and*

$$\lim_{t\to T} \|u\|_{H^1} = \infty.$$

The solution u satisfies

$$M(t) = \frac{1}{2}\int |u|^2 = \frac{1}{2}\int |u_0|^2 \tag{2.1}$$

and

$$E(t) = \frac{1}{2}\int \left(|\nabla u|^2 - 2G(u)\right) \equiv E(0).$$

Lemma 2.2 ([20]). *Let $g(u)$ satisfy* (H)*. Then*

$$u\left(ug'(u) - g(u)\right) \geq 0,$$

and the equality holds only for $u = 0$.

From [24] we have the following lemma.

Lemma 2.3. *Let u be the solution of the problem (1.1) with $u_0 \in H$. For $J(t) := \int |x|^2|u|^2$, we have*

$$J''(t) = 8\int \left(|\nabla u|^2 - |u|g(|u|) + 2G(u)\right).$$

Next, for $\varphi \in H$, we define

$$P(\varphi) := \frac{1}{2}\int \left(|\nabla\varphi|^2 + |\varphi|^2 - 2G(\varphi)\right) \tag{2.2}$$

and

$$I(\varphi) := \int \left(|\nabla\varphi|^2 + |\varphi|^2 - |\varphi|g(|\varphi|)\right). \tag{2.3}$$

When φ is the solution of problem (1.1) with $\varphi_0 \in H$, there holds

$$P(\varphi) \equiv P(\varphi_0). \tag{2.4}$$

Now we consider a constrained variational problem

$$d = \inf_{\varphi\in M} P(\varphi), \tag{2.5}$$

where

$$M = \{\varphi \in H \setminus \{0\} \mid I(\varphi) = 0\}.$$

Lemma 2.4. *If $\varphi \in M$, then $d > 0$.*

Proof. By (H), (2.2) and (2.3), we have

$$\int \left(|\nabla\varphi|^2 + |\varphi|^2\right) = \int |\varphi|g(|\varphi|)$$

and

$$\begin{aligned} P(\varphi) &= \frac{1}{2}\int \left(|\nabla\varphi|^2 + |\varphi|^2 - 2G(\varphi)\right) \\ &= \frac{1}{2}\int (|\varphi|g(|\varphi|) - 2G(\varphi)) \\ &\geq \frac{1}{2}\int \left(|\varphi|g(|\varphi|) - \frac{2}{p+1}|\varphi|g(|\varphi|)\right) \\ &> 0. \end{aligned} \tag{2.6}$$

Furthermore, combining with (2.6) and (2.5), we can obtain $d > 0$. □

Lemma 2.5. *Let $\varphi \in H$. Put $\varphi_\lambda(x) = \lambda\varphi(x)$ for $\lambda > 0$, then there exists a unique constant $\mu > 0$ (depending on φ) such that $I(\varphi_\mu) = 0$, $I(\varphi_\lambda) > 0$ for any $0 < \lambda < \mu$, and $I(\varphi_\lambda) < 0$ for any $\lambda > \mu$. Furthermore, $P(\varphi_\mu) \geq P(\varphi_\lambda)$ for any $\lambda > 0$.*

Proof. From (2.2) and (2.3), we have

$$I(\varphi_\lambda) = \lambda^2 \int \left(|\nabla\varphi|^2 + |\varphi|^2 - \left|\frac{1}{\lambda}\varphi\right| g(|\lambda\varphi|) \right)$$

and

$$P(\varphi_\lambda) = \frac{\lambda^2}{2} \int (|\nabla\varphi|^2 + |\varphi|^2) - \int G(\lambda\varphi).$$

It is easy to see that there exists a unique constant $\mu > 0$ (depending on φ) such that $I(\varphi_\mu) = 0$,

$$I(\varphi_\lambda) > 0, \ \ 0 < \lambda < \mu,$$

and

$$I(\varphi_\lambda) < 0, \ \ \lambda > \mu.$$

Combining

$$\frac{d}{d\lambda} P(\varphi_\lambda) = \lambda^{-1} I(\varphi_\lambda),$$

$$\begin{aligned} \frac{d^2}{d\lambda^2} P(\varphi_\lambda) &= -\lambda^{-2} I(\varphi_\lambda) + \lambda^{-1} \frac{d}{d\lambda} I(\varphi_\lambda) \\ &= \|\nabla u\|^2 + \|u\|^2 - \frac{1}{\lambda^2} \int \lambda^2 u^2 g'(\lambda u) \end{aligned}$$

and Lemma 2.2, we get

$$\lambda u(\lambda u g'(\lambda u) - g(\lambda u)) > 0. \tag{2.7}$$

Integrating (2.7) with respect to x in $\mathbb{R}^2$ and dividing its both sides by λ^2, we derive

$$\frac{1}{\lambda} \int_\Omega u g(\lambda u) < \frac{1}{\lambda^2} \int_\Omega \lambda^2 u^2 g'(\lambda u),$$

which, together with

$$I(\varphi_\mu) = 0,$$

yields

$$\frac{d^2}{d\lambda^2} P(\varphi_\lambda) < 0.$$

Hence

$$P(\varphi_\mu) \geq P(\varphi_\lambda), \ \ \lambda > 0.$$

□

Theorem 2.6. *Define*

$$V := \{\varphi \in H \big| P(\varphi) < d, I(\varphi) < 0\},$$

then V is an invariant manifold of (1.1), that is, if $u_0 \in V$, then the solution u of problem (1.1) also satisfies $u \in V$ for all $t \in [0, T)$.

Proof. By Proposition 2.1, problem (1.1) admits a unique solution $u \in C([0,T);H)$ with $T \leq \infty$. As (2.4) shows

$$P(u) = P(u_0), \quad t \in [0,T),$$

we conclude that $P(u_0) < d$ implies $P(u) < d$ for all $t \in [0,T)$.

Next, we demonstrate $I(u) < 0$ for all $t \in [0,T)$. If it is not true, then from the continuity of $I(u(t))$ in t, there exists a $t_1 \in [0,T)$ such that $I(u(t_1)) = 0$. By (2.2), (2.3) and

$$P(u(t_1)) > 0,$$

we have $u(t_1) \neq 0$. If it is not true, then $P(u(t_1)) = 0$, which contradicts $P(u(t_1)) > 0$. From (2.5) we get $P(u(t_1)) \geq d$. This contradicts $P(u) < d$ for all $t \in [0,T)$. Therefore, $I(u) < 0$ for all $t \in [0,T)$, i.e., $u \in V$ for all $t \in [0,T)$. So V is an invariant manifold of problem (1.1). □

By the same arguments as Theorem 2.6, we have the following theorem.

Theorem 2.7. *Define*

$$W := \{\varphi \in H \big| P(\varphi) < d, I(\varphi) > 0\} \cup \{0\}.$$

Then W is an invariant manifold of problem (1.1).

3. Sharp conditions

Theorem 3.1. *If $u_0 \in W$, then the solution u of problem (1.1) globally exists on $t \in [0,\infty)$.*

Proof. Theorem 2.7 shows that the solution u of problem (1.1) satisfies $u \in W$ for all $t \in [0,T)$. Hence $P(u) < d$ and $I(u) > 0$. By (H), (2.2) and (2.3), we get

$$\begin{aligned}
&\left(\frac{1}{2} - \frac{1}{p+1}\right) \int \left(|\nabla u|^2 + |u|^2\right) \\
=&\frac{1}{2} \int \left(|\nabla u|^2 + |u|^2\right) - \frac{1}{p+1}\left(I(u) + \int |u| g(|u|)\right) \\
<&\frac{1}{2} \int \left(|\nabla u|^2 + |u|^2 - 2G(u)\right) < d,
\end{aligned}$$

which gives

$$\int (|\nabla u|^2 + |u|^2) < \frac{2(p+1)}{p-1} d. \tag{3.1}$$

Therefore, by Proposition 2.1, (3.1) shows that u globally exists.

Let $u_0 = 0$. Thanks to (2.1), we get $u = 0$, which shows that u is the trivial solution of problem (1.1). The proof of Theorem 3.1 is completed. □

By the similar arguments in [10], we have the following lemma.

Lemma 3.2. *Let $\varphi \in H$ and $\mu > 0$ satisfy $I(\varphi_\mu) = 0$. Suppose that $\mu < 1$, then*

$$P(\varphi) - P(\varphi_\mu) \geq \frac{1}{2} I(\varphi).$$

Theorem 3.3. *If $u_0 \in V$, then the solution u of problem (1.1) blows up in finite time.*

Proof. Suppose that $T = \infty$. Since $u_0 \in V$, we conclude from Theorem 2.6 that $u \in V$, i.e., $I(u) < 0$ for all $t \in [0, \infty)$. Thus

$$I(u) < 0, \quad P(u) < d, \quad t \in [0, \infty).$$

From Lemma 2.3 we get

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \int |x|^2 |u|^2 \leq 8 \left(I(u) - \int |u_0|^2 \right). \tag{3.2}$$

Let $\mu > 0$ satisfy

$$I(u_\mu) = 0.$$

From $I(u) < 0$ and Lemma 2.5 we obtain $\mu < 1$. Note that

$$P(u_\mu) \geq d, \quad P(u) = P(u_0).$$

From Lemma 3.2 we have

$$I(u) \leq 2\,(P(u_0) - d) < 0. \tag{3.3}$$

Let

$$\delta = 2\,(d - P(u_0))$$

and $\delta > 0$ be a constant independent of t. From (3.2) and (3.3) we obtain

$$\begin{aligned} J''(t) &= \frac{\mathrm{d}^2}{\mathrm{d}t^2} \int |x|^2 |u|^2 dx \\ &\leq 8\delta - 8 \int |u_0|^2 dx \\ &= -c_0 < 0, \quad t \in [0, \infty), \end{aligned}$$

where $c_0 > 0$ is a constant. Furthermore, we get

$$J'(t) \leq -c_0 t + J'(0), \quad t \in [0, \infty).$$

Hence there exists a $t_0 \geq 0$ such that $J'(t) < J'(0) < 0$ for all $t \in (t_0, \infty)$, and so

$$J(t) < J'(t_0)(t - t_0) + J(t_0), \quad t \in (t_0, \infty). \tag{3.4}$$

Since $I(u_0) < 0$ implies $J(0) > 0$, we conclude from (3.4) that there exists a $T_1 > 0$ such that $J(t) > 0$ for all $t \in [0, T_1)$ and

$$\lim_{t \to T_1} J(t) = 0. \tag{3.5}$$

From (3.5) and

$$\|u_0\|^2 = \|u\|^2 \leq \|\nabla u\| J^{\frac{1}{2}}(t),$$

it follows that

$$\lim_{t \to T_1} \|\nabla u\| = \infty.$$

This contradicts $T = \infty$. Thus

$$\lim_{t \to T} \|u\|_{H^1} = \infty.$$

□

4. Conclusions

It is clear that

$$\{u \in H|P(u) < d\} = W \cup V, \quad W \cap V = \phi.$$

Thus, by means of the location of the initial data, Theorems 3.1 and 3.3 provide a sharp condition for global existence and blow-up of solutions of problem (1.1), i.e., $u_0 \in W$ vs $u_0 \in V$.

The fractional Schrödinger equations may have a lot of interesting phenomena like the fractional version of other partial differential equations explored in [25,26], hence we shall focus on these models to investigate the corresponding sharp conditions.

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Conflict of interest

The authors declare there is no conflict of interest.

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