Mathematical Biosciences and Engineering

http://www.aimspress.com/journal/mbe

MBE, 20(2): 3700–3720.
DOI: 10.3934/mbe.2023173
Received: 26 October 2022
Revised: 07 November 2022
Accepted: 16 November 2022
Published: 09 December 2022

Research article

Bifurcations of a prey-predator system with fear, refuge and additional food

Jinxing Zhao[1,*] **and Yuanfu Shao**[2,*]

[1] School of Mathematical Sciences, Inner Mongolia University, Hohhot 010021, China
[2] College of Science, Guilin University of Technology, Guilin 541004, China

* **Correspondence:** Email:zhjxing@imu.edu.cn, shaoyuanfu@163.com; Tel: +864714991650, +867735896179.

Abstract: In the predator-prey system, predators can affect the prey population by direct killing and inducing predation fear, which ultimately force preys to adopt some anti-predator strategies. Therefore, it proposes a predator-prey model with anti-predation sensitivity induced by fear and Holling-II functional response in the present paper. Through investigating the system dynamics of the model, we are interested in finding how the refuge and additional food supplement impact the system stability. With the changes of the anti-predation sensitivity (the refuge and additional food), the main result shows that the stability of the system will change accordingly, and it has accompanied with periodic fluctuations. Intuitively the bubble , bistability phenomena and bifurcations are found through numerical simulations. The bifurcation thresholds of crucial parameters are also established by the Matcont software. Finally, we analyze the positive and negative impacts of these control strategies on the system stability and give some suggestions to the maintaining of ecological balance, we perform extensive numerical simulations to illustrate our analytical findings.

Keywords: fear; refuge; additional food; anti-predation sensitivity; bifurcation

1. Introduction

In ecology, prey-predator interaction happens at many higher trophic levels, and due to these interactions, predators always have an impact; direct, indirect or both, on prey population. The direct effect can be seen in nature when the predators catch the prey and kill [1,2]. In resent years, more and more habitats and ecological communities have been modified by human activities, and this state will continue. Unlike in the past, the predator-prey interactions increasingly occur in other aspects [3]. The theories with respect to the evolution of predator-prey interactions are reviewed by Peter A. Abrams [4], which includes the dynamics and stability of both populations and traits, as well as the anti-predator traits to environmental changes. The crucial feature of predator-prey interactions is the predator's functional response, which is used to describe the predation of predator on prey [5, 6]. They are generally classified into two types: the prey-dependent and predator-dependent, namely, the Holling type and Bedding type respectively [7–9]. Whereas, the Holling-II type functional response is most commonly used in mathematical modeling [10, 11].

Apart from the direct killing of prey by predator, more and more experimental data show that population of prey reduced due to fear of the predator [12–17]. Due to the fear effect, the prey shows a variety of antipredator responses, including different psychological changes, habitat changes, and foraging. For example, the impact of fear has been investigated on the prey's birth rate, Zanette et al. [15] carried out an experiment on song sparrows during the whole breeding season in 2011, the experiment was conducted in a free-living area for birds and there were no predators, the experimenter have used playbacks of the predator's voice to manipulate artificial risk, they found a reduction in reproduction by 40% in the number of offsprings due to the fear of predation alone, the experiment showed that prey could respond to the danger of being preyed. Similarly, another experiment revealed the effect of predator's fear for the death rate of prey, the experiment was performed by on the prey (grasshopper nymphs) and the predator (toothless spider) [17], the results of the experiment showed that the presence of predator increased the number of the death of prey by 20% compared with no predators, if the number of killing by predators was taken into account, then the death rate of prey was enhanced only by 9%, in fact, the indirect predation rate was 2 times larger than the direct killing. Further evidence has been found about Siberian jay in Eggers et al. [18], birds in Hua et al. [19] and lambs in Krapivsky et al. [20]. In addition, the recent research shows that the fear effect can induce bifurcations and even stabilize chaos of predator-prey system dynamics [21–23]. The adult prey also shows weaker anti-predator behaviors if the cost of fear is higher, such as reducing outside activity, foraging and starving to prevent from predation. These anti-predation sensitivity of prey should be considered into the mathematical model to reveal the realistic system dynamics [24]. Almost all experiment evidences confirmed that the impact of fear is very important in studying the system dynamics, refer to Blumstein [25]. Therefore, the impact of fear and the anti-predation of prey should be added to the ecological model, we can reveal how these factors affect the system stability.

In the anti-predation behaviors of prey, the prey usually move to safer place to avoid attack from predator, so the refuge area is very important for prey during their survival [26–30]. Most of the researchers have shown that refuges can preserve the prey and stabilize the system, which is very important for the survival of prey and predator. Kar [31] proposed a prey-predator model with Holling-II type functional response, and he added shelter for prey in the model, Kar established some sufficient conditions assuring the existence and stability of the equilibrium points, he found that a stable limit cycles appeared when the equilibrium point was unstable by bifurcation analysis. Recently, the prey-predator models with refuge areas have attracted many researchers's attention [32].

Actually, due to the destruction of ecological environment, apart from the construction of refuge zones to preserve the survival of animals, human want to reduce the prey's negative effect of fear by releasing additional food to predator or prey or both of them, this is a very popular strategy in biological control [33–36]. Releasing proper additional food for predator can reduce the consumption of predator and conserve the prey, Moorland Working Group [34] made an experiment in the spring and summer of 1998 and 1999 years, they put in additional food to the hen harriers (predator) living on Langholm moor in south Scotland, the predation rate of prey reduced from 3.7 chicks to 0.5 chicks per hour. However, it is worth of pointing out that additional food supplement may increase the reproduction rate of predators causing the reduce or extinction of prey [35]. Certainly, the "quality" and "quantity" of additional food plays an key role in the system stability [36]. The researchers found that good additional food benefited the survival of predators which has led to arising high predation rate, low quality of food would do harm to the predator and benefit the prey [37, 38], so the strategy of releasing additional food should be based on some theoretical results and be done scientifically and reasonably.

In this work we have proposed a prey-predator mathematical model with the fear effect and the anti-predation. In view of the importance of system stability, we have mainly studied the local and global stability of the equilibrium states and bifurcations. To observe the impact of fear, refuge and additional food on the dynamics of preyCpredator system, we study the nonlinear dynamics of proposed mathematical model using various techniques of stability analysis, permanence, persistence and Hopf bifurcation analysis.

Our main works are as follows:

1) The fear effect and their anti-predation sensitivity to predator have been incorporated in the prey-predator system.
2) Biological control strategies such as constructing refuge area and releasing additional food for predator have been included.
3) Some theorems of the local and global asymptotical stability and bifurcations have been obtained by using characteristic equation and Sotomayor's bifurcation theorems.
4) The interaction of the effects of fear, refuge, additional food and the stability of equilibrium points have been investigated by using theoretical and numerical methods.

The paper has been organized as follows. The mathematical model is formulated in Section 2. The stability and bifurcation analysis are carried out in Section 3. The numerical simulations are executed in Section 4 to establish the bifurcation thresholds of some key parameters. Finally, we end the paper with a discussion of our findings.

2. Model construction

Let $x(t)$ and $y(t)$ (x and y for simplicity) be the density of prey and predator at time t respectively. In the absence of predator, we assume that the birth rate of prey is a, the competition rate between prey is b. The functional response of predator to prey is the Holling-II type, where the meanings of c and d refer to [11, 12]. The conversion rate of prey to predator is β, the dearth rate of predator is e, then we have the following prey-predator model:

$$\begin{cases} x' &= x\left(a - bx - \frac{cy}{1+dx}\right), \\ y' &= y\left(\frac{\beta cx}{1+dx} - e\right). \end{cases}$$

Taking into account the fear of predator and the anti-predation level of prey to protect them from outside risk, the parameter f is the fear level and the parameter κ is the anti-predation sensitivity. Meanwhile, we incorporate parameter μ to describe the refuge area for prey and α, β and F to describe the additional food supplement in biological control, where αF and β represent the quantity (effectual food level) and quality of additional food respectively (the details refer to [33,36]), then we get the following model:

$$\begin{cases} x' &= x\left(\dfrac{a}{1+f\kappa y} - bx - \dfrac{c(1-\mu)y}{1+d\kappa(1-\mu)x+\delta\alpha F}\right), \\ y' &= y\left(\dfrac{\beta c[(1-\mu)x+\alpha F]}{1+d\kappa(1-\mu)x+\delta\alpha F} - e\right). \end{cases} \tag{2.1}$$

The initial conditions are $x(0) > 0$ and $y(0) > 0$. All parameters in (2.1) are supposed to be positive so as to accord with the biological justifications.

3. Main results

3.1. Existence of fixed point

By the theory of functional differential equation [40], we can obtain that there exists a unique positive solution for system (2.1) which remains in $R_+^2 = \{(x, y) : x > 0, y > 0\}$. It is easy to obtain that the trivial fixed point of system (2.1) is $Q_0(0, 0)$, the boundary fixed point is $Q_1(\frac{a}{b}, 0)$. Now we discuss the coexistence fixed point $\widetilde{Q}(\widetilde{x}, \widetilde{y})$, by the definition of coexistence fixed point, $\widetilde{x}$ satisfy the following equation,

$$\frac{\beta c[(1-\mu)x+\alpha F]}{1+d\kappa(1-\mu)x+\delta\alpha F} - e = 0.$$

Using restrictive condition $ed\kappa < \beta c < \frac{e(1+\delta\alpha F}{\alpha F}$, then

$$\widetilde{x} = \frac{e(1+\delta\alpha F) - \beta c\alpha F}{(1-\mu)(\beta c - ed\kappa)} > 0.$$

Denote $\frac{c(1-\mu)}{1+d\kappa(1-\mu)\widetilde{x}+\delta\alpha F} = \Upsilon$ and put $\widetilde{x}$ into the first equation of Eq (2.1), we have

$$\frac{a}{1+f\kappa y} - b\widetilde{x} - \Upsilon y = 0.$$

That is

$$\Upsilon f\kappa y^2 + (\Upsilon + b\widetilde{x}f\kappa)y + b\widetilde{x} - a = 0. \tag{3.1}$$

Due to $b\widetilde{x} - a < 0$, it is clear that the Eq (3.1) has a unique positive solution, say $\widetilde{y}$. Consequently, (2.1) has a unique positive coexistence fixed point $\widetilde{Q}(\widetilde{x}, \widetilde{y})$. Next we will establish some sufficient conditions assuring the stability of these fixed points.

3.2. LAS and bifurcation analysis

In this part, we focus on the local asymptotic stability (LAS for simplicity) and bifurcation analysis of the fixed points. For (2.1), let $X = (x, y)^T$, we rewrite (2.1) as the form $X' = G(X), G = (G_1, G_2)^T$. For the convenience of the reader, we compute the partial derivatives of the operator G as follows:

$$DG = \begin{pmatrix} \dfrac{a}{1+f\kappa y} - 2bx - \dfrac{c(1-\mu)(1+\delta\alpha F)y}{(1+d\kappa(1-\mu)x+\delta\alpha F)^2} & -\dfrac{af\kappa x}{(1+f\kappa y)^2} - \dfrac{c(1-\mu)x}{1+d\kappa(1-\mu)x+\delta\alpha F} \\ \dfrac{\beta cy(1-\mu)(1+\delta\alpha F - d\kappa\alpha F)}{(1+d\kappa(1-\mu)x+\delta\alpha F)^2} & \dfrac{\beta c[(1-\mu)x+\alpha F]}{1+d\kappa(1-\mu)x+\delta\alpha F} - e \end{pmatrix},$$

$$G_{xx} = \begin{pmatrix} -2b + \dfrac{2cd\kappa(1-\mu)^2(1+\delta\alpha F)y}{(1+d\kappa(1-\mu)x+\delta\alpha F)^3} \\ \\ \dfrac{-2\beta cd\kappa y(1-\mu)^2(1+\delta\alpha F-d\kappa\alpha F)}{(1+d\kappa(1-\mu)x+\delta\alpha F)^3} \end{pmatrix},$$

$$G_{xy} = G_{yx} = \begin{pmatrix} -\dfrac{af\kappa}{(1+f\kappa y)^2} - \dfrac{c(1-\mu)(1+\delta\alpha F)}{(1+d\kappa(1-\mu)x+\delta\alpha F)^2} \\ \\ \dfrac{\beta c(1-\mu)(1+\delta\alpha F-d\kappa\alpha F)}{(1+d\kappa(1-\mu)x+\delta\alpha F)^2} \end{pmatrix},$$

$$G_{yy} = \begin{pmatrix} \dfrac{2af^2\kappa^2 x}{(1+f\kappa y)^3} \\ \\ 0 \end{pmatrix}.$$

Next, we begin to discuss the stability of above fixed points one by one.

- **Stability of Q_0**

By the representation of DG, we can easily obtain the variational matrix at Q_0 is

$$M_0|_{Q_0} = \begin{pmatrix} a & 0 \\ 0 & \dfrac{b\alpha F}{1+\delta\alpha F} - e \end{pmatrix}.$$

By using the characteristic equation, we know that there is a positive characteristic root $\lambda = a > 0$, and hence it is unstable. If $\dfrac{b\alpha F}{1+\delta\alpha F} - e = 0$, then zero is a characteristic root, the eigenvectors of $\lambda = 0$ corresponding to the matrix M_0 and M_0^T are $V_1 = (0,0)^T$ and $V_2 = (0,1)^T$ respectively. Take δ as a parameter and denote the solution of $\dfrac{b\alpha F}{1+\delta\alpha F} - e = 0$ as $\delta = \hat{\delta}$. We investigate the conditions of the existence of bifurcations [39].

(i) $V_2^T[G_\delta(Q_0, \delta = \hat{\delta})] = 0$,

(ii) $V_2^T[DG_\delta(Q_0, \delta = \hat{\delta})V_1] = (0,1)\left[\begin{pmatrix} 0 & 0 \\ 0 & \frac{\beta c\alpha F}{1+\delta\alpha F} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right] = 0$,

then there is neither transcritical bifurcation nor saddle-node bifurcation around the trivial fixed point Q_0, so we can get the following theorem.

Theorem 3.1. *System (2.1) undergoes no bifurcation around the trivial fixed point $Q_0(0,0)$, which is unstable.*

- **Stability of Q_1**

We proceed to study the dynamics of the predator free equilibrium state $Q_1(\frac{a}{b}, 0)$. By same manner as before, we obtain the variational matrix at $Q_1(\frac{a}{b}, 0)$ is,

$$M_1|_{Q_1} = \begin{pmatrix} -a & -\dfrac{a^2 f\kappa}{b} - \dfrac{ac(1-\mu)}{ad\kappa + b(1+\delta\alpha F)} \\ \\ 0 & \dfrac{\beta c[(1-\mu)a + b\alpha F]}{ad\kappa + b(1+\delta\alpha F)} - e \end{pmatrix}.$$

Denote $m_1 = e - \frac{\beta c[(1-\mu)a + b\alpha F]}{ad\kappa + b(1+\delta\alpha F)}$, then the characteristic equation corresponding to matrix M_1 is

$$(\lambda + a)(\lambda + m_1) = 0.$$

The solutions of above equation are $\lambda_1 = -a, \lambda_2 = -m_1$. If $m_1 > 0$, we obtain that the characteristic roots are all negative, and hence system (2.1) is LAS around $Q_1(\frac{a}{b}, 0)$. If $m_1 = 0$, there is a zero characteristic root, then eigenvectors of zero root w.r.t matrix M_1 and M_1^T are $V_1 = (\nu, 1)^T$ and $V_2 = (0, 1)^T$ respectively, where $\nu = -\frac{af\kappa}{b} - \frac{p(1-\mu)}{ad\kappa + b(1+\delta\alpha F)}$. Take μ as a parameter, we verify the following transversality conditions of the existence of transcritical and saddle-node bifurcations respectively. We denote the μ satisfying $m_1 = 0$ as $\overline{\mu}$. It is not difficult to compute that

$$G_\mu = \begin{pmatrix} \dfrac{c(1+\delta\alpha F)y}{(1+d\kappa(1-\mu)x+\delta\alpha F)^2} \\[2ex] \dfrac{\beta c x[\alpha F - (1+\delta\alpha F)]}{(1+d\kappa(1-\mu)x+\delta\alpha F)^2} \end{pmatrix},$$

$$DG_\mu = \begin{pmatrix} \dfrac{-2d\kappa(1-\mu)c(1+\delta\alpha F)y}{(1+d\kappa(1-\mu)x+\delta\alpha F)^3} & \dfrac{c(1+\delta\alpha F)}{(1+d\kappa(1-\mu)x+\delta\alpha F)^2} \\[2ex] \dfrac{\beta c[\alpha F - (1+\delta\alpha F)](1 - d\kappa(1-\mu)x+\delta\alpha F)}{(1+d\kappa(1-\mu)x+\delta\alpha F)^2} & 0 \end{pmatrix}.$$

By computation, we have

(i) $V_2^T[G_\mu(Q_1, \mu = \overline{\mu})] \neq 0$.

(ii) $V_2^T[DG_\mu(Q_1, \mu = \overline{\mu})V_1] = (0, 1)\left[\begin{pmatrix} 0 & D_{\mu 1} \\ D_{\mu 2} & 0 \end{pmatrix}\begin{pmatrix} \nu \\ 1 \end{pmatrix}\right] \neq 0$,

where $D_{\mu 1} = \frac{c(1+\delta\alpha F)}{(1+d\kappa(1-\mu)x+\delta\alpha F)^2}, D_{\mu 2} = \frac{\beta c[\alpha F - (1+\delta\alpha F)](1 - d\kappa(1-\mu)x+\delta\alpha F)}{(1+d\kappa(1-\mu)x+\delta\alpha F)^2}$.

(iii) By the definition of D^2G, then

$$\begin{aligned} D^2G(Q_1, \mu = \overline{\mu})(V_1, V_1) &= G_{xx}(Q_1, \mu = \overline{\mu})\nu^2 + 2G_{xy}(Q_1, \mu = \overline{\mu})\nu + G_{yy}(Q_1, \mu = \overline{\mu}) \\ &= 2\begin{pmatrix} -af\kappa - \dfrac{c(1-\mu)(1+\delta\alpha F)}{(1+d\kappa(1-\mu)a/b+\delta\alpha F)^2} \\[2ex] \dfrac{\beta c(1-\mu)(1+\delta\alpha F - d\kappa\alpha F)}{(1+d\kappa(1-\mu)a/b+\delta\alpha F)^2} \end{pmatrix}\nu + \begin{pmatrix} 2af^2\kappa^2 a/b \\ 0 \end{pmatrix}. \end{aligned}$$

Accordingly,

$$\begin{aligned} V_2^T[D^2G(Q_1, \mu = \overline{\mu})(V_1, V_1)] &= (0, 1)\begin{pmatrix} -2\nu af\kappa - 2\nu\dfrac{c(1-\mu)(1+\delta\alpha F)}{(1+d\kappa(1-\mu)a/b+\delta\alpha F)^2} + 2af^2\kappa^2 a/b \\[2ex] \dfrac{\beta c(1-\mu)(1+\delta\alpha F - d\kappa\alpha F)}{(1+d\kappa(1-\mu)a/b+\delta\alpha F)^2}\nu \end{pmatrix} \\ &= \frac{\beta c(1-\mu)(1+\delta\alpha F - d\kappa\alpha F)}{(1+d\kappa(1-\mu)a/b+\delta\alpha F)^2}\nu \end{aligned}$$

$$\neq \quad 0.$$

Therefore, by Sotomayor's bifurcation theorem, there is a transcritical bifurcation around the predator-free fixed point Q_1. It is easy to get the representation of the threshold of parameter μ as,

$$\overline{\mu} = 1 + \frac{b\alpha F}{a} - \frac{e(ad\kappa + b(1+\delta\alpha F))}{a\beta c}.$$

Similarly, we can obtain the transcritical bifurcations and get their representations of thresholds w.r.t. parameters κ, δ and β. We summarize our findings as the following theorem.

Theorem 3.2. *System (2.1) is LAS around the predator-free fixed point Q_1 if $\frac{\beta c[(1-\mu)a + b\alpha F]}{ad\kappa + b(1+\delta\alpha F)} < e$; and if $\frac{\beta c[(1-\mu)a + b\alpha F]}{ad\kappa + b(1+\delta\alpha F)} = e$, then system (2.1) undergoes transcritical bifurcations w.r.t. parameter μ, κ, δ and β with the following parameter thresholds respectively.*

$$\begin{aligned}
\overline{\mu} &= 1 + \frac{b\alpha F}{a} - \frac{e(ad\kappa + b(1+\delta\alpha F))}{a\beta c}, \\
\overline{\kappa} &= \frac{\beta c[(1-\mu)a + b\alpha F] - eb(1+\delta\alpha F)}{ead}, \\
\overline{\delta} &= \frac{\beta c[(1-\mu)a + b\alpha F] - ead\kappa - b\alpha F}{b\alpha F}, \\
\overline{\beta} &= \frac{e(ad\kappa + b(1+\delta\alpha F))}{c[(1-\mu)a + b\alpha F]}.
\end{aligned}$$

- **Stability of $\widetilde{Q}$**

We begin with the analysis of the stability of system (2.1) near the coexistence equilibrium state $\widetilde{Q}(\widetilde{x}, \widetilde{y})$. Repeating the above computation leads to the following Jacobian matrix at $\widetilde{Q}(\widetilde{x}, \widetilde{y})$ as

$$\widetilde{M}|_{\widetilde{Q}} = \begin{pmatrix} -m_{11} & -m_{12} \\ m_{21} & 0 \end{pmatrix},$$

where

$$\begin{aligned}
m_{11} &= -\frac{a}{1 + f\kappa\widetilde{y}} + 2b\widetilde{x} + \frac{c(1-\mu)(1+\delta\alpha F)\widetilde{y}}{(1 + d\kappa(1-\mu)\widetilde{x} + \delta\alpha F)^2}, \\
m_{12} &= \frac{af\kappa\widetilde{x}}{(1 + f\kappa\widetilde{y})^2} + \frac{c(1-\mu)\widetilde{x}}{1 + d\kappa(1-\mu)\widetilde{x} + \delta\alpha F}, \\
m_{21} &= \frac{\beta c\widetilde{y}[(1-\mu)(1+\delta\alpha F) - d\kappa\alpha F]}{(1 + d\kappa(1-\mu)\widetilde{x} + \delta\alpha F)^2}.
\end{aligned}$$

The characteristic equation of the matrix $\widetilde{M}$ reads,

$$\lambda^2 + m_{11}\lambda + m_{12}m_{21} = 0. \tag{3.2}$$

Obviously, if $m_{11} > 0, m_{21} > 0$, then the characteristic roots are both negative and system (2.1) is LAS around the interior fixed point $\widetilde{M}$. If $m_{11} > 0, m_{21} = 0$, then $\lambda_1 = 0, \lambda_2 = -m_{11}$. We continue to study whether the bifurcation appears around $\widetilde{M}$ or not. By computation, the eigenvectors of $\lambda = 0$ corresponding to $\widetilde{M}$ and $\widetilde{M}^T$ are $\widetilde{V}_1 = (\widetilde{\nu}, 1)^T, \widetilde{V}_2 = (0, 1)^T$ respectively, where $\widetilde{\nu} = -\frac{m_{11}}{m_{12}}$. Take μ as a parameter and denote $\widetilde{\mu}$ such that $m_{21} = 0$. By same manner as before, it is not difficult to verify that

(i) $V_2^T[G_\mu(\widetilde{Q}, \mu = \widetilde{\mu})] = \dfrac{\beta c\widetilde{x}[\alpha F - (1 + \delta\alpha F)]}{(1 + d\kappa(1 - \mu)\widetilde{x} + \delta\alpha F)^2} \neq 0.$

(ii) $V_2^T[D^2G(Q_1, \mu = \widetilde{\mu})(V_1V_1)] \neq 0.$

Therefore Sotomayor's theorem for saddle-node bifurcation [39] indicates that there is a saddle-node bifurcation around the coexistence fixed point $\widetilde{Q}$ with the threshold $\widetilde{\mu} = 1 - \frac{d\kappa\alpha F}{1+\delta\alpha F}$.

Similarly, we can find that the saddle-node bifurcation around $\widetilde{Q}$ for parameters κ, δ and β. We will make a summary in the conclusion part.

Theorem 3.3. *System (2.1) is LAS around the coexistence fixed point $\widetilde{M}$ if the following C_1 and C_2 hold.*

$$C_1 \,:\, 2b\widetilde{x} + \frac{c(1-\mu)(1+\delta\alpha F)\widetilde{y}}{(1 + d\kappa(1-\mu)\widetilde{x} + \delta\alpha F)^2} - \frac{a}{1 + f\kappa\widetilde{y}} > 0,$$

and

$$C_2 \,:\, (1-\mu)(1+\delta\alpha F) - d\kappa\alpha F > 0.$$

Further, if C_1 as well as C_3 hold,

$$C_3 \,:\, (1-\mu)(1+\delta\alpha F) - d\kappa\alpha F = 0,$$

then system (2.1) undergoes saddle-node bifurcations w.r.t. parameter μ, κ and δ with the following thresholds respectively.

$$\widetilde{\mu} = 1 - \frac{d\kappa\alpha F}{1+\delta\alpha F}, \quad \widetilde{\kappa} = \frac{(1-\mu)(1+\delta\alpha F)}{d\alpha F}, \quad \widetilde{\delta} = \frac{1 + d\kappa\alpha F - \mu}{(1-\mu)\alpha F}.$$

Next we investigate the existence of Hopf bifurcation. If $m_{11} = 0, m_{21} > 0$, then there are two imaginary roots $\lambda = \pm\sqrt{m_{12}m_{21}}i$ for Eq (3.2). By the Hopf bifurcation theorem [39], we only need to verify the transversality conditions. Take f as a bifurcation parameter and denote f^H such that $m_{11} = 0$. Suppose the root of (3.2) has the general form as $\lambda = \phi_1(f) + \phi_2(f)i$. Substituting it into (3.2) leads to

$$\phi_1^2 - \phi_2^2 + m_{11}\phi_1 + m_{12}m_{21} + \phi_2(2\phi_1 + m_{11})i = 0.$$

Differentiating it w.r.t parameter f, then

$$2\phi_1\phi_1' - 2\phi_2\phi_2' + m_{11}'\phi_1 + m_{11}\phi_1' + m_{12}'m_{21} + m_{12}m_{21}' + \phi_2'(2\phi_1 + m_{11})i + \phi_2(2\phi_1' + m_{11}')i = 0.$$

That is

$$\begin{cases} (2\phi_1 + m_{11})\phi_1' - 2\phi_2\phi_2' = -m_{11}'\phi_1 - m_{12}'m_{21} - m_{12}m_{21}', \\ 2\phi_2\phi_1' + (2\phi_1 + m_{11})\phi_2' = -\phi_2 m_{11}'. \end{cases}$$

Consequently we have

$$\phi_1' = \frac{X_1X_3 + X_2X_4}{X_1^2 + X_2^2},$$

where $X_1 = 2\phi_1 + m_{11}, X_2 = 2\phi_2, X_3 = 2\phi_2 - m'_{11}\phi_1 - m'_{12}m_{21} - m_{12}m'_{21}, X_4 = -\phi_2 m'_{11}$. By the definition of f^H and the necessary condition of bifurcation appearance, at $f = f^H$, we have $\phi_1 = 0, m_{11} = 0$, and

$$\phi'_1|_{f=f^H} = -\frac{m'_{11}}{2}|_{f=f^H} \neq 0.$$

That is, the transversality conditions of Hopf bifurcation hold, and hence we get the following result.

Theorem 3.4. *System (2.1) undergoes a Hopf bifurcation around the coexistence equilibrium* $\widetilde{M}(\widetilde{x},\widetilde{y})$ *provided the conditions* C_3 *and the following* C_4 *hold,*

$$C_4 \,:\, 2b\widetilde{x} + \frac{c(1-\mu)(1+\delta\alpha F)\widetilde{y}}{(1+d\kappa(1-\mu)\widetilde{x}+\delta\alpha F)^2} - \frac{a}{1+f\kappa\widetilde{y}} = 0,$$

and hence, the threshold is

$$f^H = \frac{(a-2b\widetilde{x})(1+d\kappa(1-\mu)\widetilde{x}+\delta\alpha F)^2 - c(1-\mu)(1+\delta\alpha F)\widetilde{y}}{ky(2b\widetilde{x}(1+d\kappa(1-\mu)\widetilde{x}+\delta\alpha F)^2) + c(1-\mu)(1+\delta\alpha F)\widetilde{y})}.$$

Similarly, we can find the Hopf bifurcation thresholds about parameters μ, δ and κ, which are to be validated in numerical analysis.

3.3. GAS analysis

In this section, by using the Lyapunov functional stability theory [34], we investigate the global asymptotic stability (GAS) of the equilibrium points of system (2.1).

Theorem 3.5. *Assume that system (2.1) is LAS near the predator-free fixed point* $Q_1(\frac{a}{b}, 0)$*, then it is GAS provided that*

$$1 + \delta\alpha F + ad\kappa(1-\mu)/b > \beta(1 + \delta\alpha F - d\kappa\alpha F).$$

Proof. Consider the following function,

$$V(x,y) = x - x_0 - x_0 \ln\frac{x}{x_0} + y,$$

where $x_0 = \frac{a}{b}$, then it is positive on the interval $x > 0, y > 0$. Computing its derivative w.r.t. t along the solution of system (2.1), we have

$$\begin{aligned}
\frac{dV}{dt}|_{(2.1)} &= (x-x_0)\left(\frac{a}{1+f\kappa y} - bx - \frac{c(1-\mu)y}{1+d\kappa(1-\mu)x+\delta\alpha F}\right) \\
&\quad + y\left(\frac{\beta c[(1-\mu)x+\alpha F]}{1+d\kappa(1-\mu)x+\delta\alpha F} - e\right) \\
&\leq -b(x-x_0)^2 - \frac{(1+\delta\alpha F+d\kappa(1-\mu)x_0) - \beta(1+\delta\alpha F-d\kappa\alpha F)}{(1+d\kappa(1-\mu)x+\delta\alpha F)(1+d\kappa(1-\mu)x_0+\delta\alpha F)} y(x-x_0) \\
&< 0.
\end{aligned}$$

Therefore, according to the stability theorem of functional differential equation [40], system (2.1) is GAS around the fixed point Q_1.■

Theorem 3.6. *Assume that system (2.1) is LAS near the coexistence fixed point $\widetilde{Q}$, then it is GAS if the following conditions hold.*

(i) $c(1-\mu)\frac{\widetilde{y}}{\widetilde{x}} < b$,
(ii) $\beta(1+\delta\alpha F - d\kappa\alpha F) < 1 + \delta\alpha F + d\kappa(1-\mu)\widetilde{x}$.

Proof. Define

$$V(x,y) = x - \widetilde{x} - \widetilde{x}\ln\frac{x}{\widetilde{x}} + y - \widetilde{y} - \widetilde{y}\ln\frac{y}{\widetilde{y}},$$

then it is positive on $x>0, y>0$. Similarly we have

$$\begin{aligned}
\frac{dV}{dt}|_{(2.1)} &= (x-\tilde{x})\left(\frac{a}{1+f\kappa y} - bx - \frac{c(1-\mu)y}{1+d\kappa(1-\mu)x+\delta\alpha F}\right) \\
&\quad +(y-\widetilde{y})\left(\frac{\beta c[(1-\mu)x+\alpha F]}{1+d\kappa(1-\mu)x+\delta\alpha F} - e\right) \\
&= (x-\tilde{x})\left(-\frac{af\kappa(y-\widetilde{y})}{(1+f\kappa y)(1+f\kappa\widetilde{y})} - b(x-\widetilde{x}) \right.\\
&\quad \left. -c(1-\mu)\frac{(1+\delta\alpha F+d\kappa(1-\mu)\widetilde{x})(y-\widetilde{y}) - d\kappa(1-\mu)\widetilde{y}(x-\widetilde{x})}{(1+d\kappa(1-\mu)\widetilde{x}+\delta\alpha F)(1+d\kappa(1-\mu)x+\delta\alpha F)}\right) \\
&\quad +(y-\widetilde{y})\left(\frac{\beta c(1-\mu)(1+\delta\alpha F-d\kappa\alpha F)(x-\widetilde{x})}{(1+d\kappa(1-\mu)x+\delta\alpha F)(1+d\kappa(1-\mu)\widetilde{x}+\delta\alpha F)}\right) \\
&\le -\left(b - c(1-\mu)\frac{\widetilde{y}}{\widetilde{x}}\right)(x-\tilde{x})^2 \\
&\quad -\frac{(1+\delta\alpha F+d\kappa(1-\mu)\widetilde{x}) - \beta(1+\delta\alpha F-d\kappa\alpha F)}{(1+d\kappa(1-\mu)x+\delta\alpha F)(1+d\kappa(1-\mu)\widetilde{x}+\delta\alpha F)}(x-\widetilde{x})(y-\widetilde{y}) \\
&< 0.
\end{aligned}$$

Using the Lyapunov stability theory of differential equation again, we conclude that system (2.1) is GAS around $\widetilde{Q}$. ■

4. Numerical analysis

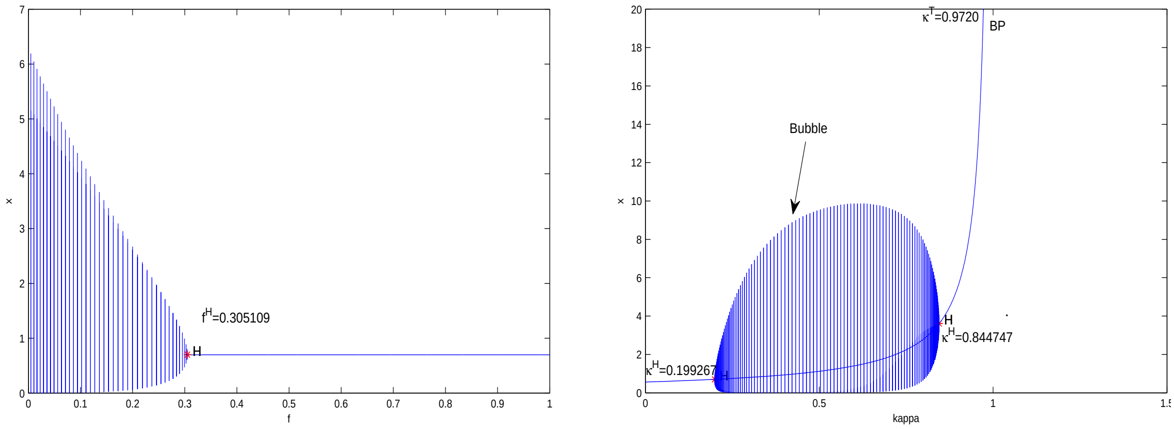
In this section, we study the bifurcations and stability of (2.1) from numerical angle intuitively by depicting some pictures.

4.1. Bifurcation analysis

We begin with our discussion of bifurcation to reveal the impacts on the system stability and explore the dynamics of (2.1) in detail. We classify it into three different scenarios as the bifurcations of fear, refuge and additional food supplement. For system (2.1), we fix a set of parameter values as below

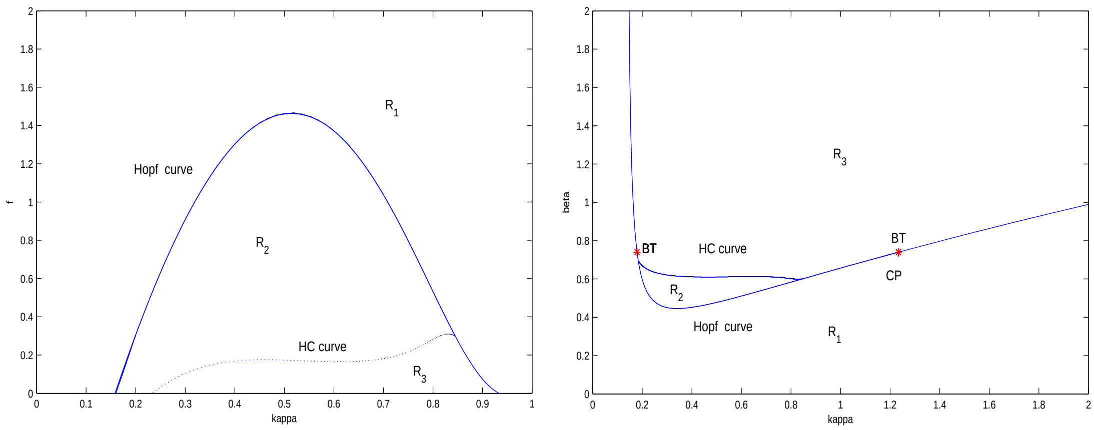
$$a = 0.8, b = 0.04, c = 0.5, d = 1, e = 0.3, F = 2, f = 0.3, \kappa = 0.2, \mu = 0.5, \beta = 0.6, \delta = 0.4, \alpha = 0.6. \tag{4.1}$$

By computation, the predator extinction equilibrium point is $\overline{Q}(20,0)$ and the interior equilibrium point is $\widetilde{Q}(0.7, 3.8547)$.



(a) (b)

Figure 1. The trajectories of equilibrium point of system (2.1). (a) The trajectory of prey about fear f. (b) The trajectory of prey about anti-predation sensitivity κ.



(a) (b)

Figure 2. The bi-parameter bifurcation graph w.r.t. fear f and anti-predation sensitivity κ or transversion rate β. (a) is about parameters κ and f, (b) is about parameters κ and β.

- Bifurcation of fear and anti-predation sensitivity

The parameter f represents the fear level of predator on prey, meanwhile, the anti-predation sensitivity parameter κ is induced by the fear of predator, which shows the sensitivity of prey that preventing themselves from the predation risk. These two indexes jointly reflect the effect of fear to the system dynamics. Consequently we analyze the bifurcations of parameters f and κ numerically. By use of Matlab2014a and continuation software Matcont [41], we get the trajectories of equilibrium point of (2.1), see Figure 1. It is clear that the prey is oscillatory if $f < 0.305109$, when $f = 0.305109$, the periodic fluctuation loses and it becomes stable when $f > 0.305109$ (see Figure 1(a)). For the parameter $\kappa < 0.19927$, the equilibrium point is stable, at $\kappa = 0.199270$, it becomes unstable and Hopf bifurcation occurs, while at $\kappa = 0.844747$, the oscillatory disappear and it becomes stable again from unstable state. A bubble formulates between the interval $\kappa \in (0.19927, 0.844747)$, which means that system (4.1) changes its stability from stable to unstable and to stable again, that is, multiple stable statuses appear, see also References [24]. Then as the increase of κ, the density of prey increases quickly, and at $\kappa = 0.9720$, a transcritical bifurcation occurs, which means the predator is extinct and the prey attains the peak value with $x = 20$ and keeps stable at the equilibrium point $\overline{Q}(20, 0)$.

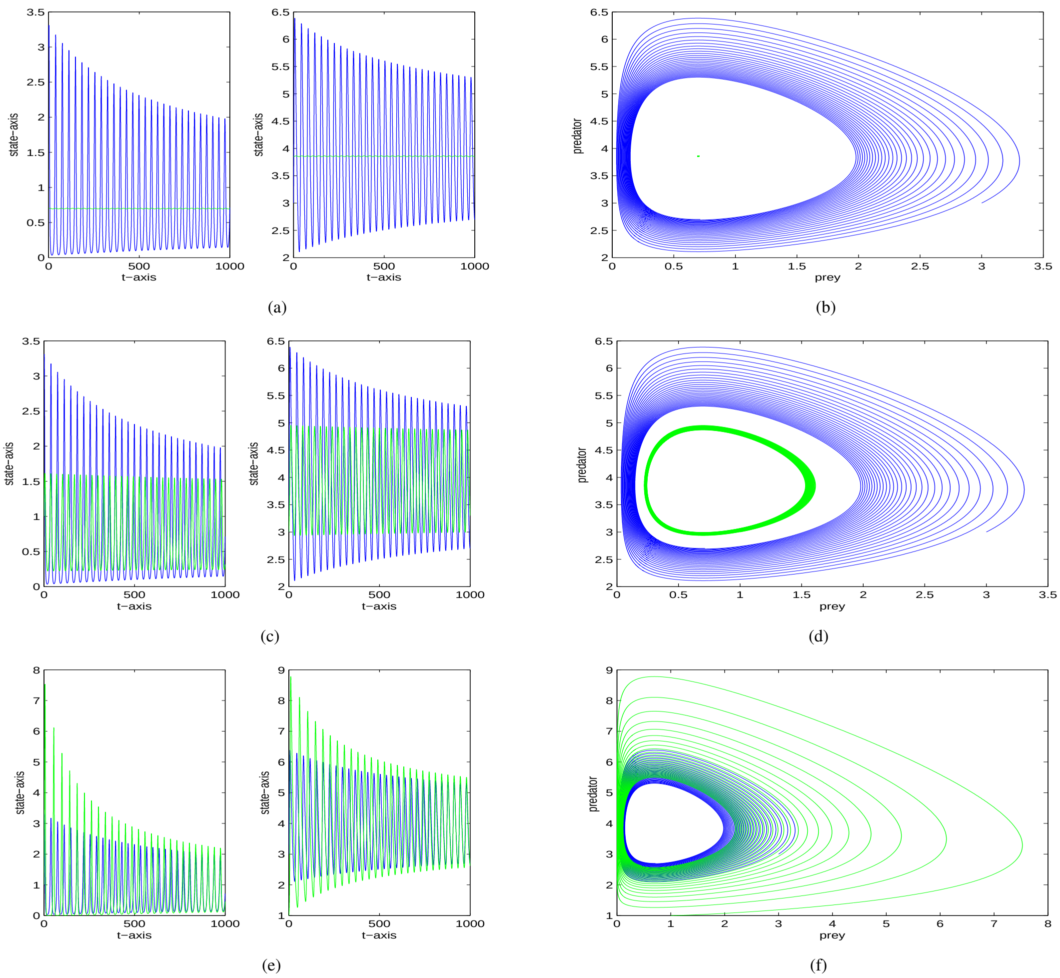


Figure 3. The bi-stability phenomena of system (2.1) with fixed parameters and different initial data. (a) is the time series graph of (2.1) with initial data (3,3) for blue line and (0.7,3.85) for green line, (b) is the phase graph of (a); (c) is the time series graph of (2.1) with initial data (3,3) for blue line and (1,3) for green line, (d) is the phase graph of (c); (e) is the time series graph of (2.1) with initial data (3,3) for blue line and (1,1) for green line, (f) is the phase graph of (e).

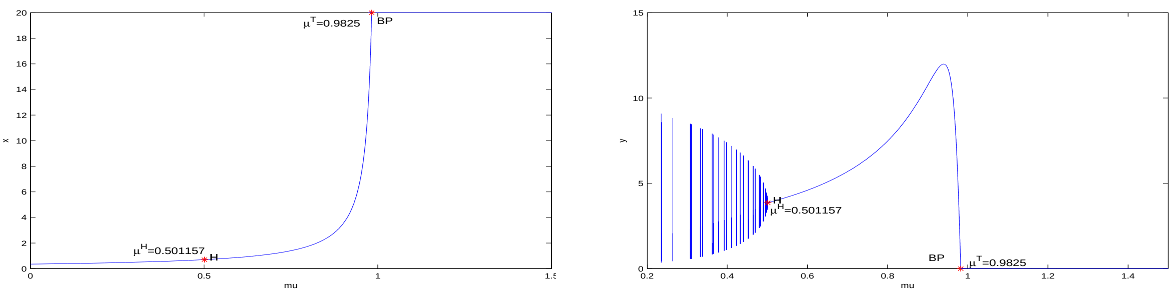
To clearly see the interactive effect of the fear level f and the anti-predation sensitivity level κ on the stability of the equilibrium state, we carry our the two-parameter bifurcation diagrams. See Figure 2(a), where HC curve represents the Homoclinic bifurcation curve produced at the BT point (Bogdanov-

Takens point for short) $(f, \kappa) = (-0.016576, 0.959873)$. The Hopf curve and the HC curve separate the whole plane into three regions denoted by R_1, R_2 and R_3 respectively, where R_1 is the region of stable coexistence, R_2 is the region of bi-stability between stable coexistence and oscillatory coexistence, R_3 is the region of oscillatory coexistence. Figure 2(b) shows the interactive effect of κ and the transfer coefficient β on the system dynamics. The BT points are (0.1791120.740001) and (1.2333380.740002) respectively which produce the same HC curve. Similarly the HC and Hopf curves separate it into three regions $R_i (i = 1, 2, 3)$ with same meanings as before. Without other stated, the meanings of $R_i (i = 1, 2, 3)$ keep fixed later.

Figure 2 shows that the bi-stability phenomena appear, which means system (2.1) will converge to different attractors under different initial data. In order to better visualize the bi-stability, we depict the graph of trajectories with $\kappa = 0.199$ (other parameter values keep unchanged) and different initial conditions, see Figure 3. With two different initial data (see the caption of Figure 3), Figure 3(a),(b) shows (2.1) converges to the stable coexistence and oscillatory coexistence; Figure 3(c),(d) indicates that it converges to two different oscillatory cycles; Figure 3(e),(f) shows that it converges to the same oscillatory cycle.

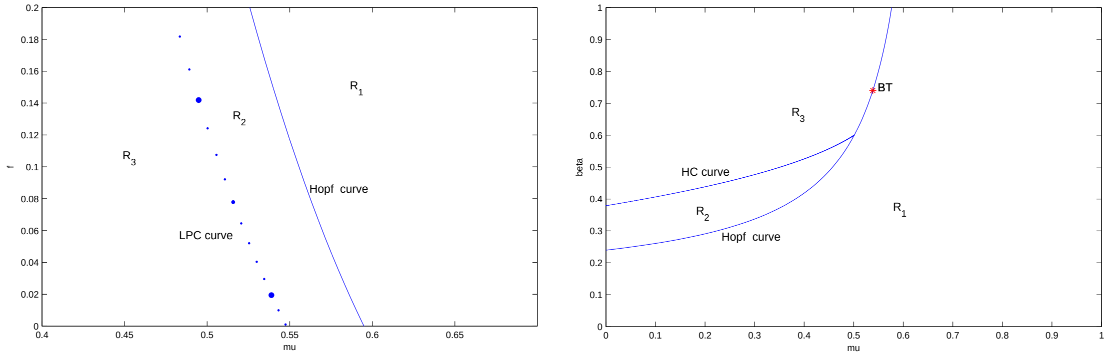
- Bifurcation of refuge

Due to the fear effect of predator, the prey will seek for some refuges to prevent from the predation of predator. Sometimes some refuge zones are constructed by people to preserve the prey from the risk of being predated. Undoubtedly, the refuge can protect the prey and keep them from extinction, so we continue to study the effect of refuge parameter μ on the stability of equilibrium point of system (2.1). Numerically we carry out the trajectories of the equilibrium point about parameter μ, see Figure 4. System (2.1) is unstable when μ is small, but at $\mu = 0.50117$, it becomes stable from periodic fluctuation. The protecting role of refuge on the prey is clearly showed. With the increasing of μ, the prey goes to the maximum value at $\mu = 0.9825$ and a transcritical bifurcation appears. When $\mu > 0.9825$, the predator is extinct and the prey keeps stable in the equilibrium state $\overline{Q}(20, 0)$. Compared with Figure 4(a), Figure 4(b) shows the oscillatory behavior of equilibrium point. From Figure 4, we can be aware the importance for refuge on the protection the prey.



(a) (a)

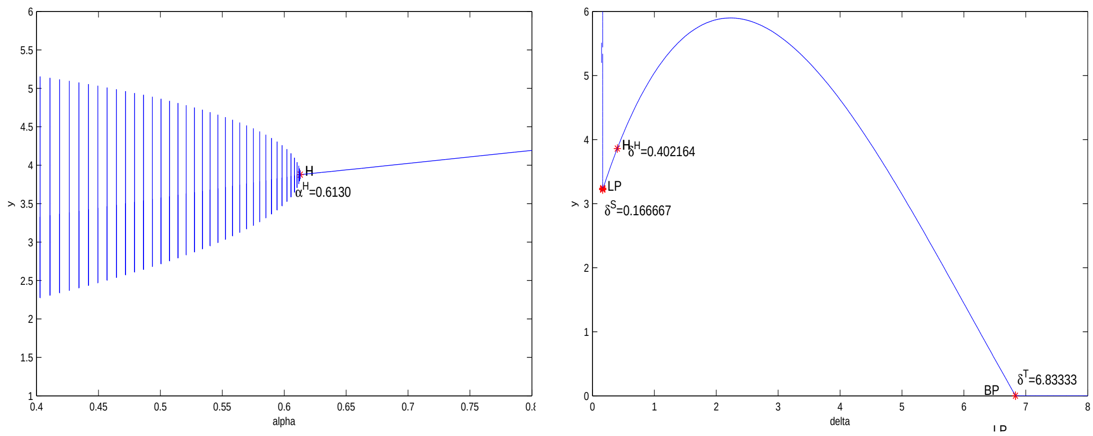
Figure 4. The trajectories of equilibrium point of system (2.1). (a) The trajectory of prey about refuge μ, (b) The trajectory of predator about refuge μ.



(a) (b)

Figure 5. The bi-parameter bifurcation graph w.r.t. refuge μ and fear f or transversion rate β. (a) is about parameters μ and f , (b) is about parameters μ and β.

Similarly we give the two-parameter graphs (Figure 3), which explain the interplay effect of μ and f or β on the system stability respectively, where the LPC curve (saddle-node bifurcation of limit cycle for simplicity) of Figure 5(a) originates from the GH point (generalized Hopf point) $(\mu, \beta) = (-0.1257070.741974)$, and the BT point of Figure 5(b) is $(\mu, \beta) = (0.5386420.740012)$, which produce the HC curve. The whole plane is divided into three same regions $R_i(i = 1, 2, 3)$ as before, and the bi-stability appears.

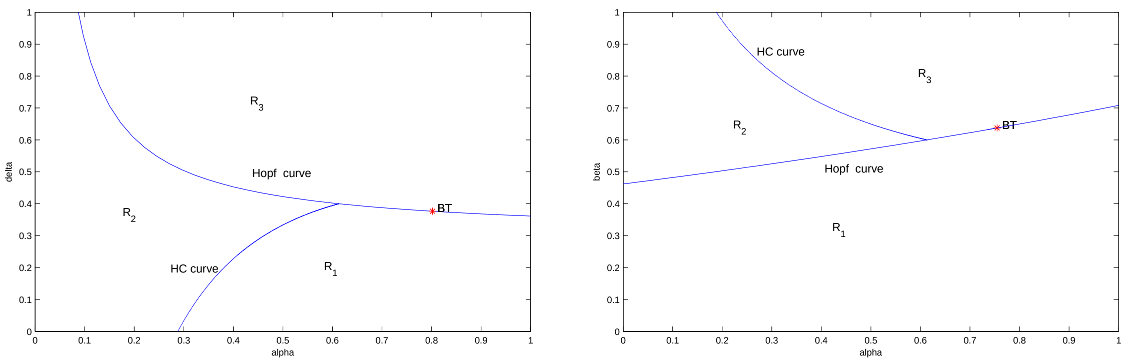


(a) (b)

Figure 6. The trajectories of equilibrium point of system (2.1). (a) The trajectory of prey about additional food α, (b) The trajectory of predator about additional food δ.

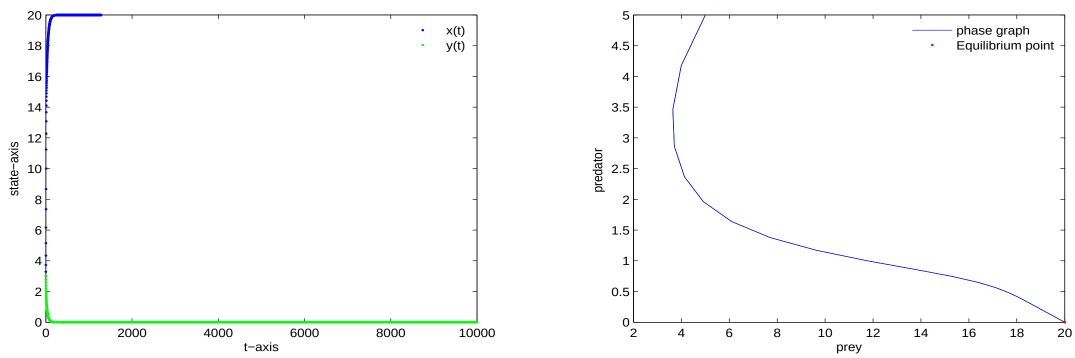
- Bifurcation of additional food supplement

For a prey-predator system with fear effect of predator, one effective way to protect the prey is to build refuge zone, another way is to supply additional food for the predator, when there are enough food for predator, they will reduce their attacks on prey so that the prey can survive naturally, the additional food supplement can not only keep the predator's survival, but also benefit to the living of prey, which is applied usually as a biological control strategy. Parameters α and δ are two crucial indexes representing the quantity and quality of additional food. So we proceed with the study of the effect of α and δ on the system dynamics. We carry out some numerical simulations about them. See Figure 6. Figure 6(a) shows the bifurcation occurs at $\alpha = 0.6130$ where $y = 3.8764$. Figure 6(b) shows Hopf bifurcation appears at the point $(y, \delta) = (3.860116, 0.402164)$. When $\delta = 0.166667$; A transcritical bifurcation occurs where the prey is extinct and the predator $y = 3.2184$, meanwhile a limit point appears at $(y, \delta) = (3.2184, 0.166667)$. At $\delta = 6.83333$, the transcritical bifurcation appears again where the predator is extinct and prey gets the peak value with $x = 20$. Figure 6 shows that when the account of additional food is relatively low, both the quantity and quality bring positive roles to the survival of predator, while Figure 6(b) shows that if δ is large enough (for example $\delta > 2$), then the quality of additional food δ produces negative role, and even leads to the extinction of predator, which reveals that the control strategy should be applied rationally in a suitable region.



(a) (b)

Figure 7. The bi-parameter bifurcation graph w.r.t. the quantity of additional food α and quality of additional food δ or transversion rate β. (a) is about parameters α and β, (b) is about parameters α and β.



(a) (b)

Figure 8. The GAS of the predator-extinction equilibrium state $\overline{Q}(20, 0)$. (a) is time series graph of prey and predator, (b) is phase graph.

To see the interference of α and δ, we depict the two parameter graphs, see Figure 7. The HC curve in $\alpha-\delta$ plane originates from the BT point $(\alpha,\delta) = (0.801951, 0.376521)$, while in $\alpha-\beta$ plane, HC curve originates from the BT point $(\alpha,\beta) = (0.75341, 0.65322)$. The meaning of $R_i(i = 1, 2, 3)$ is the same as before.

Numerically, we find that the fear f, anti-predation sensitivity κ, the refuge μ, and the additional food parameters α and δ bring large influence to the system dynamics including both positive and negative effect, which result in the system complexity of system (2.1) and should be controlled rationally.

4.2. Stability analysis

Now we give some numerical examples to validate the criteria of the global asymptotic stability. We start with the verification of Theorem 3.5. For system (2.1), we fix a set of parameter values as below:

$$a = 0.8, b = 0.04, c = 0.5, d = 1, e = 0.2, F = 2, f = 0.3, \kappa = 0.5, \mu = 0.5, \beta = 0.2, \delta = 0.4, \alpha = 0.6. \tag{4.2}$$

Then, the predator extinction equilibrium point is $\overline{Q}(20, 0)$. We compute that the conditions of Theorem 3.5 are satisfied, so it is GAS, see Figure 8.

Next we fix a set of parameter values as below:

$$a = 0.8, b = 0.04, c = 0.5, d = 1, e = 0.3, F = 2, f = 0.3, \kappa = 0.2, \mu = 0.8, \beta = 0.6, \delta = 0.4, \alpha = 0.6. \tag{4.3}$$

By computation, the interior equilibrium point of (2.1) is $\widetilde{Q}(1.75, 7.4754)$. We verify that the conditions of Theorem 3.6 hold, and it is GAS. See Figure 9.

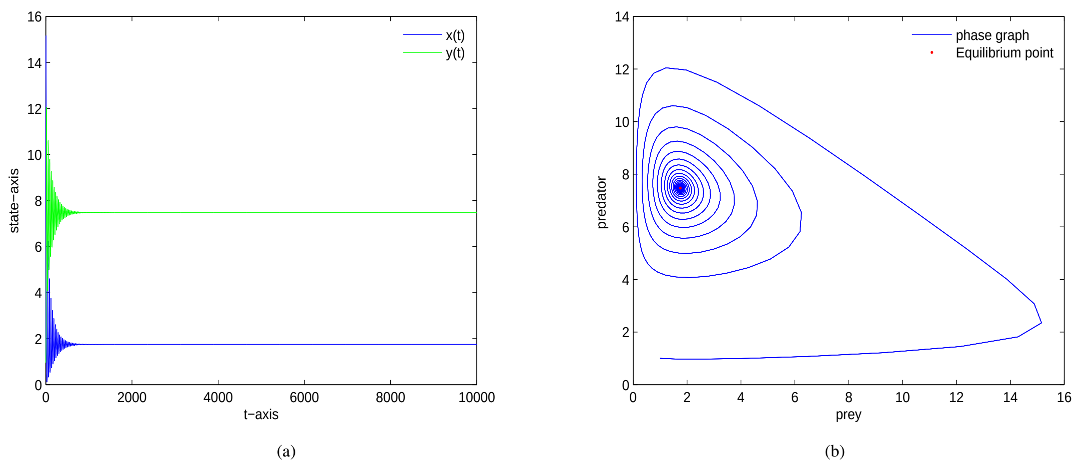


Figure 9. The GAS of the coexistence equilibrium state $\widetilde{Q}(1.75, 7.4754)$. (a) is time series graph of prey and predator, (b) is phase graph.

5. Discussion and conclusions

Considering the joint role of fear effect and direct consumption of predator populations on prey populations, we formulated prey-predator mathematical model with fear and Holling-II type functional response. Due to the existence of counter-predation behavior of prey, we introduced an anti-predation sensitivity parameter to describe the level of anti-predation behavior. In addition, in order to prevent the prey from being predated too quickly and protect the prey for biological balance, we add the additional food supplement for predator. By use of Lyapunov stability theory, we investigated the existence of equilibrium points and their stability. See Table 1. Finally, the bifurcation analysis is executed by applying Sotomayor's bifurcation theorem.

Table 1. Existence and stability of equilibria of (2.1).

Equilibrium	Existence	Stability
Q_0	always exists	unstable
Q_1	always exists	LAS under $\frac{\beta c[(1-\mu)a+b\alpha F]}{ad\kappa+b(1+\delta\alpha F)} < e$
$\widetilde{Q}$	$ed\kappa < \beta c < \frac{e(1+\delta\alpha F}{\alpha F}$	LAS under C_1 and C_2

Numerically, we explored the effects on the system stability of such parameters as fear level, anti-predation sensitivity, refuge level and the account of addition food. Particularly, we obtained the bifurcation thresholds of these parameters, which are summarized in the following Table 2.

Table 2. Bifurcation thresholds for different parameters of (2.1).

Bifurcation	Thresholds of parameters				
	f	κ	μ	α	δ
Transcritical bifurcation	–	0.9720	0.9825	0.8333	0.1667, 6.8333
Saddle-node bifurcation	−0.2565	−0.1834	–	0.8333	0.1667
Hopf bifurcation	0.3051	0.1992,0.8447	0.5012	0.6130	0.4022

The main findings show that the fear significantly affect the system stability, the construction of refuge area and the releasing of additional food for predator are two kinds of effective ways to reduce the effect induced by the fear of predator and can preserve both predator and prey for their natural survival. While these measures have both positive and negative effects, it can not only keep the continuous survival of species, but also make the species be extinct if they are wrongly used. Therefore, rational use of these measures is very crucial for people to control the ecological balance and continuous development. If we take into the complexity of systems, there are still many factors should be considered, for example, the delay exists in almost every process and the fear delay induced by predator is inevitable. On the other hand, the environmental fluctuation, like earthquake, always appear, and the disaster caused by which can affect the system dynamics seriously. All these are left for us to study in the near future.

Acknowledgments

The paper is supported by NSF of China (Nos. 11861027, 12161062), Science and Technology Program of Inner Mongolia Autonomous Region (Nos. 2022YFHH0017, 2022YFHH0063) and Natural Science Foundation of Inner Mongolia (No. 2021MS01005).

Conflict of interest

The authors declare there is no conflict of interest.

References

1. A. J. Lotka, *Elements Physical Biology*, Williams and Wilkins, Baltimore, 1924. https://doi.org/10.1038/116461b0
2. V. Volterra, Fluctuations in the abundance of a species considered mathematically, *Nature*, **118** (1926), 558–560. https://doi.org/10.1038/118558a0
3. P. A. Abrams, The evolution of predator-prey interactions: theory and evidence, *Ann. Rev. Ecol. Syst.*, **31** (2000), 79–105. https://doi.org/10.1146/annurev.ecolsys.31.1.79
4. P. W. Guiden, S. L. Bartel, N. W. Byer, A. A. Shipley, J. L. Orrock, Predator-prey interactions in the Anthropocene: reconciling multiple aspects of novelty, *Trends Ecol. Evol.*, **34** (2019), 616–627. https://doi.org/10.1016/j.tree.2019.02.017
5. J. B. Collings, The effects of the functional response on the bifurcation behavior of a mite predator-prey interaction model, *J. Math. Biol.*, **36** (1997), 149–168. https://doi.org/10.1007/s002850050095
6. K. Antwi-Fordjour, D. R. Parshad, A. M. Beauregard, Dynamics of a predator-prey model with generalized functional response and mutual interference, *Math. Biosci.*, **360** (2020), 108407. https://doi.org/10.1016/j.mbs.2020.108407
7. Y. Huang, F. Chen, L. Zhong, Stability analysis of a prey-predator model with holling type III response function incorporating a prey refuge, *Appl. Math. Comput.*, **182** (2006), 672–683. https://doi.org/10.1016/j.amc.2006.04.030
8. J. P. Tripathi, S. Abbas, M. Thakur, Dynamical analysis of a prey-predator model with Beddington-DeAngelis type function response incorporating a prey refuge, *Nonlinear Dyn.*, **80** (2015), 177–196. https://doi.org/10.1007/s11071-014-1859-2
9. C. Xu, P. Li, Oscillations for a delayed predator-prey modelwith Hassell-Varley-type functional response, *Comptes Rendus Biol.*, **338** (2015), 227–240. https://doi.org/10.1016/j.crvi.2015.01.002
10. J. P. Tripathi, S. Tyagi, S. Abbas, Global analysis of a delayed density dependent predator-prey model with Crowley-Martin functional response, *Commun. Nonlinear Sci. Numer. Simul.*, **30** (2016), 45–69. https://doi.org/10.1016/j.cnsns.2015.06.008

11. J. H. P. Dawes, M. O. Souza, A derivation of Hollings type I, II and III functional responses in predator-prey systems, *J. Theor. Biol.*, **327** (2017), 11–22. https://doi.org/10.1016/j.jtbi.2013.02.017
12. Y. Shao, Global stability of a delayed predator-prey system with fear and Holling-type II functional response in deterministic and stochastic environments, *Math. Comput. Simul.*, **200** (2022), 65–77. https://doi.org/10.1016/j.matcom.2022.04.013
13. S. Pal, N. Pal, S. Samanta, J. Chattopadhyay, Effect of hunting cooperation and fear in a predator-prey model, *Ecol. Complex*, **39** (2019), 100770. https://doi.org/10.1016/j.ecocom.2019.100770
14. S. Creel, D. Christianson, S. Liley, J. A. Winnie, Predation risk affects reproductive physiology and demography of elk, *Science*, **315** (2007), 960. http://doi.org/10.1126/science.1135918
15. L. Y. Zanette, A. F. White, M. C. Allen, M. Clinchy, Perceived predation risk reduces the number of offspring songbirds produce per year, *Science*, **334** (2011), 398–401. https://doi.org/10.1126/science.1210908
16. W. Ripple, R. L. Beschta, Wolves and the ecology of fear: can predation risk structure ecosystems?, *BioScience*, **54** (2004), 755–766. https://doi.org/10.1641/0006-3568(2004)054[0755:WATEOF]2.0.CO;2
17. L. Y. Zanette, M. Clinchy, Ecology of fear, *Curr. Biol.*, **29** (2019), 309–313. http://dx.doi.org/10.1016/j.cub.2019.02.042
18. S. Eggers, M. Griesser, J. Ekman, Predator-induced plasticity in nest visitation rates in the Siberian jay (Perisoreus infaustus), *Behav. Ecol.*, **16** (2005), 309–315. https://doi.org/10.1093/beheco/arh163
19. F. Hua, K. E. Sieving, R. J. Fletcher, C. A. Wright, Increased perception of predation risk to adults and offspring alters avian reproductive strategy and performance, *Behav. Ecol.*, **25** (2014), 509–519. https://doi.org/10.1093/beheco/aru017
20. P. L. Krapivsky, S. Redner, Kinetics of a diffusive capture process: lamb besieged by a pride of lions, *J. Phys. A Math. Gen.*, **26** (1996), 1–13. https://doi.org/10.1088/0305-4470/29/17/011
21. V. Kumar, N. Kumari, Controlling chaos in three species food chain model with fear effect, *AIMS Math.*, **5** (2020), 828–842. https://doi.org/10.3934/math.2020056
22. V. Kumar, N. Kumari, Stability and bifurcation analysis of Hassell-Varley Prey-Predator system with fear effect, *Int. J. Appl. Comput. Math.*, **6** (2020), 150. https://doi.org/10.1007/s40819-020-00899-y
23. B. Nath, N. Kumari, V. Kumar, K. P. Das, Refugia and allee effect in prey species stabilize chaos in a Tri-Trophic food chain model, *Differ. Equations Dyn. Syst.*, **30** (2022), 631–657. https://doi.org/10.1007/s12591-019-00457-z
24. P. Panday, N. Pal, S. Samanta, P. Tryjanowski, J. Chattopadhyay, Dynamics of a stage-structured predator-prey model: cost and benefit of fear-induced group defense, *J. Theor. Biol.*, **528** (2021), 110846. https://doi.org/10.1016/j.jtbi.2021.110846

25. D. T. Blumstein, *The Nature of Fear: Survival Lessons from the Wild*, Harvard University Press, 2020.

26. S. Mondal, G. P. Samanta, Dynamics of an additional food provided predator-prey system with prey refuge dependent on both species and constant harvest in predator, *Phys. A.*, **534** (2019), 122301. https://doi.org/10.1016/j.physa.2019.122301

27. S. Saha, A. Maiti, G. P. Samanta, A michaelis-menten predator-prey model with strong allee effect and disease in prey incorporating prey refuge, *Int. J. Bifurcation Chaos.*, **28** (2018), 1850073. https://doi.org/10.1142/S0218127418500736

28. S. Saha, G. P. Samanta, Analysis of a predator-prey model with herd behaviour and disease in prey incorporating prey refuge, *Int. J. Biomath.*, **12** (2019), 1950007. https://doi.org/10.1142/S1793524519500074

29. M. Das, A. Maity, G. P. Samanta, Stability analysis of a prey-predator fractional order model incorporating prey refuge, *Ecol. Genet. Genomics.*, **7–8** (2018), 33–46. http://doi.org/10.1016/j.egg.2018.05.001

30. H. Li, L. Zhang, C. Hu, Y. L. Jiang, Z. Teng, Dynamical analysis of a fractional-order predatorCprey model incorporating a prey refuge, *J. Appl. Math. Comput.*, **54** (2017), 435–449. https://doi.org/10.1007/s12190-016-1017-8

31. T. K. Kar, Stability analysis of a predator-prey model incorporating a prey refuge, *Commun. Nonlinear Sci. Numer. Simul.*, **10** (2006), 681–691. https://doi.org/10.1016/j.cnsns.2003.08.006

32. Y. X. Xia, S. L. Yuan, Survival analysis of a stochastic predator-prey model with prey refuge and fear effect, *J. BBiol. Dynam.*, **14** (2020), 871–892. https://doi.org/10.1080/17513758.2020.1853832

33. S. Mondal, A. Maiti, G. P. Samanta, Effects of fear and additional food in a delayed predator-prey model, *Biophys. Rev. Lett.*, **13** (2018), 157177. https://doi.org/10.1142/S1793048018500091

34. Moorland Working Group, *Diversionary feeding of hen harriers on grouse moors a practical guide Report*, Scottish Natural heritage, Inverness, 1999.

35. M. J. Crawley, *Plant Ecology*, Wiley, New York, 1997.

36. A. Das, G. P. Samanta, Modeling the fear effect on a stochastic prey-predator system with additional food for the predator, *J. Phys. A Math. Theor.*, **51** (2018), 465601. https://doi.org/10.1088/1751-8121/aae4c6

37. G. R. Huxel, K. McCann, G. A. Polis, Effects of partitioning allochthonous and autochthonous resources on food web stability, *Ecol. Res.*, **17** (2002), 419–432. https://doi.org/10.1046/j.1440-1703.2002.00501.x

38. P. D. N. Srinivasu, B. S. R. V. Prasad, Role of quantity of additional food to predators as a control in predator-prey systems with relevance to pest management and biological conservation, *Bull. Math. Biol.*, **73** (2011), 2249–2276. https://doi.org/10.1007/s11538-010-9601-9

39. L. Perko, *Differential Equations and Dynamical Systems*, Springer, 1996.

40. V. B. Kolmanovskii, A. D. Myshkis, *Applied Theory of Functional Differential Equations*, Kluwer Academic Publishers, Boston, 1992.

41. A. Dhooge, W. Govaerts, Y. A. Kuznetsov, H. G. E. Meijer, B. Sautois, New features of the software matcont for bifurcation analysis of dynamical systems, *Math. Comput. Model. Dyn. Syst.*, **14** (2008), 147–175. https://doi.org/10.1080/13873950701742754

AIMS Press

© 2023 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0)