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Research article

Chaos criteria and chaotification schemes on a class of first-order partial difference equations

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Abstract: This article is involved in chaos criteria and chaotification schemes on one kind of first-order partial difference equations having non-periodic boundary conditions. Firstly, four chaos criteria are achieved by constructing heteroclinic cycles connecting repellers or snap-back repellers. Secondly, three chaotification schemes are obtained by using these two kinds of repellers. For illustrating the usefulness of these theoretical results, four simulation examples are presented.

Keywords: first-order partial difference equation; heteroclinic cycle connecting repellers; snap-back repeller; chaotification; chaos

1. Introduction

The chaos problems, including chaos criteria and chaotification, are studied on one kind of first-order partial difference equations as follows

$$u(a+1,b) = h(u(a,b), u(a,b+1)), \tag{1.1}$$

with a non-periodic boundary condition

$$u(a,n+1) = \psi(u(a,m)), \quad a \geq 0, \tag{1.2}$$

and an initial condition

$$u(0,j) = \phi(j), \quad 0 \leq j \leq n+1, \tag{1.3}$$

where the map $h : F \subset \mathbf{R}^2 \to \mathbf{R}$, the integer $a \geq 0$ represents the time step, the integer b represents the lattice point satisfying $0 \leq b \leq n < \infty$, $n+1$ represents the system size for some integer $n > 0$, the map $\psi : E \subset R \to R$, the integer m satisfies $0 \leq m \leq n$, the map ϕ meets the non-periodic boundary condition, that is, $\phi(n+1) = \psi(\phi(m))$.

Equation (1.1) had been extremely studied for the engineering fields, such as imaging and digital filter, see [1,2]. The uniqueness of solutions of Eq (1.1) satisfying (1.2) and (1.3) can be easily proved by successive iterations. Hence, it is well-defined for this initial-boundary value problem.

For the study of chaos criteria about Eq (1.1) with a periodic boundary condition, which is a special case for $m = 0$ in (1.2) satisfying $\psi(u(a, 0)) = u(a, 0)$, some important results have been obtained. In 2003, Chen and Liu [3] first constructed some periodic orbits in space with a particular period in $\mathbf{R}^3$ for one certain type of Eq (1.1) and showed there existed Li-Yorke chaos. Later, stability and chaos for Eq (1.1) were studied in [4], where Eq (1.1) was reformulated into one certain discrete system. Applying snap-back repellers and this method, several chaos criteria were achieved in [5] for Eq (1.1). Motivated by the idea of [5], some chaos criteria for Eq (1.1) were also achieved in [6], where heteroclinic cycles connecting repellers were applied. As far as we know, out of the above achievements, there exist very few achievements on the chaos criteria of Eq (1.1). Especially, there are no corresponding results on chaos criteria when the boundary condition becomes non-periodic. Since the non-periodic boundary condition (1.2) for Eq (1.1) is more general and practical in real applications, it is very necessary to study the non-periodic case.

For the study of chaotification about Eq (1.1), some important results have also been obtained. On the one hand, for the case of Eq (1.1) with a periodic boundary condition, one chaotification scheme was first achieved in [7] by applying the theory of coupled expansion. Later, some chaotification schemes of Eq (1.1) were achieved in [8], where the controllers had general forms or some special forms, such as sawtooth functions and mod operations. Motivated by the ideas used in [7–9] and through constructing heteroclinic cycles connecting repellers, two chaotification theorems of Eq (1.1) were obtained in [6], where the controllers had general forms. On the other hand, for the case of Eq (1.1) with a non-periodic boundary condition, in 2014, Liang et al. [10] first studied chaotification for Eq (1.1) with a special case of the boundary condition (1.2), i.e., $m = 0$ in (1.2). Lately, Liang et al. [11–14] used the theory of coupled expansion to investigate chaotification problems of Eq (1.1) with polynomial maps or fundamental elementary functions as controllers. At the same time, Liang and Guo [15] studied chaotification of Eq (1.1) with controllers satisfying a special case of condition (1.2), i.e., $\psi(u(a, m)) = u(a, m)$. As far as we know, out of the above achievements, there exist very few achievements on chaotification of Eq (1.1). It is easy to see that all the results in the second case only studied some special forms of condition (1.2) or only used some special functions as controllers. So, it is natural to ask whether can we use a general controller to chaotify Eq (1.1) with a non-periodic boundary condition (1.2). Since the non-periodic boundary condition (1.2) for Eq (1.1) is more general and practical in real applications, it is also very necessary to study this chaotification problem which will provide theoretical foundations for applications.

Based on the above discussions, we will apply snap-back repellers or heteroclinic cycles connecting repellers to achieve some chaos criteria such that Eq (1.1) with (1.2) emerges chaos. Furthermore, we will also use the above two kinds of repellers to achieve some chaotification theorems of Eq (1.1) with (1.2).

The main contents of the study are arranged below. Some preparatory knowledge is presented in Section 2. Four chaos criteria are achieved through constructing snap-back repellers or heteroclinic cycles connecting repellers in Section 3. In the last section, two simulation examples are presented to show the usefulness of these theorems. In Section 4, three chaotification schemes for Eq (1.1) with a non-periodic boundary condition (1.2) are obtained through constructing snap-back repellers or heteroclinic cycles connecting repellers, respectively. At last of the part, two simulation examples are also supplied to exhibit the usefulness of chaotification theorems. Finally, in Section 5, some conclusions are given.

2. Preliminaries

Some notations will be first presented. Then, by using the method in [4, 7], Eq (1.1) will be transformed into one ordinary difference equation such that those theories in discrete systems can be applied.

Set

$$\mathbf{R}^{n+1} = \{u' = \{u(b)\}_{b=0}^{n} : \ u(b) \in \mathbf{R}, \ 0 \le b \le n\},$$

where $0 \le n < \infty$. Let

$$V_{n+1} = \{u' = \{u(b)\}_{b=0}^{n} \in \mathbf{R}^{n+1} : \ \|u'\|_{n+1} < \infty\},$$

where

$$\|u'\|_{n+1} = \sup\{|u(b)| : \ 0 \le b \le n\}.$$

From [7], we get $(V_{n+1}, \|\cdot\|_{n+1})$ is a real Banach space. Take an interval $E \subset \mathbf{R}$ and set

$$E^{n+1} = \{u' = \{u(b)\}_{b=0}^{n} : u(b) \in E, \ 0 \le b \le n\}.$$

Obviously, when E is bounded, we have $E^{n+1} \subset V_{n+1}$.

The Frechét derivative is used in a Banach space $(U, \|\cdot\|)$, and $Dh(u)$ represents the derivative for a map h at a point $u \in U$, see Definition 10.34 of [16]. In addition, $h_u(u, v)$, $h_v(u, v)$ represent first-order partial derivatives about u, v for a differentiable function $h(u, v)$, respectively. If $J : U \to U$ is one linear map, let

$$\|J\| = \sup\{\|Ju\| : u \in U, \ \|u\| = 1\}, \quad \|J\|^0 = \inf\{\|Ju\| : u \in U, \ \|u\| = 1\}.$$

If J is bounded and possesses a bounded inverse, we call J an invertible linear map, refer to Definition 4.17 of [16].

For convenience, for one general Banach space $(U, \|\cdot\|)$ or one general metric space (U, d), let $\bar{N}(u, s)$ and $N(u, s)$ represent closed and open spheres with center $u \in U$ and radius s, respectively. While for one special Banach space $(U, \|\cdot\|_{n+1})$ with the norm $\|\cdot\|_{n+1}$, denote the above spheres as $\bar{N}_{n+1}(u, s)$ and $N_{n+1}(u, s)$, respectively.

Now, Eq (1.1) will be transformed into a finite-dimensional discrete dynamical system by using the method in [4, 7]. Set

$$u_a = \{u(a, b)\}_{b=0}^{n} = (u(a, 0), u(a, 1), \cdots, u(a, n))^T \in \mathbf{R}^{n+1}, \ a \ge 0.$$

Then, Eq (1.1) satisfying (1.2) can be rewritten as one discrete dynamical system

$$u_{a+1} = H(u_a), \tag{2.1}$$

where

$$\begin{aligned} H(u_a) &= \{h(u(a, b), u(a, b+1))\}_{b=0}^{n} \\ &= (h(u(a, 0), u(a, 1)), \cdots, h(u(a, n-1), u(a, n)), h(u(a, n), \psi(u(a, m))))^T \end{aligned}$$

and m is some integer with $0 \le m \le n$. We call H the induced map of h, and call system (2.1) the induced system of Eq (1.1) satisfying (1.2) in $(V_{n+1}, \|\cdot\|_{n+1})$. From Section 5.1 of [7], we can obtain that the dynamical behaviors of Eq (1.1) satisfying (1.2) in regard to a correspond to the behaviors of the induced system 2.1. Hence, from Definitions 5.1 and 5.2 of [7], we can define some concepts, such as chaos, for Eq (1.1) satisfying (1.2) through the corresponding concepts of the induced system (2.1). For example, one point $u' = \{u(b)\}_{b=0}^{n}$ is named one fixed point for Eq (1.1) satisfying (1.2) when and only when it is a fixed point for system (2.1), that is, when and only when $u' \in V_{n+1}$ and meets that

$$\begin{aligned} &u(b) = h(u(b), u(b+1)),\ 0 \le b \le n-1, \\ &u(n) = h(u(n), \psi(u(m))), \end{aligned} \tag{2.2}$$

where m is one integer with $0 \le m \le n$.

Up to now, there is not a unified definition of chaos. A strict definition in mathematics for chaos, called Li-Yorke chaos, was first introduced by Li and Yorke [17]. After that, there emerged a few definitions for chaos, in this paper we will use the other two definitions, i.e., Devaney chaos and Wiggins chaos, see [18, 19] for them. One can refer to references [20–24] for further research on definitions of chaos.

To study the above three kinds of chaos for Eq (1.1) with (1.2), we will apply snap-back repellers and heterocilinc cycles connecting repellers. For brevity, we omit their definitions of them and refer readers to references [25, 26]. One can refer to [25, 27–29] for more details about the definition of snap-back repeller, and [26,30–33] for more details about the definition of heterocilinc cycle connecting repellers.

3. Chaos criteria

The chaos criterion below is obtained through constructing a snap-back repeller, which is motivated by [5].

Theorem 3.1. *If the conditions below are satisfied,*

(i) *there are two constants* $s > 0$, $\sigma > 1$ *and a point* $w_0 \in \mathbf{R}$ *such that* $h(w_0, w_0) = w_0$, $\psi(w_0) = w_0$, *h has continuous differentiability in* $[w_0 - s, w_0 + s]^2$, $\psi : [w_0 - s, w_0 + s] \to [w_0 - s, w_0 + s]$ *has continuous differentiability in* $[w_0 - s, w_0 + s]$, *and*

$$\begin{aligned} &\min\{||h_u(u,v)| - |h_v(u,v)||,\ ||h_u(u,v)| - |h_v(u,v)\psi'(t)||\} \ge \sigma, \\ &\forall (u,v) \in [w_0 - s, w_0 + s]^2 \text{ and } \forall t \in [w_0 - s, w_0 + s]; \end{aligned} \tag{3.1}$$

(ii) *there are a sequence* $\{u_0(b)\}_{b=0}^{n}$ *lying in* $(w_0 - s, w_0 + s)$ *and an integer* $e \ge 2$ *such that* $\{u_0(b)\}_{b=0}^{n} \ne \{w_0, \cdots, w_0\}$ *and for some integer m,* $0 \le m \le n$,

$$\begin{aligned} &u_q(b) = h(u_{q-1}(b), u_{q-1}(b+1)),\ \ 0 \le b \le n-1, \\ &u_q(n) = h(u_{q-1}(n), \psi(u_{q-1}(m))),\ \ 1 \le q \le e-1, \\ &w_0 = h(u_{e-1}(b), u_{e-1}(b+1)),\ \ 0 \le b \le n-1, \\ &w_0 = h(u_{e-1}(n), \psi(u_{e-1}(m))); \end{aligned} \tag{3.2}$$

(iii) *for any q,* $1 \le q \le e-1$, *h has continuous differentiability near* $(u_q(b), u_q(b+1))$, ψ *has continuous differentiability near* $u_q(b)$ *for* $0 \le b \le n$, $u_q(n+1) = \psi(u_q(m))$, *and their derivatives satisfy one of the following two conditions.*

(iiia) *When* $0 \le m \le n-1$,

$$\begin{aligned}&\prod_{b=0}^{n} h_u(u_q(b), u_q(b+1))\\&+(-1)^{n+m}\psi'(u_q(m))\prod_{b=0}^{m-1} h_u(u_q(b), u_q(b+1))\prod_{b=m}^{n} h_v(u_q(b), u_q(b+1)) \neq 0,\end{aligned} \tag{3.3}$$

where $\prod_{b=0}^{m-1} h_u(u_q(b), u_q(b+1))$ *vanishes as* $m = 0$.

(iiib) *When* $m = n$,

$$\prod_{b=0}^{n-1} h_u(u_q(b), u_q(b+1))[h_u(u_q(n), \psi(u_q(n))) + h_v(u_q(n), \psi(u_q(n)))\psi'(u_q(n))] \neq 0. \tag{3.4}$$

Then, Eq (1.1) with (1.2) has Li-Yorke chaos and Devaney chaos.

Proof. We will use Lemma 2.2 of [5] to show it. Let $w^* = (w_0, w_0, \cdots, w_0)^T \in V_{n+1}$. Then, from $h(w_0, w_0) = w_0$, $\psi(w_0) = w_0$ and (2.2), we get the induced map H of h possesses a fixed point w^*. According to assumption (i), we obtain H has continuous differentiability in $\bar{N}_{n+1}(w^*, s)$. So, by the mean value theorem and (3.1), one gets for arbitrary $u' = \{u(b)\}_{b=0}^n$, $v' = \{v(b)\}_{b=0}^n \in \bar{N}_{n+1}(w^*, s)$,

$$\begin{aligned}&\|H(u') - H(v')\|_{n+1}\\&= \sup\{|h(u(b), u(b+1)) - h(v(b), v(b+1))| : 0 \le b \le n\}\\&= \sup\{|h_u(\eta(b))(u(b) - v(b)) + h_v(\eta(b))(u(b+1) - v(b+1))|\\&\quad \text{for } 0 \le b \le n-1, \text{and } |h_u(\eta(n))(u(n) - v(n)) + h_v(\eta(n))(\psi(u(m)) - \psi(v(m)))|\}\\&\ge \sup\{||h_u(\eta(b))| \cdot |u(b) - v(b)| - |h_v(\eta(b))| \cdot |u(b+1) - v(b+1)||\\&\quad \text{for } 0 \le b \le n-1, \text{and } ||h_u(\eta(n))| \cdot |u(n) - v(n)| - |h_v(\eta(n))| \cdot |\psi'(\xi(m))| \cdot |u(m) - v(m)||\}\\&\ge \sigma\|u' - v'\|_{n+1},\end{aligned} \tag{3.5}$$

where $\eta(b) \in (w_0 - s, w_0 + s)^2$ for $0 \le b \le n$, $\xi(m) \in (w_0 - s, w_0 + s)$ for some integer m with $0 \le m \le n$. Hence, the fixed point w^* of H is expansive since $\sigma > 1$ in (3.5). By (3.5) and Lemma 2.2 of [25], we can get for arbitrary s' satisfying $0 < s' \le s$, $H(\bar{N}_{n+1}(w^*, s'))$ is closed, $H(N_{n+1}(w^*, s'))$ is open,

$$H(\bar{N}_{n+1}(w^*, s')) \supset \bar{N}_{n+1}(w^*, s'), \quad H(N_{n+1}(w^*, s')) \supset N_{n+1}(w^*, s').$$

Therefore, we get from the above relation that the fixed point w^* is also regular.

Let

$$u_q = \{u_q(b)\}_{b=0}^{n} \text{ for } 0 \le q \le e-1,$$

where $u_q(b)$ is determined in assumption (ii). Further, by assumption (ii) we obtain that $u_0 \in N_{n+1}(w^*, s)$, $u_0 \neq w^*$, $u_q \in V_{n+1}$ for $1 \le q \le e-1$, $H(u_q) = u_{q+1}$ for $0 \le q \le e-2$, $H(u_{e-1}) = w^*$. So, we have that $H^e(u_0) = w^*$ and consequently, w^* becomes a snap-back repeller.

By assumption (iii), to arbitrary q satisfying $1 \le q \le e-1$, there is a constant $\delta_q(b) > 0$ such that h has continuous differentiability in $N_2((u_q(b), u_q(b+1)), \delta_q(b))$, ψ has continuous differentiability in $N_1(u_q(b), \delta_q(b))$, where $0 \le b \le n$, $u_q(n+1) = \psi(u_q(m))$. Take $\delta_q = \min\{\delta_q(b) : 0 \le b \le n\}$, we can get that h has continuous differentiability in $N_2((u_q(b), u_q(b+1)), \delta_q)$, ψ has continuous differentiability in $N_1(u_q(b), \delta_q)$, and H has continuous differentiability in $N_{n+1}(u_q, \delta_q)$ for $1 \le q \le e-1$. For convenience, let $\mu_q(b) = (u_q(b), u_q(b+1))$ for $0 \le b \le n$, $1 \le q \le e-1$ and $u_q(n+1) = \psi(u_q(m))$. Since $0 \le m \le n$, the derivatives of H at u_q, $1 \le q \le e-1$, have two cases. One case is for $0 \le m \le n-1$, which is as follows

$$DH(u_q) = \begin{pmatrix} h_u(\mu_q(0)) & h_v(\mu_q(0)) & 0 & \cdots & 0 & \cdots & 0 \\ 0 & h_u(\mu_q(1)) & h_v(\mu_q(1)) & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & h_v(\mu_q(n))\psi'(u_q(m)) & \cdots & h_u(\mu_q(n)) \end{pmatrix}, \tag{3.6}$$

where $h_v(\mu_q(n))\psi'(u_q(m))$ in the last line lies at the $(m+1)$-th column of the matrix. From (3.6) and (3.3), we can get that for each $1 \le q \le e-1$,

$$\det DH(u_q) = \prod_{b=0}^{n} h_u(\mu_q(b)) + (-1)^{n+m}\psi'(u_q(m)) \prod_{b=0}^{m-1} h_u(\mu_q(b)) \prod_{b=m}^{n} h_v(\mu_q(b)) \neq 0, \tag{3.7}$$

where $\prod_{b=0}^{m-1} h_u(\mu_q(b))$ vanishes as $m = 0$. The other case is for $m = n$, and it is as follows

$$DH(u_q) = \begin{pmatrix} h_u(\mu_q(0)) & h_v(\mu_q(0)) & 0 & \cdots & 0 & 0 \\ 0 & h_u(\mu_q(1)) & h_v(\mu_q(1)) & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & h_u(\mu_q(n)) + h_v(\mu_q(n))\psi'(u_q(n)) \end{pmatrix}, \tag{3.8}$$

From (3.6) and (3.4), we get that for each $1 \le q \le e-1$,

$$\det DH(u_q) = \prod_{b=0}^{n-1} h_u(\mu_q(b))[h_u(\mu_q(n)) + h_v(\mu_q(n))\psi'(u_q(n))] \neq 0. \tag{3.9}$$

Moreover, by Lemma 2.2 of [7] and (3.5), we have

$$\|DH(u')\|_{n+1}^{0} \ge \sigma, \quad \forall u' \in N_{n+1}(w^*, s).$$

This concludes that

$$\|DH(u')v'\|_{n+1} \geq \sigma\|v'\|_{n+1}, \quad \forall u' \in N_{n+1}(w^*, s), \ \forall v' \in V_{n+1}. \tag{3.10}$$

By (3.10) and Theorem 4.1 of [29], we get the absolute value of each eigenvalue of $DH(u')$ is greater than 1 for arbitrary $u' \in N_{n+1}(w^*, s)$. Then, we have $\det DH(u') \neq 0$ for arbitrary $u' \in N_{n+1}(w^*, s)$. Since $u_0 \in N_{n+1}(w^*, s)$, we have $\det DH(u_0) \neq 0$. By this result and (3.7), (3.9), we get $\det DH(u_q) \neq 0$ for $0 \leq q \leq e-1$.

Hence, the conditions of Lemma 2.2 in [5] are satisfied. Then, the snap-back repeller w^* of the map H is regular and nondegenerate. Consequently, system (2.1), that is, Eq (1.1) with (1.2), has Devaney chaos and Li-Yorke chaos. □

The following result may be verified with the same method to Theorem 3.1 by using Theorem 4.2 of [24]. So, we omit the proof and only state it as follows.

Theorem 3.2. *Suppose conditions* (i) *and* (ii) *in Theorem 3.1 hold, meanwhile,*

(iv) *for arbitrary q, $1 \leq q \leq e-1$, there is a constant $\delta_q > 0$ such that h has continuous differentiability in $\bar{N}_2((u_q(b), u_q(b+1)), \delta_q)$, ψ has continuous differentiability in $\bar{N}_1(u_q(b), \delta_q)$ for $0 \leq b \leq n$, $u_q(n+1) = \psi(u_q(m))$, and their derivatives satisfy one of the following conditions.*

(iva) *When $m = 0$,*

$$\prod_{b=0}^{n} h_u(u(b), u(b+1)) + (-1)^n \psi'(u(0)) \prod_{b=0}^{n} h_v(u(b), u(b+1)) \neq 0$$

keeps a same sign for arbitrary $u' = \{u(b)\}_{b=0}^{n}$ which satisfies $(u(b), u(b+1)) \in \bar{N}_2((u_q(b), u_q(b+1)), \delta_q) \setminus (u_q(b), u_q(b+1))$ for $0 \leq b \leq n$. At the same time, there are at minimum n elements which are not equal to zero and lie either in $\{h_u(u_q(b), u_q(b+1)) : 0 \leq b \leq n\}$ or in

$$\{h_v(u_q(b), u_q(b+1)) \text{ for } 0 \leq b \leq n-1, h_v(u_q(n), \psi(u_q(0)))\psi'(u_q(0))\},$$

where $u(n+1) = \psi(u(0))$, $u_q(n+1) = \psi(u_q(0))$;

(ivb) *When $1 \leq m \leq n-1$,*

$$\prod_{b=0}^{n} h_u(u(b), u(b+1))$$
$$+(-1)^{n+m}\psi'(u(m)) \prod_{b=0}^{m-1} h_u(u(b), u(b+1)) \prod_{b=m}^{n} h_v(u(b), u(b+1)) \neq 0$$

keeps a same sign for arbitrary $u' = \{u(b)\}_{b=0}^{n}$ which satisfies $(u(b), u(b+1)) \in \bar{N}_2((u_q(b), u_q(b+1)), \delta_q) \setminus (u_q(b), u_q(b+1))$ for $0 \leq b \leq n$, either $h_u(u_q(b), u_q(b+1)) \neq 0$ for $0 \leq b \leq n-1$ or $h_v(u_q(b), u_q(b+1)) \neq 0$ for $0 \leq b \leq n-1$, where $u(n+1) = \psi(u(m))$, $u_q(n+1) = \psi(u_q(m))$;

(ivc) *When $m = n$,*

$$\prod_{b=0}^{n-1} h_u(u(b), u(b+1))[h_u(u(n), \psi(u(n))) + h_v(u(n), \psi(u(n)))\psi'(u(n))] \neq 0$$

keeps a same sign for arbitrary $u' = \{u(b)\}_{b=0}^{n}$ *which satisfies* $(u(b), u(b+1)) \in \bar{N}_2((u_q(b), u_q(b+1)), \delta_q) \setminus (u_q(b), u_q(b+1))$ *for* $0 \le b \le n$*. At the same time, either there are at minimum n elements which are not equal to zero lying in*

$$\{h_u(u_q(b), u_q(b+1)) \text{ for } 0 \le b \le n-1,$$

$$h_u(u_q(n),\ \psi(u_q(n))) + h_v(u_q(n), \psi(u_q(n)))\psi'(u_q(n))\}$$

or $h_v(u_q(b), u_q(b+1)) \neq 0$ *for* $0 \le b \le n-1$*, where* $u(n+1) = \psi(u(n))$*,* $u_q(n+1) = \psi(u_q(n))$*;*

Then, Eq (1.1) with (1.2) has Wiggins chaos and Li-Yorke chaos.

The next chaos criterion, inspired by [6], is achieved through constructing a heteroclinic cycle connecting repellers.

Theorem 3.3. *If the conditions below are satisfied,*

(i) *there are constants* $s_p > 0$*,* $\sigma_p > 1$ *and different points* $w_p \in \mathbf{R}$ *for* $1 \le p \le e$*,* $e \ge 2$*, such that* $h(w_p, w_p) = w_p$*,* $\psi(w_p) = w_p$*, h has continuous differentiability in* $[w_p - s_p, w_p + s_p]^2$ *which are disjoint with each other,* $\psi : [w_p - s_p, w_p + s_p] \to [w_p - s_p, w_p + s_p]$ *has continuous differentiability in* $[w_p - s_p, w_p + s_p]$*, and*

$$\begin{aligned} &\min\{||h_u(u,v)| - |h_v(u,v)||,\ ||h_u(u,v)| - |h_v(u,v)\psi'(t)||\} \ge \sigma_p, \\ &\forall (u,v) \in [w_p - s_p, w_p + s_p]^2 \text{ and } \forall t \in [w_p - s_p, w_p + s_p]; \end{aligned} \tag{3.11}$$

(ii) *for arbitrary p,* $1 \le p \le e$*, there are a sequence* $\{u_{p0}(b)\}_{b=0}^{n}$ *lying in* $(w_p - s_p, w_p + s_p)$ *and an integer* $l_p \ge 1$ *such that for some integer m with* $0 \le m \le n$*,*

$$\begin{aligned} &u_{pq}(b) = h(u_{p,q-1}(b), u_{p,q-1}(b+1)),\ \ 0 \le b \le n-1, \\ &u_{pq}(n) = h(u_{p,q-1}(n), \psi(u_{p,q-1}(m))),\ \ 1 \le q \le l_p - 1, \\ &w_{\tau(p)} = h(u_{p,l_p-1}(b), u_{p,l_p-1}(b+1)),\ \ 0 \le b \le n-1, \\ &w_{\tau(p)} = h(u_{p,l_p-1}(n), \psi(u_{p,l_p-1}(m))), \end{aligned} \tag{3.12}$$

where $\tau(p) = [p \bmod e] + 1$*;*

(iii) *for arbitrary p,* $1 \le p \le e$*, h has continuous differentiability near* $(u_{pq}(b),\ u_{pq}(b+1))$*,* ψ *has continuous differentiability near* $u_{pq}(b)$ *for* $0 \le b \le n$*,* $1 \le q \le l_p - 1$*,* $u_{pq}(n+1) = \psi(u_{pq}(m))$*, and their derivatives satisfy one of the following two conditions.*

(iiia) *When* $0 \le m \le n-1$*,*

$$\begin{aligned} &\prod_{b=0}^{n} h_u(u_{pq}(b), u_{pq}(b+1)) \\ &+(-1)^{n+m}\psi'(u_{pq}(m)) \prod_{b=0}^{m-1} h_u(u_{pq}(b), u_{pq}(b+1)) \prod_{b=m}^{n} h_v(u_{pq}(b), u_{pq}(b+1)) \neq 0, \end{aligned} \tag{3.13}$$

where $\prod_{b=0}^{m-1} h_u(u_{pq}(b), u_{pq}(b+1))$ *vanishes as* $m = 0$*.*

(iiib) *When $m = n$,*

$$\prod_{b=0}^{n-1} h_u(u_{pq}(b), u_{pq}(b+1))[h_u(u_{pq}(n), \psi(u_{pq}(n)) + h_v(u_{pq}(n), \psi(u_{pq}(n)))\psi'(u_{pq}(n))] \neq 0. \quad (3.14)$$

Then, Eq (1.1) with (1.2) has Li-Yorke chaos and Devaney chaos.

Proof. We will apply Corollary 4.2 of [26] to show it. The sup-norm $\|\cdot\|_{n+1}$ will be used to describe the expansivity. Without loss of generality, here it just verifies as $e = 2$, $p = 1$ or 2.

Let $w_p^* = (w_p, w_p, \cdots, w_p)^T \in V_{n+1}$. By conditions (i) and (2.2) , we get $H(w_p^*) = w_p^*$ and H has continuous differentiability in $\bar{N}_{n+1}(w_p^*, s_p)$. Similarly to obtain (3.5), based on the mean value theorem and (3.11), one obtains for arbitrary $u' = \{u(b)\}_{b=0}^n$, $v' = \{v(b)\}_{b=0}^n \in \bar{N}_{n+1}(w_p^*, s_p)$,

$$\|H(u') - H(v')\|_{n+1} \geq \sigma_p \|u' - v'\|_{n+1}, \quad (3.15)$$

From (3.15), we get the fixed point w_p^* of H is expansive due to $\sigma_p > 1$. Through an analogous proof as in Theorem 3.1, we get the fixed point w_p^* is also regular. In addition, according to Lemma 2.3 of [25], one can get that for arbitrary $u' \in N_{n+1}(w_p^*, s_p)$, $u' \neq w_p^*$, $H^{-i}(u')$ is unique in $N_{n+1}(w_p^*, s_p)$ for $i \geq 1$, $H^{-i}(u') \to w_p^*$ when $i \to \infty$. Let

$$u_{pq} = \{u_{pq}(b)\}_{b=0}^n \text{ for } 0 \leq q \leq l_p - 1.$$

Then, $u_{p0} \in N_{n+1}(w_p^*, s_p)$ due to $\{u_{p0}(b)\}_{b=0}^n$ lying in $(w_p - s_p, w_p + s_p)$, and $u_{pq} \in V_{n+1}$, $1 \leq q \leq l_p - 1$. It follows from (3.12) that $H(u_{pq}) = u_{p,q+1}$, $0 \leq q \leq l_p - 2$ as $l_p \geq 2$, and $H(u_{p,l_p-1}) = w_{\tau(p)}^*$. Hence, $H^{l_p}(u_{p0}) = w_{\tau(p)}^*$. Then, H possesses one heteroclinic cycle χ connecting repellers w_1^* and w_2^*.

By condition (iii), for arbitrary q, $1 \leq q \leq l_p - 1$, there is one constant $\delta_{pq}(b) > 0$ such that h has continuous differentiability in $N_2((u_{pq}(b), u_{pq}(b+1)), \delta_{pq}(b))$, ψ has continuous differentiability in $N_1(u_{pq}(b), \delta_{pq}(b))$, where $0 \leq b \leq n$, $1 \leq q \leq l_p - 1$ and $u_{pq}(n+1) = \psi(u_{pq}(m))$. Take $\delta_{pq} = \min\{\delta_{pq}(b) : 0 \leq b \leq n\}$, we can get that h has continuous differentiability in $N_2((u_{pq}(b), u_{pq}(b+1)), \delta_{pq})$, ψ has continuous differentiability in $N_1(u_{pq}(b), \delta_{pq})$, and H has continuous differentiability in $N_{n+1}(u_{pq}, \delta_{pq})$ for $1 \leq q \leq l_p - 1$. For convenience, let $\mu_{pq}(b) = (u_{pq}(b), u_{pq}(b+1))$, $0 \leq b \leq n$, $1 \leq q \leq l_p - 1$ and $u_{pq}(n+1) = \psi(u_{pq}(m))$. Since $0 \leq m \leq n$, the derivatives of H at u_{pq}, $1 \leq q \leq l_p - 1$, have two cases. One case is for $0 \leq m \leq n - 1$, which is as follows

$$DH(u_{pq}) = \begin{pmatrix} h_u(\mu_{pq}(0)) & h_v(\mu_{pq}(0)) & 0 & \cdots & 0 & \cdots & 0 \\ 0 & h_u(\mu_{pq}(1)) & h_v(\mu_{pq}(1)) & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & h_v(\mu_{pq}(n))\psi'(u_{pq}(m)) & \cdots & h_u(\mu_{pq}(n)) \end{pmatrix}, \quad (3.16)$$

where $h_v(\mu_{pq}(n))\psi'(u_{pq}(m))$ in the last line lies at the $(m+1)$-th column of the matrix. It follows from (3.13) and (3.16) that for each $1 \leq q \leq l_p - 1$,

$$\det DH(u_{pq}) = \prod_{b=0}^{n} h_u(\mu_{pq}(b)) + (-1)^{n+m}\psi'(u_{pq}(m)) \prod_{b=0}^{m-1} h_u(\mu_{pq}(b)) \prod_{b=m}^{n} h_v(\mu_{pq}(b)) \neq 0, \quad (3.17)$$

where $\prod_{b=0}^{m-1} h_u(\mu_{pq}(b))$ vanishes as $m = 0$. The other case is for $m = n$, and it is as follows

$$DH(u_{pq}) = \begin{pmatrix} h_u(\mu_{pq}(0)) & h_v(\mu_{pq}(0)) & 0 & \cdots & 0 & 0 \\ 0 & h_u(\mu_{pq}(1)) & h_v(\mu_{pq}(1)) & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & h_u(\mu_{pq}(n)) + h_v(\mu_{pq}(n))\psi'(u_{pq}(n)) \end{pmatrix}, \tag{3.18}$$

It follows from (3.14) and (3.18) that for arbitrary $1 \le q \le l_p - 1$,

$$\det DH(u_{pq}) = \prod_{b=0}^{n-1} h_u(\mu_{pq}(b))[h_u(\mu_{pq}(n)) + h_v(\mu_{pq}(n))\psi'(u_{pq}(n))] \neq 0. \tag{3.19}$$

Through an analogous discussion as in Theorem 3.1, we also obtain $\det DH(u') \neq 0$ for arbitrary $u' \in N_{n+1}(w_p^*, s_p)$. Hence, by this result and (3.17), (3.19), we get $\det DH(u_0) \neq 0$ for arbitrary u_0 on χ.

Hence, the conditions in Corollary 4.2 of [26] are met. Then, Eq (1.1) with (1.2) possesses Li-Yorke chaos and Devaney chaos. □

The following result may be shown with the same method as Theorem 3.3 by using Theorem 4.2 of [26]. So, we also omit the proof and only state it as follows.

Theorem 3.4. *If conditions* (i) *and* (ii) *of Theorem 3.3 hold, meanwhile,*

(iv) *for arbitrary p, $1 \le p \le e$ and for arbitrary q, $1 \le q \le l_p - 1$ as $l_p \ge 2$, there is a constant $\delta_{pq} > 0$ such that h has continuous differentiability in $\bar{N}_2((u_{pq}(b), u_{pq}(b+1)), \delta_{pq})$, ψ has continuous differentiability in $\bar{N}_1(u_{pq}(b), \delta_{pq})$ for $0 \le b \le n$, $u_{pq}(n+1) = \psi(u_{pq}(m))$, and their derivatives satisfy one of the following conditions.*

(iva) *When $m = 0$,*

$$\prod_{b=0}^{n} h_u(u(b), u(b+1)) + (-1)^n \psi'(u(0)) \prod_{b=0}^{n} h_v(u(b), u(b+1)) \neq 0$$

keeps a same sign for arbitrary $u' = \{u(b)\}_{b=0}^{n}$ which satisfies $(u(b), u(b+1)) \in \bar{N}_2((u_{pq}(b), u_{pq}(b+1)), \delta_{pq}) \setminus (u_{pq}(b), u_{pq}(b+1))$ for $0 \le b \le n$. At the same time, there are at minimum n elements which are not equal to zero and lie either in $\{h_u(u_{pq}(b), u_{pq}(b+1)) : 0 \le b \le n\}$ or in

$$\{h_v(u_{pq}(b), u_{pq}(b+1)) \text{ for } 0 \le b \le n-1, h_v(u_{pq}(n), \psi(u_{pq}(0)))\psi'(u_{pq}(0))\},$$

where $u(n+1) = \psi(u(0))$, $u_{pq}(n+1) = \psi(u_{pq}(0))$;

(ivb) *When $1 \le m \le n-1$,*

$$\prod_{b=0}^{n} h_u(u(b), u(b+1))$$
$$+(-1)^{n+m}\psi'(u(m)) \prod_{b=0}^{m-1} h_u(u(b), u(b+1)) \prod_{b=m}^{n} h_v(u(b), u(b+1)) \neq 0$$

keeps a same sign for arbitrary $u' = \{u(b)\}_{b=0}^{n}$ *which satisfies* $(u(b), u(b+1)) \in \bar{N}_2((u_{pq}(b), u_{pq}(b+1)), \delta_{pq}) \setminus (u_{pq}(b), u_{pq}(b+1))$ *for* $0 \le b \le n$*, either* $h_u(u_{pq}(b), u_{pq}(b+1)) \neq 0$ *for* $0 \le b \le n-1$ *or* $h_v(u_{pq}(b), u_{pq}(b+1)) \neq 0$ *for* $0 \le b \le n-1$*, where* $u(n+1) = \psi(u(m))$*,* $u_{pq}(n+1) = \psi(u_{pq}(m))$*;*

(ivc) *When* $m = n$*,*

$$\prod_{b=0}^{n-1} h_u(u(b), u(b+1))[h_u(u(n), \psi(u(n))) + h_v(u(n), \psi(u(n)))\psi'(u(n))] \neq 0$$

keeps a same sign for arbitrary $u' = \{u(b)\}_{b=0}^{n}$ *which satisfies* $(u(b), u(b+1)) \in \bar{N}_2((u_{pq}(b), u_{pq}(b+1)), \delta_{pq}) \setminus (u_{pq}(b), u_{pq}(b+1))$ *for* $0 \le b \le n$*. At the same time, and either there are at minimum n elements which are not equal to zero lying in*

$$\{h_u(u_{pq}(b), u_{pq}(b+1)) \text{ for } 0 \le b \le n-1,$$

$$h_u(u_{pq}(n), \psi(u_{pq}(n))) + h_v(u_{pq}(n), \psi(u_{pq}(n)))\psi'(u_{pq}(n))\}$$

or $h_v(u_{pq}(b), u_{pq}(b+1)) \neq 0$ *for* $0 \le b \le n-1$*, where* $u(n+1) = \psi(u(n))$*,* $u_{pq}(n+1) = \psi(u_{pq}(n))$*;*

Then, Eq (1.1) with (1.2) has Wiggins chaos and Li-Yorke chaos.

At last of this part, two examples are introduced to exhibit usefulness for theorems above about Eq (1.1) having one non-periodic boundary condition (1.2).

Example 3.1. Think Eq (1.1) with (1.2). At this point, let $m = n = 1$,

$$h(u, v) = \begin{cases} 5u - v, & \text{if } (u, v) \in [-1, 1]^2 \\ \sin[(u - a_1)^3 (v - a_2^3)^2 + (u - a_2)(v - a_2)^2], & \text{else,} \end{cases}$$

where $a_1 = \frac{376}{125}$, $a_2 = \frac{7780876}{1953125}$, and

$$\psi(u) = u^3, \quad u \in \mathbf{R}.$$

Theorem 3.2 is used to study this example. Clearly, $h(0, 0) = 0$, $\psi(0) = 0$, h has continuous differentiability in $[-1, 1]^2$, $\psi : [-1, 1] \to [-1, 1]$ has continuous differentiability, and

$$\min\{||h_u(u, v)| - |h_v(u, v)||,\ ||h_u(u, v)| - |h_v(u, v)\psi'(t)||\} = 2 > 1,\ \forall (u, v) \in [-1, 1]^2,\ \forall t \in [-1, 1].$$

Then, condition (i) in Theorem 3.2 holds with $w_0 = 0$, $s = 1$ and $\sigma = 2$.

Let $w^* = (w_0, w_0) = (0, 0)$ and $u_0 = (u_0(0), u_0(1)) = (\frac{1}{5}, \frac{1}{5})$. Then, by (2.2) and (3.5), we get the fixed point w^* of H is expansive. It is easily seen that $u_0 \in N_2(w^*, 1)$, and

$$\begin{aligned} u_1 &= (u_1(0), u_1(1)) = H(u_0) = (h(u_0(0), u_0(1)), h(u_0(1), u_0^3(1))) \\ &= (h(\tfrac{1}{5}, \tfrac{1}{5}), h(\tfrac{1}{5}, \tfrac{1}{125})) = (\tfrac{4}{5}, \tfrac{124}{125}) \in (-1, 1)^2, \\ u_2 &= (u_2(0), u_2(1)) = H(u_1) = (h(u_1(0), u_1(1)), h(u_1(1), u_1^3(1))) \\ &= (h(\tfrac{4}{5}, \tfrac{124}{125}), h(\tfrac{124}{125}, \tfrac{124^3}{125^3})) = (\tfrac{376}{125}, \tfrac{7780876}{1953125}) = (a_1, a_2) \notin [-1, 1]^2, \\ H(u_2) &= (h(u_2(0), u_2(1)), h(u_2(1), u_2^3(1))) = (h(a_1, a_2), h(a_2, a_2^3)) = (0, 0) = w^*. \end{aligned}$$

Consequently, $H^3(w_0) = w^*$ and condition (ii) of Theorem 3.2 agrees with $e = 3$ and $u_q(b)$ as given in the above for $0 \le b \le 1$, $0 \le q \le 2$.

Obviously, h has continuous differentiability near the points below:

$$\mu_1(0) = (u_1(0), u_1(1)) = (\tfrac{4}{5}, \tfrac{124}{125}) \in (-1,1)^2,$$

$$\mu_1(1) = (u_1(1), u_1(2)) = (u_1(1), u_1^3(1)) = (\tfrac{124}{125}, \tfrac{124^3}{125^3}) \in (-1,1)^2,$$

$$\mu_2(0) = (u_2(0), u_2(1)) = (\tfrac{376}{125}, \tfrac{7780876}{1953125}) = (a_1, a_2) \notin [-1,1]^2,$$

$$\mu_2(1) = (u_2(1), u_2(2)) = (u_2(1), u_2^3(1)) = (a_2, a_2^3) \notin [-1,1]^2.$$

For convenience, we can take a small positive constant 0.001 to make h has continuous differentiability in the following domains:

$$\bar{N}_2(\mu_1(0), 0.001) \subset (-1,1)^2, \quad \bar{N}_2(\mu_1(1), 0.001) \subset (-1,1)^2,$$

$$\bar{N}_2(\mu_2(0), 0.001) \not\subset [-1,1]^2, \quad \bar{N}_2(\mu_2(1), 0.001) \not\subset [-1,1]^2.$$

In addition, for arbitrary $(u, v) \in [-1,1]^2$, we get $h_u(u, v) \equiv 5$, $h_v(u, v) \equiv -1$. So, for arbitrary $u' = (u(0), u(1)) \in [-1,1]^2$, we have

$$\begin{aligned} \det DH(u') &= h_u(u(0), u(1))[h_u(u(1), \psi(u(1))) + h_v(u(1), \psi(u(1)))\psi'(u(1))] \\ &= 25 - 15u^2(1) \ge 10 > 0. \end{aligned}$$

For any point $(u, v) \notin [-1,1]^2$, we get that

$$h_u(u, v) = [3(u - a_1)^2(v - a_2^3)^2 + (v - a_2)^2] \cos[(u - a_1)^3(v - a_2^3)^2 + (u - a_2)(v - a_2)^2], \tag{3.20}$$

and

$$h_v(u, v) = [2(u - a_1)^3(v - a_2^3) + 2(u - a_2)(v - a_2)] \cos[(u - a_1)^3(v - a_2^3)^2 + (u - a_2)(v - a_2)^2]. \tag{3.21}$$

Hence, by (3.9), (3.20) and (3.21), we get

$$\begin{aligned} \det DH(u_1) &= h_u(\mu_1(0))[h_u(\mu_1(1)) + h_v(\mu_1(1))\psi'(u_1(1))] \\ &= h_u(\mu_1(0))[h_u(\mu_1(1)) + 3u_1^2(1)h_v(\mu_1(1))] \\ &= h_u(\tfrac{4}{5}, \tfrac{124}{125})[h_u(\tfrac{124}{125}, \tfrac{124^3}{125^3}) + 3 \times \tfrac{124^2}{125^2} \times h_v(\tfrac{124}{125}, \tfrac{124^3}{125^3})] \\ &= 5 \times [5 + 3 \times \tfrac{124^2}{125^2} \times (-1)] = \tfrac{31997}{3125} > 0. \end{aligned}$$

In addition, $h_u(u, v)$, $h_v(u, v)$ are continuous in $[-1,1]^2$. This together with the above inequality implies there is one small positive constant $\delta < 0.001$ to make $\det DH(u') > 0$ for any $u' = (u(0), u(1))$ in one small neighborhood of $u_1 = \mu_1(0)$ satisfying $(u(0), u(1)) \in \bar{N}_2(\mu_1(0), \delta) \subset \bar{N}_2(\mu_1(0), 0.001)$ and $(u(1), u^3(1)) \in \bar{N}_2(\mu_1(1), \delta) \subset \bar{N}_2(\mu_1(1), 0.001)$.

Again from (3.9), (3.20) and (3.21), we get that

$$\begin{aligned} \det DH(u_2) &= h_u(\mu_2(0))[h_u(\mu_2(1)) + h_v(\mu_2(1))\psi'(u_2(1))] \\ &= h_u(\mu_2(0))[h_u(\mu_2(1)) + 3u_2^2(1)h_v(\mu_2(1))] \\ &= h_u(a_1, a_2)[h_u(a_2, a_2^3) + 3a_2^2 h_v(a_2, a_2^3)] = 0, \end{aligned}$$

which implies that (3.4) in Theorem 3.1 cannot be satisfied. From the above equality, one gets the snap-back repeller w^* is degenerate, and consequently Theorem 3.1 cannot be applied to this example.

Suppose that $u' = (u(0), u(1))$ is an arbitrary point in a small neighborhood of $u_2 = \mu_2(0)$ satisfying $(u(0), u(1)) \in \bar{N}_2(\mu_2(0), 0.001)$ and $(u(1), u^3(1)) \in \bar{N}_2(\mu_2(1), 0.001)$. For simplicity, denote $\bar{u}_0 = u(0)$, $\bar{u}_1 = u(1)$ in the rest of the example. Then, similarly to obtain (3.9), we have that

$$\begin{aligned} \det DH(u') &= h_u(\bar{u}_0, \bar{u}_1)[h_u(\bar{u}_1, \psi(\bar{u}_1)) + h_v(\bar{u}_1, \psi(\bar{u}_1))\psi'(\bar{u}_1)] \\ &= h_u(\bar{u}_0, \bar{u}_1)[h_u(\bar{u}_1, \bar{u}_1^3)) + 3\bar{u}_1^2 h_v(\bar{u}_1, \bar{u}_1^3)] \qquad (3.22) \\ &= h_u(\bar{u}_0, \bar{u}_1) f_1(\bar{u}_1) f_2(\bar{u}_1), \end{aligned}$$

where

$$\begin{aligned} h_u(\bar{u}_0, \bar{u}_1) = &\ [3(\bar{u}_0 - a_1)^2(\bar{u}_1 - a_2^3)^2 + (\bar{u}_1 - a_2)^2] \\ &\times \cos[(\bar{u}_0 - a_1)^3(\bar{u}_1 - a_2^3)^2 + (\bar{u}_0 - a_2)(\bar{u}_1 - a_2)^2], \\ f_1(\bar{u}_1) = &\ \cos[(\bar{u}_1 - a_1)^3(\bar{u}_1^3 - a_2^3)^2 + (\bar{u}_1 - a_2)(\bar{u}_1^3 - a_2)^2], \\ f_2(\bar{u}_1) = &\ 3(\bar{u}_1 - a_1)^2(\bar{u}_1^3 - a_2^3)^2 + (\bar{u}_1^3 - a_2)^2 + 2(\bar{u}_1 - a_1)^3(\bar{u}_1^3 - a_2^3) + 2(\bar{u}_1 - a_2)(\bar{u}_1^3 - a_2). \end{aligned}$$

Since $(\bar{u}_0, \bar{u}_1) \in \bar{N}_2(\mu_2(0), 0.001)$ and $(\bar{u}_1, \bar{u}_1^3) \in \bar{N}_2(\mu_2(1), 0.001)$, we can suppose $\bar{u}_0 = a_1 + \Delta_0$, $\bar{u}_1 = a_2 + \Delta_1$ and $\bar{u}_1^3 = a_2^3 + \Delta_2$, such that

$$\max\{|\Delta_j| : 0 \le j \le 2\} \le 0.001. \qquad (3.23)$$

Then, from (3.23), we get

$$h_u(\bar{u}_0, \bar{u}_1) = [3\Delta_0^2(\Delta_1 + a_2 - a_2^3)^2 + \Delta_1^2]\cos[\Delta_0^3(\Delta_1 + a_2 - a_2^3)^2 + (\Delta_0 + a_1 - a_2)\Delta_1^2] > 0$$

for all $\Delta_0 \neq 0$ or $\Delta_1 \neq 0$, and $h_u(\bar{u}_0, \bar{u}_1) = h_u(a_1, a_2) = 0$ if and only if $\Delta_0 = \Delta_1 = 0$. In addition, from (3.23), we also have

$$f_1(\bar{u}_1) = \cos[(\Delta_1 + a_2 - a_1)^3\Delta_2^2 + \Delta_1(\Delta_2 + a_2^3 - a_2)^2] > 0,$$

and

$$f_2(\bar{u}_1) = 3(\Delta_1 + a_2 - a_1)^2\Delta_2^2 + (\Delta_2 + a_2^3 - a_2)^2 + 2(\Delta_1 + a_2 - a_1)^3\Delta_2 + 2\Delta_1(\Delta_2 + a_2^3 - a_2) > 0,$$

where the second term of $f_2(\bar{u}_1)$ is approximately equal to 3509.5 and is the main term to decide values of $f_2(\bar{u}_1)$, while the other terms are so small that can be relatively negligible.

Then, it follows from the above discussion and (3.22) that $\det DH(u') > 0$ for any $u' = (u(0), u(1))$ satisfying $(u(0), u(1)) \in \bar{N}_2(\mu_2(0), 0.001) \setminus \{\mu_2(0)\}$ and $(u(1), u^3(1)) \in \bar{N}_2(\mu_2(1), 0.001) \setminus \{\mu_2(1)\}$. Moreover, it follows from (3.20) and (3.21) that

$$\begin{aligned}
&\{h_u(u_2(0), u_2(1)), h_u(u_2(1), \psi(u_2(1))) + h_v(u_2(1), \psi(u_2(1)))\psi'(u_2(1))\} \\
&= \{h_u(u_2(0), u_2(1)), h_u(u_2(1), u_2^3(1)) + 3u_2^2(1) h_v(u_2(1), u_2^3(1))\} \\
&= \{h_u(a_1, a_2), h_u(a_2, a_2^3) + 3a_2^2 h_v(a_2, a_2^3)\} = \{0, (a_2^3 - a_2)^2\},
\end{aligned}$$

which has $n = 1$ nonzero element. Hence, condition (iv) in Theorem 3.2 holds. Then, the whole conditions in Theorem 3.2 are met. It follows that Eq (1.1) with (1.2) has Wiggins chaos and Li-Yorke chaos. One simulation is made to show complex behaviors of this example, see Figure 1.

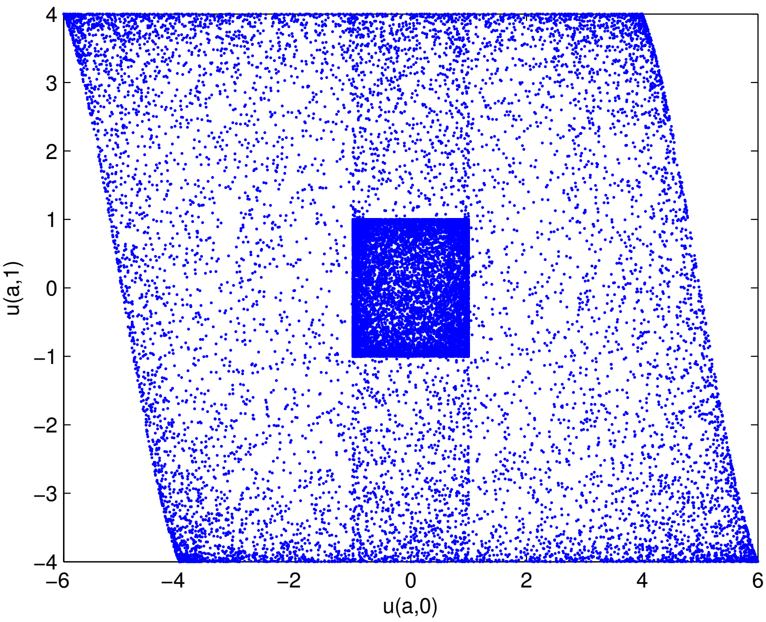

Figure 1. Behaviors in Example 3.1 for Eq (1.1) having (1.2), where $(u(0,0), u(0,1)) = (0.2, 0.3)$, $m = n = 1$, a is from 0 to 20000.

Example 3.2. Think Eq (1.1) with (1.2). At this point, let $m = n = 1$,

$$h(u, v) = \begin{cases} 5u - v, & \text{if } (u, v) \in [-1, 1]^2 \\ 4u - v - 6, & \text{if } (u, v) \in [2, 4]^2 \\ \sin[(u - 4.5)^3 (v - 11)^2 + (u - 5.5)(v - 5.5)^2], & \text{else,} \end{cases}$$

and

$$\psi(u)=\begin{cases} u^2, & \text{if } u\in[-1,1] \\ 6-u, & \text{if } u\in[2,4] \\ 2u, & \text{else,} \end{cases}$$

Theorem 3.4 is applied to study this example. It is easily seen that: $h(0,0)=0$, $h(3,3)=3$, $\psi(0)=0$, $\psi(3)=3$; h has continuous differentiability in mutually disjoint sets $[-1,1]^2$ and $[2,4]^2$, $\psi:[-1,1]\to[0,1]\subset[-1,1]$ and $\psi:[2,4]\to[2,4]$ are continuously differentiable, and

$$\min\{||h_u(u,v)|-|h_v(u,v)||,\ ||h_u(u,v)|-|h_v(u,v)\psi'(t)||\}=3>1,\ \forall(u,v)\in[-1,1]^2,\ \forall t\in[-1,1].$$

$$\min\{||h_u(u,v)|-|h_v(u,v)||,\ ||h_u(u,v)|-|h_v(u,v)\psi'(t)||\}=3>1,\ \forall(u,v)\in[2,4]^2,\ \forall t\in[2,4].$$

Hence, condition (i) in Theorem 3.4 holds for $w_1=0$, $w_2=3$, $s_1=s_2=1$ and $\sigma_1=\sigma_2=3$.

Let $w_1^*=(w_1,w_1)=(0,0)$, $w_2^*=(w_2,w_2)=(3,3)$. By (2.2) and (3.15), one gets the fixed points w_1^* and w_2^* of H are expansive. Take $u_{10}=(u_{10}(0),u_{10}(1))=(\frac{11-\sqrt{13}}{10},\frac{5-\sqrt{13}}{2})\in(-1,1)^2$ and $u_{20}=(u_{20}(0),u_{20}(1))=(3.5,3.5)\in(2,4)^2$. Then, it is easily seen the results below:

$$\begin{aligned} H(u_{10}) &= (h(u_{10}(0),u_{10}(1)),h(u_{10}(1),u_{10}(2)))=(h(u_{10}(0),u_{10}(1)),h(u_{10}(1),u_{10}^2(1))) \\ &= (h(\tfrac{11-\sqrt{13}}{10},\tfrac{5-\sqrt{13}}{2}),h(\tfrac{5-\sqrt{13}}{2},\tfrac{19-5\sqrt{13}}{2}))=(3,3)=w_2^*, \\ u_{21} &= (u_{21}(0),u_{21}(1))=H(u_{20})=(h(u_{20}(0),u_{20}(1)),h(u_{20}(1),6-u_{20}(1))) \\ &= (h(3.5,3.5),h(3.5,2.5))=(4.5,5.5)\notin[-1,1]^2\cup[2,4]^2, \\ H(u_{21}) &= (h(u_{21}(0),u_{21}(1)),h(u_{21}(1),u_{21}(2)))=(h(u_{21}(0),u_{21}(1)),h(u_{21}(1),2u_{21}(1))) \\ &= (h(4.5,5.5),h(5.5,11))=(0,0)=w_1^*. \end{aligned}$$

Consequently, $H(u_{10})=w_2^*$, $H^2(u_{20})=w_1^*$ and condition (ii) of Theorem 3.4 is satisfied for $l_1=1$, $l_2=2$ and $u_{pq}(b)$ as above. Then, the induced map H possesses one heteroclinic cycle χ connecting repellers w_1^* and w_2^*.

When $p=1$, we have $l_1=1$, so it does not need to show assumption (iv) of Theorem 3.4 for this case. When $p=2$, we have $l_2=2$ and need to verify assumption (iv) in Theorem 3.4. It is clear that h has continuous differentiability near the following points:

$$\begin{aligned} \mu_{21}(0) &= (u_{21}(0),u_{21}(1))=(4.5,5.5)\notin[-1,1]^2\cup[2,4]^2, \\ \mu_{21}(1) &= (u_{21}(1),u_{21}(2))=(u_{21}(1),2u_{21}(1)) \\ &= (5.5,11)\notin[-1,1]^2\cup[2,4]^2. \end{aligned}$$

For convenience, we take a small positive constant 0.001 to make h has continuous differentiability in the following domains: $\bar N_2(\mu_{21}(0),0.001)$ and $\bar N_2(\mu_{21}(1),0.001)$, which do not lie in $[-1,1]^2\cup[2,4]^2$. For any point $(u,v)\notin[-1,1]^2\cup[2,4]^2$, we have

$$\begin{aligned} h_u(u,v)= &\ [3(u-4.5)^2(v-11)^2+(v-5.5)^2] \\ &\times\cos[(u-4.5)^3(v-11)^2+(u-5.5)(v-5.5)^2], \end{aligned} \tag{3.24}$$

and

$$h_v(u,v) = [2(u-4.5)^3(v-11) + 2(u-5.5)(v-5.5)] \times \cos[(u-4.5)^3(v-11)^2 + (u-5.5)(v-5.5)^2]. \tag{3.25}$$

From (3.19), (3.24) and (3.25), we obtain

$$\begin{aligned}\det DH(u_{21}) &= h_u(\mu_{21}(0))[h_u(\mu_{21}(1)) + h_v(\mu_{21}(1))\psi'(u_{21}(1))]\\ &= h_u(\mu_{21}(0))[h_u(\mu_{21}(1)) + 2h_v(\mu_{21}(1))]\\ &= h_u(4.5,5.5)[h_u(5.5,11) + 2h_v(5.5,11)] = 0,\end{aligned}$$

which implies that (3.14) in Theorem 3.3 cannot be satisfied. So, χ is degenerate and Theorem 3.3 cannot be applied to this example.

Let $u' = (u(0), u(1))$ be an arbitrary point in a small neighborhood of $u_{21} = \mu_{21}(0)$ satisfying $(u(0), u(1)) \in \bar{N}_2(\mu_{21}(0), 0.001)$ and $(u(1), 2u(1)) \in \bar{N}_2(\mu_{21}(1), 0.001)$. For convenience, denote $\bar{u}_0 = u(0)$, $\bar{u}_1 = u(1)$ in the rest of the example. Then, similarly to obtain (3.19), one obtains

$$\begin{aligned}\det DH(u') &= h_u(\bar{u}_0,\bar{u}_1)[h_u(\bar{u}_1,\psi(\bar{u}_1)) + h_v(\bar{u}_1,\psi(\bar{u}_1))\psi'(\bar{u}_1)]\\ &= h_u(\bar{u}_0,\bar{u}_1)[h_u(\bar{u}_1,2\bar{u}_1)) + 2h_v(\bar{u}_1,2\bar{u}_1)]\\ &= h_u(\bar{u}_0,\bar{u}_1)f_1(\bar{u}_1)f_2(\bar{u}_1),\end{aligned} \tag{3.26}$$

where

$$\begin{aligned}h_u(\bar{u}_0,\bar{u}_1) &= [3(\bar{u}_0-4.5)^2(\bar{u}_1-11)^2 + (\bar{u}_1-5.5)^2]\\ &\quad\times\cos[(\bar{u}_0-4.5)^3(\bar{u}_1-11)^2 + (\bar{u}_0-5.5)(\bar{u}_1-5.5)^2],\\ f_1(\bar{u}_1) &= \cos[(\bar{u}_1-4.5)^3(2\bar{u}_1-11)^2 + (\bar{u}_1-5.5)(2\bar{u}_1-5.5)^2],\\ f_2(\bar{u}_1) &= 3(\bar{u}_1-4.5)^2(2\bar{u}_1-11)^2 + (2\bar{u}_1-5.5)^2\\ &\quad+4(\bar{u}_1-4.5)^3(2\bar{u}_1-11) + 4(\bar{u}_1-5.5)(\bar{u}_1-5.5).\end{aligned}$$

Since $(\bar{u}_0,\bar{u}_1) \in \bar{N}_2(\mu_{21}(0), 0.001)$ and $(\bar{u}_1, 2\bar{u}_1) \in \bar{N}_2(\mu_{21}(1), 0.001)$, we can suppose that $\bar{u}_0 = 4.5 + \Delta_0$, $\bar{u}_1 = 5.5 + \Delta_1$ such that

$$\max\{|\Delta_0|, 2|\Delta_1|\} \le 0.001. \tag{3.27}$$

Then, from (3.27), one has

$$h_u(\bar{u}_0,\bar{u}_1) = [3\Delta_0^2(\Delta_1-5.5)^2 + \Delta_1^2]\cos[\Delta_0^3(\Delta_1-5.5)^2 + (\Delta_0-1)\Delta_1^2] > 0$$

for all $\Delta_0 \neq 0$ or $\Delta_1 \neq 0$, and $h_u(\bar{u}_0,\bar{u}_1) = h_u(4.5, 5.5) = 0$ if and only if $\Delta_0 = \Delta_1 = 0$. In addition, from (3.27), we also have

$$f_1(\bar{u}_1) = \cos[4(\Delta_1+1)^3\Delta_1^2 + \Delta_1(2\Delta_1+5.5)^2] > 0,$$

and

$$f_2(\bar{u}_1) = 12(\Delta_1+1)^2\Delta_1^2 + (2\Delta_1+5.5)^2 + 8(\Delta_1+1)^3\Delta_1 + 4\Delta_1(2\Delta_1+5.5) > 0,$$

where the second term of $f_2(\bar{u}_1)$ is approximately equal to 30.2 and is the main term to decide values of $f_2(\bar{u}_1)$, while the other terms are so small that can be relatively negligible.

So, it follows from the above discussion and (3.26) that $\det DH(u') > 0$ for any $u' = (u(0), u(1))$ satisfying

$$(u(0), u(1)) \in \bar{N}_2(\mu_{21}(0), 0.001) \setminus \{\mu_{21}(0)\}, \quad (u(1), u^3(1)) \in \bar{N}_2(\mu_{21}(1), 0.001) \setminus \{\mu_{21}(1)\}.$$

Moreover, it follows from (3.24) and (3.25) that

$$\{h_u(u_{21}(0), u_{21}(1)), h_u(u_{21}(1), \psi(u_{21}(1))) + h_v(u_{21}(1), \psi(u_{21}(1)))\psi'(u_{21}(1))\}$$
$$= \{h_u(u_{21}(0), u_{21}(1)), h_u(u_{21}(1), 2u_{21}(1)) + 2h_v(u_{21}(1), 2u_{21}(1))\}$$
$$= \{h_u(4.5, 5.5), h_u(5.5, 11) + 2h_v(5.5, 11)\} = \{0, 30.25\},$$

which has $n = 1$ nonzero element. Then, assumption (iv) in Theorem 3.4 is met. Hence, the whole conditions in Theorem 3.4 hold, and Eq (1.1) and (1.2) have Wiggins chaos and Li-Yorke chaos. One simulation is made to show the complex behaviors of this example, see Figure 2.

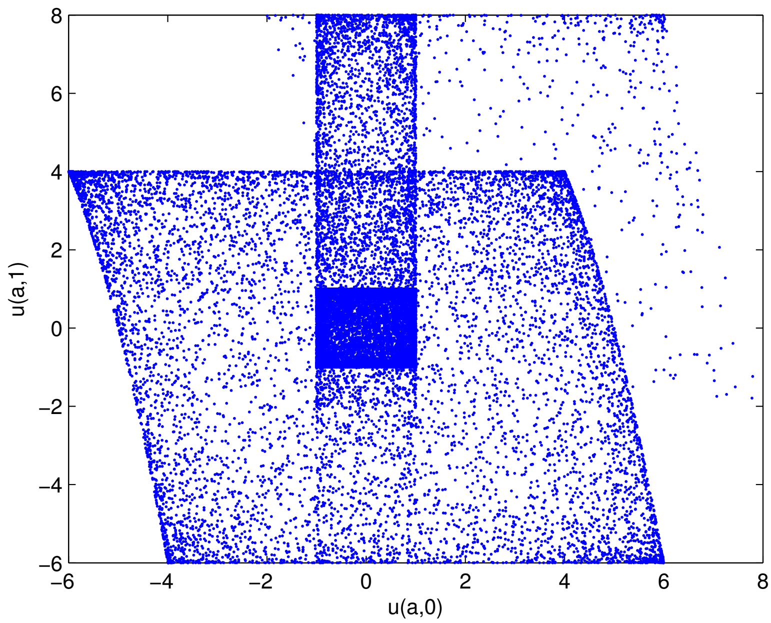

Figure 2. Behaviors in Example 3.2 for Eq (1.1) having (1.2), where $(u(0,0), u(0,1)) = (0.2, 0.3)$, $m = n = 1$, a is from 0 to 20000.

It is remarked that the criteria obtained in [5] and [6] cannot be applied to Examples 3.1 and 3.2 since the boundary conditions here are not periodic.

4. Chaotification schemes

The feedback control method was traditionally applied to study the stability, tracking and robustness of dynamical systems, see [34, 35] and references therein. However, Chen and Lai [36, 37] first used this method to study chaotification problems of dynamical systems. Here, we will apply this method and two kinds of repellers used in the former section to investigate chaotification for Eq (1.1) satisfying a non-periodic boundary condition condition (1.2). The ideas are motivated by [6, 8, 10], which had used the above theories to study chaotification schemes of Eq (1.1) with a periodic boundary condition.

Here, the sequence $\{\mu(a,b)\}$ is used as a controller to make the following controlled system

$$u(a+1,b) = h(u(a,b), u(a,b+1)) + \mu(a,b),\ a \geq 0,\ 0 \leq b \leq n \tag{4.1}$$

has Li-Yorke chaos and Devaney chaos, where the integer $n > 0$ is finite, $u(a, n+1) = \psi(u(a,m))$ for some integer m with $0 \leq m \leq n$. Here, one concrete form of controllers is taken as one of the followings:

$$\mu(a,b) = r(\alpha u(a,b)), \tag{4.2}$$

or

$$\mu(a,b) = \alpha r(u(a,b)), \tag{4.3}$$

where $\alpha > 0$ is the control argument, $r : E \subset \mathbf{R} \to \mathbf{R}$ is one map.

By preparation of Section 2, let $u_a = \{u(a,b)\}_{b=0}^{n} = (u(a,0), u(a,1), \cdots, u(a,n))^T \in V_{n+1}$ for $a \geq 0$, the controlled system (4.1) having controllers (4.2) or (4.3) can be transformed into the following forms, respectively,

$$u_{a+1} = H(u_a) + R(\alpha u_a), \tag{4.4}$$

or

$$u_{a+1} = H(u_a) + \alpha R(u_a), \tag{4.5}$$

where H is given in system (2.1), $R(u_a) = (r(u(a,0)), r(u(a,1)), \cdots, r(u(a,n)))^T$.

Now, we will first use snap-back repellers to achieve two chaotification schemes. Without loss of generality, the origin is taken as a fixed point. Otherwise, by a coordinate translation, a nonzero fixed point can become the origin.

Theorem 4.1. *Think of the controlled system (4.1) having a controller (4.2). If the conditions below hold,*

- (i) *there is one constant $s > 0$ to make h has continuous differentiability in $[-s,s]^2$ with $h(0,0) = 0$, and $\psi : [-s,s] \to [-s,s]$ has continuous differentiability in $[-s,s]$ with $\psi(0) = 0$;*
- (ii) *r meets the conditions below:*
  - (iia) *r has continuous differentiability in $[-s,s] \cup [c,d]$ satisfying $r'(u) \neq 0$ for arbitrary $u \in [-s,s] \cup [c,d]$, where $s < c < d$;*
  - (iib) *$r(0) = 0$, there is one $\eta \in (c,d)$ satisfying $r(\eta) = 0$;*

Then, for arbitrary constant α meeting the requirement

$$\alpha > \alpha_0 := \max\{\frac{d}{s}, \frac{Bs+d}{Ms}, \frac{Bd}{M(\eta - c)}, \frac{Bd}{M(d-\eta)}\}, \tag{4.6}$$

system (4.1) having (4.2) possesses Li-Yorke chaos and Devaney chaos, where

$$\begin{aligned} B := \ & \max\{|h_u(u,v)| + |h_v(u,v)|, |h_u(u,v)| \\ & +|h_v(u,v)\psi'(t)| : (u,v) \in [-s,s]^2,\ t \in [-s,s]\}, \end{aligned} \tag{4.7}$$

$$M := \min\{|r'(u)| : u \in [-s, s] \cup [c, d]\}. \tag{4.8}$$

Proof. We will use Theorem 2.1 of [8] to prove it. As stated in the preliminaries, H, R are induced maps for h, r, respectively, and (4.4) is an induced system for (4.1) having (4.2). Hence, it only needs to prove H, R meet the whole conditions of the theorem. To facilitate, set $\alpha > \alpha_0$ in the rest of the proof.

Firstly, we prove that H meets condition (i) of the theorem. Let $w^* = (0, 0, \cdots, 0)^T \in V_{n+1}$. By condition (i) and (2.2), H possesses one fixed point w^* and has continuous differentiability in $[-s, s]^{n+1}$. Since $0 \le m \le n$, the derivatives of H have two cases at a point $u' = (u(0), u(1), \cdots, u(n))^T \in (-s, s)^{n+1}$. One case is for $0 \le m \le n-1$, which is as follows

$$DH(u') = \begin{pmatrix} h_u(\mu(0)) & h_v(\mu(0)) & 0 & \cdots & 0 & \cdots & 0 \\ 0 & h_u(\mu(1)) & h_v(\mu(1)) & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & h_v(\mu(n))\psi'(u(m)) & \cdots & h_u(\mu(n)) \end{pmatrix}, \tag{4.9}$$

where $\mu(b) = (u(b), u(b+1))$ for $0 \le b \le n$, $u(n+1) = \psi(u(m))$, $h_v(\mu(n))\psi'(u(m))$ in the last line lies at the $(m+1)$-th column of the matrix. Then, it follows from (4.9) that for arbitrary fixed $u' \in (-s, s)^{n+1}$, for arbitrary $v' = (v(0), v(1), \cdots, v(n))^T \in (-s, s)^{n+1}$,

$$\begin{aligned} DH(u')v' = \ & (h_u(\mu(0))v(0) + h_v(\mu(0))v(1), h_u(\mu(1))v(1) + h_v(\mu(1))v(2), \\ & \cdots, h_u(\mu(n))v(n) + h_v(\mu(n))\psi'(u(m))v(m))^T, \end{aligned}$$

which together with (4.7) implies that for each $u' \in (-s, s)^{n+1}$,

$$\|DH(u')\|_{n+1} = \sup\{\|DH(u')v'\|_{n+1} : v' \in V_{n+1} \text{ with } \|v'\|_{n+1} = 1\} \le B. \tag{4.10}$$

The other case is for $m = n$, and it is

$$DH(u') = \begin{pmatrix} h_u(\mu(0)) & h_v(\mu(0)) & 0 & \cdots & 0 & 0 \\ 0 & h_u(\mu(1)) & h_v(\mu(1)) & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & h_u(\mu(n)) + h_v(\mu(n))\psi'(u(n)) \end{pmatrix}. \tag{4.11}$$

So, from (4.11), we can get that for each fixed $u' \in (-s, s)^{n+1}$ and for arbitrary $v' \in (-s, s)^{n+1}$,

$$\begin{aligned} DH(u')v' = \ & (h_u(\mu(0))v(0) + h_v(\mu(0))v(1), h_u(\mu(1))v(1) + h_v(\mu(1))v(2), \\ & \cdots, h_u(\mu(n))v(n) + h_v(\mu(n))\psi'(u(n))v(n))^T, \end{aligned}$$

which also together with (4.7) implies that for arbitrary $u' \in (-s, s)^{n+1}$, the inequality (4.10) still holds. Hence, the above two cases show that for each m, $0 \le m \le n$,

$$\|DH(u')\|_{n+1} \le B, \ \ \forall u' \in (-s, s)^{n+1},$$

which implies that condition (i) of that theorem is satisfied for H.

Secondly, we prove R meets condition (ii) of that theorem. By (iia), we obtain R has continuous differentiability in $[-s,s]^{n+1} \cup [c,d]^{n+1}$. Let $\eta^* = (\eta,\eta,\cdots,\eta)^T \in (c,d)^{n+1}$. Then, by condition (iib), we get $R(w^*) = R(\eta^*) = 0$. Because $r'(u) \neq 0$ for all $u \in [-s,s] \cup [c,d]$, we can obtain that for any $u' \in (-s,s)^{n+1} \cup (c,d)^{n+1}$,

$$DR(u') = diag\{r'(u(0)), r'(u(1)), \cdots, r'(u(n))\}, \tag{4.12}$$

is invertible. With a similar discussion to (4.10), we have $\|DR(u')\|_{n+1} \leq K$ for arbitrary $u' \in (-s,s)^{n+1} \cup (c,d)^{n+1}$, where

$$K = \max\{|r'(u)| : u \in [-s,s] \cup [c,d]\}.$$

So, $DR(u')$ is a bounded linear map for arbitrary $u' \in (-s,s)^{n+1} \cup (c,d)^{n+1}$. By (4.12), we have

$$(DR(u'))^{-1} = diag\{(r'(u(0)))^{-1}, (r'(u(1)))^{-1}, \cdots, (r'(u(n)))^{-1}\}, \tag{4.13}$$

which together with (4.8) implies that

$$\|(DR(u'))^{-1}\|_{n+1} \leq M^{-1}, \ \ \forall u' \in (-s,s)^{n+1} \cup (c,d)^{n+1}. \tag{4.14}$$

Then, from the above discussion and (4.14), we get $DR(u')$ is an invertible linear map for arbitrary $u' \in (-s,s)^{n+1} \cup (c,d)^{n+1}$. Moreover, from (4.8) and the mean value theorem, we have for arbitrary $u', v' \in [-s,s]^{n+1} \cup [c,d]^{n+1}$,

$$\begin{aligned} \|R(u') - R(v')\|_{n+1} &= \sup\{|r(u(b)) - r(v(b))| : 0 \leq b \leq n\} \\ &= \sup\{|r'(\beta(b))(u(b) - v(b))| : 0 \leq b \leq n\} \\ &\geq M\|u' - v'\|_{n+1}, \end{aligned}$$

where $\beta(b)$ falls in between $u(b)$ and $v(b)$. So, H meets condition (ii) of the theorem.

In a summary, H, R meet the conditions of Theorem 2.1 in [8]. Consequently, system (4.1) having (4.2) has Li-Yorke chaos and Devaney chaos for each $\alpha > \alpha_0$. □

The following result may be verified with the same method to Theorem 4.1 by using Theorem 2.2 of [8]. So, we omit its proof and only state it as follows.

Theorem 4.2. *Think the controlled system (4.1) having controller (4.3). If the conditions below hold,*

(i) *there is one constant $s > 0$ to make h has continuous differentiability in $[-s,s]^2$ with $h(0,0) = 0$, and $\psi : [-s,s] \to [-s,s]$ has continuous differentiability in $[-s,s]$ with $\psi(0) = 0$;*

(ii) *r meets the conditions below:*

  (iia) *r has continuous differentiability in $[-c,c] \cup [d,s]$ satisfying $r'(u) \neq 0$ for arbitrary $u \in [-c,c] \cup [d,s]$, where $0 < c < d < s$;*

  (iib) *$r(0) = 0$, there is one $\eta \in (d,s)$ satisfying $r(\eta) = 0$;*

Then, for arbitrary constant α meeting the requirement

$$\alpha > \alpha_0 := \max\{\frac{Bc+s}{Mc}, \ \frac{Bs}{M(\eta - d)}, \ \frac{Bs}{M(s-\eta)}\}, \tag{4.15}$$

system (4.1) having (4.3) possesses Devaney chaos and Li-Yorke chaos, where B is defined in (4.7), $M := \min\{|r'(u)| : u \in [-c,c] \cup [d,s]\}$.

Next, we will use one heteroclinic cycle connecting repellers to conclude one chaotification scheme for Eq (1.1) satisfying (1.2). As pointed out in [6, 9] there may be more invariant sets of chaos when using this method to chaotify a system than that only using a snap-back repeller. So, it will be very useful to use this method to establish some chaotification schemes in practical problems. When this method is used, there will be at least two fixed points and we can choose some or all of the fixed points to chaotify a system. For simplicity and without loss of generality, two arbitrary fixed points in Eq (1.1) are chosen to set up the chaotification scheme. With a similar method, one can set up similar chaotification schemes in the case of more than two fixed points.

Theorem 4.3. *Think the controlled system (4.1) having controller (4.3). If the conditions below hold,*

(i) *for $1 \leq p \leq 2$, there are constants $s_p > 0$ and points $w_p \in \mathbf{R}$ to make that $h(w_p, w_p) = w_p$, h has continuous differentiability in $[w_p - s_p, w_p + s_p]^2$ which are disjoint with each other, and $\psi : [w_p - s_p, w_p + s_p] \to [w_p - s_p, w_p + s_p]$ has continuous differentiability in $[w_p - s_p, w_p + s_p]$ with $\psi(w_p) = w_p$;*

(ii) *r meets the conditions below for $1 \leq p \leq 2$:*

(iia) *r has continuous differentiability in $[w_p - c_p, w_p + c_p] \cup [w_p + d_p, w_p + s_p]$ satisfying $r'(u) \neq 0$ for arbitrary $u \in [w_p - c_p, w_p + c_p] \cup [w_p + d_p, w_p + s_p]$, where $0 < c_p < d_p < s_p$;*

(iib) *$r(w_p) = 0$, there is one $\eta_p \in (w_p + d_p, w_p + s_p)$ satisfying $r(\eta_p) = 0$;*

Then, for arbitrary constant α meeting the requirement

$$\alpha > \alpha_0 := \max_{1 \leq q \leq 2}\{\frac{B_q s_q}{M_q(|\eta_q| - w_q - d_q)}, \frac{B_q s_q}{M_q(w_q + s_q - |\eta_q|)}, \frac{B_q c_q + |w_1 - w_2| + s_{\tau(q)}}{M_q c_q}\}, \tag{4.16}$$

system (4.1) having (4.3) possesses Li-Yorke chaos and Devaney chaos, where

$$\begin{aligned} B_q := \ & \max\{|h_u(u, v)| + |h_v(u, v)|, |h_u(u, v)| + |h_v(u, v)\psi'(t)| : \\ & (u, v) \in [w_q - s_q, w_q + s_q]^2, \ t \in [w_q - s_q, w_q + s_q]\}, \end{aligned} \tag{4.17}$$

$$M_q := \min\{|r'(u)| : u \in [w_q - c_q, w_q + c_q] \cup [w_q + d_q, w_q + s_q]\}. \tag{4.18}$$

Proof. We will use Theorem 4.1 of [6] to prove it. Obviously, system (4.5) is induced by (4.1) having (4.3). So, it only needs to verify the induced maps H, R meet the whole conditions of the theorem. For convenience, set $\alpha > \alpha_0$ and $p = 1$ or 2 within the residual proof.

Firstly, it will show H meets condition (i) of that theorem. Set $w_p^* = (w_p, w_p, \cdots, w_p)^T \in V_{n+1}$. Then, by (2.2) and condition (i), we obtain H has two fixed points w_p^*, and H has continuous differentiability in $[w_p - s_p, w_p + s_p]^{n+1}$ satisfying $[w_1 - s_1, w_1 + s_1]^{n+1} \cap [w_2 - s_2, w_2 + s_2]^{n+1} = \emptyset$. Since $0 \leq m \leq n$, the derivatives of H also have two cases at any point $u' = (u(0), u(1), \cdots, u(n))^T \in [w_p - s_p, w_p + s_p]^{n+1}$, which have the same forms as (4.9) and (4.11). So, we can get for arbitrary $u' \in (w_p - s_p, w_p + s_p)^{n+1}$ for arbitrary m, $0 \leq m \leq n$,

$$||DH(u')||_{n+1} = \sup\{||DH(u')v'||_{n+1} : v' \in V_{n+1} \text{ with } ||v'||_{n+1} = 1\} \leq B_p,$$

where B_p is as defined in (4.17). Hence, condition (i) in the theorem holds.

Secondly, it will show R meets condition (ii) of that theorem. By (iia), we have R has continuous differentiability in $[w_p - c_p, w_p + c_p]^{n+1} \cup [w_p + d_p, w_p + s_p]^{n+1}$. Take $\eta_p^* = (\eta_p, \eta_p, \cdots, \eta_p)^T \in (w_p + d_p, w_p + s_p)^{n+1}$. Then, by (iib), we get $R(w_p^*) = R(\eta_p^*) = 0$. Since $r'(u) \neq 0$ for arbitrary $u \in [w_p - c_p, w_p + c_p] \cup [w_p + d_p, w_p + s_p]$, we can obtain for arbitrary $u' \in (w_p - c_p, w_p + c_p)^{n+1} \cup (w_p + d_p, w_p + s_p)^{n+1}$,

$$DR(u') = diag\{r'(u(0)), r'(u(1)), \cdots, r'(u(n))\}, \tag{4.19}$$

becomes invertible and satisfies that $\|DR(u')\|_{n+1} \leq K_p$, where

$$K_p = \max\{|r'(u)| : u \in [w_p - c_p, w_p + c_p] \cup [w_p + d_p, w_p + s_p]\}.$$

Hence, for arbitrary $u' \in (w_p - c_p, w_p + c_p)^{n+1} \cup (w_p + d_p, w_p + s_p)^{n+1}$, $DR(u')$ is a bounded linear map. By (4.19), for arbitrary $u' \in (w_p - c_p, w_p + c_p)^{n+1} \cup (w_p + d_p, w_p + s_p)^{n+1}$, the inverse of $DR(u')$ exists and satisfies

$$\|(DR(u'))^{-1}\|_{n+1} = \sup\{\|(DR(u'))^{-1}v'\|_{n+1} : v' \in V_{n+1} \text{ with } \|v'\|_{n+1} = 1\} \leq M_p^{-1},$$

where M_p is as defined in (4.18). So, for arbitrary $u' \in (w_p - c_p, w_p + c_p)^{n+1} \cup (w_p + d_p, w_p + s_p)^{n+1}$, $DR(u')$ is an invertible linear map. In addition, for arbitrary $u', v' \in [w_p - c_p, w_p + c_p]^{n+1} \cup [w_p + d_p, w_p + s_p]^{n+1}$, by (4.18) and the mean value theorem, one obtains

$$\begin{aligned} \|R(u') - R(v')\|_{n+1} &= \sup\{|r(u(b)) - r(v(b))| : 0 \leq b \leq n\} \\ &= \sup\{|r'(\beta(b))(u(b) - v(b))| : 0 \leq b \leq n\} \\ &\geq M_p \|u' - v'\|_{n+1}, \end{aligned}$$

where $\beta(b)$ falls in between $u(b)$ and $v(b)$. So, R meets condition (ii) of the theorem.

Therefore, H, R meet the whole conditions of the theorem. Consequently, system (4.1) having (4.3) possesses Li-Yorke chaos and Devaney chaos for each $\alpha > \alpha_0$. □

Remark 4.1. Here, we remark the map h of Theorems 4.1, 4.2 and 4.3 could be chosen to possess a greatly brief form which is similar to that chosen in Remark 4.2 of [6]. So, this type of controller can be easily used in practice.

At the end of this part, we introduce two examples to exhibit usefulness of these chaotification schemes. While the chaotification schemes obtained in [6] and [8] cannot be applied to the examples.

The following example is for the controlled system (4.1) having a controller (4.2) or (4.3) by using Theorems 4.1 or 4.2.

Example 4.1. Think Eq (1.1) with (1.2). At this point, let $n = 2$, $m = 1$,

$$h(u, v) = \frac{1}{6}u + \frac{1}{6}v, \quad (u, v) \in \mathbf{R},$$

and

$$\psi(u) = u^2, \quad u \in \mathbf{R}.$$

Obviously, in $[-1,1]^2$, h has continuous differentiability satisfying $h(0,0) = 0$; $\psi : [-1,1] \to [0,1] \subset [-1,1]$ is continuously differentiable and satisfies $\psi(0) = 0$. So, by (2.2), we get the induced map H of h has a fixed point $W^* = (0,0)$. By (4.9), the derivative of H at W^* is as follows

$$DH(W^*) = \begin{pmatrix} h_u(0,0) & h_v(0,0) & 0 \\ 0 & h_u(0,0) & h_v(0,0) \\ 0 & h_v(0,0)\psi'(0) & h_u(0,0) \end{pmatrix} = \begin{pmatrix} \frac{1}{6} & \frac{1}{6} & 0 \\ 0 & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & \frac{1}{6} \end{pmatrix},$$

which implies that W^* is asymptotically stable. Then Eq (1.1) with (1.2) possesses oversimplified dynamics.

Clearly, condition (i) of Theorem 4.1 or Theorem 4.2 holds with $s = 1$. We first use Theorem 4.1 to chaotify Eq (1.1) with (1.2). The controller in the form (4.2) is

$$r(u) = \begin{cases} 4u, & \text{if } u \in [-1,1] \\ 5u - 20, & \text{if } u \in [3,5] \\ \frac{1}{6}\sin u, & \text{else.} \end{cases}$$

Obviously, r is continuously differentiable in $[-1,1] \cup [3,5]$ satisfying $r(0) = r(4) = 0$, and

$$M = \min\{|r'(u)| : u \in [-1,1] \cup [3,5]\} = 4.$$

So, $c = 3$, $d = 5$ and $\eta = 4$ in condition (ii) of the theorem. Furthermore, from (4.7), we have

$$\begin{aligned} B &= \max\{|h_u(u,v)| + |h_v(u,v)|, |h_u(u,v)| + |h_v(u,v)\psi'(t)| : \\ &\quad (u,v) \in [-1,1]^2,\ t \in [-1,1]\} \\ &= \max\{\tfrac{1}{3},\ \tfrac{1}{6} + |\tfrac{1}{3}t| :\ t \in [-1,1]\} = \tfrac{1}{2}. \end{aligned}$$

So, the conditions in Theorem 4.1 hold. Consequently, we get $\alpha_0 = 5$ satisfying (4.6) to make the system (4.1) having (4.2) possesses Li-Yorke chaos and Devaney chaos for arbitrary $\alpha > \alpha_0 = 5$. Here, we take $\alpha = 5.1$ for computer simulation. Figure 3 shows complicated behaviors of the controlled (4.1) having controller (4.2).

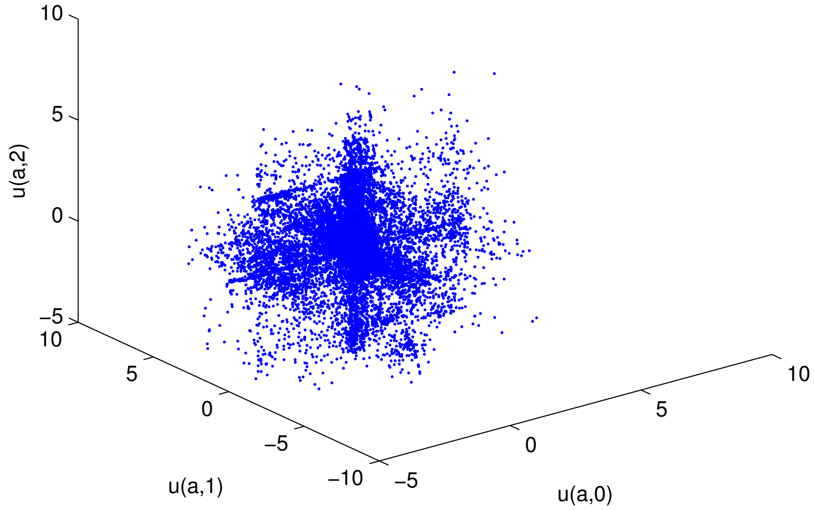

Figure 3. Behaviors in Example 4.1 for Eq (4.1) having (4.2), where $(u(0,0), u(0,1), u(0,2)) = (0.1, 0.1, 0.1)$, $m = 1$, $n = 2$, $\alpha = 5.1$, a is from 0 to 20000.

Next, we will use Theorem 4.2 to chaotify Eq (1.1) with (1.2). The controller in the form (4.3) is

$$r(u) = \begin{cases} 4u, & \text{if } u \in [-\frac{1}{4}, \frac{1}{4}] \\ 5u - 3, & \text{if } u \in [\frac{1}{3}, 1] \\ \frac{1}{6}\sin u, & \text{else.} \end{cases}$$

Obviously, r has continuous differentiability in $[-\frac{1}{4}, \frac{1}{4}] \cup [\frac{1}{3}, 1]$ satisfying $r(0) = r(\frac{3}{5}) = 0$, and

$$M = \min\{|r'(u)| : u \in [-\frac{1}{4}, \frac{1}{4}] \cup [\frac{1}{3}, 1]\} = 4.$$

Then, $c = \frac{1}{4}$, $d = \frac{1}{3}$ and $\eta = \frac{3}{5}$ in assumption (ii) of Theorem 4.2. Similarly, we also have $B = \frac{1}{2}$ from (4.7). So, the whole conditions of Theorem 4.2 hold. Consequently, we get $\alpha_0 = 1.125$ satisfying (4.15) to make system (4.1) having (4.3) possesses Li-Yorke chaos and Devaney chaos for arbitrary $\alpha > \alpha_0 = 1.125$. Here, we take $\alpha = 1.2$ to simulate. Figure 4 shows complicated behaviors of the controlled (4.1) having controller (4.3).

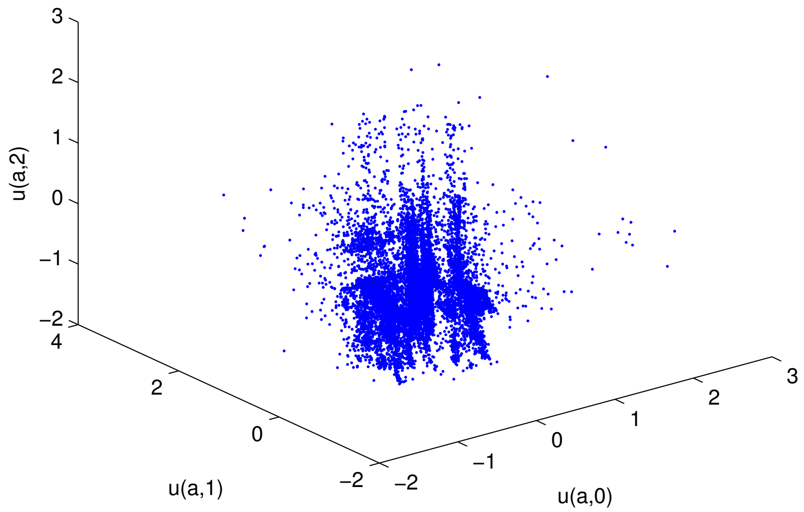

Figure 4. Behaviors in Example 4.1 for Eq (4.1) having (4.3), where $(u(0,0), u(0,1), u(0,2)) = (0.1, 0.1, 0.1)$, $m = 1$, $n = 2$, $\alpha = 1.2$, a is from 0 to 20000.

The following example is for the controlled system (4.1) having controller (4.3) by using Theorem 4.3.

Example 4.2. Think Eq (1.1) with (1.2). At this point, let $n = 2$, $m = 1$,

$$h(u,v) = \begin{cases} \frac{1}{4}u + \frac{1}{4}v, & \text{if } (u,v) \in [-1,1]^2 \\ \frac{1}{5}u + \frac{2}{5}v + 2, & \text{if } (u,v) \in [4,6]^2 \\ \frac{1}{3}\sin(u+v), & \text{else.} \end{cases}$$

and

$$\psi(u) = \begin{cases} u^3, & \text{if } u \in [-1,1] \\ u, & \text{if } u \in [4,6] \\ 3u, & \text{else.} \end{cases}$$

Clearly, h is continuously differentiable in $[-1,1]^2$ and $[4,6]^2$ satisfying $h(0,0) = 0$, $h(5,5) = 5$; $\psi : [-1,1] \to [-1,1]$ and $\psi : [4,6] \to [4,6]$ are continuously differentiable satisfying $\psi(0) = 0$, $\psi(5) = 5$. So, condition (i) of Theorem 4.3 agrees with $w_1 = 0$, $w_2 = 5$ and $s_1 = s_2 = 1$.

Obviously, the induced map H has fixed points $w_1^* = (0,0)$ and $w_2^* = (5,5)$. It follows from (4.9) that the derivatives of H at w_1^* and w_2^* are as follows

$$DH(w_1^*) = \begin{pmatrix} h_u(0,0) & h_v(0,0) & 0 \\ 0 & h_u(0,0) & h_v(0,0) \\ 0 & h_v(0,0)\psi'(0) & h_u(0,0) \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} \end{pmatrix}, \tag{4.20}$$

and

$$DH(w_2^*) = \begin{pmatrix} h_u(5,5) & h_v(5,5) & 0 \\ 0 & h_u(5,5) & h_v(5,5) \\ 0 & h_v(5,5)\psi'(5) & h_u(5,5) \end{pmatrix} = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} & 0 \\ 0 & \frac{1}{5} & \frac{2}{5} \\ 0 & \frac{2}{5} & \frac{1}{5} \end{pmatrix}. \tag{4.21}$$

It is easy to calculate that the absolute value of each eigenvalue of (4.20) or (4.21) is less than 1. So, w_1^* and w_2^* are asymptotically stable. Then Eq (1.1) with (1.2) possesses oversimplified dynamics.

The controller in the form (4.3) is taken as the following

$$r(u) = \begin{cases} 2u, & \text{if } u \in [-\frac{1}{5}, \frac{1}{5}] \\ 3u-2, & \text{if } u \in [\frac{1}{4}, 1] \\ 2u-10, & \text{if } u \in [\frac{14}{3}, \frac{16}{3}] \\ u - \frac{17}{3}, & \text{if } u \in [\frac{11}{2}, 6] \\ \frac{1}{6}\sin u, & \text{else.} \end{cases}$$

It is clear that r is continuously differentiable in $[-\frac{1}{5}, \frac{1}{5}] \cup [\frac{1}{4}, 1] \cup [\frac{14}{3}, \frac{16}{3}] \cup [\frac{11}{2}, 6]$ satisfying $r(0) = r(5) = r(\frac{2}{3}) = r(\frac{17}{3}) = 0$,

$$M_1 = \min\{|r'(u)| : u \in [-\frac{1}{5}, \frac{1}{5}] \cup [\frac{1}{4}, 1]\} = 2,$$

and

$$M_2 = \min\{|r'(u)| : u \in [\frac{14}{3}, \frac{16}{3}] \cup [\frac{11}{2}, 6]\} = 1.$$

Then, condition (ii) of Theorem 4.3 agrees with $c_1 = \frac{1}{5}$, $d_1 = \frac{1}{4}$, $\eta_1 = \frac{2}{3}$, $c_2 = \frac{1}{3}$, $d_2 = \frac{1}{2}$, $\eta_2 = \frac{17}{3}$. In addition, it follows from (4.17) that

$$\begin{aligned} B_1 &= \max\{|h_u(u,v)| + |h_v(u,v)|, |h_u(u,v)| + |h_v(u,v)\psi'(t)| : \\ &\quad (u,v) \in [-1,1]^2,\ t \in [-1,1]\} \\ &= \max\{\tfrac{1}{2},\ \tfrac{1}{4} + |\tfrac{3}{4}t^2| :\ t \in [-1,1]\} = 1, \end{aligned}$$

and

$$\begin{aligned} B_2 &= \max\{|h_u(u,v)| + |h_v(u,v)|, |h_u(u,v)| + |h_v(u,v)\psi'(t)| : \\ &\quad (u,v) \in [4,6]^2,\ t \in [4,6]\} = \tfrac{3}{5}. \end{aligned}$$

So, the whole conditions in Theorem 4.3 are met. Consequently, we get $\alpha_0 = 18.6$ satisfying (4.16) to make system (4.1) having (4.3) possesses Li-Yorke chaos and Devaney chaos for arbitrary $\alpha > \alpha_0 =$

18.6. Here, we take $\alpha = 18.7$ for simulation. Figure 5 shows complicated behaviors of the controlled (4.1) having controller (4.3).

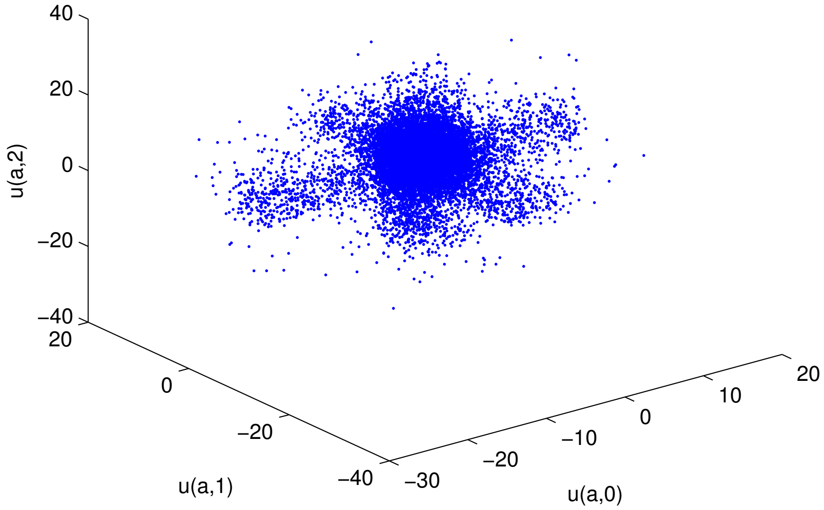

Figure 5. Behaviors in Example 4.2 for Eq (4.1) having (4.3), where $(u(0,0), u(0,1), u(0,2)) = (0.1, 0.1, 0.1)$, $m = 1$, $n = 2$, $\alpha = 18.7$, a is from 0 to 20000.

5. Conculsions

This article investigates chaos criteria and chaotification schemes on one kind of first-order partial difference equations, which have non-periodic boundary conditions. The results are achieved by constructing two kinds of repellers. Firstly, four chaos criteria are gained to ensure this kind of partial difference equations to have snap-back repellers or heteroclinic cycles connecting repellers. These repellers may be regular and nondegenerate or only regular. These maps in the theorems are showed to possess both Li-Yorke chaos and Devaney chaos or both Wiggins chaos and Li-Yorke chaos. Two simulation examples are introduced to exhibit the usefulness of our theorems. Those chaos criteria generalize the existing literature on these kind of equations having periodic boundary conditions to more general situations, which will be more practical in applications. Secondly, three chaotification schemes are gained by constructing two kinds of repellers. Those controllers may be devised to possess greatly brief forms. To exhibit the usefulness of our results, two simulation examples are also provided. Those chaotification schemes also generalize the existing literature on these kind of equations having periodic boundary conditions to more general situations, which will also be more practical in applications. In a word, the results on chaos criteria and chaotification schemes obtained in this paper generalize the existing results and will provide an important theoretical basis for studying chaos problems for these kind of equations. However, the boundary conditions discussed in this paper are only concerned with one-variable functions. What are the cases for functions with multiple variables? It is worth studying and will be our further study.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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