Mathematical Biosciences and Engineering

http://www.aimspress.com/journal/mbe

MBE, 20(2): 2628–2650.
DOI: 10.3934/mbe.2023123
Received: 18 October 2022
Revised: 12 November 2022
Accepted: 16 November 2022
Published: 25 November 2022

Research article

Observer-based adaptive fuzzy output feedback control for functional constraint systems with dead-zone input

Tianqi Yu, Lei Liu*and Yan-Jun Liu

College of Science, Liaoning University of Technology, Jinzhou 121001, China

* **Correspondence:** Email: liuleill@live.cn; Tel: +8618840176734.

Abstract: This paper develops an adaptive output feedback control for a class of functional constraint systems with unmeasurable states and unknown dead zone input. The constraint is a series of functions closely linked to state variables and time, which is not achieved in current research results and is more general in practical systems. Furthermore, a fuzzy approximator based adaptive backstepping algorithm is designed and an adaptive state observer with time-varying functional constraints (TFC) is constructed to estimate the unmeasurable states of the control system. Relying on the relevant knowledge of dead zone slopes, the issue of non-smooth dead-zone input is successfully solved. The time-varying integral barrier Lyapunov functions (iBLFs) are employed to guarantee that the states of the system remain within the constraint interval. By Lyapunov stability theory, the adopted control approach can ensure the stability of the system. Finally, the feasibility of the considered method is conformed via a simulation experiment.

Keywords: fuzzy state observer; time-varying functional constraints (TFC); dead-zone input; adaptive fuzzy control; backstepping algorithm

1. Introduction

Over the past few decades, a lot of attention has been raised to handle the stability of nonlinear control systems [1, 2]. It is worth noting that adaptive control is favored by many scholars because of its ability to update adaptive parameters online. For dealing with the unknown nonlinear characteristics, fuzzy logic systems (FLSs) [3, 4] and neural networks (NNs) [5–7] are widely employed. To list a few, utilizing FLSs control approach, a state feedback adaptive fuzzy method is presented in [8]. The work [9] develops an adaptive fuzzy control scheme to overcome the actuator faults of stochastic nonlinear systems. Moreover, a fixed time tracking control is investigated in [10], where an adaptive fuzzy controller is devised via backstepping technique. In [11], a Lyapunov stability strategy is addressed based on event-triggered mechanism. Considering fault-tolerant control problem, a suitable neural controller combining with backstepping method is proposed in [12], which ensures the stability of the system in finite-time. Nevertheless, the mentioned adaptive control schemes don't take the constraint problem into consideration.

As the main factor affecting system performance, constraint problem always appears in most practical systems. Hence, it is a challenging task to construct a suitable controller to maintain the stability of such systems. The barrier Lyapunov functions (BLFs) and backstepping algorithm are selected to stop the signal of the system from exceeding the constraint compact set in [13–16]. According to fuzzy approximate approach and BLFs, the output constraints related to constants are developed in [17, 18]. In addition, full-state constant constraints are achieved in [19–21], where all signals are not transgressed the constraint boundary. Under the frame of NNs, a neural network control scheme with external disturbances and uncertainties is introduced in [22]. In particular, Zhao and Song [23] develop a unique approach (nonlinear state-dependent function) to achieve asymmetric state constraint, which completely removes the feasibility conditions that current BLFs exist. Subsequently, the time-varying constraints have attracted scholar's attention because of its generally. In [24], a neural approach is presented to prevent arms to move to the desired position. Furthermore, Liu et al. [25] address a backstepping feedback control strategy with uncertain parameters, preventing the constraint boundaries from being violated and achieving full state constraints. Differently, a unified barrier function (UBF) with time-varying state constraints is established in [26], where novel coordinate transformations are introduced into the backstepping technique. Remarkably, only a small number of scholars have devoted themselves to the study of complex functional constraints. As far as we know, this breakthrough is only completed in [27]. However, the aforementioned results are realized under the assumption that the system is in good working condition.

In practical systems, the non-smooth input characteristics such as hysteresis, dead zone, saturation signal, etc. are always inevitable, which can lead to system instability. It is emphasized that dead zone regarded as a significant input nonlinearity continually occurs in actual systems. Therefore, the performance of the system will also be greatly affected when dead-zone inputs exist in the system, which should not be ignored. To ensure tracking performance, an adaptive compensation algorithm subject to dead-zone characteristics is proposed in [28]. Considering continuous-time nonlinear dynamic systems, Wang et al. [29] employ an adaptive control scheme by relying on the method of establishing dead zone model. An adaptive asymptotic control is analyzed in [30] where the unknown dead-zone and event trigger input are considered simultaneously. For nonlinear discrete-time systems, a fuzzy approximation combining with backstepping algorithm is constructed in [31]. Especially, not only the above-mentioned nonlinear systems, but also the dead zone input has been introduced into the constraint control systems. To just name a few, a full-state constraint tracking control approach based adaptive backstepping technique is addressed in [32]. The stability of feedback control systems subjected to dead-zone is outlined in [33], while barrier Lyapunov-Krasovskii functional (LKF) is introduced to overcome time-delay terms. It is noteworthy that these dead-zone inputs are investigated under the condition of state constraints, ignoring the problem of immeasurable states.

In addition to the state measurable systems of the above-mentioned researches, there are still a number of states that cannot be directly obtained in many practical systems, which encourages scholars to construct state observers to estimate the unmeasurable states. In [34], a sliding-mode observer is addressed to cope with unmeasurable states of stochastic polynomial systems. According to the approximation of FLSs, various state observer control approaches are achieved in [35–37] via employing backstepping algorithm. Subsequently, the control strategy has been further developed to stabilize other nonlinear systems, such as discrete-time fuzzy systems [38] and input delay systems [39]. Yoo [40] proposes an output-feedback control scheme considering fault detection and accommodation, where a neural state observer is constructed. Under the framework of constraint control systems, a neural-based output constraints control [41] and an adaptive fuzzy observer with time-varying full state constraints (TFSC) [42] are developed. By relying on BLF, a fuzzy tracking control strategy about backlash-like hysteresis and TFSC is established in [43]. Liu et al. [44] present a constraint control of multi-input-multi-output systems, where the problem of unmeasurable states is well solved. Despite remarkable achievements have been made in nonlinear constrained control systems, the situation of unmeasurable states in functional constrained systems need to be further studied.

Inspired by aforementioned approaches, this paper addresses an output feedback control scheme with functional constraints and dead-zone input, where a state observer is constructed to estimate the unmeasurable states. The major contributions are summarized as follows.

(1) The time-varying functional constraints (TFC) are considered by adopting integral BLF. In particular, this paper specifically investigates the impact on system performance when the state variables and time exist simultaneously in the constraint boundary.

(2) Most studies tend to develop state measurable systems, but neglect the situation of state unmeasurable. In order to handle this issue, an adaptive fuzzy state observer combining with backstepping technique is presented in this paper. Currently, the output feedback control with functional constraints has not been developed.

(3) As a significant input nonlinearity affecting the stability of the system, dead-zone input is successfully solved in the controller design. Finally, an observer based adaptive backstepping algorithm with TFC and dead zone input is achieved in this paper.

The remainder of this paper is organized as follows. Some basic knowledge and system descriptions are elaborated in Section 2. In Section 3, a fuzzy state observer is constructed. The process of controller construction is provided in Section 4. Section 5 gives the simulation results. At last, Section 6 concludes the work of this paper.

2. System description and preliminaries

2.1. System descriptions

Take the following strict feedback nonlinear systems into consideration:

$$\begin{cases} \dot{x}_1 = f_1(x_1) + x_2, \\ \dot{x}_2 = f_2(X_2) + x_3, \\ \vdots \\ \dot{x}_i = f_i(X_i) + x_{i+1}, \\ \dot{x}_n = f_n(X) + u, \\ y = x_1 \end{cases} \tag{2.1}$$

where $X_i = [x_1, x_2, ..., x_i]^T$ denotes immeasurable state vectors with $i \geq 2$, $X = [x_1, x_2, ..., x_n]^T$ denotes the state variables, and $y \in R$ represents the system output. $f_i(X_i)$ stands for unknown nonlinear smooth functions. In addition, choose the known functional constraints $\xi_{c_i}(X_{i-1}, t)$, $(i = 1, 2, ..., n)$ with

$x_0 = y_r$, so that the states in this paper are constrained in predefined compact sets $\Delta_x = \{x_i | |x_i(t)| < \xi_{c_i}(X_{i-1}, t), \forall t \geq 0\}$, where $\xi_{c_i}(X_{i-1}, t)$ is a designable function. $u \in R$ denotes the input of the dead-zone, which is described as:

$$u(t) = D(\upsilon(t)) = \begin{cases} m_r(\upsilon(t) - k_r), & if \quad \upsilon(t) \geq k_r \\ 0, & if - k_l < \upsilon(t) < k_r \\ m_l(\upsilon(t) + k_l), & if \quad \upsilon(t) \leq -k_l \end{cases} \tag{2.2}$$

where $\upsilon(t)$ denotes the input of the dead zone, m_r, m_l represent right and left slopes, $m_r = m_l = m$. k_r, $k_l > 0$ are the break points. The dead zone Eq (2.2) can be expressed as:

$$D(\upsilon(t)) = m\upsilon(t) + k(t) \tag{2.3}$$

where

$$k(t) = \begin{cases} -mk_r, & \upsilon(t) \geq k_r \\ -m\upsilon(t), & -k_l < \upsilon(t) < k_r \\ \quad mk_l, & \upsilon(t) \geq k_r \end{cases}$$

with $\bar{k} = \max\{mk_r, mk_l\}$ is the upper bounded of $|k(t)|$.

Transforming system Eq (2.1) into the following state space form:

$$\begin{cases} \dot{X} = AX + \eta y + \sum_{i=1}^{n} B_i f_i(X_i) + \beta u \\ y = CX \end{cases} \tag{2.4}$$

where

$$A = \begin{bmatrix} -\eta_1 & 1 & \cdots & \cdots & 0 \\ -\eta_2 & 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -\eta_{n-1} & 0 & \cdots & \ddots & 1 \\ -\eta_n & 0 & \cdots & \cdots & 0 \end{bmatrix}_{n\times n}, \beta = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}_{n\times 1},$$

$\eta = [\eta_1, \eta_2, ..., \eta_n]^T$, $B_i = [\ 0 \ \ \ldots \ \ 1 \ \ \ldots \ \ 0\ ]^T$, $C = [\ 1 \ \ \ldots \ \ 0 \ \ \ldots \ \ 0\ ]_{1\times n}$, and vector η is selected such that A denotes a strict Hurwitz matrix. Thus, given a matrix $Q = Q^T > 0$, there exists a matrix $P = P^T > 0$ satisfying:

$$A^T P + PA = -2Q \tag{2.5}$$

Remark 1. A large number of achievements investigated nonlinear constraint systems whose boundary was a constant [19–23] or a time-varying function [24–26]. Differently, this paper takes functional constraints relying on state variables and time into account, which has not achieved in current research. In addition, the states of this system are unmeasurable, leading us to construct a fuzzy observer to estimate the former. The non-smooth input dead-zone is also considered in this paper, which is a challenging task to design a reasonable controller.

Control objective: The control objective is to develop an output feedback control strategy to achieve the following points: a) the output of this system can follow desired signal $y_r(t)$ and the constructed fuzzy state observer can estimate the unmeasurable states commendably; b) the functional constraints are never violated; c) all signals in the closed-loop system remain within bounds.

Assumption 1 [25]: There exist unknown constants $\Im_i^0$ and $\Im_i^q$ $(i = 1, ..., n,\ q = 1, ..., n)$ satisfying $\left|\xi_{c_i}(X_{i-1}, t)\right| \le \Im_i^0$ and $\left|\xi_{c_i}^{(q)}(X_{i-1}, t)\right| \le \Im_i^q$, where $\xi_{c_i}^{(q)}(X_{i-1}, t)$ denotes the *qth*-order derivative of $\xi_{c_i}(X_{i-1}, t),\ \forall t \ge 0$.

Assumption 2 [18]: For any functional constraints $\xi_{c_i}(X_{i-1}, t) > 0$, there exist positive constants C_0 and C_i such that the desired signal $y_r(t)$ and its *ith*-order derivative $y_r^{(i)}(t)$ satisfy $|y_r(t)| \le C_0 < \xi_{c_1}(y_r, t)$ and $\left|y_r^{(i)}(t)\right| \le C_i$.

Remark 2. To make this paper more rigorous, we introduced Assumptions 1 and 2. Assumption 1 indicates that the selected boundary function and its *qth* derivative are bounded. Obviously, it is more meaningful to construct an appropriate controller to maintain the states in a closed set. Assumption 2 guarantees the boundedness of the desired signal $y_r(t)$, which facilitates the theorem proving. Similar assumptions have also been introduced in existing researches [18,25].

Lemma 1 [14,15]: For $|x_i(t)| < \xi_{c_i}(X_{i-1}, t),\ i = 1, ..., n$, the function V_{z_i} satisfies the following inequality:

$$V_{z_i} \le z_i^2 \xi_{c_i}^2(X_{i-1}, t) \Big/ \left(\xi_{c_i}^2(X_{i-1}, t) - x_i^2\right).$$

2.2. FLSs

A fuzzy approximator is constructed to estimate uncertain nonlinear functions which exists in the function-constrained systems with unmeasurable states. The detailed characteristics are as follows.

Lemma 2: An unknown continuous function $f(x)$ defined on a compact set Δ satisfies the following inequality:

$$\sup_{x \in \Delta} \left| f(x) - \vartheta^T \phi(x) \right| \le \varepsilon \tag{2.6}$$

In this paper, the unknown continuous functions are described as:

$$f_i(X_i | \vartheta_i) = \vartheta_i^T \psi_i(X_i) \tag{2.7}$$

$$\hat{f}_i(\hat{X}_i \big| \vartheta_i) = \vartheta_i^T \psi_i(\hat{X}_i) \tag{2.8}$$

where $\hat{X}_i = [\hat{x}_1, \hat{x}_2, ..., \hat{x}_n]^T$ stands for the estimation of $X_i = [x_1, x_2, ..., x_n]^T$.

Define

$$\delta_i = f_i(X_i) - \hat{f}_i(\hat{X}_i \big| \vartheta_i^*) \tag{2.9}$$

$$\zeta_i = f_i(X_i) - \hat{f}_i(\hat{X}_i \big| \vartheta_i), i = 1, ..., n \tag{2.10}$$

where δ_i denotes the fuzzy minimum approximation error, ζ_i is the approximation error, and ϑ_i^* denotes the optional parameter vector. Moreover, there exist positive constants $\bar{\delta}_i$ and $\bar{\zeta}_i$, which satisfy $|\delta_i| \le \bar{\delta}_i$, $|\zeta_i| \le \bar{\zeta}_i, (i = 1, ..., n)$.

According to above analysis, select $\varpi_i = \delta_i - \zeta_i$, we can obtain $|\varpi_i| \le \bar{\varpi}_i$ with constant $\bar{\varpi}_i > 0$.

3. Observer-based adaptive controller construction and stability analysis

To handle the unmeasurable state problem, an adaptive observer is developed by combining back-stepping technique in this section. And the stability of this system is analyzed at the end.

Constructing the following fuzzy state observer:

$$\begin{cases} \dot{\hat{X}} = A\hat{X} + \eta y + \sum_{i=1}^{n} B_i \hat{f}_i(\hat{X}_i \big| \vartheta_i) + \beta u \\ \hat{y} = C\hat{X} \end{cases} \tag{3.1}$$

Let $\breve{X} = X - \hat{X} = [\breve{x}_1, \breve{x}_2, ..., \breve{x}_n]^T$ be the observer errors, based on Eqs (2.4) and (3.1), one obtains

$$\begin{aligned} \dot{\breve{X}} &= A\breve{X} + \sum_{i=1}^{n} B_i \left[f_i(X_i) - \hat{f}_i(\hat{X}_i \big| \vartheta_i) \right] \\ &= A\breve{X} + \zeta \end{aligned} \tag{3.2}$$

where $\zeta = [\zeta_1, \zeta_2, ..., \zeta_n]^T$.

Take the following Lyapunov function candidate into account:

$$V_0^X = \frac{1}{2}\breve{X}^T P \breve{X} \tag{3.3}$$

The time derivative of V_0^X along Eq (3.2) is given as

$$\dot{V}_0^X = \frac{1}{2}\dot{\breve{X}}^T P \breve{X} + \frac{1}{2}\breve{X}^T P \dot{\breve{X}} \tag{3.4}$$

Substituting Eq (2.5) into Eq (3.4) and combining Eq (3.2), one acquires

$$\begin{aligned} \dot{V}_0^X &= \frac{1}{2}\breve{X}^T (A^T P + PA)\breve{X} + \breve{X}^T P\xi \\ &= -\breve{X}^T Q\breve{X} + \breve{X}^T P\xi \\ &\le -\lambda_{\min}(Q)\left\|\breve{X}\right\|^2 + \breve{X}^T P\xi \end{aligned} \tag{3.5}$$

Utilizing the Young's inequality, we have

$$\breve{X}^T P\xi \le \frac{1}{2}\left\|\breve{X}\right\|^2 + \frac{1}{2}\|P\xi\|^2 \tag{3.6}$$

Then, the following inequality holds

$$\dot{V}_0^X \le -(\lambda_{\min}(Q) - \frac{1}{2})\left\|\breve{X}\right\|^2 + \frac{1}{2}\|P\xi\|^2 \tag{3.7}$$

To realize the control objectives, the following coordinate transformation are given:

$$z_1 = x_1 - y_r \tag{3.8}$$

$$z_i = \hat{x}_i - \alpha_{i-1}, i = 2, ..., n \tag{3.9}$$

where z_i represents the tracking error, and α_{i-1} denotes a virtual controller with $\alpha_0 = y_r$.

Selecting the following Lyapunov function candidate:

$$V_{z_i} = \int_0^{z_i} \frac{\gamma\xi_{c_i}^2\left(\hat{X}_{i-1}, t\right)}{\xi_{c_i}^2\left(\hat{X}_{i-1}, t\right) - (\gamma + \alpha_{i-1})^2} d\gamma \tag{3.10}$$

where V_{z_i} is positive definite and continuously differentiable. The state vectors x_1 and $\hat{X}_i, (i \geq 2)$ are confined to $|x_1| < \xi_{c_1}(y_r, t)$ and $\left|\hat{X}_i(t)\right| < \xi_{c_i}(\hat{X}_{i-1}, t)$, respectively.

Define $\gamma = \varepsilon z_i$, one gets

$$V_{z_i} = z_i^2 \int_0^1 \frac{\varepsilon\xi_{c_i}^2(\hat{X}_{i-1}, t)}{\xi_{c_i}^2(\hat{X}_{i-1}, t) - (\varepsilon z_i + \alpha_{i-1})^2} d\varepsilon \geq \frac{1}{2} z_i^2 \tag{3.11}$$

which is applied in stability analysis.

Step 1: Based on Eq (3.8), one acquires

$$\dot{z}_1 = \tilde{\vartheta}_1^T \psi_1(\hat{X}_1) + \vartheta_1^T \psi_1(\hat{X}_1) + \varpi_1 + z_2 + \alpha_1 + \breve{x}_2 - \dot{y}_r \tag{3.12}$$

in which ϑ_1 represents the estimation of ϑ_1^*, $\tilde{\vartheta}_1$ denotes the estimation error, and $\tilde{\vartheta}_1 = \vartheta_1^* - \vartheta_1$.

According to Eq (3.12), $\dot{V}_{z_1}$ is obtained as

$$\begin{aligned} \dot{V}_{z_1} =& \frac{z_1\xi_{c_1}^2(y_r, t)}{\xi_{c_1}^2(y_r, t) - (z_1 + y_r)^2}\dot{z}_1 + \dot{y}_r \int_0^{z_1} \frac{\partial}{\partial y_r} \frac{\gamma\xi_{c_1}^2(y_r, t)}{\xi_{c_1}^2(y_r, t) - (\gamma + y_r)^2} d\gamma \\ &+ \dot{\xi}_{c_1}(y_r, t) \int_0^{z_1} \frac{\partial}{\partial \xi_{c_1}(y_r, t)} \frac{\gamma\xi_{c_1}^2(y_r, t)}{\xi_{c_1}^2(y_r, t) - (\gamma + y_r)^2} d\gamma \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} &\int_0^{z_1} \frac{\partial}{\partial y_r} \frac{\gamma\xi_{c_1}^2(y_r, t)}{\xi_{c_1}^2(y_r, t) - (\gamma + y_r)^2} d\gamma \\ &= z_1 \left(\frac{\xi_{c_1}^2(y_r, t)}{\xi_{c_1}^2(y_r, t) - x_1^2} + \frac{\partial\xi_{c_1}(y_r, t)}{\partial y_r} M_1(\xi_{c_1}, y_r, z_1) - N_1(\xi_{c_1}, y_r, z_1) \right) \end{aligned}$$

with

$$\begin{aligned} M_1(\xi_{c_1}, y_r, z_1) =& -\frac{(z_1 + y_r)\xi_{c_1}(y_r, t)}{\xi_{c_1}^2(y_r, t) - x_1^2} + \int_0^1 \frac{(2\varepsilon z_1 + y_r)\xi_{c_1}(y_r, t)}{\xi_{c_1}^2(y_r, t) - (\varepsilon z_1 + y_r)^2} d\varepsilon \\ =& -\frac{(z_1 + y_r)\xi_{c_1}(y_r, t)}{\xi_{c_1}^2(y_r, t) - x_1^2} - \frac{\xi_{c_1}(y_r, t)}{z_1} \ln \frac{\xi_{c_1}^2(y_r, t) - x_1^2}{\xi_{c_1}^2(y_r, t) - y_r^2} \\ &+ \frac{y_r}{2z_1} \ln \frac{\left(\xi_{c_1}(y_r, t) - x_1\right)\left(\xi_{c_1}(y_r, t) + y_r\right)}{\left(\xi_{c_1}(y_r, t) - y_r\right)\left(\xi_{c_1}(y_r, t) + x_1\right)} \end{aligned}$$

$$\begin{aligned} N_1(\xi_{c_1}, y_r, z_1) =& \int_0^1 \frac{\xi_{c_1}^2(y_r, t)}{\xi_{c_1}^2(y_r, t) - (\varepsilon z_1 + y_r)^2} d\varepsilon \\ =& \frac{\xi_{c_1}(y_r, t)}{2z_1} \ln \frac{\left(\xi_{c_1}(y_r, t) + x_1\right)\left(\xi_{c_1}(y_r, t) - y_r\right)}{\left(\xi_{c_1}(y_r, t) + y_r\right)\left(\xi_{c_1}(y_r, t) - x_1\right)}. \end{aligned}$$

The following part of Eq (3.13) is expressed as

$$\begin{aligned}
&\int_0^{z_1} \frac{\partial}{\partial \xi_{c_1}(y_r,t)} \frac{\gamma \xi_{c_1}^2(y_r,t)}{\xi_{c_1}^2(y_r,t) - (\gamma + y_r)^2} d\gamma \\
&= \int_0^{z_1} -\gamma(\gamma + y_r) d \frac{\xi_{c_1}(y_r,t)}{\xi_{c_1}^2(y_r,t) - (\gamma + y_r)^2} \\
&= z_1 \left(-\frac{z_1 \xi_{c_1}(y_r,t)}{\xi_{c_1}^2(y_r,t) - x_1^2} + P_1(\xi_{c_1}, y_r, z_1) \right)
\end{aligned} \tag{3.14}$$

where

$$\begin{aligned}
P_1(\xi_{c_1}, y_r, z_1) &= -\frac{y_r \xi_{c_1}(y_r,t)}{\xi_{c_1}^2(y_r,t) - x_1^2} + \int_0^1 \frac{(2\varepsilon z_1 + y_r)\xi_{c_1}(y_r,t)}{\xi_{c_1}^2(y_r,t) - (\varepsilon z_1 + y_r)^2}\, d\varepsilon \\
&= -\frac{y_r \xi_{c_1}(y_r,t)}{\xi_{c_1}^2(y_r,t) - x_1^2} - \frac{\xi_{c_1}(y_r,t)}{z_1} \ln\left(\frac{\xi_{c_1}^2(y_r,t) - x_1^2}{\xi_{c_1}^2(y_r,t) - y_r^2} \right) \\
&\quad + \frac{y_r}{2z_1} \ln\left(\frac{(\xi_{c_1}(y_r,t) - x_1)(\xi_{c_1}(y_r,t) + y_r)}{(\xi_{c_1}(y_r,t) + x_1)(\xi_{c_1}(y_r,t) - y_r)} \right)
\end{aligned}$$

Remark 3. For convenience of description, this paper rewrites $M_1(\xi_{c_1}, y_r, z_1)$, $N_1(\xi_{c_1}, y_r, z_1)$, and $P_1(\xi_{c_1}, y_r, z_1)$ as M_1, N_1, P_1, respectively. Applying L'Hôpital's rule, we get $\lim_{z_1 \to 0} M_1 = \lim_{z_1 \to 0} P_1 = 0$, $\lim_{z_1 \to 0} N_1 = \xi_{c_1}^2(y_r,t) \Big/ (\xi_{c_1}^2(y_r,t) - y_r^2)$. Assumption 2 supposes that y_r is bounded satisfying $|y_r(t)| \le C_0$, so the boundedness of N_1 is guaranteed when $z_1 \to 0$. These rules are also established in below steps.

Choose a Lyapunov function as

$$V_1 = V_0^X + V_{z_1} + \frac{1}{2\rho_1} \tilde{\vartheta}_1^T \tilde{\vartheta}_1 \tag{3.15}$$

where ρ_1 is a designable parameter.

Then, the derivative of V_1 becomes

$$\begin{aligned}
\dot{V}_1 =& \dot{V}_0^X + \frac{z_1 \xi_{c_1}^2(y_r,t)}{\xi_{c_1}^2(y_r,t) - x_1^2} \left(\varpi_1 + z_2 + \alpha_1 + \breve{x}_2 - \dot{y}_r \right) \\
&+ \frac{z_1 \xi_{c_1}^2(y_r,t)}{\xi_{c_1}^2(y_r,t) - x_1^2} \left(\vartheta_1^T \psi_1(\hat{X}_1) + \tilde{\vartheta}_1^T \psi_1(\hat{X}_1) + \dot{y}_r \right) \\
&+ \frac{\partial \xi_{c_1}(y_r,t)}{\partial y_r} z_1 M_1 \dot{y}_r + z_1 P_1 \dot{\xi}_{c_1}(y_r,t) \\
&- z_1 N_1 \dot{y}_r - \frac{z_1^2 \xi_{c_1}(y_r,t)}{\xi_{c_1}^2(y_r,t) - x_1^2} \dot{\xi}_{c_1}(y_r,t) - \frac{1}{\rho_1} \tilde{\vartheta}_1^T \dot{\vartheta}_1
\end{aligned} \tag{3.16}$$

where

$$\dot{\xi}_{c_1}(y_r,t) = \frac{\partial \xi_{c_1}(y_r,t)}{\partial y_r} \dot{y}_r + \frac{\partial \xi_{c_1}(y_r,t)}{\partial t}.$$

The first virtual controller α_1 and adaption law $\dot{\vartheta}_1$ are constructed as

$$
\begin{aligned}
\alpha_1 =& -\iota_1 z_1 - \bar{c}_1 z_1 - \vartheta_1^T \psi_1(\hat{X}_1) - \frac{z_1 \xi_{c_1}(y_r,t)}{\xi_{c_1}^2(y_r,t) - x_1^2} - \frac{\partial \xi_{c_1}(y_r,t)}{\partial y_r}\theta_1 M_1 \dot{y}_r \\
&+ \theta_1 N_1 \dot{y}_r - \frac{\partial \xi_{c_1}(y_r,t)}{\partial y_r}\theta_1 P_1 \dot{y}_r - \frac{\partial \xi_{c_1}(y_r,t)}{\partial t}\theta_1 P_1
\end{aligned} \tag{3.17}
$$

$$
\dot{\vartheta}_1 = \frac{\rho_1 z_1 \xi_{c_1}^2(y_r,t)}{\xi_{c_1}^2(y_r,t) - x_1^2}\psi_1(\hat{X}_1) - \beta_1 \vartheta_1 \tag{3.18}
$$

where $\iota_1 > 0, \beta_1 > 0$ are designable parameters, and $\theta_1 = \left(\xi_{c_1}^2(y_r,t) - x_1^2\right)\Big/\xi_{c_1}^2(y_r,t)$. $\bar{c}_1$ is a time-varying function described as $\bar{c}_1 = \left((\dot{\xi}_{c_1}(y_r,t)\Big/\xi_{c_1}(y_r,t))^2 + o_1\right)^{\frac{1}{2}}$ with $o_1 > 0$.

Utilizing Young's inequality, one yields

$$
\frac{z_1 \xi_{c_1}^2(y_r,t)}{\xi_{c_1}^2(y_r,t) - x_1^2}\left(\varpi_1 + \breve{x}_2\right) \le \left(\frac{z_1 \xi_{c_1}^2(y_r,t)}{\xi_{c_1}^2(y_r,t) - x_1^2}\right)^2 + \frac{1}{2}\left\|\breve{x}_2\right\|^2 + \frac{1}{2}\bar{\varpi}_1^2 \tag{3.19}
$$

Substituting Eqs (3.7), (3.17), (3.18) and (3.19) into Eq (3.16) gets

$$
\begin{aligned}
\dot{V}_1 \le& -(\lambda_{\min}(Q) - 1)\left\|\breve{X}\right\|^2 - \iota_1 \frac{z_1^2 \xi_{c_1}^2(y_r,t)}{\xi_{c_1}^2(y_r,t) - x_1^2} + \frac{\beta_1}{\rho_1}\tilde{\vartheta}_1^T \vartheta_1 \\
&+ \frac{z_1 z_2 \xi_{c_1}^2(y_r,t)}{\xi_{c_1}^2(y_r,t) - x_1^2} + \frac{1}{2}\bar{\varpi}_1^2 + \frac{1}{2}\|P\zeta\|^2
\end{aligned} \tag{3.20}
$$

Step $i(2 \le i \le n-1)$: In view of Eq (3.9), $\dot{z}_i$ is calculated as

$$
\begin{aligned}
\dot{z}_i &= \dot{\hat{x}}_i - \dot{\alpha}_{i-1} \\
&= \eta_i \breve{x}_1 + z_{i+1} + \alpha_i + \tilde{\vartheta}_i^T \psi_i(\hat{X}_i) + \vartheta_i^T \psi_i(\hat{X}_i) + \varpi_i - \dot{\alpha}_{i-1}
\end{aligned} \tag{3.21}
$$

where

$$
\begin{aligned}
\dot{\alpha}_{i-1} =& \sum_{m=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial \hat{x}_m}\left(\hat{x}_{m+1} - \eta_m(\hat{x}_1 - y) + \vartheta_m^T \psi_m(\hat{X}_m)\right) \\
&+ \frac{\partial \alpha_{i-1}}{\partial x_1}\left(\hat{x}_2 + \breve{x}_2 + \vartheta_1^T \psi_1(\hat{X}_1) + \zeta_1\right) + \sum_{m=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial y_r^{(m-1)}} y_r^{(m)} \\
&+ \sum_{m=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial \vartheta_m}\dot{\vartheta}_m + \sum_{m=1}^{i-1}\sum_{j=0}^{i-m} \frac{\partial \alpha_{i-1}}{\partial \xi_{c_m}^{(j)}(\hat{X}_{m-1},t)}\xi_{c_m}^{(j+1)}(\hat{X}_{m-1},t)
\end{aligned} \tag{3.22}
$$

Choose the following Lyapunov function

$$
V_i = V_{i-1} + \int_0^{z_i} \frac{\gamma \xi_{c_i}^2\left(\hat{X}_{i-1},t\right)}{\xi_{c_i}^2\left(\hat{X}_{i-1},t\right) - (\gamma + \alpha_{i-1})^2} d\gamma + \frac{1}{2\rho_i}\tilde{\vartheta}_i^T \tilde{\vartheta}_i \tag{3.23}
$$

The time derivative of V_i is

$$
\begin{aligned}
\dot{V}_i =& \dot{V}_{i-1} + \frac{z_i \xi_{c_i}^2(\hat{X}_{i-1}, t)}{\xi_{c_i}^2(\hat{X}_{i-1}, t) - \hat{x}_i^2} \dot{z}_i + \dot{\alpha}_{i-1} \int_0^{z_i} \frac{\partial}{\partial \alpha_{i-1}} \frac{\gamma \xi_{c_i}^2(\hat{X}_{i-1}, t)}{\xi_{c_i}^2(\hat{X}_{i-1}, t) - (\gamma + \alpha_{i-1})^2} d\gamma \\
&+ \dot{\xi}_{c_i}(\hat{X}_{i-1}, t) \int_0^{z_i} \frac{\partial}{\partial \xi_{c_i}(\hat{X}_{i-1}, t)} \frac{\gamma \xi_{c_i}^2(\hat{X}_{i-1}, t)}{\xi_{c_i}^2(\hat{X}_{i-1}, t) - (\gamma + \alpha_{i-1})^2} d\gamma - \frac{1}{\rho_i} \tilde{\vartheta}_i^T \dot{\vartheta}_i
\end{aligned}
\tag{3.24}
$$

Substituting Eq (3.21) into Eq (3.24), one yields

$$
\begin{aligned}
\dot{V}_i =& \dot{V}_{i-1} + \frac{z_i \xi_{c_i}^2(\hat{X}_{i-1}, t)}{\xi_{c_i}^2(\hat{X}_{i-1}, t) - \hat{x}_i^2} (\eta_i \breve{x}_1 + z_{i+1} + \alpha_i + \tilde{\vartheta}_i^T \psi_i(\hat{X}_i) \\
&+ \vartheta_i^T \psi_i(\hat{X}_i) + \varpi_i - \dot{\alpha}_{i-1}) + \dot{\alpha}_{i-1} \frac{z_i \xi_{c_i}^2(\hat{X}_{i-1}, t)}{\xi_{c_i}^2(\hat{X}_{i-1}, t) - \hat{x}_i^2} \\
&+ \frac{\partial \xi_{c_i}(\hat{X}_{i-1}, t)}{\partial \alpha_{i-1}} z_i M_i \dot{\alpha}_{i-1} - z_i N_i \dot{\alpha}_{i-1} + z_i P_i \dot{\xi}_{c_i}(\hat{X}_{i-1}, t) \\
&- \frac{z_i^2 \xi_{c_i}(\hat{X}_{i-1}, t)}{\xi_{c_i}^2(\hat{X}_{i-1}, t) - \hat{x}_i^2} \dot{\xi}_{c_i}(\hat{X}_{i-1}, t) - \frac{1}{\rho_i} \tilde{\vartheta}_i^T \dot{\vartheta}_i
\end{aligned}
\tag{3.25}
$$

where M_i, N_i and P_i is similar to step 1, and the detailed calculations of them are provided in the Appendix (a). Besides, $\dot{\xi}_{c_i}(\hat{X}_{i-1}, t)$ is expressed as

$$
\dot{\xi}_{c_i}(\hat{X}_{i-1}, t) = \sum_{m=1}^{i-1} \frac{\partial \xi_{c_i}(\hat{X}_{i-1}, t)}{\partial \hat{X}_m} \dot{\hat{X}}_m + \frac{\partial \xi_{c_i}(\hat{X}_{i-1}, t)}{\partial t}
\tag{3.26}
$$

Further, Eq (3.25) is rewritten in the following form

$$
\begin{aligned}
\dot{V}_i =& \dot{V}_{i-1} + \frac{z_i \xi_{c_i}^2(\hat{X}_{i-1}, t)}{\xi_{c_i}^2(\hat{X}_{i-1}, t) - \hat{x}_i^2} (z_{i+1} + \alpha_i + \varpi_i) + \frac{z_i \xi_{c_i}^2(\hat{X}_{i-1}, t)}{\xi_{c_i}^2(\hat{X}_{i-1}, t) - \hat{x}_i^2} (\eta_i \breve{x}_1 \\
&+ \tilde{\vartheta}_i^T \psi_i(\hat{X}_i) + \vartheta_i^T \psi_i(\hat{X}_i)) + \frac{\partial \xi_{c_i}(\hat{X}_{i-1}, t)}{\partial \alpha_{i-1}} z_i M_i \dot{\alpha}_{i-1} - z_i N_i \dot{\alpha}_{i-1} \\
&- \frac{z_i^2 \xi_{c_i}(\hat{X}_{i-1}, t)}{\xi_{c_i}^2(\hat{X}_{i-1}, t) - \hat{x}_i^2} \left(\sum_{m=1}^{i-1} \frac{\partial \xi_{c_i}(\hat{X}_{i-1}, t)}{\partial \hat{X}_m} \dot{\hat{X}}_m + \frac{\partial \xi_{c_i}(\hat{X}_{i-1}, t)}{\partial t} \right) \\
&+ z_i P_i \sum_{m=1}^{i-1} \frac{\partial \xi_{c_i}(\hat{X}_{i-1}, t)}{\partial \hat{X}_m} \dot{\hat{X}}_m + z_i P_i \frac{\partial \xi_{c_i}(\hat{X}_{i-1}, t)}{\partial t} - \frac{1}{\rho_i} \tilde{\vartheta}_i^T \dot{\vartheta}_i
\end{aligned}
\tag{3.27}
$$

Construct the intermediate virtual controller α_i and adaption law $\dot{\vartheta}_i$ as

$$
\begin{aligned}
\alpha_i =& -\iota_i z_i - \bar{c}_i z_i - \eta_i \breve{x}_1 - \vartheta_i^T \psi_i(\hat{X}_i) - \frac{\partial \xi_{c_i}(\hat{X}_{i-1}, t)}{\partial \alpha_{i-1}} \theta_i M_i \dot{\alpha}_{i-1} \\
&+ \theta_i N_i \dot{\alpha}_{i-1} - \theta_i P_i \left(\sum_{m=1}^{i-1} \frac{\partial \xi_{c_i}(\hat{X}_{i-1}, t)}{\partial \hat{X}_m} \dot{\hat{X}}_m \right) - \theta_i P_i \frac{\partial \xi_{c_i}(\hat{X}_{i-1}, t)}{\partial t} \\
&- \frac{1}{2} \frac{z_i \xi_{c_i}^2(\hat{X}_{i-1}, t)}{\xi_{c_i}^2(\hat{X}_{i-1}, t) - \hat{x}_i^2} - \frac{z_{i-1} \xi_{c_{i-1}}^2(\hat{X}_{i-2}, t)(\xi_{c_i}^2(\hat{X}_{i-1}, t) - \hat{x}_i^2)}{(\xi_{c_{i-1}}^2(\hat{X}_{i-2}, t) - \hat{x}_{i-1}^2)\xi_{c_i}^2(\hat{X}_{i-1}, t)}
\end{aligned}
\tag{3.28}
$$

$$\dot{\vartheta}_i = \frac{\rho_i z_i \xi_{c_i}^2(\hat{X}_{i-1},t)}{\xi_{c_i}^2(\hat{X}_{i-1},t)-\hat{x}_i^2}\psi_i(\hat{X}_i) - \beta_i\vartheta_i \tag{3.29}$$

where $\iota_i > 0$, $\beta_i > 0$ are designable parameters, and $\theta_i = \left(\xi_{c_i}^2(\hat{X}_{i-1},t)-\hat{x}_i^2\right)\Big/\xi_{c_i}^2(\hat{X}_{i-1},t)$. $\bar{c}_i$ is a time-varying function described as $\bar{c}_i = \left((\dot{\xi}_{c_i}(\hat{X}_{i-1},t)\Big/\xi_{c_i}(\hat{X}_{i-1},t))^2 + o_i\right)^{\frac{1}{2}}$ with $o_i > 0$.

According to Young's inequality, one has

$$\frac{z_i\xi_{c_i}^2(\hat{X}_{i-1},t)}{\xi_{c_i}^2(\hat{X}_{i-1},t)-\hat{x}_i^2}\varpi_i \le \frac{1}{2}\left(\frac{z_i\xi_{c_i}^2(\hat{X}_{i-1},t)}{\xi_{c_i}^2(\hat{X}_{i-1},t)-\hat{x}_i^2}\right)^2 + \frac{1}{2}\bar{\varpi}_i^2 \tag{3.30}$$

Thus, we have

$$\begin{aligned}\dot{V}_i \le& -(\lambda_{\min}(Q)-1)\left\|\breve{X}\right\|^2 - \sum_{m=1}^{i}\iota_m\frac{z_m^2\xi_{c_m}^2(\hat{X}_{m-1},t)}{\xi_{c_m}^2(\hat{X}_{m-1},t)-\hat{x}_m^2} + \sum_{m=1}^{i}\frac{\beta_m}{\rho_m}\tilde{\vartheta}_m^T\vartheta_m \\ &+ \frac{\xi_{c_i}^2(\hat{X}_{i-1},t)}{\xi_{c_i}^2(\hat{X}_{i-1},t)-\hat{x}_i^2}z_i z_{i+1} + \sum_{m=1}^{i}\frac{1}{2}\bar{\varpi}_m^2 + \frac{1}{2}\|P\zeta\|^2\end{aligned} \tag{3.31}$$

Step n: Form Eq (3.9), $\dot{z}_n$ is calculated as

$$\begin{aligned}\dot{z}_n &= \dot{\hat{x}}_n - \dot{\alpha}_{n-1} \\ &= \eta_n\breve{x}_1 + m\upsilon(t) + \kappa(t) + \tilde{\vartheta}_n^T\psi_n(\hat{X}_n) + \vartheta_n^T\psi_n(\hat{X}_n) + \varpi_n - \dot{\alpha}_{n-1}\end{aligned} \tag{3.32}$$

where

$$\begin{aligned}\dot{\alpha}_{n-1} =& \sum_{m=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\hat{x}_m}\left(\hat{x}_{m+1} - \eta_m(\hat{x}_1 - y) + \vartheta_m^T\psi_m(\hat{X}_m)\right) \\ &+ \frac{\partial\alpha_{n-1}}{\partial x_1}\left(\hat{x}_2 + \breve{x}_2 + \vartheta_1^T\psi_1(\hat{X}_1) + \zeta_1\right) + \sum_{m=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial y_r^{(m-1)}}y_r^{(m)} \\ &+ \sum_{m=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\vartheta_m}\dot{\vartheta}_m + \sum_{m=1}^{n-1}\sum_{j=0}^{n-m}\frac{\partial\alpha_{n-1}}{\partial\xi_{c_m}^{(j)}(\hat{X}_{m-1},t)}\xi_{c_m}^{(j+1)}(\hat{X}_{m-1},t)\end{aligned} \tag{3.33}$$

Choose the following Lyapunov function

$$V_n = V_{n-1} + \int_0^{z_n}\frac{\gamma\xi_{c_n}^2\left(\hat{X}_{n-1},t\right)}{\xi_{c_n}^2\left(\hat{X}_{n-1},t\right)-(\gamma+\alpha_{n-1})^2}d\gamma + \frac{1}{2\rho_n}\tilde{\vartheta}_n^T\tilde{\vartheta}_n \tag{3.34}$$

The time derivative of V_n is

$$\begin{aligned}\dot{V}_n =& \dot{V}_{n-1} + \frac{z_n\xi_{c_n}^2(\hat{X}_{n-1},t)}{\xi_{c_n}^2(\hat{X}_{n-1},t)-\hat{x}_n^2}\dot{z}_n + \dot{\alpha}_{n-1}\int_0^{z_n}\frac{\partial}{\partial\alpha_{n-1}}\frac{\gamma\xi_{c_n}^2(\hat{X}_{n-1},t)}{\xi_{c_n}^2(\hat{X}_{n-1},t)-(\gamma+\alpha_{n-1})^2}d\gamma \\ &+ \dot{\xi}_{c_n}(\hat{X}_{n-1},t)\int_0^{z_n}\frac{\partial}{\partial\xi_{c_n}(\hat{X}_{n-1},t)}\frac{\gamma\xi_{c_n}^2(\hat{X}_{n-1},t)}{\xi_{c_n}^2(\hat{X}_{n-1},t)-(\gamma+\alpha_{n-1})^2}d\gamma - \frac{1}{\rho_n}\tilde{\vartheta}_n^T\dot{\vartheta}_n\end{aligned} \tag{3.35}$$

Replacing Eq (3.35) by Eq (3.32) results in

$$\begin{aligned}\dot{V}_n =&\dot{V}_{n-1} + \frac{z_n\xi^2_{c_n}(\hat{X}_{n-1},t)}{\xi^2_{c_n}(\hat{X}_{n-1},t)-\hat{x}^2_n}(\eta_n\breve{x}_1 + m\upsilon(t) + k(t) + \vartheta^T_n\psi_n(\hat{X}_n) + \varpi_n)\\ &+ \frac{\partial\xi_{c_n}(\hat{X}_{n-1},t)}{\partial\alpha_{n-1}}z_nM_n\dot{\alpha}_{n-1} - z_nN_n\dot{\alpha}_{n-1} + z_nP_n\dot{\xi}_{c_n}(\hat{X}_{n-1},t)\\ &- \frac{z^2_n\xi_{c_n}(\hat{X}_{n-1},t)}{\xi^2_{c_n}(\hat{X}_{n-1},t)-\hat{x}^2_n}\dot{\xi}_{c_n}(\hat{X}_{n-1},t) - \frac{1}{\rho_n}\tilde{\vartheta}^T_n\left(\frac{\rho_nz_n\xi^2_{c_n}(\hat{X}_{n-1},t)}{\xi^2_{c_n}(\hat{X}_{n-1},t)-\hat{x}^2_n}\psi_n(\hat{X}_n) - \beta_n\vartheta_n\right)\end{aligned} \tag{3.36}$$

where ρ_n, β_n are positive constants, the definition of M_n, N_n and P_n will be explained in the Appendix (b). Besides, $\dot{\xi}_{c_n}(\hat{X}_{n-1},t)$ is expressed as

$$\dot{\xi}_{c_n}(\hat{X}_{n-1},t) = \sum_{m=1}^{n-1}\frac{\partial\xi_{c_n}(\hat{X}_{n-1},t)}{\partial\hat{X}_m}\dot{\hat{X}}_m + \frac{\partial\xi_{c_n}(\hat{X}_{n-1},t)}{\partial t} \tag{3.37}$$

The real controller $\upsilon(t)$ and adaption law $\dot{\vartheta}_n$ are given by

$$\begin{aligned}\upsilon(t) =&\frac{1}{m}\Bigg[-\iota_nz_n - \bar{c}_nz_n - \eta_n\breve{x}_1 - \vartheta^T_n\psi_n(\hat{X}_n) - \frac{\partial\xi_{c_n}(\hat{X}_{n-1},t)}{\partial\alpha_{n-1}}\theta_nM_n\dot{\alpha}_{n-1}\\ &+ \theta_nN_n\dot{\alpha}_{n-1} - \theta_nP_n\left(\sum_{m=1}^{n-1}\frac{\partial\xi_{c_n}(\hat{X}_{n-1},t)}{\partial\hat{X}_m}\dot{\hat{X}}_m\right) - \theta_nP_n\frac{\partial\xi_{c_n}(\hat{X}_{n-1},t)}{\partial t}\\ &-\frac{z_n\xi^2_{c_n}(\hat{X}_{n-1},t)}{\xi^2_{c_n}(\hat{X}_{n-1},t)-\hat{x}^2_n} - \frac{z_{n-1}\xi^2_{c_{n-1}}(\hat{X}_{n-2},t)(\xi^2_{c_n}(\hat{X}_{n-1},t)-\hat{x}^2_n)}{(\xi^2_{c_{n-1}}(\hat{X}_{n-2},t)-\hat{x}^2_{n-1})\xi^2_{c_n}(\hat{X}_{n-1},t)}\Bigg]\end{aligned} \tag{3.38}$$

$$\dot{\vartheta}_n = \frac{\rho_nz_n\xi^2_{c_n}(\hat{X}_{n-1},t)}{\xi^2_{c_n}(\hat{X}_{n-1},t)-\hat{x}^2_n}\psi_n(\hat{X}_n) - \beta_n\vartheta_n \tag{3.39}$$

where $\iota_n > 0$ is desingable paremeter and $\theta_n = \left(\xi^2_{c_n}(\hat{X}_{n-1},t)-\hat{x}^2_n\right)\Big/\xi^2_{c_n}(\hat{X}_{n-1},t)$. $\bar{c}_n$ is a time-varying function described as $\bar{c}_n = \left((\dot{\xi}_{c_n}(\hat{X}_{n-1},t)\big/\xi_{c_n}(\hat{X}_{n-1},t))^2 + o_n\right)^{\frac{1}{2}}$ with $o_n > 0$.

Based on Young's inequality, it has

$$\frac{z_n\xi^2_{c_n}(\hat{X}_{n-1},t)}{\xi^2_{c_n}(\hat{X}_{n-1},t)-\hat{x}^2_n}\kappa(t) \le \frac{1}{2}\left(\frac{z_n\xi^2_{c_n}(\hat{X}_{n-1},t)}{\xi^2_{c_n}(\hat{X}_{n-1},t)-\hat{x}^2_n}\right)^2 + \frac{1}{2}\bar{\kappa}^2 \tag{3.40}$$

$$\frac{z_n\xi^2_{c_n}(\hat{X}_{n-1},t)}{\xi^2_{c_n}(\hat{X}_{n-1},t)-\hat{x}^2_n}\varpi_n \le \frac{1}{2}\left(\frac{z_n\xi^2_{c_n}(\hat{X}_{n-1},t)}{\xi^2_{c_n}(\hat{X}_{n-1},t)-\hat{x}^2_n}\right)^2 + \frac{1}{2}\bar{\varpi}^2_n \tag{3.41}$$

Finally, we obtain

$$\dot{V}_n \le -(\lambda_{\min}(Q)-1)\left\|\breve{X}\right\|^2 - \sum_{m=1}^{n}\iota_m\frac{z^2_m\xi^2_{c_m}(\hat{X}_{m-1},t)}{\xi^2_{c_m}(\hat{X}_{m-1},t)-\hat{x}^2_m} + \sum_{m=1}^{n}\frac{\beta_m}{\rho_m}\tilde{\vartheta}^T_m\vartheta_m \tag{3.42}$$

Remark 4. It is worth emphasized that a direct constraint is adopted on the states of this system according to the constraint Eq (3.10). But, through [19] and [24], the boundedness of the virtual controller must be obtained firstly, then from error transformation $z_i = x_i - \alpha_{i-1}$, the boundedness of states can be known. The integral BLF-approach introduced in this paper successfully overcomes the aforementioned conservation conditions.

Theorem 1. Under the condition of functional constraints $|\hat{x}_i| < \xi_{c_i}(\hat{X}_{i-1}, t)$, consider strict-feedback control systems Eq 2.1, actual fuzzy controller Eq (3.38), and adaptation laws Eqs (3.18), (3.29) and (3.39), if the initial condition satisfies $x(0) \in \Delta_x = \{\hat{x}_i | |\hat{x}_i(0)| < \xi_{c_i}(\hat{X}_{i-1}(0), 0)\}$, the following properties can be guaranteed:

- the constructed fuzzy state observer can estimate the unmeasurable states commendably;
- the functional constraints are never violated;
- the boundedness of all the closed-loop signals are ensured.

Proof. Applying Young's inequality, the third term $\sum_{m=1}^{n} \frac{\beta_m}{\rho_m} \tilde{\vartheta}_m^T \vartheta_m$ of Eq (3.42) is transformed into the following form:

$$\sum_{m=1}^{n} \frac{\beta_m}{\rho_m} \tilde{\vartheta}_m^T \vartheta_m \leq -\frac{1}{2} \sum_{m=1}^{n} \frac{\beta_m}{\rho_m} \tilde{\vartheta}_m^T \tilde{\vartheta}_m + \frac{1}{2} \sum_{m=1}^{n} \frac{\beta_m}{\rho_m} \vartheta_m^{*T} \vartheta_m^* \tag{3.43}$$

Substituting Eq (3.43) into Eq (3.42), one acquires

$$\dot{V}_n \leq -\chi V_n + \Lambda \tag{3.44}$$

where

$$\chi = \{2(\lambda_{\min}(Q) - 1)/\lambda_{\min}(P), 2\iota_m, 2\beta_m\}, m = 1, ..., n,$$

and

$$\Lambda = \sum_{m=1}^{n} \frac{1}{2} \bar{\varpi}_m^2 + \frac{1}{2} \|P\zeta\|^2 + \sum_{m=1}^{n} \frac{1}{2} \bar{\kappa}^2 + \frac{1}{2} \sum_{m=1}^{n} \frac{\beta_m}{\rho_m} \vartheta_m^{*T} \vartheta_m^*.$$

Multiplying Eq (3.44) by $e^{\pi_1 t}$, one obtains

$$d(e^{\pi_1 t} V_n)/dt \leq e^{\pi_1 t} \Lambda \tag{3.45}$$

Integrating Eq (3.45) over $[0, t]$ yields

$$V_n(t) \leq (V_n(0) - \Lambda/\chi) e^{-\pi_1 t} + \Lambda/\chi \tag{3.46}$$

Employing Eqs (3.11), (3.15) and (3.46), the following inequalities are acquired

$$z_i^2 \leq 2 \left[(V_n(0) - \Lambda/\chi) e^{-\pi_1 t} + \Lambda/\chi \right] \tag{3.47}$$

$$\left\| \tilde{\vartheta}_i \right\|^2 \leq 2 \left[(V_n(0) - \Lambda/\chi) e^{-\pi_1 t} + \Lambda/\chi \right] \tag{3.48}$$

According to Eqs (3.42) and (3.44), one has

$$\breve{X}^T P \breve{X} \leq 2 V_n(0) e^{-\pi_1 t} + 2\Lambda/\chi \tag{3.49}$$

Further, we can obtain the following form

$$\left\|\breve{X}\right\| \leq \sqrt{\lambda_{\min}(P^{-1})(2V_n(0)e^{-\pi_1 t} + 2\Lambda/\chi)}. \tag{3.50}$$

Based on the inequality Eq (3.47), the boundedness of z_1 is obtained. From Assumption 1 and 2, it can be known that $|y_r(t)| \leq C_0 < \xi_{c_1}(y_r, t)$. Thus, we conclude that the boundedness of x_1 is guaranteed. Since $\breve{x}_i = x_i - \hat{x}_i$, it's obvious that $\breve{x}_i$ and $\hat{x}_i$ are within limits. Therefore, it can be proved that x_i is bounded. According to $z_i = \hat{x}_i - \alpha_{i-1}$ and inequality Eq (3.47), the boundedness of α_{i-1} is implied. Depending on the inequality Eq (3.48), the boundedness of ϑ_i is ensured. In the same way, it can be deduced that the controller $\upsilon(t)$ is bounded.

Depending on the above analyses, we can draw a conclude that all the signals of this system are within bounds.

4. Simulation example

To further verify the feasibility of this scheme, the following simple pendulum system is considered:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\dfrac{m_1 g L \sin(x_1)}{2M} - \dfrac{D x_2}{M} + \dfrac{u}{M} \\ y = x_1 \end{cases} \tag{4.1}$$

where $m_1 = 1$, $g = 9.8$, $L = 1.5$, $M = 0.5$, $D = 0.4$. The state variables x_1 and x_2 are constrained by $-\xi_{c_1}(y_r, t) < x_1 < \xi_{c_1}(y_r, t)$ and $-\xi_{c_2}(x_1, t) < x_2 < \xi_{c_2}(x_1, t)$. In this paper, the constraint boundaries are selected as $\xi_{c_1}(y_r, t) = 0.2\sin(2t) + 0.8e^{-0.1y_r} + 1$ and $\xi_{c_2}(x_1, t) = 5\sin(0.3x_1) + 0.1e^{-0.5t} + 4$. In addition, u denotes the dead-zone input which is described by:

$$u = D(\upsilon(t)) = \begin{cases} 0.5(\upsilon(t) - 2.5) & \upsilon(t) \geq 2.5 \\ 0 & -1.5 < \upsilon(t) < 2.5 \\ 0.5(\upsilon(t) + 1.5) & \upsilon(t) \leq -1.5 \end{cases} \tag{4.2}$$

The state observer is constructed as:

$$\begin{cases} \dot{\hat{x}}_1 = \hat{x}_2 + \eta_1 \breve{x}_1 + \vartheta_1^T \psi_1(\hat{x}_1) \\ \dot{\hat{x}}_2 = \eta_2 \breve{x}_1 + \vartheta_2^T \psi_2(\hat{x}_2) + u \end{cases} \tag{4.3}$$

where the initial values are chosen as $x_1(0) = 0.6$, $x_2(0) = 0.2$, $\hat{x}_1(0) = 0.6$ and $\hat{x}_2(0) = 0.2$.

Select the following fuzzy membership functions:

$$\mu_{F_1^i}(\hat{x}_1) = \exp\left((-\hat{x}_1 + 12 - 0.5i)^2\right),$$

$$\mu_{F_2^i}(\hat{x}_1, \hat{x}_2) = \exp\left((-\hat{x}_1 + 12 - 0.5i)^2\right) \times \exp\left((-0.5\hat{x}_2 + 12 - 0.6i)^2\right).$$

where $i = 1, 2, ..., 8$.

The fuzzy basis functions are defined as:

$$\psi_{1j}(\hat{x}_1) = \frac{\mu_{F_1^i}(\hat{x}_1)}{\sum_{j=1}^{8} \mu_{F_1^j}(\hat{x}_1)}, \quad \psi_{2j}(\hat{x}_1, \hat{x}_2) = \frac{\mu_{F_1^i}(\hat{x}_1)\mu_{F_2^i}(\hat{x}_2)}{\sum_{j=1}^{8} \mu_{F_1^j}(\hat{x}_1)\mu_{F_2^j}(\hat{x}_2)}.$$

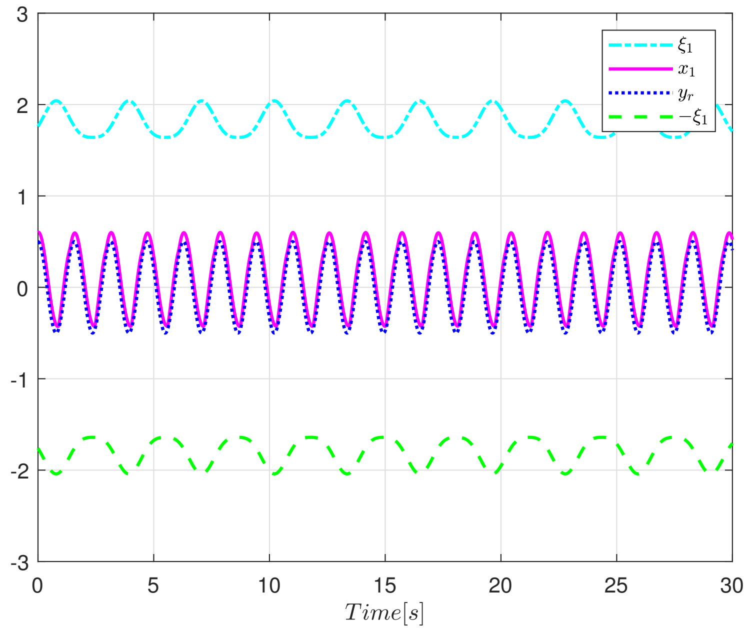

Figure 1. Trajectories of x_1 and y_r.

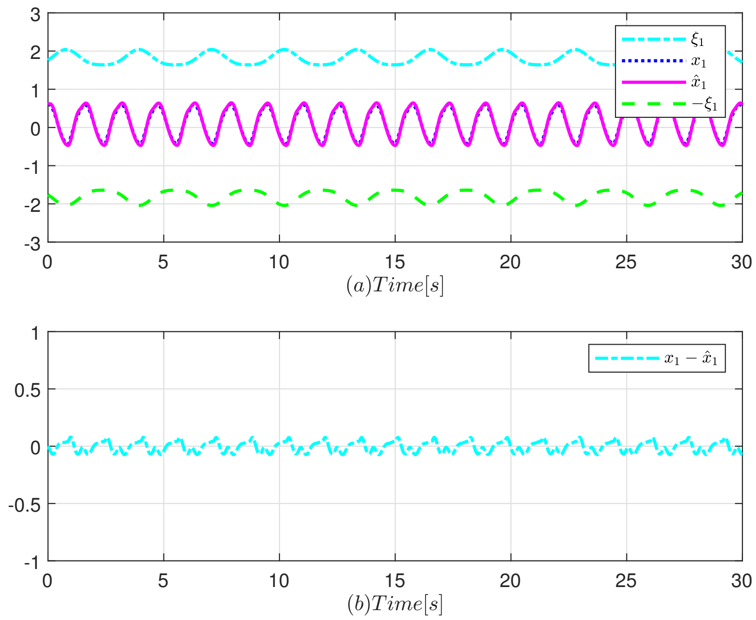

Figure 2. Trajectories of x_1, $\hat{x}_1$ and estimation error.

where $j = 1, 2, ..., 8$.

In this simulation, the tracking signal is designed as $y_r = 0.5\cos(4t)$, $\dot{y}_r = -2\sin(4t)$. The initial values of the adaptation laws are $\vartheta_1(0) = [0.2, 0, 0.2, 0.2, 0.1, 0.1, 0.1, 0.1]^T$, $\vartheta_2(0) =$

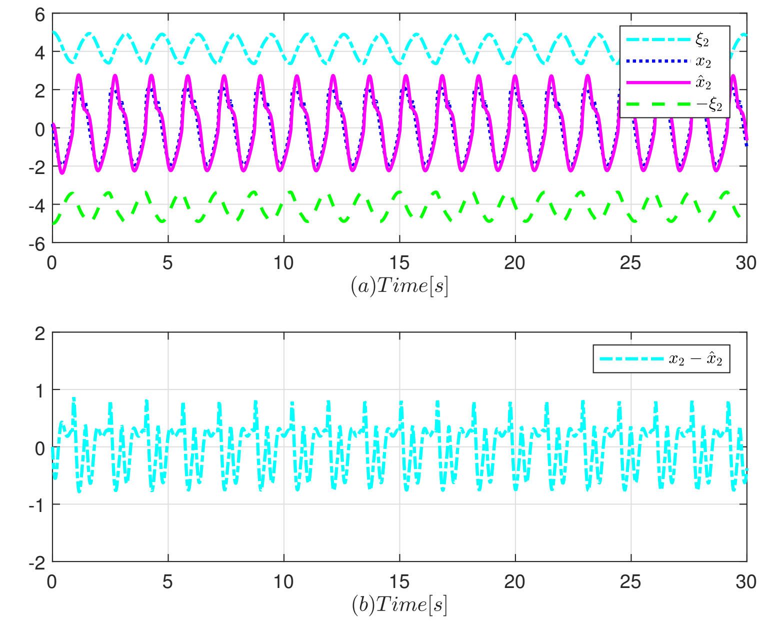

Figure 3. Trajectories of x_2, $\hat{x}_2$ and estimation error.

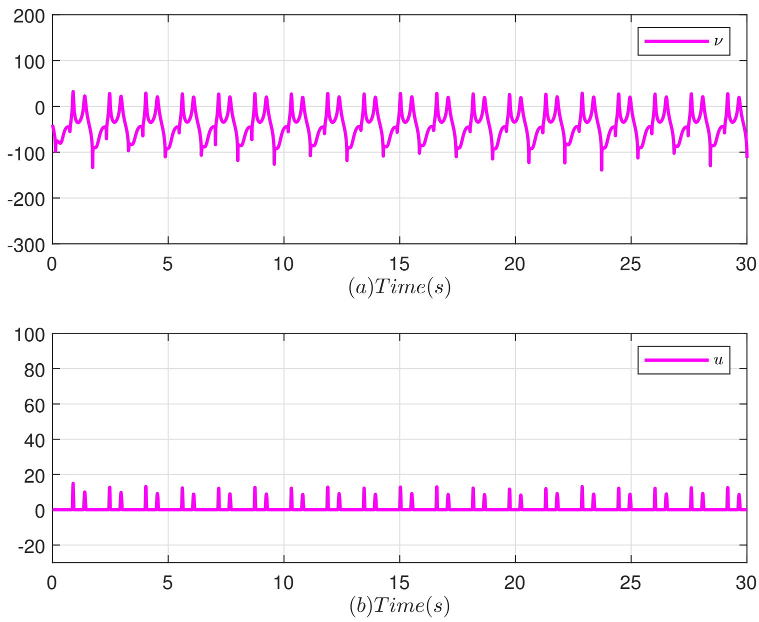

Figure 4. Trajectories of real controller υ and dead-zone input u.

$[0.2, 0.2, 0.2, 0.1, 0.1, 0.1, 0.1, 0.1]^T$. The relevant parameters in this paper are chosen as $\rho_1 = 0.5$, $\rho_2 = 0.2$, $\beta_1 = 5$, $\beta_2 = 16$, $\iota_1 = 30$, $\iota_2 = 25$, $o_1 = o_2 = 0.2$, $\eta_1 = 5$ and $\eta_2 = 155$.

According to the mentioned above parameter values, the corresponding simulation results are

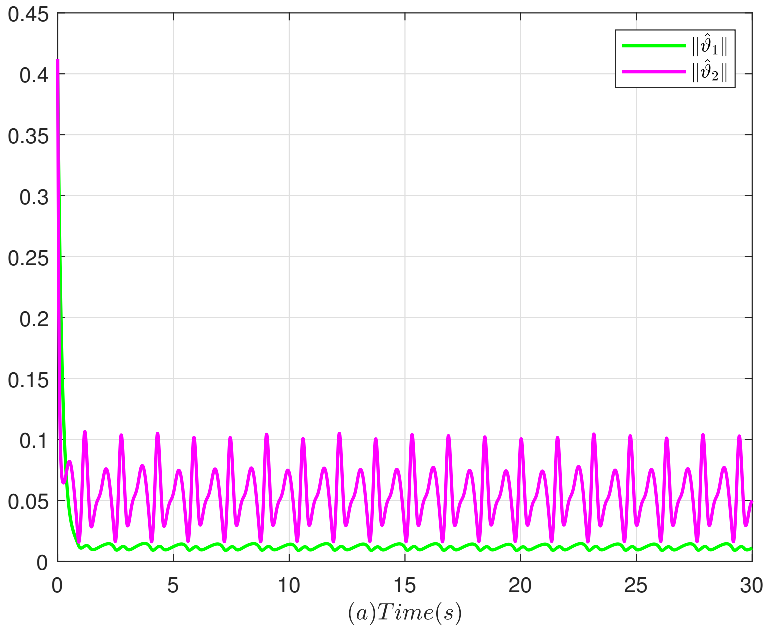

Figure 5. Trajectories of ϑ_1 and ϑ_2.

demonstrated in Figures 1–5. The change curves of state x_1 and reference signal $y_r(t)$ are described in Figure 1, where two curves can approximate overlap and the tracking effect is good. In Figures 2 and 3, the constructed observer is able to estimate the system states x_1 and x_2 well, and their estimation errors are relatively small. Figures 1–3 indicates that system states are strictly restricted in the predetermined ranges, and the TFSC are achieved. The trajectories of actual controller υ and dead-zone input u are plotted in Figure 4, and their curves tend to be stable. Finally, Figure 5 diagrams the boundedness of adaptation parameters ϑ_1 and ϑ_2.

5. Conclusions

A state observer and a fuzzy controller for a class of functional constraint systems subject to unknown dead-zone have been constructed in this paper. The former is applied to estimate unmeasurable states, while the latter is established to approximate uncertain nonlinear functions. Relying on backstepping algorithm and iBLFs, the full state TFC are accomplished and the issue of non-smooth dead-zone input is successfully handled. The simulation diagrams further indicate that the developed control scheme is feasible. In the future, how to choose an appropriate barrier function to settle the application of constraint control in practical systems is a key problem.

Acknowledgments

This work is supported in part by the National Natural Science Foundation of China under Grants 62025303 and 62173173.

Conflict of interest

The authors declare that there is no conflict of interest.

References

1. Z. Sabir, M. A. Z. Raja, A. Kamal, J. L. G. Guirao, D. Le, T. Saeed, et al., Neuro-swarm heuristic using interior-point algorithm to solve a third kind of multi-singular nonlinear system, *Math. Biosci. Eng.*, **18** (2021), 5285–5308. https://doi.org/10.3934/mbe.2021268
2. X. Li, D. W. C. Ho, J. Cao, Finite-time stability and settling-time estimation of nonlinear impulsive systems, *Automatica*, **99** (2019), 361–368. https://doi.org/10.1016/j.automatica.2018.10.024
3. W. Chen, L. Jiao, R. Li, J. Li, Adaptive backstepping fuzzy control for nonlinearly parameterized systems with periodic disturbances, *IEEE Trans. Fuzzy Syst.*, **18** (2010), 674–685. https://doi.org/10.1109/TFUZZ.2010.2046329
4. H. Liang, L. Chen, Y. Pan, H. K. Lam, Fuzzy-based robust precision consensus tracking for uncertain networked systems with cooperative-antagonistic interactions, *IEEE Trans. Fuzzy Syst.*, https://doi:10.1109/TFUZZ.2022.3200730S
5. W. Wang, J. Dong, D. Xu, Z. Yan, J. Zhou, Synchronization control of time-delay neural networks via event-triggered non-fragile cost-guaranteed control, *Math. Biosci. Eng.*, **20** (2022), 52–75. http://dx.doi.org/10.3934/mbe.2023004
6. H. Liang, Z. Du, T. Huang, Y. Pan, Neuroadaptive performance guaranteed control for multiagent systems with power integrators and unknown measurement sensitivity, *IEEE Trans. Neural Networks Learn. Syst.*, **2022** (2022). https://doi:10.1109/TNNLS.2022.3160532
7. Z. Sabir, M. A. Z. Raja, A. S. Alnahdi, M. B. Jeelani, M. A. Abdelkawy, Numerical investigations of the nonlinear smoke model using the gudermannian neural networks, *Math. Biosci. Eng.*, **18** (2021), 5285–5308. https://doi.org/10.3934/mbe.2022018
8. B. Chen, X. P. Liu, S. S. Ge, C. Lin, Adaptive fuzzy control of a class of nonlinear systems by fuzzy approximation approach, *IEEE Trans. Fuzzy Syst.*, **20** (2012), 1012–1021. https://doi.org/10.1109/TFUZZ.2012.2190048
9. H. Su, W. Zhang, Adaptive fuzzy tracking control for a class of nonstrict-feedback stochastic nonlinear systems with actuator faults, *IEEE Trans. Syst. Man Cybern. Syst.*, **50** (2020), 3456–3469. https://doi.org/10.1109/TSMC.2018.2883414
10. F. Wang, B. Chen, X. Liu, C. Lin, Finite-time adaptive fuzzy tracking control design for nonlinear systems, *IEEE Trans. Fuzzy Syst.*, **26** (2018), 1207–1216. https://doi.org/10.1109/TFUZZ.2017.2717804
11. X. Li, X. Yang, J. Cao, Event-triggered impulsive control for nonlinear delay systems, *Automatica*, **117** (2020), 108981. https://doi.org/10.1016/j.automatica.2020.108981
12. L. Liu, Y. J. Liu , S. C. Tong, Neural networks-based adaptive finite-time fault-tolerant control for a class of strict-feedback switched nonlinear systems, *IEEE Trans. Cybern,* , **49** (2018), 2536–2545. https://doi.org/10.1109/TCYB.2018.2828308

13. S. Vutukuri, R. Padhi, Quaternion constrained robust attitude control using barrier Lyapunov function based back-stepping, *IFAC-PapersOnLine*, **55** (2022), 522–527. https://doi.org/10.1016/j.ifacol.2022.04.086

14. Y. H. Liu, Y. Liu, Y. F. Liu, C. Y. Su, Adaptive fuzzy control with global stability guarantees for unknown strict-feedback systems using novel integral barrier Lyapunov functions, *IEEE Trans. Syst. Man Cybern. Syst.*, **52** (2022), 4336–4348. https://doi.org/10.1109/TSMC.2021.3094975

15. K. P. Tee, S. S. Ge, Control of state-constrained nonlinear systems using integral barrier Lyapunov functionals, in *2012 IEEE 51st IEEE Conference on Decision and Control (CDC)*, (2012), 3239–3244. https://doi.org/10.1109/CDC.2012.6426196

16. D. Yang, X. Gao, E. Cui, Z. Ma, State-constraints adaptive backstepping control for active magnetic bearings with parameters nonstationarities and uncertainties, *IEEE Transa. Ind. Electron.*, **68** (2021), 9822–9831. https://doi.org/10.1109/TIE.2020.3020034

17. Q. Zhou, L. Wang, C. Wu, H. Li, H. Du, Adaptive fuzzy control for nonstrict-feedback systems with input saturation and output constraint, *IEEE Trans. Syst. Man Cybern. Syst.*, **47** (2017), 1–12. https://doi.org/10.1109/TSMC.2016.2557222

18. K. P. Tee, S. S. Ge, E. H. Tay, Barrier Lyapunov functions for the control of output-constrained nonlinear systems, *Automatica*, **45** (2009), 918–927. https://doi.org/10.1016/j.automatica.2008.11.017

19. C. Enchang, Y. Jing, X. Gao, Full state constraints control of switched complex networks based on time-varying barrier Lyapunov functions, *IET Control Theory Appl.*, **14** (2020), 2419–2428. https://doi.org/10.1049/iet-cta.2020.0165

20. K. Yang, L. Zhao, Command-filter-based backstepping control for flexible joint manipulator systems with full-state constrains, *Int. J. Control Autom. Syst.*, **20** (2022), 2231–2238. https://doi.org/10.1007/s12555-020-0810-1

21. P. Seifi, S. K. H. Sani, Barrier Lyapunov functions-based adaptive neural tracking control for non-strict feedback stochastic nonlinear systems with full-state constraints: A command filter approach, *Math. Control Relat. Fields*, **2022** (2022). https://doi: 10.3934/mcrf.2022024

22. W. He, Y. Chen, Z. Yin, Adaptive neural network control of an uncertain robot with full-state constraints, *IEEE Trans. Cybern.*, **46** (2016), 620–629. https://doi.org/10.1109/TCYB.2015.2411285

23. K. Zhao, Y. Song, Removing the feasibility conditions imposed on tracking control designs for state-constrained strict-feedback systems, *IEEE Trans. Autom. Control*, **64** (2019), 1265–1272. https://doi.org/10.1109/TAC.2018.2845707

24. Z. Zhang, Z. Li, Y. Zhang, Y. Luo, Y. Li, Neural-dynamic-method-based dual-arm CMG scheme with time-varying constraints applied to humanoid robots, *IEEE Trans. Neural Networks Learn. Syst.*, **26** (2015), 3251–3262. https://doi.org/10.1109/TNNLS.2015.2469147

25. Y. J. Liu, S. Lu, D. Li, S. Tong, Adaptive controller design-based ABLF for a class of nonlinear time-varying state constraint systems, *IEEE Trans. Syst. Man Cybern. Syst.*, **47** (2017), 1546–1553. https://doi.org/10.1109/TSMC.2016.2633007

26. K. Zhao, Y. D. Song, C. L. P. Chen, L. Chen, Control of nonlinear systems under dynamic constraints: A unified barrier function-based approach, *Automatica*, **119** (2020). https://doi.org/10.1016/j.automatica.2020.109102

27. Y. J. Liu, W. Zhao, L. Liu, D. Li, S. C. Tong, C. L. P. Chen, Adaptive neural network control for a class of nonlinear systems with function constraints on states, *IEEE Trans. Neural Networks Learn. Syst.*, **2021** (2021). https://doi: 10.1109/TNNLS.2021.3107600

28. S. Ibrir, W. F. Xie, C. Su, Adaptive tracking of nonlinear systems with non-symmetric dead-zone input, *Automatica*, **43** (2007), 522–530. https://doi.org/10.1016/j.automatica.2006.09.022

29. X. Wang, C. Su, H. Hong, Robust adaptive control of a class of nonlinear systems with unknown dead-zone, *Automatica*, **40** (2004), 407–413. https://doi.org/10.1016/j.automatica.2003.10.021

30. L. Wu, J. H. Park, X. Xie, Y. Liu, Z. Yang, Event-triggered adaptive asymptotic tracking control of uncertain nonlinear systems with unknown dead-zone constraints, *Appl. Math. Comput.*, **386** (2020). https://doi.org/10.1016/j.amc.2020.125528

31. Y. J. Liu, Y. Gao, S. C. Tong, Y. Li, Fuzzy approximation-based adaptive backstepping optimal control for a class of nonlinear discrete-time systems with dead-zone, *IEEE Trans. Fuzzy Syst.*, **24** (2016), 16–28. https://doi.org/10.1109/TFUZZ.2015.2418000

32. H. Li, S. Zhao, W. He, R. Lu, Adaptive finite-time tracking control of full state constrained nonlinear systems with dead-zone, *Automatica*, **100** (2019), 99–107. https://doi.org/10.1016/j.automatica.2018.10.030

33. W. Xiao, L. Cao, G. Dong, Q. Zhou, Adaptive fuzzy control for pure-feedback systems with full state constraints and unknown nonlinear dead zone, *Appl. Math. Comput.*, **343** (2019), 354–371. https://doi.org/10.1016/j.amc.2018.09.016

34. M. V. Basin, P. C. Rodríguez-Ramírez, Sliding mode controller design for stochastic polynomial systems with unmeasured states, *IEEE Trans. Ind. Electron.*, **61** (2014), 387–396. https://doi.org/10.1109/TIE.2013.2240641

35. S. C. Tong, Y. Li, Y. M. Li, Y. J. Liu, Observer-based adaptive fuzzy backstepping control for a class of stochastic nonlinear strict-feedback systems, *IEEE Trans. Syst. Man Cybern. Part B Cybern.*, **41** (2011), 1693–1704. https://doi.org/10.1109/TSMCB.2011.2159264

36. J. Yu, P. Shi, W. Dong, H. Yu, Observer and command-filter-based adaptive fuzzy output feedback control of uncertain nonlinear systems, *IEEE Trans. Ind. Electron.*, **62** (2015), 5962–5970. https://doi.org/10.1109/TIE.2015.2418317

37. C. Wang, C. Zhang, D. He, J. Xiao, L. Liu, Observer-based finite-time adaptive fuzzy backstepping control for MIMO coupled nonlinear systems, *Math. Biosci. Eng.*, **19** (2022), 10637–10655. https://doi.org/10.3934/mbe.2022497

38. X. Xie, D. Yue, C. Peng, Multi-instant observer design of discrete-time fuzzy systems: a ranking-based switching approach, *IEEE Trans. Fuzzy Syst.*, **25** (2017), 1281–1292. https://doi.org/10.1109/TFUZZ.2016.2612260

39. N. Wang, S. C. Tong, Y. Li, Observer-based adaptive fuzzy control of nonlinear nonstrict feedback system with input delay, *IEEE Trans. Fuzzy Syst.*, **20** (2018), 236–245. https://doi.org/10.1007/s40815-017-0388-9

40. S. J. Yoo, Output-feedback fault detection and accommodation of uncertain interconnected systems with time-delayed nonlinear faults, *IEEE Trans. Syst. Man Cybern. Syst.*, **47** (2017), 758–766. https://doi.org/10.1109/TSMC.2016.2523900

41. B. Ren, S. S. Ge, K. P. Tee, T. H. Lee, Adaptive neural control for output feedback nonlinear systems using a barrier Lyapunov function, *IEEE Trans. Neural Network*, **21** (2010), 1339–1345. https://doi.org/10.1109/TNN.2010.2047115

42. L. Liu, A. Chen, Y. J. Liu, Adaptive fuzzy output-feedback control for switched uncertain nonlinear systems with full-state constraints, *IEEE Trans. Cybern.*, **52** (2022), 7340–7351. https://doi.org/10.1109/TCYB.2021.3050510

43. Y. Liu, Q. Zhu, N. Zhao, L. Wang, Adaptive fuzzy backstepping control for non-strict feedback nonlinear systems with time-varying state constraints and backlash-like hysteresis, *Inf. Sci.*, **574** (2021), 606–624. https://doi.org/10.1016/j.ins.2021.07.068

44. Y. J. Liu, M. Gong, L. Liu, S. C. Tong, C. L. P. Chen, Fuzzy observer constraint based on adaptive control for uncertain nonlinear MIMO systems with time-varying state constraints, *IEEE Trans. Cybern.*, **51** (2021), 1380–1389. https://doi.org/10.1109/TCYB.2019.2933700

Appendix A

In this part, the specific steps of Eq (3.24) are demonstrated as follows. From Eq (3.24), one has

$$\begin{aligned}&\int_0^{z_i} \frac{\partial}{\partial \alpha_{i-1}} \frac{\gamma \xi_{c_i}^2(\hat{X}_{i-1}, t)}{\xi_{c_i}^2(\hat{X}_{i-1}, t) - (\gamma + \alpha_{i-1})^2} d\gamma \\ &= z_i \left(\frac{\xi_{c_i}^2(\hat{X}_{i-1}, t)}{\xi_{c_i}^2(\hat{X}_{i-1}, t) - \hat{x}_i^2} + \frac{\partial \xi_{c_i}(\hat{X}_{i-1}, t)}{\partial \alpha_{i-1}} M_i(\xi_{c_i}, \alpha_{i-1}, z_i) - N_i(\xi_{c_i}, \alpha_{i-1}, z_i) \right)\end{aligned}$$

where

$$\begin{aligned}M_i(\xi_{c_i}, \alpha_{i-1}, z_i) &= -\frac{(z_i + \alpha_{i-1})\xi_{c_i}(\hat{X}_{i-1}, t)}{\xi_{c_i}^2(\hat{X}_{i-1}, t) - \hat{x}_i^2} + \int_0^1 \frac{(2\varepsilon z_i + \alpha_{i-1})\xi_{c_i}(\hat{X}_{i-1}, t)}{\xi_{c_i}^2(\hat{X}_{i-1}, t) - (\varepsilon z_i + \alpha_{i-1})^2} d\varepsilon \\ &= -\frac{(z_i + \alpha_{i-1})\xi_{c_i}(\hat{X}_{i-1}, t)}{\xi_{c_i}^2(\hat{X}_{i-1}, t) - \hat{x}_i^2} - \frac{\xi_{c_i}(\hat{X}_{i-1}, t)}{z_i} \ln \frac{\xi_{c_i}^2(\hat{X}_{i-1}, t) - \hat{x}_i^2}{\xi_{c_i}^2(\hat{X}_{i-1}, t) - \alpha_{i-1}^2} \\ &\quad + \frac{\alpha_{i-1}}{2z_i} \ln \frac{\left(\xi_{c_i}(\hat{X}_{i-1}, t) - \hat{x}_i\right)\left(\xi_{c_i}(\hat{X}_{i-1}, t) + \alpha_{i-1}\right)}{\left(\xi_{c_i}(\hat{X}_{i-1}, t) - \alpha_{i-1}\right)\left(\xi_{c_i}(\hat{X}_{i-1}, t) + \hat{x}_i\right)}\end{aligned}$$

and

$$\begin{aligned}N_i(\xi_{c_i}, \alpha_{i-1}, z_i) &= \int_0^1 \frac{\xi_{c_i}^2(\hat{X}_{i-1}, t)}{\xi_{c_i}^2(\hat{X}_{i-1}, t) - (\varepsilon z_i + \alpha_{i-1})^2} d\varepsilon \\ &= \frac{\xi_{c_i}(\hat{X}_{i-1}, t)}{2z_i} \ln \frac{\left(\xi_{c_i}(\hat{X}_{i-1}, t) + \hat{x}_i\right)\left(\xi_{c_i}(\hat{X}_{i-1}, t) - \alpha_{i-1}\right)}{\left(\xi_{c_i}(\hat{X}_{i-1}, t) + \alpha_{i-1}\right)\left(\xi_{c_i}(\hat{X}_{i-1}, t) - \hat{x}_i\right)}.\end{aligned}$$

The following part of Eq (3.24) is expressed as

$$\int_0^{z_i} \frac{\partial}{\partial \xi_{c_i}(\hat{X}_{i-1}, t)} \frac{\gamma \xi_{c_i}^2(\hat{X}_{i-1}, t)}{\xi_{c_i}^2(\hat{X}_{i-1}, t) - (\gamma + \alpha_{i-1})^2} d\gamma$$
$$= \int_0^{z_i} -\gamma(\gamma + \alpha_{i-1}) d \frac{\xi_{c_i}(\hat{X}_{i-1}, t)}{\xi_{c_i}^2(\hat{X}_{i-1}, t) - (\gamma + \alpha_{i-1})^2}$$
$$= z_i \left(-\frac{z_i \xi_{c_i}(\hat{X}_{i-1}, t)}{\xi_{c_i}^2(\hat{X}_{i-1}, t) - \hat{x}_i^2} + P_i(\xi_{c_i}, \alpha_{i-1}, z_i) \right)$$

where

$$\begin{aligned} P_i(\xi_{c_i}, \alpha_{i-1}, z_i) =& -\frac{\alpha_{i-1} \xi_{c_i}(\hat{X}_{i-1}, t)}{\xi_{c_i}^2(\hat{X}_{i-1}, t) - \hat{x}_i^2} + \int_0^1 \frac{(2\varepsilon z_i + \alpha_{i-1}) \xi_{c_i}(\hat{X}_{i-1}, t)}{\xi_{c_i}^2(\hat{X}_{i-1}, t) - (\varepsilon z_i + \alpha_{i-1})^2} \, d\varepsilon \\ =& -\frac{\alpha_{i-1} \xi_{c_i}(\hat{X}_{i-1}, t)}{\xi_{c_i}^2(\hat{X}_{i-1}, t) - \hat{x}_i^2} - \frac{\xi_{c_i}(\hat{X}_{i-1}, t)}{z_i} \ln \left(\frac{\xi_{c_i}^2(\hat{X}_{i-1}, t) - \hat{x}_i^2}{\xi_{c_i}^2(\hat{X}_{i-1}, t) - \alpha_{i-1}^2} \right) \\ &+ \frac{\alpha_{i-1}}{2z_i} \ln \frac{\left(\xi_{c_i}(\hat{X}_{i-1}, t) - \hat{x}_i\right)\left(\xi_{c_i}(\hat{X}_{i-1}, t) + \alpha_{i-1}\right)}{\left(\xi_{c_i}(\hat{X}_{i-1}, t) - \alpha_{i-1}\right)\left(\xi_{c_i}(\hat{X}_{i-1}, t) + \hat{x}_i\right)} \end{aligned}$$

Appendix B

Partial calculations of the final step will be described in the following contents. According to Eq (3.35), one has

$$\int_0^{z_n} \frac{\partial}{\partial \alpha_{n-1}} \frac{\gamma \xi_{c_n}^2(\hat{X}_{n-1}, t)}{\xi_{c_n}^2(\hat{X}_{n-1}, t) - (\gamma + \alpha_{n-1})^2} d\gamma$$
$$= z_n \left(\frac{\xi_{c_n}^2(\hat{X}_{n-1}, t)}{\xi_{c_n}^2(\hat{X}_{n-1}, t) - \hat{x}_n^2} + \frac{\partial \xi_{c_n}(\hat{X}_{n-1}, t)}{\partial \alpha_{n-1}} M_n(\xi_{c_n}, \alpha_{n-1}, z_n) - N_n(\xi_{c_n}, \alpha_{n-1}, z_n) \right)$$

where

$$\begin{aligned} M_n(\xi_{c_n}, \alpha_{n-1}, z_n) =& -\frac{(z_n + \alpha_{n-1}) \xi_{c_n}(\hat{X}_{n-1}, t)}{\xi_{c_n}^2(\hat{X}_{n-1}, t) - \hat{x}_n^2} + \int_0^1 \frac{(2\varepsilon z_n + \alpha_{n-1}) \xi_{c_n}(\hat{X}_{n-1}, t)}{\xi_{c_n}^2(\hat{X}_{n-1}, t) - (\varepsilon z_n + \alpha_{n-1})^2} \, d\varepsilon \\ =& -\frac{(z_n + \alpha_{n-1}) \xi_{c_n}(\hat{X}_{n-1}, t)}{\xi_{c_n}^2(\hat{X}_{n-1}, t) - \hat{x}_n^2} - \frac{\xi_{c_n}(\hat{X}_{n-1}, t)}{z_n} \ln \frac{\xi_{c_n}^2(\hat{X}_{n-1}, t) - \hat{x}_n^2}{\xi_{c_n}^2(\hat{X}_{n-1}, t) - \alpha_{n-1}^2} \\ &+ \frac{\alpha_{n-1}}{2z_n} \ln \frac{\left(\xi_{c_n}(\hat{X}_{n-1}, t) - \hat{x}_n\right)\left(\xi_{c_n}(\hat{X}_{n-1}, t) + \alpha_{n-1}\right)}{\left(\xi_{c_n}(\hat{X}_{n-1}, t) - \alpha_{n-1}\right)\left(\xi_{c_n}(\hat{X}_{n-1}, t) + \hat{x}_n\right)} \end{aligned}$$

and

$$\begin{aligned} N_n(\xi_{c_n}, \alpha_{n-1}, z_n) =& \int_0^1 \frac{\xi_{c_n}^2(\hat{X}_{n-1}, t)}{\xi_{c_n}^2(\hat{X}_{n-1}, t) - (\varepsilon z_n + \alpha_{n-1})^2} \, d\varepsilon \\ =& \frac{\xi_{c_n}(\hat{X}_{n-1}, t)}{2z_n} \ln \frac{\left(\xi_{c_n}(\hat{X}_{n-1}, t) + \hat{x}_n\right)\left(\xi_{c_n}(\hat{X}_{n-1}, t) - \alpha_{n-1}\right)}{\left(\xi_{c_n}(\hat{X}_{n-1}, t) + \alpha_{n-1}\right)\left(\xi_{c_n}(\hat{X}_{n-1}, t) - \hat{x}_n\right)}. \end{aligned}$$

The following part of Eq (3.35) is changed as

$$
\begin{aligned}
&\int_0^{z_n} \frac{\partial}{\partial \xi_{c_n}(\hat{X}_{n-1}, t)} \frac{\gamma \xi_{c_n}^2(\hat{X}_{n-1}, t)}{\xi_{c_n}^2(\hat{X}_{n-1}, t) - (\gamma + \alpha_{n-1})^2} d\gamma \\
&= \int_0^{z_n} -\gamma(\gamma + \alpha_{n-1}) d \frac{\xi_{c_n}(\hat{X}_{n-1}, t)}{\xi_{c_n}^2(\hat{X}_{n-1}, t) - (\gamma + \alpha_{n-1})^2} \\
&= z_n \left(-\frac{z_n \xi_{c_n}(\hat{X}_{n-1}, t)}{\xi_{c_n}^2(\hat{X}_{n-1}, t) - \hat{x}_n^2} + P_n(\xi_{c_n}, \alpha_{n-1}, z_n) \right)
\end{aligned}
$$

where

$$
\begin{aligned}
P_n(\xi_{c_n}, \alpha_{n-1}, z_n) &= -\frac{\alpha_{n-1} \xi_{c_n}(\hat{X}_{n-1}, t)}{\xi_{c_n}^2(\hat{X}_{n-1}, t) - \hat{x}_n^2} + \int_0^1 \frac{(2\varepsilon z_n + \alpha_{n-1}) \xi_{c_n}(\hat{X}_{n-1}, t)}{\xi_{c_n}^2(\hat{X}_{n-1}, t) - (\varepsilon z_n + \alpha_{n-1})^2} d\varepsilon \\
&= -\frac{\alpha_{n-1} \xi_{c_n}(\hat{X}_{n-1}, t)}{\xi_{c_n}^2(\hat{X}_{n-1}, t) - \hat{x}_n^2} - \frac{\xi_{c_n}(\hat{X}_{n-1}, t)}{z_n} \ln \left(\frac{\xi_{c_n}^2(\hat{X}_{n-1}, t) - \hat{x}_n^2}{\xi_{c_n}^2(\hat{X}_{n-1}, t) - \alpha_{n-1}^2} \right) \\
&\quad + \frac{\alpha_{n-1}}{2z_n} \ln \frac{\left(\xi_{c_n}(\hat{X}_{n-1}, t) - \hat{x}_n\right)\left(\xi_{c_n}(\hat{X}_{n-1}, t) + \alpha_{n-1}\right)}{\left(\xi_{c_n}(\hat{X}_{n-1}, t) - \alpha_{n-1}\right)\left(\xi_{c_n}(\hat{X}_{n-1}, t) + \hat{x}_n\right)}
\end{aligned}
$$