Mathematical Biosciences and Engineering

MBE, 20(2): 1674–1694.
DOI: 10.3934/mbe.2023076
Received: 21 June 2022
Revised: 16 October 2022
Accepted: 24 October 2022
Published: 04 November 2022

http://www.aimspress.com/journal/MBE

Research article

Parametric inference on partially accelerated life testing for the inverted Kumaraswamy distribution based on Type-II progressive censoring data

Manal M. Yousef[1,*], Rehab Alsultan[2,*] and Said G. Nassr[3,4,*]

[1] Department of Mathematics, Faculty of Science, New Valley University, El-Khargah 72511, Egypt
[2] Department of Mathematical Science, Faculty of Applied Science, Umm AL-Qura University, Makkah 24382, Saudi Arabia
[3] Department of Statistics and Insurance, Faculty of Commerce, Arish University, Egypt
[4] Higher Institute for Administrative Sciences, Belbis, El-Sharquia, Egypt

* **Correspondence:** Email: manal.mansour@sci.nvu.edu.eg, rasultan@uqu.edu.sa, dr.saidstat@gmail.com.

Abstract: This article discusses the problem of estimation with step stress partially accelerated life tests using Type-II progressively censored samples. The lifetime of items under use condition follows the two-parameters inverted Kumaraswamy distribution. The maximum likelihood estimates for the unknown parameters are computed numerically. Using the property of asymptotic distributions for maximum likelihood estimation, we constructed asymptotic interval estimates. The Bayes procedure is used to calculate estimates of the unknown parameters from symmetrical and asymmetric loss functions. The Bayes estimates cannot be obtained explicitly, therefor the Lindley's approximation and the Markov chain Monte Carlo technique are used to obtaining the Bayes estimates. Furthermore, the highest posterior density credible intervals for the unknown parameters are calculated. An example is presented to illustrate the methods of inference. Finally, a numerical example of March precipitation (in inches) in Minneapolis failure times in the real world is provided to illustrate how the approaches will perform in practice.

Keywords: step-stress partially accelerated life test; Type-II progressive censoring; inverted Kumaraswamy distribution; maximum likelihood method; Bayesian inference; Lindley's approximation; Markov chain Monte Carlo; Monte Carlo simulation

1. Introduction

As a result of significant advancements in high technology, today's products are becoming more and more reliable, and product lifetimes are increasing. A product's failure may take a long period, such as several years, making it difficult, if not impossible, to gather failure information for products that are as reliable under normal settings. While running at a higher stress level shortens the product's life, the accelerated life test (ALT) is utilized to induce more failures and then derive the reliability information under normal conditions. ALT enables the researcher to change the stress level factors to in order to gain information on the parameters of lifetime distributions more quickly than under regular operating conditions. The main assumption in ALT is that the mathematical model in which species the relationship between the average lifetime and the stress is known or the acceleration factor is known. In some cases, such a model does not exist or is very difficult to suppose. So, partially ALT (PALT) is a good nominee to carry out the life test in such cases. Various types of stress loading may be applied when performing PALT. Constant-stress and step-stress are the two most common types. A test unit runs under constant-stress PALT (CSPALT) in one of two modes: normal use or accelerated use. However, with step-stress PALT (SSPALT), a test item is run under normal conditions first, if it does not fail, it is then run under accelerated conditions until it fails or the observation is censored. The various types of PALT models have been the interest of many researchers see [1–17].

In many lifetime studies, it is common for the lifetime of test units to be inaccurately recorded. In practice, investigators need to process the censored data, as they rarely have the time to record and watch all of the people involved in the experience during their course of lifetime. There are various censoring patterns. Type-I and Type-II censoring are the most prevalent censoring techniques used in life testing or reliability experiments. Lately, the Type-II progressive censoring scheme has become popular enough to analyze highly reliable data. This kind of censoring scheme can be described as: suppose n identical items are put to test, the integer $m \le n$ is a prespecified number of failures and $R_1, R_2, \dots, R_m$ are m prefixed integers satisfying $R_1 + R_2 + \cdots + R_m + m = n$. At the time of the first failure $t_{1:m:n}, R_1$ of the surviving units are randomly withdrawn. Likewise, at the time of the second failure $t_{2:m:n}, R_2$ of the surviving units are randomly withdrawn, and so on. At the time of the mth failure $t_{m:m:n}$, the experiment is stopped and all surviving $R_m = n - R_1 - R_2 - \cdots - R_{m-1} - m$ units are withdrawn. Conventional Type-II censoring is a special case when $R_1 = R_2 = \cdots = R_{m-1} = 0$ and $R_m = n - m$. For more details about Type-II progressive censoring, see [18–21].

The inverted Kumaraswamy (IKum) distribution with the parameters $\alpha, \beta > 0$, will be denoted by IKum (α, β). IKum distribution was derived from Kumaraswamy distribution (Kum) using the transformation $X = 1/Y - 1$, when Y has a Kum distribution. Three special cases of IKum (α, β) distribution are Lomax distribution (when $\beta = 1$), inverted beta Type-II distribution (when $\alpha = 1$) and log-logistic (Fisk) distribution (when $\alpha = \beta = 1$). The corresponding cumulative density distribution (CDF), probability density function (PDF) and hazard rate function (HRF) are given, respectively, by

$$F(x) = [1 - (1 + x)^{-\alpha}]^{\beta}, \quad x > 0 \ , \alpha, \beta > 0 \ ,$$

$$f(x) = \alpha\beta(1 + x)^{-(\alpha+1)}[1 - (1 + x)^{-\alpha}]^{\beta-1},$$

$$h(x) = \alpha\beta(1 + x)^{-(\alpha+1)}[1 - (1 + x)^{-\alpha}]^{\beta-1}\left[1 - [1 - (1 + x)^{-\alpha}]^{\beta}\right]^{-1}.$$

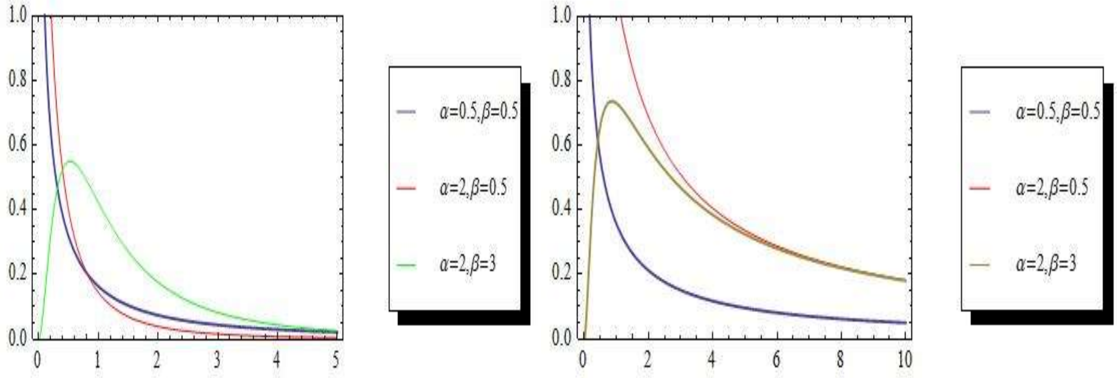

Figure 1. The PDF and HRFplots for the IKum distribution.

Figure 1 shows that the IKum distribution has a lengthy right tail when compared to other commonly used distributions based on the PDF and HRF curves. As a result, it will influence long-term reliability predictions, producing optimistic predictions of uncommon events that occur in the right tail of the distribution when compared to other distributions. Furthermore, the IKum distribution fits various data sets from the literature quite well. IKum distribution was introduced by reference [22]. They investigated various structural properties with the application. They also addressed the problem of estimation of parameters of the IKum distribution based on Type-II censoring. Reference [23] used general progressive censored samples to evaluate the unknown parameters of the IKum distribution. Reference [24] studied relations for moments of dual generalized order statistics for IKum. For more details about IKum distribution see [25,26]

The motivation of this paper is to apply SSPALT to items whose lifetimes under normal stress conditions follow the IKum distribution under Type-II progressive censoring, and to estimate the involved parameters using ML and Bayes methods (under squared error (SE) and linear exponential (LINEX) loss functions). To demonstrate and evaluate the performance of the given estimating methods, an actual data set is investigated. The rest of this article is planned as follows. Section 2 presents the description of the model. In Section 3, both the ML estimates (MLEs) and observed Fisher information matrix are presented. In addition, Lindely's approximation and the Markov chain Monte Carlo (MCMC) technique are used to get the Bayes estimates (Bes) and the highest posterior density (HPD) credible intervals of the model parameters as given in Section 4. illustrative example and Monte Carlo (MC) simulation results are presented in Sections 5 and 6, respectively. Section 7 presents a numerical example to illustrate all methods of inference established in the article in hand. Finally, we make some concluding remarks in Section 8.

2. Model description

In this Section, for Type-II progressive censoring, we develop the following assumptions under SSPALT:

1) The SSPALT is composed of two stress levels, s_0 and s_1 $(s_0 < s_1)$, where $s_0$represents normal stress conditions and $s_1$represents accelerated stress conditions.

2) Suppose n independent and identical distribution units are placed on a life test and at least one failure must be observed for each stress s_0 and s_1.

3) All n units are subjected to an initial stress level s_0. At the fixed pre-specified time τ the stress level is increased to s_1.

4) The lifetime distributions at the stress levels s_0 and s_1 are assumed to be IKum distribution with shape parameters α_1 and α_2 respectively, and a common additional shape parameter β.

Under the assumption of the cumulative exposure model (CEM), the CDF of the lifetime of a test unit under SSPALT is given by

$$G(t)=\begin{cases}G_1(t)=F_1(t), & 0\le t<\tau,\\ G_2(t)=F_2(s+t-\tau), & t\ge\tau,\end{cases}\tag{1}$$

where s is the solution of the equation $F_1(\tau)=F_2(s)$ (see [27]). So, it is evident that $s=(1+\tau)^{\alpha_1/\alpha_2}-1$.

The corresponding PDF of the lifetime of a test unit is

$$g(t)=\begin{cases}\alpha_1\ \beta\ (1+t)^{-(\alpha_1+1)}\ [1-(1+t)^{-\alpha_1}]^{\beta-1}, & 0\le t<\tau,\\ \alpha_2\ \beta\ \left[(1+\tau)^{\alpha_1/\alpha_2}+t-\tau\right]^{-(\alpha_2+1)}\ \left\{1-\left[(1+\tau)^{\alpha_1/\alpha_2}+t-\tau\right]^{-\alpha_2}\right\}^{\beta-1}, & t\ge\tau.\end{cases}\tag{2}$$

3. Maximum likelihood estimation

This part derives the MLEs of unknown model parameters. Also, we obtain the observed Fisher information matrix. Based on the Type-II progressively censored sample, we have n identical units under an initial stress level s_0. The stress level is changed to s_1 at a pre-fixed time τ, and the life-testing experiment is terminated when the mth failure time $t_{m:m:n}$ occurs, where $2\le m\le n$. Let n_1 be the number of units that fail before time τ at stress levelss_0. With these notations the observed progressive censored data is $t_{1:m:n}<t_{2:m:n}<\cdots<t_{n_1:m:n}<\tau<t_{n_1+1:m:n}<\cdots<t_{m:m:n}$. with the corresponding progressive censoring scheme $R=(R_1,\dots,R_m)$, where $\sum_{j=1}^{m}R_j=n-m$.

From the CEM in (1) and the corresponding PDF in (2), the likelihood function (LF) of α_1,α_2 and β are obtained based on the Type-II progressively censored sample as follows:

$$L(\alpha_1,\alpha_2,\beta|t)=C\prod_{i=1}^{n_1}\alpha_1\beta(1+t_{i:m:n})^{-(\alpha_1+1)}\left(\psi_1(t_{i:m:n})\right)^{\beta-1}\left[1-\left(\psi_1(t_{i:m:n})\right)^{\beta}\right]^{R_i}$$
$$\times\prod_{i=n_1+1}^{m}\alpha_2\beta\left(\psi(t_{i:m:n})\right)^{-(\alpha_2+1)}\left(\psi_2(t_{i:m:n})\right)^{\beta-1}\left[1-\left(\psi_2(t_{i:m:n})\right)^{\beta}\right]^{R_i},\tag{3}$$

where $\psi(t_{i:m:n})=(1+\tau)^{\alpha_1/\alpha_2}+t_{i:m:n}-\tau$, $\psi_1(t_{i:m:n})=1-(1+t_{i:m:n})^{-\alpha_1}$, $\psi_2(t_{i:m:n})=1-\left(\psi(t_{i:m:n})\right)^{-\alpha_2}$ and $C=n(n-1-R_1)(n-2-R_1-R_2)\dots\ (n-m-1-\sum_{k=1}^{m-1}R_k)$.

The logarithm of LF may be written as

$$l(\alpha_1,\alpha_2,\beta)=\ln C+n_1\ln\alpha_1+(m-n_1)\ln\alpha_2+m\ln\beta$$
$$-\sum_{i=1}^{n_1}\left[(\alpha_1+1)\ln(1+t_{i:m:n})-(\beta-1)\ln\psi_1(t_{i:m:n})-R_i\ln\left(1-\left(\psi_1(t_{i:m:n})\right)^{\beta}\right)\right]$$
$$-\sum_{i=n_1+1}^{m}\left[(\alpha_2+1)\ln\psi(t_{i:m:n})-(\beta-1)\ln\psi_2(t_{i:m:n})-R_i\ln\left(1-\left(\psi_2(t_{i:m:n})\right)^{\beta}\right)\right].\tag{4}$$

The likelihood equations of α_1, α_1 and β as

$$\frac{\partial L}{\partial\alpha_1}=\frac{n_1}{\alpha_1}-\sum_{i=1}^{n_1}\left[\ln(1+t_{i:m:n})+\left(1-\beta+\beta R_i\psi_3(t_{i:m:n})\right)\frac{1}{\psi_1(t_{i:m:n})}\frac{\partial\psi_1(t_{i:m:n})}{\partial\alpha_1}\right]$$

$$-\sum_{i=n_1+1}^{m}\left[\left((\alpha_2+1)+\left(1-\beta+\beta R_i\psi_4(t_{i:m:n})\right)\frac{\alpha_2\left(\psi(t_{i:m:n})\right)^{-\alpha_2}}{\psi_2(t_{i:m:n})}\right)\frac{1}{\psi(t_{i:m:n})}\frac{\partial\psi(t_{i:m:n})}{\partial\alpha_1}\right], \qquad (5)$$

$$\frac{\partial L}{\partial\alpha_2}=\frac{m-n_1}{\alpha_2}-\sum_{i=n_1+1}^{m}\left[\ln\psi(t_{i:m:n})+(\alpha_2-1)\frac{1}{\psi(t_{i:m:n})}\frac{\partial\psi(t_{i:m:n})}{\partial\alpha_2}\right]$$

$$-\sum_{i=n_1+1}^{m}\left[\left(1-\beta+\beta R_i\psi_4(t_{i:m:n})\right)\frac{1}{\psi_2(t_{i:m:n})}\frac{\partial\psi_2(t_{i:m:n})}{\partial\alpha_2}\right], \qquad (6)$$

$$\frac{\partial L}{\partial\beta}=\frac{m}{\beta}+\sum_{i=1}^{n_1}\left[\left(1-R_i\psi_3(t_{i:m:n})\right)\ln\psi_1(t_{i:m:n})\right]+\sum_{i=n_1+1}^{m}\left[\left(1-R_i\psi_4(t_{i:m:n})\right)\ln\psi_2(t_{i:m:n})\right], \qquad (7)$$

where; $\psi_3(t_{i:m:n})=\frac{(\psi_1(t_{i:m:n}))^{\beta}}{1-(\psi_1(t_{i:m:n}))^{\beta}}$, $\psi_4(t_{i:m:n})=\frac{(\psi_2(t_{i:m:n}))^{\beta}}{1-(\psi_1(t_{i:m:n}))^{\beta}}$, $\frac{\partial\psi(t_{i:m:n})}{\partial\alpha_1}=\frac{1}{\alpha_2}(1+\tau)^{\alpha_1/\alpha_2}\ln(1+\tau)$, $\frac{\partial\psi_1(t_{i:m:n})}{\partial\alpha_1}=(1+t_{i:m:n})^{-\alpha_1}\ln(1+t_{i:m:n})$, $\frac{\partial\psi_2(t_{i:m:n})}{\partial\alpha_1}=\alpha_2\left(\psi(t_{i:m:n})\right)^{-(\alpha_2+1)}\frac{\partial\psi(t_{i:m:n})}{\partial\alpha_1}$, $\frac{\partial\psi(t_{i:m:n})}{\partial\alpha_2}=-\frac{\alpha_1}{\alpha_2}\frac{\partial\psi(t_{i:m:n})}{\partial\alpha_1}$ and $\frac{\partial\psi_2(t_{i:m:n})}{\partial\alpha_2}=\left(\psi(t_{i:m:n})\right)^{-\alpha_2}\ln\psi(t_{i:m:n})\frac{\partial\psi(t_{i:m:n})}{\partial\alpha_2}$.

It can be seen that (5)–(7) cannot be solved explicitly, hence the MLEs of α_1, α_2 and β must be obtained using an appropriate numerical method. The iterative algorithm such as the Newton–Raphson (NR) can be utilized to obtain $\hat{\alpha}_1$, $\hat{\alpha}_2$ and $\hat{\beta}$.

Asymptotic confidence interval

The observed Fisher information matrix on α_1, α_2 and β, $\boldsymbol{I}$, can be obtained by using (5)–(7). If $\Theta=(\alpha_1,\alpha_2,\beta)$, then

$$\boldsymbol{I}=-\left(\frac{\partial^2 l}{\partial\Theta_i\Theta_j}\right),i,j=1,2,3,$$

where the information matrix $\boldsymbol{I}$ is calculated at $\left(\hat{\alpha}_1,\hat{\alpha}_2,\hat{\beta}\right)$. The asymptotic variance-covariance matrix may be approximated as the inverse of $\boldsymbol{I}$. That is,

$$\boldsymbol{I}^{-1}=\left(cov(\Theta_i\Theta_j)\right). \qquad (8)$$

Based on the asymptotic theory of MLEs, the sampling distribution of $\left(\widehat{\Theta}_i-\Theta_i\right)/\sqrt{\hat{\sigma}_{ij}}$ is asymptotically standard normal distribution, where $\hat{\sigma}_{ij}=1/\sqrt{var(\widehat{\Theta}_i)}$ is calculated from (8). Therefore, the $100\ (1-\gamma)\%$ approximate confidence interval (ACI) of Θ_i can then be constructed as

$$\left(\widehat{\Theta}_i-Z_{1-\gamma/2}\ \hat{\sigma}_{ij}\ ,\widehat{\Theta}_i+Z_{1-\gamma/2}\ \hat{\sigma}_{ij}\right), \qquad i=1,2,3.$$

where $Z_{1-\gamma/2}$ is the upper $(\gamma/2)$ percentile of the standard normal distribution.

4. Bayes estimation

This section used SE and LINEX loss functions to obtain BEs of the parameters α_1, α_2 and β. Unfortunately, in many cases, the BEs are not always able to be described explicitly forms. As a result, using Lindley's approximation and MCMC approach, approximate BEs are obtained under informative prior.

Suppose all the unknown parameters are stochastically independent. Assume that the prior distribution for the parameters α_1 and α_2 are taken $Gamma(\mu_1,\nu_1)$ and $Gamma(\mu_2,\nu_2)$ receptively. While the prior distribution of the parameter β is taken $Gamma(\mu_3,\nu_3)$. Hence, the joint prior distribution for α_1, α_2 and β is

$$\pi(\alpha_1,\alpha_2,\beta)=\alpha_1^{\mu_1-1}\alpha_2^{\mu_2-1}\beta^{\mu_3-1}\exp\left[-\left(\frac{\alpha_1}{\nu_1}+\frac{\alpha_2}{\nu_2}+\frac{\beta}{\nu_3}\right)\right]. \quad (9)$$

Combining (3) and (9) to obtain the joint posterior density function of the parameters α_1, α_2 and β as

$$\pi^*(\alpha_1,\alpha_2,\beta)=\alpha_1^{\mu_1+n_1-1}\alpha_2^{\mu_2+m-n_1-1}\beta^{\mu_3+m-1}\exp\left[-\left(\frac{\alpha_1}{\nu_1}+\frac{\alpha_2}{\nu_2}+\frac{\beta}{\nu_3}\right)\right]$$
$$\prod_{i=1}^{n_1}\quad(1+t_{i:m:n})^{-(\alpha_1+1)}\left(\psi_1(t_{i:m:n})\right)^{\beta-1}\left[1-\left(\psi_1(t_{i:m:n})\right)^{\beta}\right]^{R_i}$$
$$\prod_{i=n_1+1}^{m}\quad\left(\psi(t_{i:m:n})\right)^{-(\alpha_2+1)}\left(\psi_2(t_{i:m:n})\right)^{\beta-1}\left[1-\left(\psi_2(t_{i:m:n})\right)^{\beta}\right]^{R_i}. \quad (10)$$

The BEs of the function of the parameters $U(\Theta)=(\alpha_1,\alpha_2,\beta)$ denting by $\widetilde{U}_{BSL}$, we observe that under SE loss function the BE of $U(\Theta)$ is the posterior mean given by

$$\widetilde{U}_{BSL}=E(U(\Theta)|t)=\int_{\Theta}U(\Theta)\ \pi^*(\Theta|t)\ \mathrm{d}\Theta. \quad (11)$$

The SE loss is an asymmetric loss function that puts equal weight to the underestimation and overestimation. In many cases, underestimating a problem is more significant than overestimation a problem, and vice versa. In these circumstances, a LINEX loss can be recommended as an alternative to the SE loss which is given by reference [28]

$$\left(U(\Theta)\ ,\widetilde{U}(\Theta)\right)=e^{\widetilde{U}(\Theta)-U(\Theta)}-C\left(\widetilde{U}(\Theta)-U(\Theta)\right)-1.$$

where $c\neq 0$ is a shape parameter. Here $c>1$ proposes that an overestimation is more serious than the underestimation, and vice versa for $c<0$. Further c approaching to zero replicates the SE loss function itself. One may refer to references [28] and [29] for more details in this regard. The BE of $U(\Theta)$ under this loss can be derived as

$$\widetilde{U}_{BLL}=E(e^{-cU}|t)=-\frac{1}{c}\ln\left[\int_{\Theta}e^{-cU}\ \pi^*(\Theta|t)\quad d\Theta\right]. \quad (12)$$

It is seen that estimates given by (11) and (12) cannot be simplified into closed form expressions. Therefore, we next apply Lindley's approximation method and MCMC technique to obtain the desired BEs.

4.1. Lindley's approximation

Reference [30] proposed an approximation procedure to evaluate the expressions like (11) and (12). Reference [31] applied this method to obtain BEs under the considered prior distribution. For the three-parameter case $U(\Theta|t)$, we observe that $E\left(U(\Theta|t)\right)$ can be approximated as

$$E\left(U(\Theta|t)\right)=U+[U_1a_1+U_2a_2+U_3a_3+a_4+a_5]$$
$$+\frac{1}{2}[\phi_1(U_1\sigma_{11}+U_2\sigma_{12}+U_3\sigma_{13})+\phi_2(U_1\sigma_{21}+U_2\sigma_{22}+U_3\sigma_{23})+\phi_3(U_1\sigma_{31}+U_2\sigma_{32}+U_3\sigma_{33})]. \quad (13)$$

where; $U_i=\frac{\partial U}{\partial \xi_i}$, σ_{ij} is the element (i,j) in the variance-covariance matrix $\left(-L_{ij}\right)$, $i,j=1,2,3$, and

$$a_i = \rho_1\sigma_{i1} + \rho_2\sigma_{i2} + \rho_3\sigma_{i3}, i = 1,2,3,$$

$$a_4 = U_{12}\sigma_{12} + U_{13}\sigma_{13} + U_{23}\sigma_{23},\ a_5 = \frac{1}{2}(U_{11}\sigma_{11} + U_{22}\sigma_{22} + U_{33}\sigma_{33}),$$

$$\phi_1 = \sigma_{11}L_{111} + 2(\sigma_{12}L_{121} + \sigma_{13}L_{131} + \sigma_{23}L_{231}) + \sigma_{22}L_{221} + \sigma_{33}L_{331},$$

$$\phi_2 = \sigma_{11}L_{112} + 2(\sigma_{12}L_{122} + \sigma_{13}L_{132} + \sigma_{23}L_{232}) + \sigma_{22}L_{222} + \sigma_{33}L_{332},$$

$$\phi_3 = \sigma_{11}L_{113} + 2(\sigma_{12}L_{123} + \sigma_{13}L_{133} + \sigma_{23}L_{233}) + \sigma_{22}L_{223} + \sigma_{33}L_{333},$$

$$\rho_i = \frac{\partial\rho}{\partial\xi_i},\ U_{ij} = \frac{\partial^2 U}{\partial\xi_i\ \partial\xi_j},\ L_{ijk} = \frac{\partial^3 L}{\partial\xi_i\ \partial\xi_j\ \partial\xi_k}.$$

Form the prior distribution in (9) and (13), the values of the BEs of various parameters under SE loss function are

$$\tilde{\alpha}_{1BSL} = \hat{\alpha}_1 + a_1 + \frac{1}{2}(\phi_1\sigma_{11} + \phi_2\sigma_{12} + \phi_3\sigma_{13}), \tag{14}$$

$$\tilde{\alpha}_{2B} \quad = \hat{\alpha}_2 + a_2 + \frac{1}{2}(\phi_1\sigma_{12} + \phi_2\sigma_{22} + \phi_3\sigma_{23}), \tag{15}$$

$$\tilde{\beta}_{BSL} = \hat{\beta} + a_3 + \frac{1}{2}(\phi_1\sigma_{13} + \phi_2\sigma_{23} + \phi_3\sigma_{33}). \tag{16}$$

the BEs of various parameters under LINEX loss function are

$$\tilde{\alpha}_{1BLL} = -\frac{1}{c}\ln\left[e^{-c\hat{\alpha}_1}\left(1 + \frac{c^2}{2}\sigma_{11} - c\hat{\alpha}_1 - \frac{c}{2}(\phi_1\sigma_{11} + \phi_2\sigma_{12} + \phi_3\sigma_{13})\right)\right], \tag{17}$$

$$\tilde{\alpha}_{2BLL} = -\frac{1}{c}\ln\left[e^{-c\hat{\alpha}_2}\left(1 + \frac{c^2}{2}\sigma_{22} - c\hat{\alpha}_2 - \frac{c}{2}(\phi_1\sigma_{12} + \phi_2\sigma_{22} + \phi_3\sigma_{23})\right)\right], \tag{18}$$

$$\tilde{\beta}_{BLL} = -\frac{1}{c}\ln\left[e^{-c\hat{\beta}}\left(1 + \frac{c^2}{2}\sigma_{33} - c\hat{\beta} - \frac{c}{2}(\phi_1\sigma_{13} + \phi_2\sigma_{23} + \phi_3\sigma_{33})\right)\right]. \tag{19}$$

The forms (14)–(16) and (17)–(19) are evaluated at the MLEs of the parameters α_1, α_2 and β respectively.

4.2. Markov chain Monte Carlo

The MCMC techniques are a general simulation method for sampling from posterior distributions and computing posterior quantities of interest. Indeed, the MCMC samples may be used to completely summarize the posterior uncertainty about the parameters α_1, α_2 and β through a kernel estimate of the posterior distribution. From the joint posterior density function in (10), the conditional posterior distributions of α_1, α_2 and β can be written, respectively, as

$$\pi^*(\alpha_1|\alpha_2,\beta,t) \propto \alpha_1^{\mu_1+n_1-1}e^{-\left(\frac{\alpha_1}{\nu_1}\right)}\prod_{i=1}^{n_1}(1+t_{i:m:n})^{-\alpha_1}\left(\psi_1(t_{i:m:n})\right)^{\beta-1}\left[1-\left(\psi_1(t_{i:m:n})\right)^{\beta}\right]^{R_i}$$

$$\prod_{i=n_1+1}^{m}\quad\left(\psi(t_{i:m:n})\right)^{-(\alpha_2+1)}\left(\psi_2(t_{i:m:n})\right)^{\beta-1}\left[1-\left(\psi_2(t_{i:m:n})\right)^{\beta}\right]^{R_i},$$

$$\pi^*(\alpha_2|\alpha_1,\beta,t) \propto \alpha_2^{\mu_2+m-n_1-1}e^{-\left(\frac{\alpha_2}{\nu_2}\right)}\exp\left[-\left(\frac{\alpha_2}{\nu_2}\right)\right]$$

$$\prod_{i=n_1+1}^{m} \left(\psi(t_{i:m:n})\right)^{-(\alpha_2+1)} \left(\psi_2(t_{i:m:n})\right)^{\beta-1} \left[1-\left(\psi_2(t_{i:m:n})\right)^{\beta}\right]^{R_i},$$

$$\pi^*(\beta|\alpha_1,\alpha_2,t) \propto \beta^{\mu_3+m-1} e^{-\left(\frac{\beta}{\nu_3}\right)} \prod_{i=1}^{n_1} \left(\psi_1(t_{i:m:n})\right)^{\beta} \left[1-\left(\psi_1(t_{i:m:n})\right)^{\beta}\right]^{R_i}$$

$$\prod_{i=n_1+1}^{m} \left(\psi_2(t_{i:m:n})\right)^{\beta} \left[1-\left(\psi_2(t_{i:m:n})\right)^{\beta}\right]^{R_i}.$$

It can be seen that the conditional posterior distributions of α_1, α_2 and β cannot be reduced analytically to well-known distribution, but the plot of them shows that they are similar to normal distribution see Figures 2–7. So, the Metropolis-Hastings (MH) method is used to generate random samples from this distribution, with normal proposal distribution.

The following MCMC procedure is proposed to compute BEs for the function $U \equiv U(\alpha_1 \ ,\alpha_2,\beta)$

Step 1: Start with $\alpha_1^{(0)} = \hat{\alpha}_1 \ , \alpha_2^{(0)} = \hat{\alpha}_2$ and $\beta^{(0)} = \hat{\beta}$.

Step 2: Set $i = 1$.

Step 3: Generate α_1^* from proposal distribution $N\left(\alpha_1^{(i-1)}, var\left(\alpha_1^{(i-1)}\right)\right)$.

Step 4: Calculate the acceptance probability

$$S\left(\alpha_1^{(i-1)}\middle|\alpha_1^*\right) = \min\left(1, \frac{\pi^*\left(\alpha_1^*\middle|\alpha_2^{(i-1)}, \beta^{(i-1)}, t\right)}{\pi^*\left(\alpha_1^*\middle|\alpha_2^{(i-1)}, \beta^{(i-1)}, t\right)}\right).$$

Step 5: Generate $U \sim U(0,1)$.

Step 6: If $U \leq S\left(\alpha_1^{(i-1)}\middle|\alpha_1^*\right)$, accept the proposal distribution and set $\alpha_1^{(i)} = \alpha_1^*$. Otherwise, reject the proposal distribution and set $\alpha_1^{(i)} = \alpha_1^{(i-1)}$.

Step 7: To generate α_2^* and β^* do the Steps 2–6 for α_2 and β.

Step 8: Set $i = i + 1$.

Step 9: Repeat Steps 3–8 N times.

Step 10: Obtain the BEs of $U(\alpha_1 \ ,\alpha_2,\beta)$ using MCMC under SE and LINEX loss functions as

$$\widetilde{U}_{BSM} = E\left(U(\alpha_1 \ ,\alpha_2,\beta|t)\right) = \frac{1}{N-M} \sum_{i=M+1}^{N} U\left(\alpha_1^{(i)}, \alpha_2^{(i)}, \beta^{(i)}\right),$$

$$\widetilde{U}_{BLM} = E\left(e^{-cU(\alpha_1 \ ,\alpha_2,\beta|t)}\right) = -\frac{1}{c}\ln\left(\frac{1}{N-M}\sum_{i=M+1}^{N} e^{-cU\left(\alpha_1^{(i)},\alpha_2^{(i)},\beta^{(i)}\right)}\right),$$

where M is the burn-in period.

Step 11: $\left[U_{(N-M)\gamma/2}\left(\alpha_1^{(i)}, \alpha_2^{(i)}, \beta^{(i)}\right),\ U_{(N-M)(1-\gamma/2)}\left(\alpha_1^{(i)}, \alpha_2^{(i)}, \beta^{(i)}\right)\right]$ yields an approximate $100(1-\gamma)\%$ credible interval for $U\left(\alpha_1^{(i)}, \alpha_2^{(i)}, \beta^{(i)}\right)$.

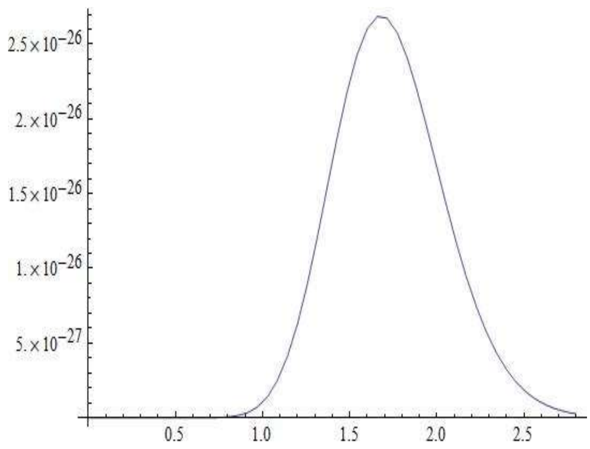

Figure 2. Posterior density function of α_1.

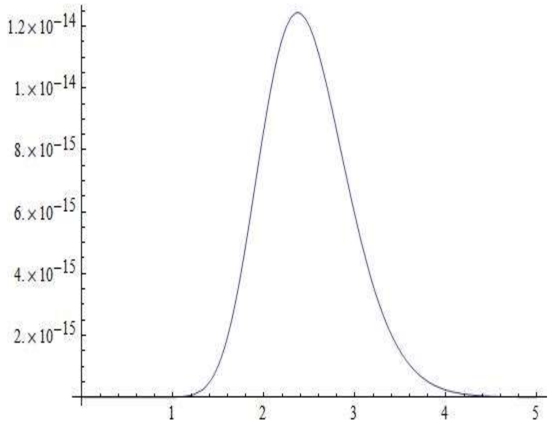

Figure 3. Posterior density function of α_2.

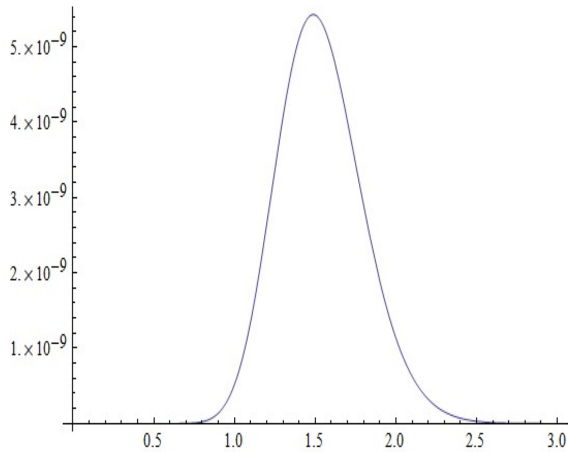

Figure 4. Posterior density function of β.

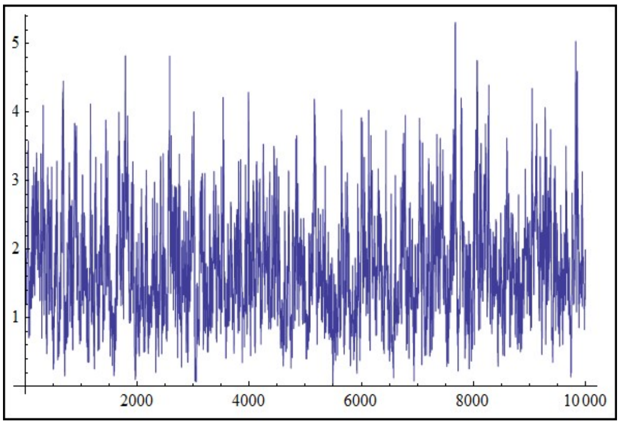

Figure 5. Simulation number of α_1 generated by the MCMC method.

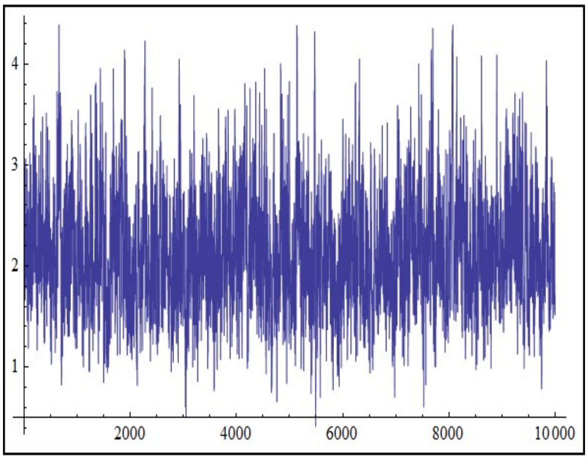

Figure 6. Simulation number of α_2 generated by the MCMC method.

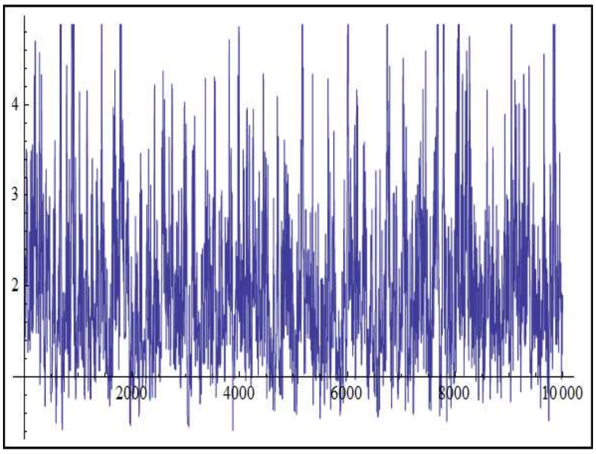

Figure 7. Simulation number of β generated by the MCMC method.

5. Illustrative example

This section presents an example to illustrate the estimation procedure and the CIs for the parameters $(\alpha_1, \alpha_2, \beta)$. Generate the Type-II progressive censoring from (3) with true value for parameters $(\alpha_1, \alpha_2, \beta)$ as $(1.50, 2.51, 1.71)$ when $(n, m, \tau) = (20, 15, 0.8)$ with censoring scheme $(m_1, m_2, \ldots, m_{15}) = (5, 0, \ldots, 10)$. The simulated data is listed in Table 1.

Table 1. SSPALT simulation data.

Failure times under normal condition				Failure times under accelerated condition			
0.161522	0.221959	0.236070	0.329451	0.809535	0.883767	0.95693	0.967955
0.522292	0.524396	0.566979		1.942770	2.307990	4.18468	5.613810

We obtain the average and mean square error (MSE) for $\Theta = (\alpha_1, \alpha_2, \beta)$ in Table 2. Also, Approximate and HPD CIs and their lengths are listed in Table 3.

Table 2. The average and MSE for $\Theta = (\alpha_1, \alpha_2, \beta)$.

Θ	ML	Bayes					
		Lindley's			MCMC		
		BSL	BLL1	BLL2	BSM	BLM1	BLM2
α_1	2.37129	0.97264	1.12533	1.44004	1.72924	1.58455	1.26443
	(0.75915)	(0.2781)	(0.14038)	(0.00359)	(0.05255)	(0.00715)	(0.05549)
α_2	2.37717	1.9153	1.87413	1.86157	2.13492	2.0524	1.84584
	(0.01765)	(0.35367)	(0.40433)	(0.42046)	(0.14068)	(0.20939)	(0.44111)
β	2.49709	1.15864	1.21243	1.49745	1.96639	1.80225	1.50235
	(0.61951)	(0.3040)	(0.24757)	(0.04518)	(0.06573)	(0.00851)	(0.04312)

Table 3. Approximate and HPD CIs and their lengths of the parameters $(\alpha_1, \alpha_2, \beta)$.

(n, m)	Θ	Cis		Lengths of CIs	
		Appr. CI	HPD CI	Appr. CI	HPD CI
$(40, 30)$	α_1	(0.11836, 4.62423)	(0.45268, 3.49434)	4.50588	3.04166
	α_2	(1.07654, 3.67779)	(1.12724, 3.43897)	2.60125	2.31173
	β	(-0.17125, 5.16542)	(0.75421, 4.22058)	5.33667	3.46637

6. Simulation study

This part conducts a simulation analysis to examine the performance of proposed estimates for the Type-II progressive censoring schemes in terms of average estimates and MSE values. We mention that Mathematica 7 software is used, and the simulation is based on 1000 repetitions. The schemes used are as follows:

Scheme I: $R_1 = n - m$ and $R_i = 0$ for $i \neq 1$.

Scheme II: $R_2 = n - m$ and $R_i = 0$ for $i \neq 2$.

Scheme III: $R_m = n - m$ and $R_i = 0$ for $i \neq m$.

The recommended estimates for each scheme are computed using the Type-II progressive censored samples obtained from the IKum distribution using the algorithm given by reference [32]. We select the hyper-parameters $(\mu_1 = 0.5, \nu_1 = 1.2, \mu_2 = 0.9, \nu_2 = 1.8, \mu_3 = 1.5, \nu_3 = 1.5)$ in (9) that allow us to generate the values of α_1, α_2 and β. These generated values are $(\alpha_1, \alpha_2, \beta\ ) = (1.50, 2.51, 1.71)$. The stress change time τ is chosen to be equal to 0.8.

The MLEs and BEs of α_1, α_2 and β are obtained based on these censoring schemes. Also, the 95% CIs are computed based on the asymptotic distribution of the MLEs and Bayesian CIs for α_1, α_2 and β. We replicate the process 1000 times and then compute the average and MSEs of the resulting estimates as well as the average lengths Appr. and HPD CIs. We consider BLL1, BLL2, BLM1 and BLM2 with a notation that $c = 0.5, 2$. The respective results are reported up to 5 decimal places in Tables 4–9.

Table 4. The average and MSE of $\alpha_1, \alpha_2, \beta$ for scheme I.

(n, m)	Θ	ML	Bayes					
			Lindley's			MCMC		
			BSL	BLL1	BLL2	BSM	BLM1	BLM2
$(40, 30)$	α_1	1.65543	1.28705	1.2519	1.21992	1.42057	1.3590	1.20177
		(0.42999)	(0.21481)	(0.24574)	(0.32018)	(0.21822)	(0.21371)	(0.23893)
	α_2	2.65326	2.44264	2.38662	2.3049	2.50495	2.43849	2.26092
		(0.33867)	(0.22183)	(0.22122)	(0.25676)	(0.23316)	(0.21504)	(0.22047)
	β	1.97409	1.57305	1.54014	1.5136	1.74602	1.66684	1.49469
		(0.70691)	(0.14638)	(0.16539)	(0.30586)	(0.25685)	(0.20506)	(0.17016)
$(40, 35)$	α_1	1.64862	1.30962	1.27755	1.24229	1.42521	1.36892	1.22295
		(0.39323)	(0.19826)	(0.22037)	(0.28567)	(0.20355)	(0.19970)	(0.22091)
	α_2	2.64346	2.45204	2.40356	2.32678	2.50473	2.44703	2.28995
		(0.31550)	(0.21374)	(0.21219)	(0.23924)	(0.22513)	(0.21044)	(0.21137)
	β	1.99358	1.58682	1.57038	1.54711	1.75867	1.68156	1.51260
		(0.81505)	(0.20465)	(0.18208)	(0.34996)	(0.28678)	(0.22754)	(0.17760)
$(60, 50)$	α_1	1.62312	1.41280	1.37516	1.31596	1.45844	1.41501	1.29831
		(0.26361)	(0.15324)	(0.16263)	(0.19733)	(0.16181)	(0.15786)	(0.16666)
	α_2	2.62169	2.49885	2.46070	2.38683	2.51963	2.47823	2.36218
		(0.20820)	(0.15756)	(0.15314)	(0.16136)	(0.16234)	(0.15376)	(0.15071)
	β	1.89337	1.68729	1.63844	1.57146	1.74089	1.68618	1.55662
		(0.43261)	(0.15993)	(0.15977)	(0.20976)	(0.21772)	(0.18392)	(0.14811)
$(60, 55)$	α_1	1.59519	1.40350	1.36644	1.30551	144190	1.40112	1.29076
		(0.21686)	(0.13908)	(0.14671)	(0.17629)	(0.14217)	(0.14075)	(0.15416)
	α_2	2.57916	2.46693	2.43251	2.36330	2.48441	2.44729	2.34256
		(0.18811)	(0.15289)	(0.15072)	(0.16101)	(0.15414)	(0.14883)	(0.15115)
	β	1.86461	168288	1.63200	1.55915	1.72537	1.67403	1.55043
		(0.32666)	(0.14426)	(0.13634)	(0.16560)	(0.17773)	(0.15357)	(0.13120)

Table 5. The average and MSE of $\alpha_1, \alpha_2, \beta$ for scheme II.

(n,m)	Θ	ML	Bayes					
			Lindley's			MCMC		
			BSL	BLL1	BLL2	BSM	BLM1	BLM2
$(40,30)$	α_1	1.68862	1.29947	1.26764	1.241690	1.43982	1.37699	1.21666
		(0.43057)	(0.21428)	(0.23551)	(0.30259)	(0.21065)	(0.20411)	(0.22597)
	α_2	2.66861	2.45134	2.39578	2.31587	2.51442	2.44717	2.26779
		(0.33133)	(0.21266)	(0.21032)	(0.24253)	(0.22553)	(0.20601)	(0.20951)
	β	2.01308	1.57484	1.56057	1.54242	1.76780	1.68805	151427
		(0.77083)	(0.30166)	(0.16461)	(0.32316)	(0.25854)	(0.20274)	(0.16110)
$(40,35)$	α_1	1.65135	1.31154	1.27931	1.24215	1.42709	1.37048	1.22409
		(0.34735)	(0.19456)	(0.20292)	(0.26069)	(0.17793)	(0.17578)	(0.20172)
	α_2	2.62027	2.43070	2.38256	2.30568	2.48129	2.42432	2.26923
		(0.27903(	(0.20208)	(0.20078)	(0.22693)	(0.20463)	(0.19348)	(0.20321)
	β	1.96691	1.56982	1.55748	1.52932	1.73960	1.66554	1.50225
		(0.69979)	(0.38616)	(0.15731)	(0.30310)	(0.23953)	(0.19178)	(0.15897)
$(60,50)$	α_1	1.60625	1.39965	1.36213	1.30233	1.44417	1.40106	1.20507
		(0.26531)	(0.16239)	(0.17236)	(0.20804)	(0.16985)	(0.16675)	(0.17770)
	α_2	2.60057	2.48035	2.44261	2.36890	2.50041	2.45968	2.34547
		(0.18288)	(0.14292)	(0.14011)	(0.15048)	(0.14607)	(0.13983)	(0.14263)
	β	1.88494	1.68365	1.63859	1.56933	1.73673	1.68342	1.55592
		(0.41470)	(0.15605)	(0.15761)	(0.21082)	(0.20395)	(0.17375)	(0.14270)
$(60,55)$	α_1	1.61851	1.42172	1.38574	1.32649	1.46185	1.42079	1.30954
		(0.23619)	(0.13926)	(0.14718)	(0.17751)	(0.14668)	(0.14335)	(0.15183)
	α_2	2.59645	2.48323	2.44889	2.37962	2.50084	2.46374	2.35885
		(0.16439)	(0.12894)	(0.12704)	(0.13638)	(0.13043)	(0.12576)	(0.12920)
	β	1.89490	1.69924	1.65377	1.58544	1.74823	1.69593	1.57028
		(0.40487)	(0.14542)	(0.14846)	(0.19649)	(0.19938)	(0.16804)	(0.13378)

Table 6. The average and MSE of $\alpha_1, \alpha_2, \beta$ for scheme III.

(n,m)	Θ	ML	Bayes					
			Lindley's			MCMC		
			BSL	BLL1	BLL2	BSM	BLM1	BLM2
$(40,30)$	α_1	1.67468	1.33963	1.30954	1.27582	1.44789	1.39401	1.25246
		(0.38210)	(0.18888)	(0.20886)	(0.26779)	(0.19642)	(0.19114)	(0.20608)
	α_2	2.71109	2.45218	2.38609	2.30586	2.52052	2.43673	2.22214
		(0.52014)	(0.28634)	(0.28811)	(0.35025)	(0.31761)	(0.27747)	(0.27014)
	β	2.00940	1.59348	1.58277	1.56234	1.76239	1.68673	1.51879
		(0.84333)	(0.23874)	(0.19153)	(0.36994)	(0.28759)	(0.22777)	(0.17522)
$(40,35)$	α_1	1.68589	1.35414	1.32299	1.28701	1.45995	1.40578	1.26368
		(0.37445)	(0.17471)	(0.19429)	(0.25425)	(0.18654)	(0.18062)	(0.19413)

Continued on next page

(n,m)	Θ	ML	Bayes					
			Lindley's			MCMC		
			BSL	BLL1	BLL2	BSM	BLM1	BLM2
	α_2	2.66805	2.45790	2.40602	2.32949	2.51081	2.44812	2.27892
		(0.35275)	(0.23424)	(0.23040)	(0.25831)	(0.24459)	(0.22541)	(0.22312)
	β	2.02579	1.61455	1.60452	1.57674	1.78200	1.70581	1.53676
		(0.83181)	(0.34268)	(0.18228)	(0.36377)	(0.27424)	(0.21447)	(0.16139)
$(60,50)$	α_1	1.60030	1.41001	1.37509	1.31679	1.44653	1.40723	1.30017
		(0.21965)	(0.14099)	(0.14861)	(0.17702)	(0.14551)	(0.14381)	(0.15526)
	α_2	2.61698	2.48318	2.44130	2.36411	2.50299	2.45715	2.32968
		(0.24871)	(0.19030)	(0.18436)	(0.19556)	(0.19353)	(0.18235)	(0.17888)
	β	1.87625	1.68946	1.641710	1.57152	1.73156	1.68095	1.55817
		(0.34566)	(0.14527)	(0.14214)	(0.17674)	(0.18279)	(0.15742)	(0.13209)
$(60,55)$	α_1	1.61398	1.42225	1.38779	1.32984	1.45989	1.42045	1.31294
		(0.21681)	(0.13114)	(0.13831)	(0.16568)	(0.13753)	(0.13484)	(0.14381)
	α_2	2.60790	2.48732	2.45145	2.38085	2.50462	2.46549	2.35508
		(0.18854)	(0.14604)	(0.14322)	(0.15220)	(0.14838)	(0.14218)	(0.14344)
	β	1.88892	1.69278	1.65098	1.58316	1.74135	1.69007	1.56630
		(0.40552)	(0.14335)	(0.15024)	(0.20478)	(0.19462)	(0.16359)	(0.13086)

Table 7. Approximate and HPD CIs and their lengths of the parameters $(\alpha_1, \alpha_2, \beta)$ for scheme I.

(n,m)	Θ	Cis		Lengths of CIs	
		Appr. CI	HPD CI	Appr. CI	HPD CI
$(40,30)$	α_1	(0.44842, 2.86243)	(0.57881, 2.51126)	2.41400	1.93245
	α_2	(1.57289, 3.73362)	(1.58029, 3.59153)	2.16073	2.01124
	β	(0.56166, 3.38652)	(0.89460, 3.05901)	2.82485	2.16441
$(40,35)$	α_1	(0.51150, 2.78575)	(0.61392, 2.46083)	2.27425	1.84690
	α_2	(1.64491, 3.64201)	(1.63511, 3.50903)	1.99710	1.87392
	β	(0.61090, 3.37626)	(0.91712, 3.03803)	2.76536	2.12091
$(60,50)$	α_1	(0.68488, 2.56136)	(0.72896, 2.35349)	1.87648	1.62453
	α_2	(1.79284, 3.45054)	(1.77264, 3.36206)	1.65770	1.58942
	β	(0.83455, 2.95220)	(1.00713, 2.79542)	2.11765	1.78829
$(60,55)$	α_1	(0.69241, 2.49797)	(0.73149, 2.30604)	1.80556	1.57456
	α_2	(1.79774, 3.36058)	(1.77490, 3.27798)	1.56284	1.50307
	β	(0.84745, 2.88178)	(1.00683, 2.7615)	2.03434	1.73932

Table 8. Approximate and HPD CIs and their lengths of the parameters $(\alpha_1, \alpha_2, \beta)$ for scheme II.

(n,m)	Θ	Cis		Lengths of CIs	
		Appr. CI	HPD CI	Appr. CI	HPD CI
$(40,30)$	α_1	(0.46780, 2.90944)	(0.58824, 2.5402)	2.44164	1.95197
	α_2	(1.58403, 3.75319)	(1.58524, 3.60722)	2.16915	2.02198

Continued on next page

(n, m)	Θ	Cis		Lengths of CIs	
		Appr. CI	HPD CI	Appr. CI	HPD CI
	β	(0.58767, 3.43848)	(0.90980, 3.07903)	2.85081	2.16922
$(40, 35)$	α_1	(0.50841, 2.79430)	(0.61227, 2.46790)	2.28589	1.85563
	α_2	(1.62795, 3.61259)	(1.61686, 3.47999)	1.98464	1.86313
	β	(0.61274, 3.32109)	(0.91109, 2.99448)	2.70834	2.08340
$(60, 50)$	α_1	(0.67377, 2.53872)	(0.71706, 2.33422)	1.86496	1.61716
	α_2	(1.77842, 3.42272)	(1.75907, 3.33550)	1.64431	1.57643
	β	(0.84327, 2.92661)	(1.00736, 2.77472)	2.08334	1.76736
$(60, 55)$	α_1	(0.71446, 2.52256)	(0.74751, 2.32754)	1.80809	1.58002
	α_2	(1.81359, 3.37931)	(1.78876, 3.29593)	1.56572	1.50717
	β	(0.86825, 2.92154)	(1.02278, 2.77349)	2.05329	1.75071

Table 9. Approximate and HPD CIs and their lengths of the parameters $(\alpha_1, \alpha_2, \beta)$ for scheme III.

(n, m)	Θ	Cis		Lengths of CIs	
		Appr. CI	HPD CI	Appr. CI	HPD CI
$(40, 30)$	α_1	(0.57373, 2.77563)	(0.64412, 2.45263)	2.20190	1.80851
	α_2	(1.48642, 3.93577)	(1.50360, 3.74638)	2.44935	2.24278
	β	(0.64792, 3.37089)	(0.92108, 3.01884)	2.72297	2.09776
$(40, 35)$	α_1	(0.58083, 2.79095)	(0.65407, 2.46931)	2.21012	1.81524
	α_2	(1.62504, 3.71106)	(1.60794, 3.55714)	2.08601	1.94920
	β	(0.65645, 3.39514)	(0.93527, 3.05102)	2.73868	2.11575
$(60, 50)$	α_1	(0.72122, 2.47937)	(0.74526, 2.29158)	1.75815	1.54631
	α_2	(1.74578, 3.48818)	(1.72035, 3.38819)	1.74240	1.66784
	β	(0.87150, 2.88100)	(1.01345, 2.7403)	2.00950	1.72685
$(60, 55)$	α_1	(0.73253, 2.49543)	(0.75623, 2.30571)	1.76290	1.54948
	α_2	(1.80380, 3.41201)	(1.77484, 3.32044)	1.60821	1.54560
	β	(0.87613, 2.90170)	(1.02087, 2.75243)	2.02557	1.73156

7. Real data analysis

This section analyzes a real data set given by reference [33]. It consists of thirty successive values of March precipitation (in inches) in Minneapolis/St Paul. The data are as follows:

0.77, 1.74, 0.81, 1.20, 1.95, 1.20, 0.47, 1.43, 3.37, 2.20, 3.00, 3.09, 1.51, 2.10, 0.52,
1.62, 1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.90, 2.05.

Reference [22], verified that the IKum distribution provides a good fit for the given data set. The calculated Kolmogorov-Smirnov $(K-S)$ distance between the empirical and the fitted for the IKum distribution was 0.1105 and its p-value is 0.8571 where $\hat{\alpha} = 3.0038$ and $\hat{\beta} = 8.7984$ which indicates that this distribution can be considered as an adequate model for the given data set.

Now, using SSPALT, we will analyze the supplied data by setting the value of τ to be 1.3. From the original data, three Type-II progressive censored schemes are generated with number of stages $m = 20$ from a total of $n = 30$ observations and removed items R_j, where $j = 1, 2, \ldots, m$. These different schemes can be described as follows:

Scheme I: $R_1 = n - m$ and $R_2 = R_3 = \cdots = R_m = 0$.
Scheme II: $R_1 = R_2 = R_3 = \cdots = R_{m-1} = 0$ and $R_m = n - m$.
Scheme III: $R_1 = R_2 = R_3 = \cdots = R_m = 0$ and $n = m$.

Note that Type-II censoring (Scheme II) and complete sampling (Scheme III) can be considered as a special case of Type-II progressive censoring when $n = m$ and $R_1 = R_2 = R_3 = \cdots = R_m = 0$.

We calculate the MLEs of the parameters α and β and their associated 95% asymptotic CIs. We also compute BEs utilizing the MH algorithm under the non-informative prior. Note that the non-informative prior is assumed where $\mu_i = \nu_i = 0,\ i = 1, 2, 3$. It is indicated that, while generating samples from the posterior distribution utilizing the MH algorithm, initial values of (α, β) are considered as $\left(\alpha^{(0)}, \beta^{(0)}\right) = (\hat{\alpha}, \hat{\beta})$, where $\hat{\alpha}$ and $\hat{\beta}$ are the MLEs of the parameters α and β respectively. Thus, we considered the variance–covariance matrix S_Θ of $\left(\ln(\hat{\alpha}), \ln(\hat{\beta})\right)$, that can be easily obtained utilizing the delta method. Finally, we discarded 2000 burn-in samples among the total 10000 samples created from the posterior density, and subsequently obtained Bayes estimates, and HPD interval estimates utilizing the technique of [34].

All the estimated values of MLEs and associated standard errors (St.Er) are presented in Table 10. Also, Bayesian estimation using Lindley's approximations and MCMC by applying MH algorithm and its St.Er are computed. Approximate CIs for MLEs and HPD for Bayesian estimates using MCMC are presented in Table 11 under SE loss function.

Table 10. ML and Bayesian estimates with associated St.Er (in practices) based on different Type-II progressive censoring schemes scheme for a given real data set.

Scheme	Θ	ML	Bayesian					
			Lindley's			MCMC		
			BSL	BLL1	BLL2	BSM	BLM1	BLM2
Scheme I	α_1	1.9162	1.9288	1.9365	1.9540	1.9137	1.9103	1.9032
		(0.6442)				(0.0924)		
	α_2	3.0628	3.0634	3.0684	3.0706	3.0623	3.0583	3.0498
		(0.6638)				(0.0995)		
	β	4.7644	4.7605	4.7652	4.7646	4.7602	4.7562	4.7478
		(2.2619)				(0.1007)		
Scheme II	α_1	24812	2.4862	2.4904	2.998	2.4771	2.4737	2.4666
		(0.6410)				(0.0924)		
	α_2	2.5023	2.5041	2.5066	2.5103	2.5015	2.4979	2.4901
		(0.4429)				(0.0934)		
	β	7.8787	7.8778	7.8788	7.8787	7.8810	7.8770	7.8685
		(3.7634)				(0.9982)		
Scheme III	α_1	2.9300	2.9331	2.9367	2.9443	2.9259	2.9224	2.9149
		(0.6427)				(0.0851)		
	α_2	3.3399	3.3405	3.3420	3.3442	3.3358	3.3319	3.3237
		(0.5232)				(0.0724)		
	β	7.7583	7.7577	7.7582	7.7582	7.7558	7.7518	7.7433
		(3.3666)				(0.0905)		

Table 11. Approximate and HPD CIs and their lengths based on different Type-II progressive censoring schemes for a given real data set.

Scheme	Θ	CIs		Lengths of CIs	
		Appr. CI	HPD CI	Appr. CI	HPD CI
Scheme I	α_1	(0.6536, 3.1788)	(1.7411, 2.1002)	2.5252	0.3591
	α_2	(1.7617, 4.3638)	(2.8737, 3.2645)	2.6021	0.3908
	β	(0.3312, 9.1975)	(4.5711, 4.9614)	8.8663	0.3903
Scheme II	α_1	(1.2249, 3.7376)	(2.2932, 2.6537)	2.5127	0.3605
	α_2	(1.6341, 3.3706)	(2.3091, 2.6851)	1.7365	0.3760
	β	(0.5025, 15.2549)	(7.6931, 8.0797)	14.7524	0.3866
Scheme III	α_1	(1.6702, 4.1897)	(2.7439, 3.1108)	2.5195	0.3669
	α_2	(2.3144, 4.3655)	(3.1494, 3.5284)	2.0510	0.3790
	β	(1.15987, 14.3568)	(7.5501, 7.9403)	13.1969	0.3902

The convergence of MCMC estimation in case of scheme II of Type-II progressive censoring for the given real data set. As shown in Figure8, the Bayesian estimates using MCMC are convergence through three sub-graphs: scatter plot, histogram, and cumulative mean of the 10,000 estimates.

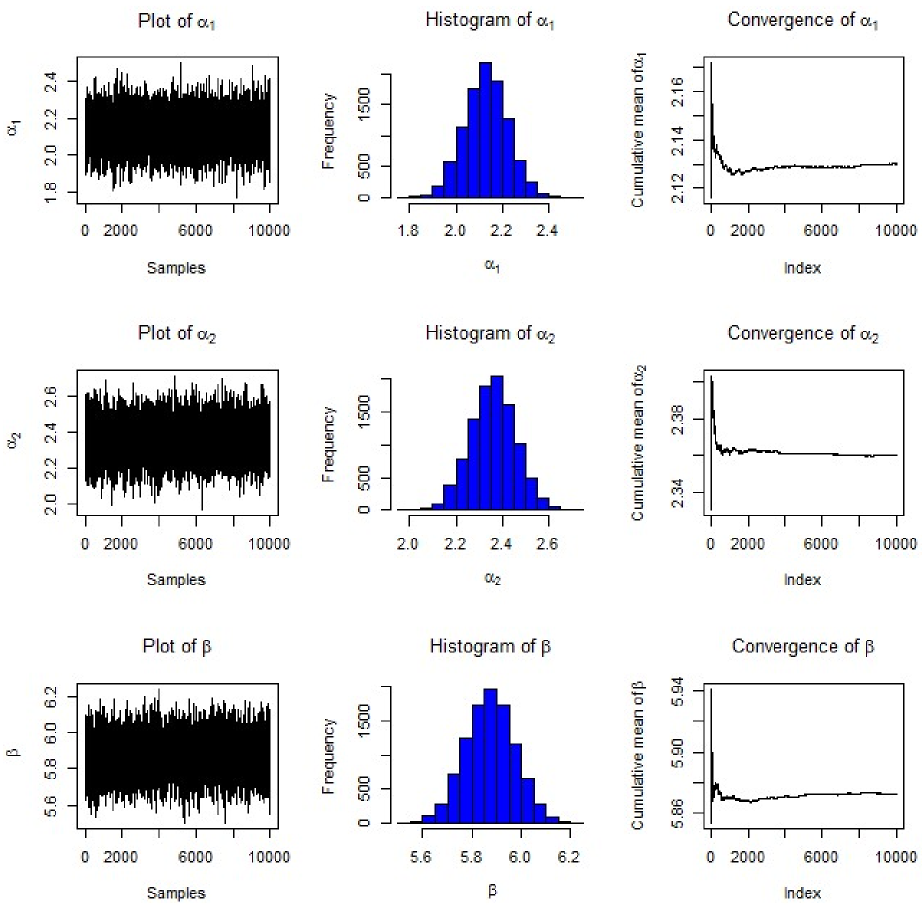

Figure 8. Convergence of MCMC estimates for α_1, α_2 and β using MH algorithm for given real data set under Type-II progressive and SSPALT model.

8. Conclusions

In this study, the statistical inferences procedure for the unknown parameters of the IKum distribution and the acceleration factor, when the data are Type-II progressive censored from SSPALT were considered. We studied this problem under CEM. Since it is impossible to compute the MLEs in closed form, the Newton-Raphson approach is suggested as an alternative. We develop the approximate confidence interval length of the parameters and acceleration factor based on the asymptotic distribution of MLEs. We investigated the Bayes estimation approach in order to obtain an alternate estimate procedure. Bayesian estimates are achieved by Lindley's approximation the MCMC method, based on SE and LINEX loss functions. Under the premise that the priors of α_1, α_2 and β are Gamma density, the Metropolis-Hastings sampling technique is shown to produce Bayesian estimates. In addition, HPD CIs have been acquired. The Monte Carlo simulation study was utilized to get numerical point and interval estimates of the parameters.

From the results in Tables 4–9, we observe the following:

1) when the sample size increases, the MSEs of MLEs and BEs of the considered parameters decrease.

2) The BEs of the considered parameters obtained from both Lindley's approximation and MCMC method give more accurate results through the MSEs than MLEs.

3) The BEs of β obtained from Lindley's approximation give more accurate results through the MSEs than the BEs obtained from MCMC method.

4) The BEs of the considered parameters based on LINEX loss function $(c = 2)$ are smaller than that based on SE loss function.

5) In most cases, the HPD CIs give more accurate results than the approximate CIs since the lengths of the former are less than the lengths of latter, for different sample sizes, observed failures, and censoring schemes.

A real-life numerical example of March precipitation (in inches) in Minneapolis failure times is used to demonstrate the usefulness of the recommended estimation technique under SSPALT based on Type-II progressive censored. As per the K–S distance and the p-value, the data set presents a good match for the IKum distribution. To confirm that the distribution matches the data.

Acknowledgements

The authors would like to thank the Deanship of Scientific Research at Umm Al-Qura University for supporting this work by Grant Code: (22UQU4340290DSR03). Additionally, the authors thank the editor and the reviewers for their careful reading of the research article and for their constructive comments that greatly improved this paper.

Conflict of interest

All authors declare that there is no conflict of interest in this paper.

References

1. E. K. Al-Hussaini, A. H. Abdel-Hamid, Bayesian estimation of the parameters, reliability and hazard rate functions of mixtures under accelerated life tests, *Commun. Stat. Simul. Comput.*, **33** (2004) ,963–982. https://doi.org/10.1081/SAC-200040703
2. E. K. Al-Hussaini, A. H. Abdel-Hamid, Accelerated life tests under finite mixture models, *J. Stat. Comput. Simul.*, **76** (2006), 673–690. https://doi.org/10.1080/10629360500108087
3. A. M. Abd-Elfattah, A. S. Hassan, S. G. Nassr, Estimation in step-stress partially accelerated life tests for the Burr type XII distribution using type I censoring, *Stat. Methodol.*, 5 (2008), 502–514. https://doi.org/10.1016/j.stamet.2007.12.001
4. M. M. Mohie El-Din, S. E. Abu-Youssef, N. S. A. Ali, A. M. Abd El-Raheem, Estimation in step-stress accelerated life tests for Weibull distribution with progressive first-failure censoring, *J. Stat. Appl. Probab.*, **3** (2015), 403–411. http://dx.doi.org/10.1155/2015/319051
5. A. E. B. A. Ahmad, A. A. Soliman, M. M. Yousef, Bayesian Bayesian estimation of exponentiated Weibull distribution under partially accelerated life tests, *Bull. Malays. Math. Sci. Soc.*, 39 (2016), 227–244. https://doi.org/10.1007/s40840-015-0170-9
6. A. A. Ismail, Likelihood Inference for a step stress partially accelerated life test model with type-I progressive hybrid censored data from Weibull distribution, *J. Stat. Comput. Simul.*, **84** (2016), 2486–2494. https://doi.org/10.1080/00949655.2013.836195
7. B. Liu, Y. Shi, J. Cai, R. Wang, Reliability analysis of masked data in adaptive step-stress partially accelerated lifetime tests with progressive removal, *Commun. Stat. Theory Methods*, **46** (2017), 6174–6191. https://doi.org/10.1080/03610926.2015.1122058
8. S. A. Lone, A. Rahman, A. Islam, Step stress partially accelerated life testing plan for competing risk using adaptive Type-I progressive hybrid censoring, *Pak. J. Stat.*, **33** (2017), 237–248.
9. M. Nassar, S. G. Nassr, S. Dey, Analysis of Burr type XII distribution under step stress partially accelerated life tests with Type I and adaptive Type II progressively hybrid censoring schemes, *Ann. Data Sci.*, **4** (2017), 227–248. https://doi.org/10.1007/s40745-017-0101-8
10. S. G. Nassr, N. Elharoun, Inference for exponentiated Weibull distribution under constant stress partially accelerated life tests with multiple censored, *Commun. Stat. Appl. Methods*, **26** (2019), 131–148. https://doi.org/10.29220/CSAM.2019.26.2.131
11. A. S. Hassan, S. G. Nassr, S. Pramanik, S. S. Maiti, Estimation in constant stress partially accelerated life tests for Weibull distribution based on censored competing risks data, *Ann. Data Sci.*, **7** (2020), 45–62. https://doi.org/10.1007/s40745-019-00226-3
12. Ç. ÇETİNKAYA, Estimation in step-stress partially accelerated life tests for the power Lindley distribution under progressive censoring, *Gazi Univ. J. Sci.*, **34** (2021), 579–590. https://doi.org/10.35378/gujs.682499
13. I. Alam, A. Ahmed, Inference on maintenance service policy under step-stress partially accelerated life tests using progressive censoring, *J. Stat. Comput. Simul.*, **92** (2022), 813–829. https://doi.org/10.1080/00949655.2021.1975282
14. Y. Wang, W. Wang, Y. Tang, A Bayesian semiparametric accelerate failure time mixture cure model, *Int. J. Biostat.*, **2021** (2021). https://doi.org/10.1515/ijb-2021-0012
15. A. Xu, S. Zhou, Y. Tang, A unified model for system reliability evaluation under dynamic operating conditions, *IEEE Trans. Reliab.*, **70** (2019), 65–72. https://doi.org/10.1109/TR.2019.2948173

16. C. Luo, L. Shen, A. Xu, Modelling and estimation of system reliability under dynamic operating environments and lifetime ordering constraints, *Reliab. Eng. Syst. Saf.*, **218** (2022), 108136. https://doi.org/10.1016/j.ress.2021.108136
17. M. M. Yousef, S. A. Alyami, A. F. Hashem, Statistical inference for a constant-stress partially accelerated life tests based on progressively hybrid censored samples from inverted Kumaraswamy distribution, *PloS One*, **17** (2022), e0272378. https://doi.org/10.1371/journal.pone.0272378
18. N. Balakrishnan, R. Aggarwala, *Progressive Censoring: Theory, Methods, and Applications*, Birkhauser, Boston, 2000.
19. N. Balakrishnan, E. Cramer, *The Art of Progressive Censoring*, Birkhauser, New York, 2014. https://doi.org/10.1007/978-0-8176-4807-7
20. F. Zhang, X. Shi, H. K. T. Ng, Information geometry of the exponential family of distributions with progressive Type-II censoring, *Entropy*, **23** (2021), 687. https://doi.org/10.3390/e23060687
21. S. Dey, A. Elshahhat, Analysis of Wilson-Hilferty distribution under progressive Type-II censoring, *Qual. Reliab. Eng. Int.*, **2022** (2022). https://doi.org/10.1002/qre.3173
22. A. M. Abd AL-Fattah, A. A. EL-Helbawy, G. R. AL-Dayian, Inverted Kumaraswamy distribution: Properties and estimation, *Pak. J. Stat.*, **33** (2017), 37–61.
23. M. Mohie El-Din, M. Abu-Moussa, On estimation and prediction for the inverted Kumaraswamy distribution based on general progressive censored samples, *Pak. J. Stat. Oper. Res.*, **14** (2018), 717–736. https://doi.org/10.18187/pjsor.v14i3.2103
24. A. M. Daghistani, B. Al-Zahrani, M. Q. Shahbaz, Relations for moments of dual generalized order statistics for a new inverse Kumaraswamy distribution, *Pak. J. Stat. Oper. Res.*, **15** (2019), 989–997. https://doi.org/10.18187/pjsor.v15i4.3079
25. K. Bagci, T. Arslan, H. E. Celik, Inverted Kumarswamy distribution for modeling the wind speed data: Lake Van, Turkey, *Renewable Sustainable Energy Rev.*, **135** (2021), 110110. https://doi.org/10.1016/j.rser.2020.110110
26. F. Noor, S. Masood, M. Zaman, M. Siddiqa, R. A. Wagan, I. U. Khan, et al., Bayesian analysis of inverted Kumaraswamy mixture model with application to burning velocity of chemicals, *Math. Prob. Eng.*, **2021** (2021). https://doi.org/10.1155/2021/5569652
27. V. Bagdonavicius, M. Nikulin, *Accelerated Life Models: Modeling and Statistical Analysis. Chapman and Hall/CRC Press*, Boca Raton, Florida, 2021.
28. H. R. Varian, A Bayesian approach to real estate assessment, in *Variants in Economic Theory: Selected Works of H. R. Varian*, Edward Elgar Publishing, (2000), 144–155.
29. M. Doostparast, S. Deepak, A. Zangoie, Estimation with the lognormal distribution on the basis of records, *J. Stat. Comput. Simul.*, **83** (2013), 2339–2351. https://doi.org/10.1080/00949655.2012.691973
30. D. V. Lindley, Approximate Bayesian methods, *Trabajos de Estadistica y de investigacin operativa*, **31** (1980), 223–245. https://doi.org/10.1007/BF02888353
31. M. Doostparast, M. G. Akbari, N. Balakrishna, Bayesian analysis for the two-parameter Pareto distribution based on record values and times, *J. Stat. Comput. Simul.*, **81** (2011), 1393–1403. https://doi.org/10.1080/00949655.2010.486762
32. N. Balakrishnan, R. A. Sandhu, A simple simulational algorithm for generating progressive Type-II censored samples, *Am. Stat.*, **49** (1985), 229–230.

33. D. Hinkley, On quick choice of power transformations, *J. R. Stat. Ser. C*, **26** (1977), 67–69. https://doi.org/10.2307/2346869
34. M. H. Chen, Q. M. Shao, Monte Carlo estimation of Bayesian credible and HPD intervals, *J. Comput. Graphical Stat.*, **8** (1999), 69–92.

AIMS Press

©2023 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0)