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Research article

A novel weighted family of probability distributions with applications to world natural gas, oil, and gold reserves

Amal S. Hassan[1,*], Najwan Alsadat[2], Christophe Chesneau[3] and Ahmed W. Shawki[1,4,*]

[1] Faculty of Graduate Studies for Statistical Research, Cairo University, Giza 12613, Egypt
[2] Department of Quantitative Analysis, College of Business Administration, King Saud University, P.O. Box 71115, Riyadh 11587, Saudi Arabia
[3] Department of Mathematics, University de Caen Normandie, Campus II, Caen 14032, France
[4] Central Agency for Public Mobilization & Statistics (CAPMAS), Cairo, Egypt

* **Correspondence:** Email: amal52_soliman@cu.edu.eg, ahmed23484@hotmail.com.

Abstract: Recent innovations have focused on the creation of new families that extend well-known distributions while providing a huge amount of practical flexibility for data modeling. Weighted distributions offer an effective approach for addressing model building and data interpretation problems. The main objective of this work is to provide a novel family based on a weighted generator called the length-biased truncated Lomax-generated (LBTLo-G) family. Discussions are held about the characteristics of the LBTLo-G family, including expressions for the probability density function, moments, and incomplete moments. In addition, different measures of uncertainty are determined. We provide four new sub-distributions and investigated their functionalities. Subsequently, a statistical analysis is given. The LBTLo-G family's parameter estimation is carried out using the maximum likelihood technique on the basis of full and censored samples. Simulation research is conducted to determine the parameters of the LBTLo Weibull (LBTLoW) distribution. Four genuine data sets are considered to illustrate the fitting behavior of the LBTLoW distribution. In each case, the application outcomes demonstrate that the LBTLoW distribution can, in fact, fit the data more accurately than other rival distributions.

Keywords: weighted distribution; incomplete moments; Tsallis measure; maximum likelihood estimation; censored samples

1. Introduction

Weighted distributions (WDs) provide an approach to deal with model specification and data interpretations problems. They adjust the probabilities of the actual occurrence of events to arrive at a specification of the probabilities when those events are recorded. Reference [1] extended the basic ideas of the methods of ascertainment upon the estimation of frequencies in [2]. The author defined a unifying concept of the WDs and described several sample conditions that the WDs can model. The usefulness and applications of the WDs in various areas, including medicine, ecology, reliability, and branching processes, can also be seen in [3–5]. Important findings on the WDs have been reported by several research. For examples, reference [6] suggested a weighted x-gamma distribution, reference [7] derived a new generalized weighted Weibull distribution, reference [8] introduced the weighted exponential-Gompertz distribution, reference [9] studied the new weighted inverse Rayleigh distribution, reference [10] introduced a weighted version of the generalized inverse Weibull distribution, reference [11] proposed a bounded weighted exponential distribution, reference [12] derived a new weighted exponential distribution, reference [13] proposed a weighted power Lomax distribution, reference [14] derived a new generalized weighted exponential distribution, reference [15] introduced a new version of the weighted Weibull distribution, reference [16] proposed the modified weighted exponential distribution, and reference [17] proposed a weighted Nwikpe distribution, reference [18] introduced a new version of the double weighted quasi Lindley distribution and reference [19] proposed the modified length-biased weighted Lomax distribution.

In contrast, statistical models have the capacity to depict and predict real-world phenomena. Over the past few decades, numerous extended distributions have been extensively utilized in data modeling. Recent progress has been centered on the development of novel distribution families that not only enhance existing distributions but also offer significant versatility in practical data modeling. Engineering, economics, biology, and environmental science are particular examples of this. Regarding this, a number of writers suggested some of the created families of continuous distributions, (see for example [20–22]). Our interest here is in the same scheme used for the beta-G (B-G) family prepared in [23]. The following is the cumulative distribution function (cdf) for the B-G family:

$$F(x) = \int_0^{G(x)} r(t)dt, \tag{1.1}$$

where $G(x)$ is a cdf of a continuous distribution and $r(t)$ is the probability density function (pdf) of the beta distribution. Naturally, any new family can be created by taking another pdf for $r(t)$ with support $[0, 1]$ (see reference [23]).

As a matter of fact, few works about the weighted-G family have been proposed in the literature. For example, reference [24] studied the weighted exponential-G family, reference [25] introduced the weighted exponentiated family, reference [26] proposed a weighted general family, and reference [27] developed a weighted Topp-Leone-G family.

The primary purpose of this study is to introduce the length-biased truncated Lomax-G (LBTLo-G) family. The following arguments give enough motivation to study it:

1) The LBTLo-G family is very flexible and simple.
2) The LBTLo-G family contains some new distributions.
3) The shapes of the pdfs of the generated distributions can be unimodal, decreasing, bathtub, right-skewed, and symmetric. Also, the hazard rate function (hrf) shapes for these distributions can be increasing, decreasing, U-shaped, upside-down-shaped, or J- shaped.

After emphasizing these important aspects, some statistical and mathematical properties of the newly suggested family are discussed. The maximum likelihood (ML) method of estimation is used to estimate the LBTLo Weibull (LBTLoW) model parameters based on complete and type II censoring (T2C).

The variability of the LBTLoW distribution is demonstrated through four authentic data sets. The first data set describes age data on rest times (in minutes) for analgesic patients. The second data set shows the percentage of natural gas reserves in 44 countries in 2020. The third authentic data set listed the top 20 countries by oil reserves. Proven reserves refer to the quantities of petroleum that can be predicted as commercially recoverable from known reservoirs, based on the analysis of geological and engineering data. These estimates are made considering existing economic conditions and are projected from a specific period onwards. The fourth data set displays the top 100 central banks in terms of gold reserves. This gold reserve data, collected from IMF IFS figures, tracks central banks' reported gold purchases and sales as a percentage of their international reserves. The application results show that the LBTLoW distribution can indeed match the data better than other competing distributions.

The following is the structure for this article: Section 2 defines the crucial functions of the LBTLo-G family and provides four special distributions of the family. In Section 3, some statistical properties of the LBTLo-G family are provided. Section 4 deals with the ML estimates (MLEs) of the unknown parameters. A simulation study to examine the theoretical performance of MLEs for the LBTLoW distribution is studied in Section 5. Section 6 presents the applicability and goodness of fit of the proposed models using four real data sets. The paper ends with a few last observations, as may be seen in Section 7.

2. Construction of the LBTLo-G family

Here, we suggest a new weighted family based on the weighted version of the truncated Lomax distribution, which is called the LBTLo distribution [28]. The cdf and pdf of the LBTLo distribution are, respectively, given by

$$G(t;\alpha) = \Lambda(\alpha)\left[(1+t)^{-\alpha}\quad(1+\alpha t) - 1\right], \quad 0 < t < 1, \quad \alpha > 0, \tag{2.1}$$

$$g(t;\alpha) = \alpha(1-\alpha)\Lambda(\alpha)\, t(1+t)^{-(\alpha+1)}, \tag{2.2}$$

where $\Lambda(\alpha) = [2^{-\alpha}\quad(1+\alpha) - 1]^{-1}$. For these functions, it is assumed the standard complementary values for $t \leq 0$ and $t \geq 1$.

As mentioned in [28], the following advantages of the LBTLo distribution are outlined: (i) It depends on only one parameter; (ii) the pdf has only one maximum point with a relatively sharp peak and a heavy tail; (iii) the hrf has increasing behavior or is N-shaped; and (iv) it outperforms some other competing models in real-world applications to medical data and the percentage of household spending on education out of total household expenditure from the Household Income, Expenditure, and Consumption Survey data for North Sinai Governorate.

In light of these merits, the LBTLo distribution is a great choice to use in various fields. As a consequence, we present a novel generated family that is based on the LBTLo distribution. In order to define the LBTLo-G family, let $G(x;\zeta)$ and $g(x;\zeta)$ be the baseline cdf and pdf, respectively, of a continuous distribution, and ζ is a vector of parameters. The generalized B-G generator specified in (1.1) and the LBTLo distribution (2.2) are combined to generate the cdf of the LBTLo-G family:

$$\begin{aligned} F(x;\alpha,\zeta) &= \alpha(1-\alpha)\Lambda(\alpha)\int_0^{G(x;\zeta)} t(1+t)^{-\alpha-1}\,dt \\ &= \Lambda(\alpha)\left[(1+G(x;\zeta))^{-\alpha}(1+\alpha G(x;\zeta))-1\right], \quad x\in\mathbb{R},\ \ \alpha>0, \end{aligned} \tag{2.3}$$

where α is a shape parameter. Therefore, the pdf of the LBTLo-G family is given by

$$f(x;\alpha,\zeta) = \alpha(1-\alpha)\Lambda(\alpha)\,g(x;\zeta)\,G(x;\zeta)(1+G(x;\zeta))^{-\alpha-1},\ x\in\mathbb{R},\ \alpha>0. \tag{2.4}$$

A random variable X with the pdf (2.4) is designated as $X \sim$ LBTLo-G from here on out. The complementary cdf (ccdf), and hrf, are, provided by

$$S(x;\alpha,\zeta) = 1-\Lambda(\alpha)\left[(1+G(x;\zeta))^{-\alpha}(1+\alpha G(x;\zeta))-1\right],$$

$$h(x;\alpha,\zeta) = \frac{\alpha(1-\alpha)\Lambda(\alpha)\,g(x)G(x)(1+G(x))^{-\alpha-1}}{1-\Lambda(\alpha)\left[(1+G(x;\zeta))^{-\alpha}(1+\alpha G(x;\zeta))-1\right]}.$$

We create four new LBTLo-G family sub-distributions in the subsections that follow: LBTLo-inverse exponential, LBTLo-uniform, LBTLo-Weibull, and LBTLo-Kumaraswamy distributions.

2.1. LBTLo-inverse exponential model

The cdf and pdf of the LBTLo-inverse exponential (LBTLoIE) distribution are obtained from (2.3) and (2.4) for $G(x;\beta) = e^{-(\beta/x)}, \beta, x > 0$, as follows:

$$F(x;\alpha,\beta) = \Lambda(\alpha)\left[\left(1+e^{-(\beta/x)}\right)^{-\alpha}\left(1+\alpha e^{-(\beta/x)}\right)-1\right], x>0\ ,\ \ \alpha,\beta>0,$$

$$f(x;\alpha,\beta) = \alpha(1-\alpha)\Lambda(\alpha)\beta x^{-2}e^{-2(\beta/x)}\left(1+e^{-(\beta/x)}\right)^{-\alpha-1}.$$

Further, the hrf is as follows:

$$h(x;\alpha,\beta) = \frac{\alpha(1-\alpha)\Lambda(\alpha)\beta x^{-2}e^{-2(\beta/x)}\left(1+e^{-(\beta/x)}\right)^{-\alpha-1}}{1-\Lambda(\alpha)\left[(1+e^{-(\beta/x)})^{-\alpha}(1+\alpha e^{-(\beta/x)})-1\right]}.$$

2.2. LBTLo-uniform distribution

The cdf and pdf of the LBTLo-uniform (LBTLoU) distribution are derived from (2.3) and (2.4) by taking $G(x;\beta) = \beta^{-1}x, 0 < x < \beta$, as follows:

$$F(x;\alpha,\beta) = \left[\left(1+\alpha\beta^{-1}x\right)\left(1+\beta^{-1}x\right)^{-\alpha}-1\right]\Lambda(\alpha), \quad 0<x<\beta,\ \alpha,\beta>0,$$

$$f(x;\alpha,\beta) = \alpha\beta^{-2}x\left(1+\beta^{-1}x\right)^{-\alpha-1}(1-\alpha)\Lambda(\alpha).$$

Further, the hrf is as follows:

$$h(x;\alpha,\beta) = \frac{\alpha\beta^{-2}x\left(1+\beta^{-1}x\right)^{-\alpha-1}(1-\alpha)\Lambda(\alpha)}{1-\Lambda(\alpha)\left[(1+\alpha\beta^{-1}x)(1+\beta^{-1}x)^{-\alpha}-1\right]}.$$

2.3. LBTLo-Weibull distribution

The cdf and pdf of the LBTLoW distribution are derived from (2.3) and (2.4) taking $G(x;\beta,\gamma) = 1 - e^{-\beta x^{\gamma}},\ x,\beta,\gamma > 0$, as follows:

$$F(x;\alpha,\beta,\gamma) = \left[\left(2 - e^{-\beta x^{\gamma}}\right)^{-\alpha}\left(1 + \alpha - \alpha e^{-\beta x^{\gamma}}\right) - 1\right]\Lambda(\alpha),\ x > 0,\ \ \alpha,\beta,\gamma > 0, \tag{2.5}$$

$$f(x;\alpha,\beta,\gamma) = \alpha\beta\gamma(1-\alpha)\,x^{\gamma-1}e^{-\beta x^{\gamma}}\left(1 - e^{-\beta x^{\gamma}}\right)\left(2 - e^{-\beta x^{\gamma}}\right)^{-\alpha-1}\Lambda(\alpha). \tag{2.6}$$

Further, the hrf is:

$$h(x;\alpha,\beta,\gamma) = \frac{\alpha\beta\gamma(1-\alpha)\,x^{\gamma-1}e^{-\beta x^{\gamma}}\left(1 - e^{-\beta x^{\gamma}}\right)\left(2 - e^{-\beta x^{\gamma}}\right)^{-\alpha-1}\Lambda(\alpha)}{1 - \Lambda(\alpha)\left[(2 - e^{-\beta x^{\gamma}})^{-\alpha}(1 + \alpha - \alpha e^{-\beta x^{\gamma}}) - 1\right]}.$$

2.4. LBTLo-Kumaraswamy distribution

The cdf and pdf of the LBTLo- Kumaraswamy (LBTLoKw) distribution are obtained from (2.3) and (2.4) by taking $G(x;\mu,b) = 1 - (1 - x^{\mu})^{b},\ 0 < x < 1, b,\mu > 0$, as follows:

$$\begin{aligned} F(x;\alpha,\mu,b) = &\left[\left(2 - (1 - x^{\mu})^{b}\right)^{-\alpha}\left(1 + \alpha - \alpha(1 - x^{\mu})^{b}\right) - 1\right]\Lambda(\alpha), \\ &0 < x < 1,\ \alpha,\mu,b > 0, \end{aligned}$$

$$\begin{aligned} f(x;\alpha,\mu,b) = \alpha\mu b(1-\alpha)\,x^{\mu-1}(1 - x^{\mu})^{b-1}&\left(1 - (1 - x^{\mu})^{b}\right) \\ \times&\left(2 - (1 - x^{\mu})^{b}\right)^{-\alpha-1}\Lambda(\alpha). \end{aligned}$$

Further, the hrf is as follows:

$$h(x;\alpha,\mu,b) = \frac{\alpha\mu b(1-\alpha)\,x^{\mu-1}(1 - x^{\mu})^{b-1}\left(1 - (1 - x^{\mu})^{b}\right)\Lambda(\alpha)\left(2 - (1 - x^{\mu})^{b}\right)^{-\alpha-1}}{1 - \Lambda(\alpha)\left[\left(2 - (1 - x^{\mu})^{b}\right)^{-\alpha}\left(1 + \alpha - \alpha(1 - x^{\mu})^{b}\right) - 1\right]}.$$

The plots of pdf and hrf for the LBTLoIE, LBTLoU, LBTLoW and LBTLoKw distributions are given in Figures 1 and 2, respectively.

The pdfs of the investigated distributions can have a variety of forms, including right- and left-skewed, bathtub, uni-modal, declining, and symmetric shapes, as shown in Figure 1. The corresponding hrf can take any form, including U, J, reverse J, growing, or decreasing, as seen in Figure 1.

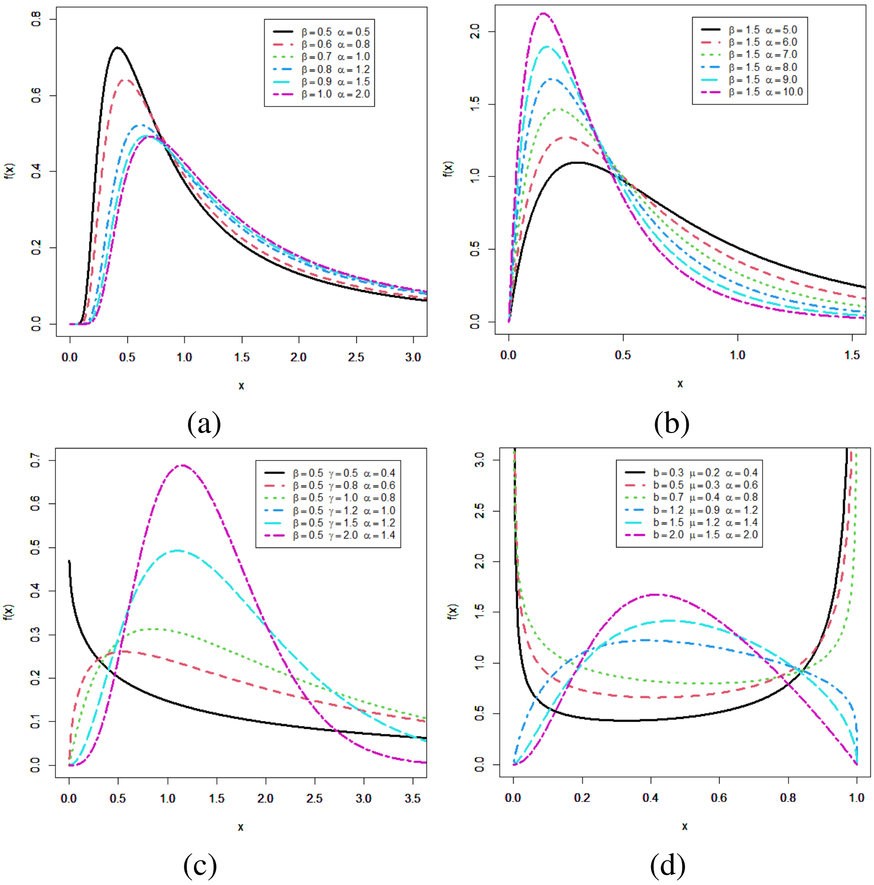

Figure 1. The pdfs of the (a) LBTLoIE, (b) LBTLoU, (c) LBTLoW, and (d) LBTLoKw distributions.

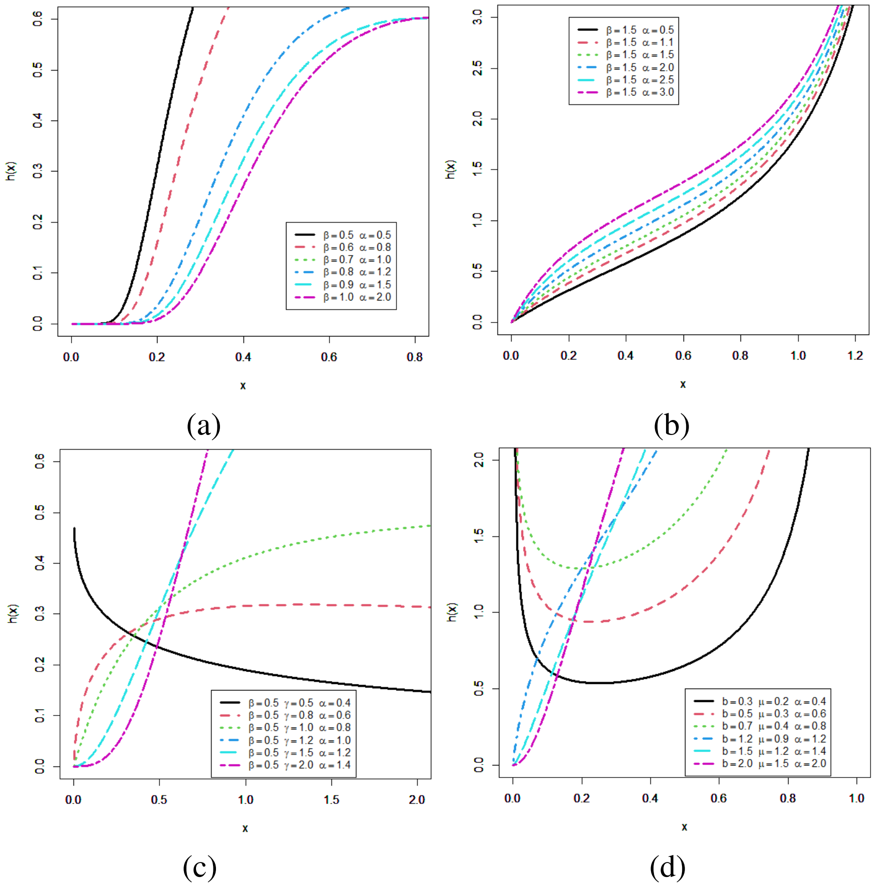

Figure 2. The hrfs of the (a) LBTLoIE, (b) LBTLoU, (c) LBTLoW, and (d) LBTLoKw distributions.

3. Some statistical properties of LBTLo-G family

In this part, we give some statistical properties of the LBTLo-G family.

3.1. Important expansion

The LBTLo-G family representations in pdf and cdf format are displayed here. The generalized binomial theorem says that

$$(1 + z)^{-\beta} = \sum_{i=0}^{\infty} (-1)^i \binom{\beta + i - 1}{i} z^i, \tag{3.1}$$

for $|z| < 1$. Hence, by using (3.1) in (2.4), the pdf of the LBTLo-G family can be written as follows:

$$f(x; \alpha, \zeta) = \sum_{i=0}^{\infty} \vartheta_i g(x; \zeta) G(x; \zeta)^{i+1}, \quad x \in \mathbb{R}, \tag{3.2}$$

where $\vartheta_i = (-1)^i \alpha (1 - \alpha) \Lambda(\alpha) \binom{\alpha + i}{i}$. For example, the expansion of pdf of the LBTLoW distribution is derived from (3.2) as follows:

$$f(x;\alpha,\beta,\gamma)=\beta\gamma\sum_{i=0}^{\infty}\vartheta_i x^{\gamma-1}e^{-\beta x^{\gamma}}\left(1-e^{-\beta x^{\gamma}}\right)^{i+1},\quad x>0,\ \alpha,\beta,\ \gamma>0. \tag{3.3}$$

But, in the special case where b is a positive integer, the standard generalized binomial theorem says that

$$(1-z)^b=\sum_{\nu=0}^{b}(-1)^{\nu}\binom{b}{\nu}z^{\nu}. \tag{3.4}$$

Then using the binomial expansion (3.4) in (3.3), we get

$$f(x;\alpha,\beta,\gamma)=\sum_{i=0}^{\infty}\sum_{\nu=0}^{i+1}\vartheta_{i,\nu}x^{\gamma-1}e^{-\beta(\nu+1)x^{\gamma}}, \tag{3.5}$$

where $\vartheta_{i,\nu}=\beta\gamma\vartheta_i(-1)^{\nu}\binom{i+1}{\nu}$. In what follows, an expansion for $F(x;\alpha,\zeta)^h$is derived, for h is an integer, again, the exponential and the binomial expansions are worked out:

$$F(x;\alpha,\zeta)^h=\Lambda(\alpha)^h\left[(1+G(x;\zeta))^{-\alpha}(1+\alpha G(x;\zeta))-1\right]^h. \tag{3.6}$$

Using the binomial expansion (3.4) in (3.6), we get

$$F(x;\alpha,\zeta)^h=\Lambda(\alpha)^h\sum_{j=0}^{h}(-1)^{h-j}\binom{h}{j}(1+G(x;\zeta))^{-\alpha j}(1+\alpha G(x;\zeta))^j. \tag{3.7}$$

Using the binomial expansion (3.1), we obtain

$$\begin{aligned}F(x;\alpha,\zeta)^h=\Lambda(\alpha)^h\sum_{d=0}^{\infty}\sum_{j=0}^{h}(-1)^{d+h-j}\binom{h}{j}\binom{\alpha j+d-1}{d}\\ \times G(x;\zeta)^d(1+\alpha G(x;\zeta))^j.\end{aligned} \tag{3.8}$$

By using (3.4) in (3.8), we obtain

$$F(x;\alpha,\zeta)^h=\sum_{d=0}^{\infty}\varpi_{d,j,m}G(x;\zeta)^{d+m}, \tag{3.9}$$

where $\varpi_{d,j,m}=\sum_{j=0}^{h}\sum_{m=0}^{j}(-1)^{d+m+h-j}\alpha^m\binom{h}{j}\binom{j}{m}\binom{\alpha j+d-1}{d}\Lambda(\alpha)^h$.

For example, the expansion of the cdf of the LBTLoW distribution is derived from (3.9), where $G(x,\zeta)=1-e^{-\beta x^{\gamma}}$, as follows:

$$F(x;\alpha,\beta,\gamma)^h=\sum_{d=0}^{\infty}\varpi_{d,j,m}\left(1-e^{-\beta x^{\gamma}}\right)^{d+m}.$$

By using the binomial expansion (3.4) in the last term of the previous equation, we get

$$F(x;\alpha,\beta,\gamma)^h = \sum_{d=0}^{\infty}\sum_{l=0}^{d+m}(-1)^l \binom{d+m}{l} \varpi_{d,j,m} e^{-\beta l x^{\gamma}}. \tag{3.10}$$

The above representations are of interest to express various important moment measures as series. By truncating the index of summation, we can have a precise approximation with a reasonable computation cost.

3.2. The Probability Weighted Moments

As a special class of moments, the probability weighted moments (PWMs) have been proposed in [29]. This class is used to derive estimates of the parameters and quantiles of distributions expressible in inverse form. Let X be a random variable with pdf and cdf $f(x)$ and $F(x)$, respectively, and r and q be non-negative integers. Then, the $(r,q)^{th}$ PWM of X, denoted by $\pi_{r,q}$, can be calculated through the following relation:

$$\pi_{r,q} = E[X^r F(X)^q] = \int_{-\infty}^{\infty} x^r f(x) F(x)^q dx. \tag{3.11}$$

On this basis, the $(r,q)^{th}$ PMW of X with pdf and cdf of the LBTLo-G family is obtained by substituting (3.2) and (3.9) into (3.11), as follows:

$$\pi_{r,q} = E[X^r F(X;\alpha,\zeta)^q] = \int_{-\infty}^{\infty}\sum_{i,d=0}^{\infty}\vartheta_i \varpi_{d,j,m} x^r g(x;\zeta)\left[G(x;\zeta)\right]^{i+d+m+1} dx.$$

Then, provided that the interchange of the integral and sum is valid, depending on the definitions of $g(x;\zeta)$ and $G(x;\zeta)$, we have

$$\pi_{r,q} = \sum_{i,d=0}^{\infty}\vartheta_i \varpi_{d,j,m}\rho_{r,i+d+m+1},$$

where

$$\rho_{r,i+d+m+1} = \int_{-\infty}^{\infty} x^r g(x;\zeta)\left[G(x;\zeta)\right]^{i+d+m+1} dx.$$

For example, the $(r,q)^{th}$ of a random variable X that follows the LBTLoW distribution can be obtained by substituting (3.5) and (3.10) into (3.11), and replacing h with q. We thus obtain

$$\pi_{r,q} = \sum_{i,d=0}^{\infty}\sum_{\nu=0}^{i+1}\sum_{l=0}^{d+m}\frac{(-1)^l\,\vartheta_{i,\nu}\varpi_{d,j,m}}{(\beta(\nu+l+1))^{\frac{r}{\gamma}+1}}\binom{d+m}{l}\Gamma\left(\frac{r}{\gamma}+1\right),$$

where $\Gamma(.)$ stands for gamma function.

3.3. Moments and incomplete moments

In this part, for any non-negative integer r, the r^{th} moment associated with the LBTLo-G family is derived.

Let X be a random variable having the pdf of the LBTLo-G family. Then, the r^{th} moment of X is obtained as follows:

$$\mu_r' = E(X^r) = \int_{-\infty}^{\infty} \sum_{i=0}^{\infty} \vartheta_i x^r g(x;\zeta) [G(x;\zeta)]^{i+1} dx = \sum_{i=0}^{\infty} \vartheta_i \upsilon_{r,i+1},$$

where $\upsilon_{r,i+1}$ is the $(r, i+1)^{th}$ PWM of the baseline distribution. For example, after some developments, the r^{th} moment associated with LBTLoW distribution is given by

$$\mu_r' = \sum_{i=0}^{\infty} \sum_{\nu=0}^{i+1} \frac{\vartheta_{i,\nu}}{(\beta(\nu+1))^{\frac{r}{\gamma}+1}} \Gamma\left(\frac{r}{\gamma}+1\right).$$

Tables 1–3 show the numerical values of the first four moments μ_1', μ_2', μ_3', μ_4', also the numerical values of variance (σ^2), coefficient of skewness (CS), coefficient of kurtosis (CK) and coefficient of variation (CV) associated with the LBTLoW and LBTLoIE distribution.

Table 1. Results of some moments, σ^2, CS, CK, and CV associated with the LBTLo-W distribution at $\beta = 1.8$.

γ	α	μ_1'	μ_2'	μ_3'	μ_4'	σ^2	CS	CK	CV
0.4	0.2	2.629	8.323	30.531	126.387	1.413	0.729	3.596	0.452
0.6	0.5	1.558	2.937	6.430	15.922	0.508	0.746	3.627	0.457
0.8	0.8	1.184	1.700	2.847	5.403	0.300	0.765	3.660	0.463
1.1	1.2	0.925	1.044	1.380	2.074	0.189	0.793	3.717	0.470
1.3	1.5	0.805	0.794	0.920	1.216	0.146	0.814	3.758	0.475
1.7	1.8	0.705	0.615	0.634	0.748	0.117	0.856	3.853	0.486
1.9	2.0	0.655	0.532	0.513	0.568	0.103	0.878	3.906	0.491
2.4	2.3	0.582	0.425	0.371	0.374	0.086	0.937	4.058	0.503
2.7	2.6	0.530	0.355	0.285	0.266	0.073	0.974	4.162	0.511
3.2	3.0	0.469	0.280	0.203	0.172	0.060	1.038	4.359	0.522

Table 2. Results of some moments, σ^2, CS, CK, and CV associated with the LBTLo-W distribution at $\beta = 2.5$.

γ	α	μ'_1	μ'_2	μ'_3	μ'_4	σ^2	CS	CK	CV
0.4	0.2	1.964	4.277	10.154	25.945	0.420	0.380	3.017	0.330
0.6	0.5	1.347	2.017	3.298	5.813	0.202	0.395	3.026	0.334
0.8	0.8	1.105	1.360	1.829	2.658	0.139	0.411	3.037	0.337
1.1	1.2	0.924	0.954	1.081	1.325	0.100	0.435	3.058	0.343
1.3	1.5	0.836	0.783	0.805	0.897	0.084	0.453	3.075	0.346
1.7	1.8	0.760	0.649	0.612	0.627	0.072	0.489	3.116	0.353
1.9	2.0	0.720	0.584	0.524	0.511	0.066	0.508	3.140	0.356
2.4	2.3	0.661	0.494	0.411	0.374	0.058	0.557	3.213	0.365
2.7	2.6	0.617	0.433	0.339	0.290	0.052	0.588	3.265	0.369
3.2	3.0	0.565	0.364	0.263	0.209	0.045	0.641	3.366	0.377

Table 3. Results of some moments, σ^2, CS, CK, and CV associated with the LBTLoIE distribution.

β	α	μ'_1	μ'_2	μ'_3	μ'_4	σ^2	CS	CK	CV
1.5	0.2	0.029	0.026	0.023	0.021	0.025	5.273	29.151	5.386
	0.5	0.032	0.028	0.025	0.023	0.027	5.014	26.463	5.138
	0.8	0.035	0.031	0.028	0.025	0.030	4.772	24.069	4.906
	1.2	0.039	0.035	0.031	0.028	0.033	4.472	21.275	4.622
	1.5	0.043	0.038	0.033	0.03	0.036	4.264	19.441	4.424
	1.8	0.046	0.041	0.036	0.033	0.038	4.069	17.801	4.241
	2	0.049	0.043	0.038	0.034	0.040	3.946	16.806	4.125
	2.3	0.053	0.046	0.041	0.037	0.043	3.771	15.445	3.961
	2.6	0.056	0.049	0.044	0.039	0.046	3.607	14.225	3.809
	3	0.062	0.054	0.048	0.043	0.050	3.405	12.794	3.621
2.5	0.2	0.006	0.005	0.005	0.004	0.005	12.420	156.328	12.403
	0.5	0.007	0.006	0.006	0.005	0.006	11.672	138.195	11.665
	0.8	0.007	0.007	0.006	0.006	0.007	10.981	122.434	10.984
	1.2	0.009	0.008	0.007	0.007	0.008	10.140	104.548	10.156
	1.5	0.01	0.009	0.008	0.007	0.009	9.563	93.121	9.589
	1.8	0.011	0.010	0.009	0.008	0.010	9.030	83.140	9.066
	2	0.012	0.011	0.010	0.009	0.010	8.697	77.190	8.739
	2.3	0.013	0.012	0.011	0.010	0.012	8.227	69.194	8.279
	2.6	0.014	0.013	0.012	0.011	0.013	7.792	62.179	7.853
	3	0.017	0.015	0.014	0.013	0.015	7.261	54.128	7.334

It can be seen from Tables 1–3 that, when the value of α, γ increases for a fixed value of β, the first four moments and σ^2 decrease, while the CS, CK, and CV measures increase. When the value of β increases for a fixed value of α and γ, we observe that the first four moments and σ decrease and then increase, while the CS, CK, and CV measures increase. The LBTLoW distribution is skewed to the right by leptokurtic curves.

Furthermore, if X is a random variable having the pdf of the LBTLo-G family, then the r^{th} incomplete moment of X is obtained as follows:

$$\varphi_r(t) = E(X^r I_{\{X \leq t\}}) = \int_{-\infty}^{t} x^r f(x; \alpha, \zeta)\, dx = \int_{-\infty}^{t} \sum_{i=0}^{\infty} \vartheta_i x^r g(x; \zeta)\, G(x; \zeta)^{i+1}\, dx.$$

For example, after some developments, the r^{th} incomplete moment associated with the LBTLoW distribution is given by

$$\varphi_r(t) = \sum_{i=0}^{\infty} \sum_{\nu=0}^{i+1} \frac{\vartheta_{i,\nu}}{[\beta(\nu+1)]^{\frac{r}{\gamma}+1}} \Gamma\left(\frac{r}{\gamma} + 1, \beta(\nu+1)t^{\gamma}\right),$$

where $\Gamma(., x)$ is the lower incomplete gamma function.

3.4. Some information measures

Here, some uncertainty measures of the LBTLo-G family are derived. Then, these measures are specialized to the LBTLoW distribution. To begin, the Rényi entropy (RE), presented in [30], associated with a distribution with pdf $f(x)$, is defined by

$$I_R(\varepsilon) = \frac{1}{1-\varepsilon} \log\left[\int_{-\infty}^{\infty} f(x)^{\varepsilon} dx\right], \qquad \varepsilon \neq 1, \qquad \varepsilon > 0.$$

A numerical study with integral calculus is possible; here, we focus on a series expansion. In what follows, an expansion for $f(x; \alpha, \zeta)^{\varepsilon}$ is derived, for ε is a non-integer (again, the generalized binomial expansion is worked out):

$$f(x; \alpha, \zeta)^{\varepsilon} = \sum_{i=0}^{\infty} \Delta_i g(x; \zeta)^{\varepsilon} G(x; \zeta)^{i+\varepsilon},$$

where

$$\Delta_i = (-1)^i [\alpha(1-\alpha)\Lambda(\alpha)]^{\varepsilon} \binom{\varepsilon(\alpha+1)+i-1}{i}.$$

Then, the RE associated with the LBTLo-G family is given by

$$I_R(\varepsilon) = (1-\varepsilon)^{-1} \quad \log\left\{\int_{-\infty}^{\infty} \sum_{i=0}^{\infty} \Delta_i g(x; \zeta)^{\varepsilon} G(x; \zeta)^{i+\varepsilon}\, dx\right\}.$$

For example, the RE associated with the LBTLoW distribution can be obtained as follows:

$$I_R(\varepsilon) = (1-\varepsilon)^{-1} \log\left\{\sum_{i,j=0}^{\infty} \frac{\Delta_{i,j}}{\gamma} [\beta(\varepsilon+j)]^{-\varepsilon+(\varepsilon/\gamma)-(1/\gamma)} \Gamma\left(\varepsilon - \frac{\varepsilon}{\gamma} + \frac{1}{\gamma}\right)\right\}.$$

The Havrda and Charvát entropy (HaCE) (see [31]) associated with a distribution with pdf $f(x)$ is defined by

$$HC_R(\varepsilon) = \frac{1}{2^{1-\varepsilon}-1}\left(\left\{\int_{-\infty}^{\infty} f(x)^{\varepsilon} dx\right\}^{1/\varepsilon} - 1\right), \qquad \varepsilon \neq 1, \qquad \varepsilon > 0.$$

Hence, the HaCE of the LBTLo-G family is given by

$$HC_R(\varepsilon) = \frac{1}{2^{1-\varepsilon}-1}\left(\left\{\int_{-\infty}^{\infty} \sum_{i=0}^{\infty} \Delta_i g(x;\zeta)^{\varepsilon} G(x;\zeta)^{i+\varepsilon} dx\right\}^{1/\varepsilon} - 1\right).$$

For example, the HaCE of the LBTLoW distribution can be obtained as follows:

$$HC_R(\varepsilon) = \frac{1}{2^{1-\varepsilon}-1}\left(\left\{\sum_{i,j=0}^{\infty} \frac{\Delta_{i,j}}{\gamma}\left[\beta(\varepsilon+j)\right]^{-\varepsilon+(\varepsilon/\gamma)-(1/\gamma)} \Gamma\left(\varepsilon - \frac{\varepsilon}{\gamma} + \frac{1}{\gamma}\right)\right\}^{1/\varepsilon} - 1\right).$$

The Arimoto entropy (ArE) (see [32]) associated with a distribution with pdf $f(x)$ is defined by

$$A_R(\varepsilon) = \frac{\varepsilon}{1-\varepsilon}\left(\left\{\int_{-\infty}^{\infty} f(x)^{\varepsilon} dx\right\}^{1/\varepsilon} - 1\right), \qquad \varepsilon \neq 1, \qquad \varepsilon > 0.$$

Hence, the ArE of the LBTLo-G family is given by

$$A_R(\varepsilon) = \frac{\varepsilon}{1-\varepsilon}\left(\left\{\int_{-\infty}^{\infty} \sum_{i=0}^{\infty} \Delta_i g(x;\zeta)^{\varepsilon} G(x;\zeta)^{i+\varepsilon} dx\right\}^{1/\varepsilon} - 1\right).$$

For example, the ArE of the LBTLoW distribution can be obtained as follows:

$$A_R(\varepsilon) = \frac{\varepsilon}{1-\varepsilon}\left(\left\{\sum_{i,j=0}^{\infty} \frac{\Delta_{i,j}}{\gamma}\left[\beta(\varepsilon+j)\right]^{-\varepsilon+(\varepsilon/\gamma)-(1/\gamma)} \Gamma\left(\varepsilon - \frac{\varepsilon}{\gamma} + \frac{1}{\gamma}\right)\right\}^{1/\varepsilon} - 1\right).$$

The Tsallis entropy (TsE) (see [33]) associated with a distribution with pdf $f(x)$, is defined by

$$T_R(\varepsilon) = \frac{1}{\varepsilon-1}\left\{1 - \int_{-\infty}^{\infty} f(x)^{\varepsilon} dx\right\}, \qquad \varepsilon \neq 1, \qquad \varepsilon > 0.$$

Hence, the TsE of the LBTLo-G family is obtained as follows:

$$T_R(\varepsilon) = \frac{1}{\varepsilon-1}\left\{1 - \int_{-\infty}^{\infty} \sum_{i=0}^{\infty} \Delta_i g(x;\zeta)^{\varepsilon} G(x;\zeta)^{i+\varepsilon} dx\right\}.$$

For example, the TsE of the LBTLoW distribution can be obtained as follows:

$$T_R(\varepsilon) = \frac{1}{\varepsilon-1}\left\{1 - \sum_{i,j=0}^{\infty} \frac{\Delta_{i,j}}{\gamma}\left[\beta(\varepsilon+j)\right]^{-\varepsilon+(\varepsilon/\gamma)-(1/\gamma)} \Gamma\left(\varepsilon - \frac{\varepsilon}{\gamma} + \frac{1}{\gamma}\right)\right\}.$$

Some numerical values for the proposed entropy measures are obtained for the LBTLoW and LBTLoIE distribution in Tables 4 and 5.

We can see from these tables that, as the value of ε rises, all entropy values decrease, providing more information. For a fixed value of β, as the values of α and γ rise, we infer that all entropy metrics decrease, indicating that there is less fluctuation. Additionally, we deduce that all entropies have less variability as the values of α, γ and β increase. When compared to other measures, the TsE measure values typically have the smallest values.

Table 4. Numerical values of entropy measures of the LBToW distribution.

ε	β	α	γ	**RE**	**HaCE**	**ArE**	**TsE**
1.5	0.25	0.2	0.4	3.331	3.340	2.767	1.957
		0.5	0.6	3.252	3.333	2.753	1.953
		0.8	0.8	3.099	3.318	2.722	1.944
		1.2	1.1	2.875	3.290	2.670	1.927
		1.5	1.3	2.716	3.264	2.627	1.912
		1.8	1.7	2.524	3.227	2.568	1.891
		2.0	1.9	2.415	3.203	2.53	1.876
		2.3	2.4	2.229	3.152	2.458	1.846
		2.6	2.7	2.084	3.104	2.394	1.818
		3.0	3.2	1.889	3.026	2.296	1.773
	0.5	0.2	0.4	1.930	3.044	2.318	1.783
		0.5	0.6	1.924	3.042	2.315	1.782
		0.8	0.8	1.824	2.996	2.260	1.755
		1.2	1.1	1.660	2.909	2.161	1.704
		1.5	1.3	1.541	2.835	2.081	1.661
		1.8	1.7	1.392	2.726	1.969	1.597
		2.0	1.9	1.308	2.657	1.901	1.556
		2.3	2.4	1.161	2.517	1.770	1.475
		2.6	2.7	1.051	2.397	1.661	1.404
		3.0	3.2	0.903	2.207	1.500	1.293
2.0	0.25	0.2	0.4	2.180	1.987	1.837	0.993
		0.5	0.6	2.167	1.986	1.835	0.993
		0.8	0.8	2.053	1.982	1.812	0.991
		1.2	1.1	1.876	1.973	1.769	0.987
		1.5	1.3	1.753	1.965	1.734	0.982
		1.8	1.7	1.595	1.949	1.681	0.975
		2.0	1.9	1.510	1.938	1.648	0.969
		2.3	2.4	1.358	1.912	1.581	0.956
		2.6	2.7	1.247	1.887	1.524	0.943
		3.0	3.2	1.098	1.840	1.435	0.920
	0.5	0.2	0.4	1.210	1.877	1.503	0.938
		0.5	0.6	1.279	1.895	1.541	0.947
		0.8	0.8	1.210	1.877	1.503	0.938
		1.2	1.1	1.080	1.834	1.423	0.917
		1.5	1.3	0.987	1.794	1.358	0.897
		1.8	1.7	0.861	1.725	1.258	0.862
		2.0	1.9	0.795	1.679	1.199	0.840
		2.3	2.4	0.672	1.574	1.077	0.787
		2.6	2.7	0.587	1.482	0.982	0.741
		3.0	3.2	0.470	1.323	0.836	0.661

Table 5. Numerical values of entropy measures of the LBToIE distribution.

ε	β	α	**RE**	**HaCE**	**ArE**	**TsE**
1.5	0.25	0.2	7.001	3.311	2.709	1.94
		0.5	7.075	3.315	2.716	1.942
		0.8	7.156	3.319	2.724	1.944
		1.2	7.274	3.324	2.735	1.947
		1.5	7.372	3.329	2.743	1.95
		1.8	7.476	3.333	2.752	1.952
		2	7.55	3.336	2.758	1.954
		2.3	7.667	3.34	2.767	1.957
		2.6	7.792	3.345	2.777	1.959
		3	7.971	3.351	2.79	1.963
	0.4	0.2	6.441	3.278	2.65	1.92
		0.5	6.452	3.279	2.651	1.921
		0.8	6.469	3.28	2.653	1.921
		1.2	6.503	3.282	2.657	1.923
		1.5	6.536	3.284	2.66	1.924
		1.8	6.577	3.287	2.665	1.925
		2	6.608	3.289	2.668	1.927
		2.3	6.661	3.292	2.674	1.928
		2.6	6.721	3.296	2.681	1.931
		3	6.813	3.301	2.69	1.934
2.0	0.25	0.2	4.376	1.975	1.776	0.987
		0.5	4.429	1.976	1.782	0.988
		0.8	4.487	1.977	1.788	0.989
		1.2	4.57	1.979	1.796	0.99
		1.5	4.639	1.981	1.803	0.99
		1.8	4.713	1.982	1.81	0.991
		2	4.765	1.983	1.815	0.991
		2.3	4.847	1.984	1.823	0.992
		2.6	4.934	1.986	1.83	0.993
		3	5.058	1.987	1.841	0.994
	0.4	0.2	3.975	1.962	1.726	0.981
		0.5	3.987	1.963	1.728	0.981
		0.8	4.003	1.963	1.73	0.982
		1.2	4.031	1.965	1.734	0.982
		1.5	4.058	1.965	1.737	0.983
		1.8	4.09	1.967	1.741	0.983
		2	4.114	1.967	1.744	0.984
		2.3	4.154	1.969	1.749	0.984
		2.6	4.199	1.97	1.755	0.985
		3	4.267	1.972	1.763	0.986

4. Maximum likelihood estimates via T2C

Let $x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(n)}$ be a T2C of size r resulting from a life test on n items whose lifetimes are described by the LBTLo-G family with a given set of parameters α and ζ, see [34–37]. The log-likelihood function of r failures and $(n-r)$ censored values, is given by

$$\log L(\alpha,\zeta) = r\log\alpha + r\log(1-\alpha) + r\log\Lambda(\alpha) + \sum_{i=1}^{r}\log g(x_i;\zeta) + \sum_{i=1}^{r}\log G(x_i;\zeta)$$
$$-(\alpha+1)\sum_{i=1}^{r}\log(1+G(x_i;\zeta)) + (n-r)\log\left[A_r(\alpha,\zeta)\right],$$

where $A_r(\alpha,\zeta) = 1-\Lambda(\alpha)\left[(1+G(x_r;\zeta))^{-\alpha}(1+\alpha G(x_r;\zeta))-1\right]$, and we write $x_{(i)} = x_i$ for simplified form.

By maximizing the previous likelihood function, the MLEs of unknown parameters are determined. To achieve this, we can first compute the first derivative of the score function $\left(U_\alpha,\, U_{\zeta_k}\right)$, given as follows:

$$U_\alpha = \frac{r}{\alpha} - \frac{r}{1-\alpha} + \frac{r}{\Lambda(\alpha)}\left(\frac{\partial}{\partial\alpha}\Lambda(\alpha)\right) - \sum_{i=1}^{r}\log(1+G(x_i;\zeta)) + \frac{(n-r)}{A_r(\alpha,\zeta)}\left(\frac{\partial}{\partial\alpha}A_r(\alpha,\zeta)\right),$$
$$U_{\zeta_k} = -(\alpha+1)\sum_{i=1}^{r}\frac{1}{1+G(x_i;\zeta)}\frac{\partial}{\partial_{\zeta_k}}(G(x_i;\zeta)) + \frac{(n-r)}{A_r(\alpha,\zeta)}\frac{\partial}{\partial_{\zeta_k}}A_r(\alpha,\zeta),$$

where

$$\frac{\partial}{\partial\alpha}\Lambda(\alpha) = [\Lambda(\alpha)]^2\, 2^{-\alpha}\left[(1+\alpha)\log 2 - 1\right],$$

$$\frac{\partial}{\partial\alpha}A_r(\alpha,\zeta) = -\frac{\partial}{\partial\alpha}\Lambda(\alpha)\left[(1+G(x_r;\zeta))^{-\alpha}(1+\alpha G(x_r;\zeta))-1\right]$$
$$+\Lambda(\alpha)\left[(1+G(x_r;\zeta))^{-\alpha}\left\{(1+\alpha G(x_r;\zeta))\log(1+G(x_r;\zeta)) - G(x_r;\zeta)\right\}\right],$$

and

$$\frac{\partial}{\partial_{\zeta_k}}(A_r(\alpha,\zeta)) = \alpha(\alpha-1)\Lambda(\alpha)G(x_r;\zeta)(1+G(x_r;\zeta))^{-\alpha-1}\frac{\partial}{\partial_{\zeta_k}}G(x_r;\zeta).$$

By putting U_α and U_{ζ_k} equal to zero and solving these equations simultaneously, the MLEs of the LBTLo-G family are found. These equations are not amenable to analytical solution, however they are amenable to numerical solution by iterative techniques utilizing statistical software.

The confidence interval (CI) of the vector of the unknown parameters $\xi = (\alpha,\, \zeta)$ could be obtained from the asymptotic distribution of the MLEs of the parameters as $\left(\hat{\xi}_{MLE} - \xi\right) \rightarrow N_2\left(0, I^{-1}\left(\hat{\xi}_{MLE}\right)\right)$, where $I(\xi)$ is the Fisher information matrix. Under particular regularity conditions, the two-sided $100(1-v)\%, 0 < v < 1$, asymptotic CI for the vector of unknown parameters ξ can be acquired in the following ways: $\hat{\xi}_{MLE} \pm z_{v/2}\sqrt{var(\hat{\xi})}$, where $var(\hat{\xi})$ is the element of the main diagonal of the asymptotic variance-covariance matrix $I^{-1}\left(\hat{\xi}_{MLE}\right)$ and $z_{v/2}$ is the upper $v^{th}\big/2$ percentile of the standard normal distribution.

5. Numerical results

This section includes a simulation study to evaluate the performance of the MLEs for the LBTLoW model (α,β,γ), for complete and T2C. The Mathematica 9 package is used to get the mean squared error (MSE), lower bound (LB) of CI, upper bound (UB) of CI, average length (AL) of 95%, and coverage probability (CP) of 95% of the estimated values of α, β and γ. The algorithm is developed in the way described below:

1) From the LBTLoW distribution, 5000 random samples of sizes n = 50, 100, 150, and 200 are created.
2) Values of the unknown parameters (α,β,γ) are selected as Set 1 = $(\alpha = 0.5, \beta = 0.5, \gamma = 0.5)$, Set 2 = $(\alpha = 0.7, \beta = 0.5, \gamma = 0.25)$, Set 3 = $(\alpha = 0.7, \beta = 0.7, \gamma = 0.5)$, and Set 4 = $(\alpha = 0.6, \beta = 0.3, \gamma = 0.5)$.
3) Three levels of censorship are chosen: r = 70%, 80% (T2C), and 100% (complete sample).
4) The MLEs, Biases, and MSEs for all sample sizes and for all selected sets of parameters are computed. Furthermore, the LB, UB, AL, and CP with a confidence level of 0.95 for all sample sizes and for all selected sets of parameters are calculated.
5) Numerical outcomes are reported in Table 6. Based on complete and T2C samples, we can detect the following about the performance of the estimated parameters.

A. For almost all the true values, the MSE of all the estimates decreases as the sample sizes and the censoring level r increase, demonstrating that the various estimates are consistent (see Table 6 and Figure 3).

B. For all true parameter values, the ALs of all the estimates decrease as the sample sizes and the censoring level r increase (see Table 6 and Figure 4).

C. For all true parameter values, the CP of all the estimates increases as the sample sizes and the censoring level r increase (see Table 6).

D. The MSE of the estimate of α at the true value of Set1 yields the lowest values in comparison to the other actual parameter values for all sample sizes (see Table 6 and Figure 5).

E. At all actual values, the MSE of the estimate of β produces the largest results for all sample sizes (see Table 6 and Figure 6). Also, it is evident that except for n = 50 and 200, the MSE of β estimates obtains the smallest values for the actual value of Set1 compared to the other actual sets at the censoring level 70%. At the censoring level 80%, the MSE of β estimates gets the smallest values at all sets of parameters except at n = 50.

F. The MSE of the estimate of γ at the true value of Set2 gets the smallest values in comparison to the other actual parameter values for all sample sizes (see Table 6 and Figure 7).

G. The MSEs, biases, and ALs of γ are smaller than the other estimates of α and β in almost all of the cases.

H. As n rises, the CI's lengths get shorter.

I. As n increases, parameter estimates grow increasingly accurate, suggesting that they are asymptotically unbiased.

J. For the parameter values examined, the CI's overall performance is fairly strong.

6. Data analysis

Here, we provide applications to four real data sets to illustrate the importance and potentiality of the LBTLoW distribution. The goodness-of-fit statistics for these distributions and other competitive distributions are compared, and the MLEs of their parameters are provided.

The first real data set [38] on the relief times of twenty patients receiving an analgesic is 1.1, 1.4, 3, 1.7, 2.3, 1.4, 1.3, 1.7, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.9, 1.8, 1.6, 1.2, 1.6, 2.

The second dataset illustrates the proportion of global reserves of natural gas in various countries as of the year 2020. In contrast to other nations, Russia possesses the largest natural gas reserves globally and maintains its position as the leading exporter of natural gas. Iran, on the other hand, ranks second in terms of natural gas reserves worldwide. Qatar, although holding slightly over 13% of the total global natural gas reserves, also plays a significant role in the natural gas market. Lastly, Saudi Arabia possesses the fifth-largest natural gas reserves globally. The electronic address from which it was taken is as follows: https://worldpopulationreview.com/. The data set is reported in Table 7.

The third dataset pertains to the Top 20 Countries with the Largest Oil Reserves, measured in thousand million barrels. Crude oil serves as the predominant fuel source globally and is the primary source of energy on a wide scale. In the year 2020, global oil consumption reached around 88.6 million barrels per day, or 30.1% of the overall primary energy consumption. Venezuela possesses the largest oil reserves globally, over 300 billion barrels in total. Saudi Arabia holds the world's second-largest oil reserves, with 297.5 billion barrels. The United States is the world's leading producer of oil as well as the world's greatest user of oil, necessitating additional imports from dozens of other oil-producing countries. Despite having the world's highest oil production, the United States is only 9th in the world in terms of available oil reserves. It was obtained from the following electronic address: https://worldpopulationreview.com/. The data set is reported in Table 8.

The fourth data set represents the Top 100 central banks that owned the largest gold Reserves (in thousand tons). Because of its safety, liquidity, and return qualities-the three major investment objectives for central banks-gold is an essential component of central bank reserves. As such, they are significant gold holders, accounting for around one-fifth of all gold extracted throughout history. They present gold reserve data derived using IMF IFS figures to help comprehend this sector of the gold market, which records central banks' (and other official institutions, when appropriate) reported purchases and sales of gold as a percentage of their international reserves. It was obtained from the following electronic address: https://www.gold.org/. The data set is reported in Table 9.

The descriptive analysis of all the data sets is reported in Table 10.

These real data sets are utilized to assess the goodness of fit of the LBTLoW distribution. The suggested model is compared with exponentiated transmuted generalized Rayleigh (ETGR) [39], beta Weibull (BW) [40], transmuted Lindley (T-Li) [41], McDonald log-logistic (McLL) [42], new modified Weibull (NMW) [43], weighted exponentiated inverted Weibull (WEIW) [44], transmuted complementary Weibull geometric (TCWG) [45], transmuted modified Weibull (TMW) [46], exponentiated Kumaraswamy Weibull (EKW) [47] and Weibull (W) models.

The maximum likelihood estimators (MLEs) and standard errors (SEs) of the model parameters are computed. In order to assess the distribution models, various criteria are taken into account, including the Akaike information criterion (A_{IC}), correct A_{IC} (C_{AIC}), Bayesian IC (B_{IC}), Hannan-Quinn IC (H_{QIC}), Kolmogorov-Smirnov (K_S) test, and p-value (P_V) test. In contrast, the broader dissemination is associated with reduced values of A_{IC}, C_{AIC}, B_{IC}, H_{QIC}, K_S, and the highest magnitude of P_V. The maximum likelihood estimators (MLEs) of the competitive models, along with their standard errors (SEs) and values of A_{IC}, C_{AIC}, B_{IC}, H_{QIC}, P_V, and K_S for the suggested data sets, are displayed in Tables 11-18. It has been observed that the LBTLoW distribution, characterized by three parameters, exhibits superior goodness of fit compared to alternative models. This distribution exhibits the lowest values of A_{IC}, C_{AIC}, B_{IC}, H_{QIC}, and K_S, and the highest value of P_V among the distributions under consideration in this analysis. Furthermore, Figures 8-15 exhibit the graphical representations of the estimated pdf, cdf, ccdf, and probability-probability (PP) plots for the competitive model applied to the given data sets.

From the previous figures, we conclude that the LBTLoW model clearly gives the best overall fit and so may be picked as the most appropriate model for explaining data.

7. Conclusions

The LBTLo-G family of distributions is explored in this article. The LBTLo-G family of probability distributions has a number of desirable characteristics, including being very flexible and simple, containing a number of new distributions, the ability for the generated distributions' pdfs to be unimodal, decreasing, bathtub-shaped, right-skewed, and symmetric, and the ability for their hrf shapes to be increasing, decreasing, U-shaped, upside-down-shaped, or J-shaped. These include discussion of the characteristics of the LBTLo-G family, including expansion for the density function, moments, incomplete moments, and certain entropy metrics. Estimating the model parameters is done using the ML technique. A simulation study demonstrated that the estimates of the model parameters are not far from their true values. Also, the biases and mean squared errors of estimates based on censored samples are larger than those based on complete samples. As the censoring levels and sample sizes increase, the coverage probability of estimates increases in approximately most cases.

As one distribution of the LBTLo-G family, the real datasets for global reserves of oil, gold, and natural gas were chosen to fit the LBTLoW distribution. The first data set proposed was the lifetime data relating to relief times (in minutes) of patients receiving an analgesic. The second data set provides the percent of global reserves of natural gas for 44 countries. We have considered the third real data analysis of the countries with the largest oil reserves in 20 countries. We consider another real-data analysis of the central bank owning the largest gold reserves in 100 countries. This gold reserve data, compiled using international monetary funds and international financial statistics, tracks central banks' reported purchases and sales of gold as a percentage of their international reserves. The LBTLoW model typically provides superior fits in comparison to certain other alternative models, as shown by real-world data applications.

Table 6. Accuracy measures of the LBTLoW estimates under T2C and complete samples.

n	r		Set1 ($\alpha = 0.5, \beta = 0.5, \gamma = 0.5$)						
			MLE	Bias	MSE	LB	UB	AL	CP
50	70%	α	0.4204	0.0796	0.0064	0.0019	0.839	0.8370	97.4%
		β	0.7041	0.2041	0.0471	0.5036	0.9046	0.4010	96.9%
		γ	0.4201	0.0799	0.0069	0.3023	0.5379	0.2356	96.0%
	80%	α	0.4218	0.0782	0.0061	0.0191	0.8245	0.8053	94.8%
		β	0.6382	0.1382	0.0242	0.4508	0.8256	0.3748	95.8%
		γ	0.4386	0.0614	0.0053	0.3282	0.5490	0.2208	97.1%
	100%	α	0.4234	0.0766	0.0059	0.0357	0.8111	0.7754	95.4%
		β	0.5177	0.0177	0.0056	0.3661	0.6694	0.3033	95.5%
		γ	0.5316	0.0316	0.0027	0.4303	0.6328	0.2025	96.0%
100	70%	α	0.4213	0.0787	0.0062	0.0844	0.7583	0.6740	96.2%
		β	0.6750	0.1750	0.0312	0.5375	0.8125	0.2750	95.9%
		γ	0.4237	0.0763	0.0065	0.3389	0.5084	0.1694	96.0%
	80%	α	0.4230	0.0770	0.0061	0.2099	0.6360	0.4262	96.2%
		β	0.6099	0.1099	0.0127	0.4819	0.7379	0.2560	96.1%
		γ	0.4487	0.0513	0.0033	0.3652	0.5321	0.1669	97.3%
	100%	α	0.4238	0.0762	0.0058	0.2501	0.5975	0.3473	95.6%
		β	0.4683	0.0317	0.0027	0.3558	0.5807	0.2249	95.8%
		γ	0.4967	0.0033	0.0025	0.4199	0.5734	0.1535	96.0%
150	70%	α	0.4217	0.0783	0.0061	0.2710	0.5725	0.3015	95.2%
		β	0.6626	0.1626	0.0281	0.5577	0.7675	0.2097	95.6%
		γ	0.4277	0.0723	0.0058	0.3571	0.4983	0.1412	97.3%
	80%	α	0.4236	0.0764	0.0059	0.3005	0.5466	0.2461	95.7%
		β	0.5977	0.0977	0.0113	0.4957	0.6997	0.2040	96.2%
		γ	0.4649	0.0351	0.0022	0.3972	0.5325	0.1353	97.0%
	100%	α	0.4238	0.0762	0.0058	0.3010	0.5467	0.2457	95.6%
		β	0.4766	0.0234	0.0023	0.3784	0.5749	0.1965	96.4%
		γ	0.5277	0.0277	0.0015	0.4659	0.5894	0.1236	96.9%
200	70%	α	0.4219	0.0781	0.0061	0.3154	0.5285	0.2132	96.1%
		β	0.6592	0.1592	0.0268	0.5675	0.7510	0.1835	96.3%
		γ	0.4375	0.0625	0.0046	0.3789	0.4962	0.1173	96.7%
	80%	α	0.4239	0.0761	0.0058	0.3236	0.5242	0.2006	96.3%
		β	0.5912	0.0912	0.0099	0.5074	0.6750	0.1676	97.0%
		γ	0.4667	0.0333	0.0020	0.4101	0.5233	0.1132	97.5%
	100%	α	0.4240	0.0760	0.0058	0.3372	0.5109	0.1737	96.5%
		β	0.4905	0.0095	0.0009	0.4167	0.5642	0.1475	96.7%
		γ	0.5035	0.0035	0.0006	0.4524	0.5546	0.1022	97.1%

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n	r		Set2 ($\alpha = 0.7, \beta = 0.5, \gamma = 0.25$)						
			MLE	Bias	MSE	LB	UB	AL	CP
50	70%	α	0.4206	0.2794	0.0782	0.0055	0.8358	0.8304	97.7%
		β	0.7016	0.2016	0.0462	0.5015	0.9016	0.4001	96.5%
		γ	0.2172	0.0328	0.0017	0.1568	0.2776	0.1208	100%
	80%	α	0.4214	0.2786	0.0776	0.2084	0.6345	0.4261	97.9%
		β	0.6361	0.1361	0.0243	0.4490	0.8231	0.3741	98.5%
		γ	0.2297	0.0203	0.0010	0.1719	0.2874	0.1155	100%
	100%	α	0.4234	0.2766	0.0765	0.2497	0.5970	0.3473	98.3%
		β	0.5161	0.0161	0.0062	0.3649	0.6673	0.3025	97.6%
		γ	0.2565	0.0065	0.0007	0.2044	0.3086	0.1043	100%
100	70%	α	0.4210	0.2790	0.0779	0.0208	0.8212	0.8004	96.4%
		β	0.7006	0.2006	0.0431	0.5593	0.8419	0.2826	98.0%
		γ	0.2141	0.0359	0.0016	0.1720	0.2562	0.0842	100%
	80%	α	0.4215	0.2785	0.0776	0.2708	0.5721	0.3013	97.2%
		β	0.6357	0.1357	0.0214	0.5035	0.7680	0.2645	97.7%
		γ	0.2270	0.0230	0.0008	0.1866	0.2673	0.0807	100%
	100%	α	0.4234	0.2766	0.0765	0.3006	0.5462	0.2456	97.3%
		β	0.5158	0.0158	0.0033	0.4088	0.6227	0.2140	98.2%
		γ	0.2540	0.0040	0.0003	0.2176	0.2905	0.0729	100%
150	70%	α	0.4212	0.2788	0.0778	0.0330	0.8093	0.7763	97.7%
		β	0.7000	0.2000	0.0419	0.5847	0.8153	0.2306	97.7%
		γ	0.2122	0.0378	0.0016	0.1781	0.2463	0.0682	100%
	80%	α	0.4215	0.2785	0.0776	0.2985	0.5445	0.2460	98.8%
		β	0.6350	0.1350	0.0203	0.5271	0.7430	0.2159	98.1%
		γ	0.2259	0.0241	0.0008	0.1931	0.2587	0.0656	96.0%
	100%	α	0.4234	0.2766	0.0765	0.3232	0.5237	0.2005	97.2%
		β	0.5151	0.0151	0.0023	0.4278	0.6024	0.1746	97.0%
		γ	0.2529	0.0029	0.0002	0.2232	0.2825	0.0593	95.4%
200	70%	α	0.4209	0.2791	0.0779	0.0849	0.7569	0.6720	100%
		β	0.6981	0.1981	0.0405	0.5984	0.7978	0.1994	97.2%
		γ	0.2118	0.0382	0.0016	0.1823	0.2412	0.0589	97.3%
	80%	α	0.4215	0.2785	0.0776	0.3150	0.5280	0.2131	100%
		β	0.6331	0.1331	0.0191	0.5398	0.7265	0.1867	98.2%
		γ	0.2256	0.0244	0.0007	0.1973	0.2540	0.0567	98.0%
	100%	α	0.4234	0.2766	0.0765	0.3366	0.5103	0.1737	100%
		β	0.5136	0.0136	0.0016	0.4381	0.5891	0.1510	98.8%
		γ	0.2523	0.0023	0.0002	0.2267	0.2779	0.0512	100%

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n	r		Set3 ($\alpha = 0.7, \beta = 0.7, \gamma = 0.5$)						
			MLE	Bias	MSE	LB	UB	AL	CP
50	70%	α	0.4178	0.2822	0.0797	0.1937	0.6419	0.4482	96.2%
		β	0.8994	0.1994	0.0477	0.6064	1.1923	0.5859	95.9%
		γ	0.6151	0.1151	0.0239	0.3763	0.8540	0.4776	95.0%
	80%	α	0.4193	0.2807	0.0788	0.2255	0.6131	0.3875	95.9%
		β	0.8238	0.1238	0.0218	0.5471	1.1006	0.5535	95.9%
		γ	0.5695	0.0695	0.0182	0.3447	0.7943	0.4496	96.7%
	100%	α	0.4211	0.2789	0.0778	0.2480	0.5942	0.3461	96.8%
		β	0.7612	0.0612	0.0104	0.5012	1.0213	0.5201	97.0%
		γ	0.5425	0.0425	0.0163	0.3395	0.7456	0.4061	95.0%
100	70%	α	0.4174	0.2826	0.0798	0.2439	0.5910	0.3470	95.0%
		β	0.8787	0.1787	0.0353	0.6426	1.1148	0.4722	96.3%
		γ	0.5696	0.0696	0.0201	0.3697	0.7696	0.3998	96.0%
	80%	α	0.4191	0.2809	0.0789	0.2690	0.5691	0.3001	95.5%
		β	0.7802	0.0802	0.0134	0.5600	1.0004	0.4404	95.7%
		γ	0.5479	0.0479	0.0091	0.3814	0.7144	0.3330	96.0%
	100%	α	0.4206	0.2794	0.0781	0.2866	0.5546	0.2680	95.6%
		β	0.7146	0.0146	0.0068	0.5069	0.9223	0.4154	95.7%
		γ	0.5597	0.0597	0.0074	0.4063	0.7131	0.3067	96.0%
150	70%	α	0.4176	0.2824	0.0798	0.2949	0.5403	0.2454	95.8%
		β	0.8697	0.1697	0.0305	0.7056	1.0338	0.3282	96.2%
		γ	0.5924	0.0924	0.0174	0.4449	0.7399	0.2950	97.1%
	80%	α	0.4193	0.2807	0.0788	0.3131	0.5254	0.2122	96.2%
		β	0.8023	0.1023	0.0124	0.6492	0.9555	0.3063	96.1%
		γ	0.5524	0.0524	0.0058	0.4332	0.6716	0.2384	97.0%
	100%	α	0.4209	0.2791	0.0779	0.3261	0.5156	0.1895	95.8%
		β	0.7374	0.0374	0.0032	0.5932	0.8817	0.2885	96.3%
		γ	0.5507	0.0507	0.0041	0.4447	0.6568	0.2121	96.9%
200	70%	α	0.4175	0.2825	0.0798	0.3173	0.5177	0.2004	96.1%
		β	0.8406	0.1406	0.0265	0.7075	0.9736	0.2661	97.2%
		γ	0.5683	0.0683	0.0069	0.4569	0.6798	0.2230	96.9%
	80%	α	0.4193	0.2807	0.0788	0.3327	0.5059	0.1733	96.3%
		β	0.8043	0.1043	0.0123	0.6790	0.9295	0.2505	96.6%
		γ	0.5502	0.0502	0.0041	0.4533	0.6471	0.1938	97.0%
	100%	α	0.4209	0.2791	0.0779	0.3435	0.4983	0.1548	96.1%
		β	0.7411	0.0411	0.0030	0.6229	0.8592	0.2362	97.0%
		γ	0.5463	0.0463	0.0041	0.4589	0.6338	0.1749	96.2%

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n	r		Set4 ($\alpha = 0.6, \beta = 0.3, \gamma = 0.5$)						
			MLE	Bias	MSE	LB	UB	AL	CP
50	70%	α	0.4197	0.1803	0.0325	0.2456	0.5939	0.3482	98.1%
		β	0.5719	0.2719	0.0744	0.3742	0.7696	0.3954	97.0%
		γ	0.2990	0.2010	0.0406	0.1769	0.4211	0.2442	98.0%
	80%	α	0.4221	0.1779	0.0317	0.2716	0.5725	0.3009	98.4%
		β	0.4934	0.1934	0.0384	0.3126	0.6742	0.3616	97.4%
		γ	0.3593	0.1407	0.0200	0.2383	0.4803	0.2421	98.2%
	100%	α	0.4246	0.1754	0.0308	0.2903	0.5589	0.2686	97.9%
		β	0.4198	0.1198	0.0150	0.2576	0.5820	0.3244	97.4%
		γ	0.4172	0.0828	0.0078	0.2968	0.5376	0.2408	98.4%
100	70%	α	0.4198	0.1802	0.0325	0.2966	0.5429	0.2463	97.2%
		β	0.5674	0.2674	0.0717	0.4278	0.7069	0.2791	97.7%
		γ	0.3137	0.1863	0.0351	0.2278	0.3995	0.1717	98.0%
	80%	α	0.4222	0.1778	0.0316	0.3157	0.5286	0.2128	97.9%
		β	0.4857	0.1857	0.0350	0.3584	0.6130	0.2546	97.9%
		γ	0.3669	0.1331	0.0179	0.2818	0.4519	0.1701	98.7%
	100%	α	0.4248	0.1752	0.0307	0.3242	0.5253	0.2011	98.1%
		β	0.4082	0.1082	0.0125	0.2944	0.5221	0.2277	97.8%
		γ	0.4171	0.0829	0.0074	0.3349	0.4993	0.1644	98.3%
150	70%	α	0.4198	0.1802	0.0325	0.3248	0.5148	0.1899	97.2%
		β	0.5653	0.2653	0.0706	0.4518	0.6788	0.2270	98.3%
		γ	0.3140	0.1860	0.0350	0.2421	0.3860	0.1439	99.3%
	80%	α	0.4222	0.1778	0.0316	0.3353	0.5091	0.1738	97.7%
		β	0.4832	0.1832	0.0338	0.3797	0.5866	0.2069	98.2%
		γ	0.3677	0.1323	0.0178	0.2996	0.4358	0.1363	99.0%
	100%	α	0.4248	0.1752	0.0307	0.3377	0.5119	0.1742	97.3%
		β	0.4018	0.1018	0.0107	0.3032	0.5004	0.1972	97.9%
		γ	0.4248	0.0752	0.0061	0.3608	0.4888	0.1280	99.7%
200	70%	α	0.4198	0.1802	0.0325	0.3423	0.4973	0.1551	99.0%
		β	0.5650	0.2650	0.0705	0.4738	0.6561	0.1823	99.1%
		γ	0.3256	0.1744	0.0309	0.2637	0.3875	0.1239	98.7%
	80%	α	0.4222	0.1778	0.0316	0.3470	0.4975	0.1505	99.6%
		β	0.4781	0.1781	0.0321	0.3882	0.5679	0.1798	99.3%
		γ	0.3733	0.1267	0.0173	0.3129	0.4336	0.1206	99.5%
	100%	α	0.4250	0.1750	0.0306	0.3578	0.4921	0.1343	98.7%
		β	0.3904	0.0904	0.0088	0.3106	0.4703	0.1596	99.6%
		γ	0.4262	0.0738	0.0060	0.3681	0.4843	0.1162	100%

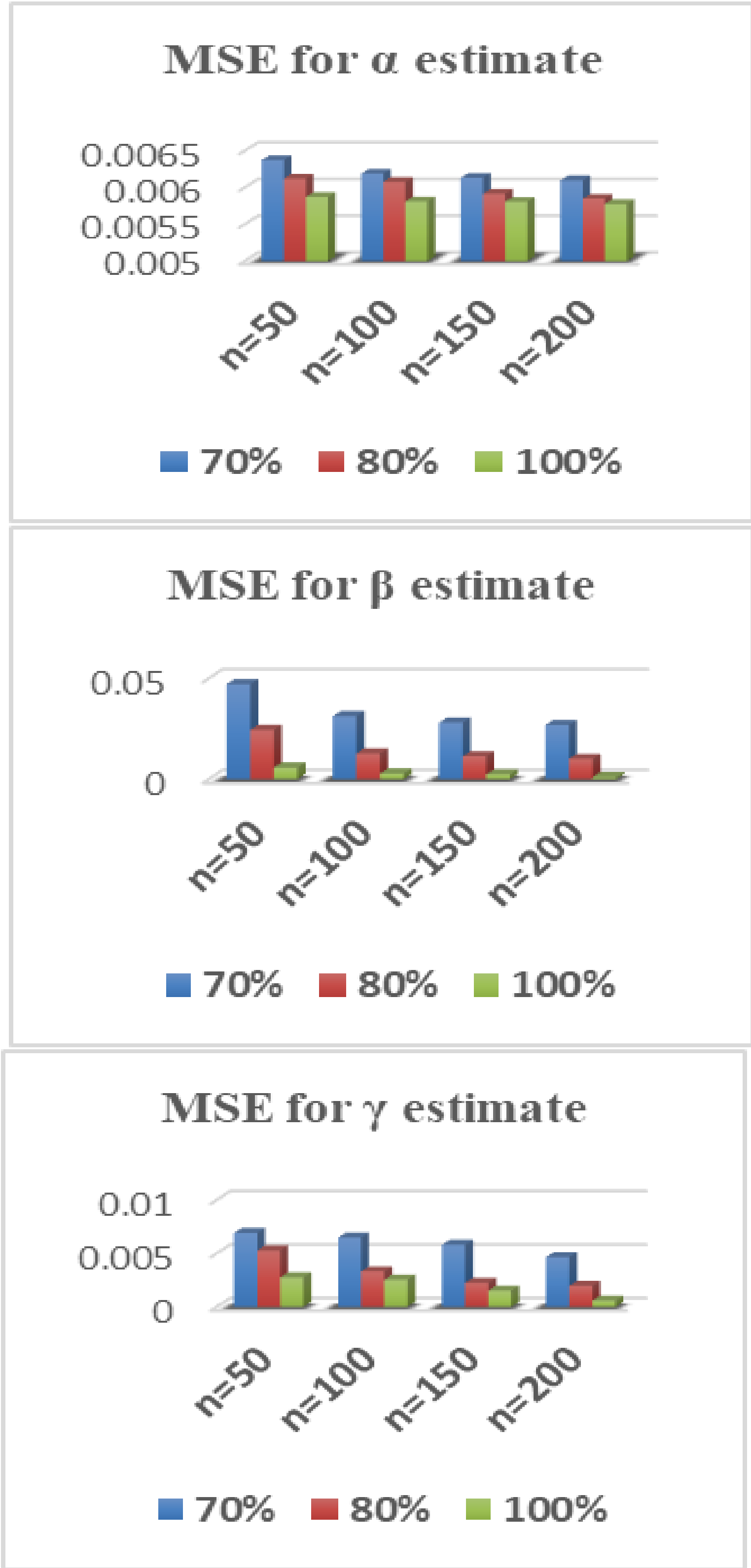

Figure 3. MSE of the estimates at the true value of Set1.

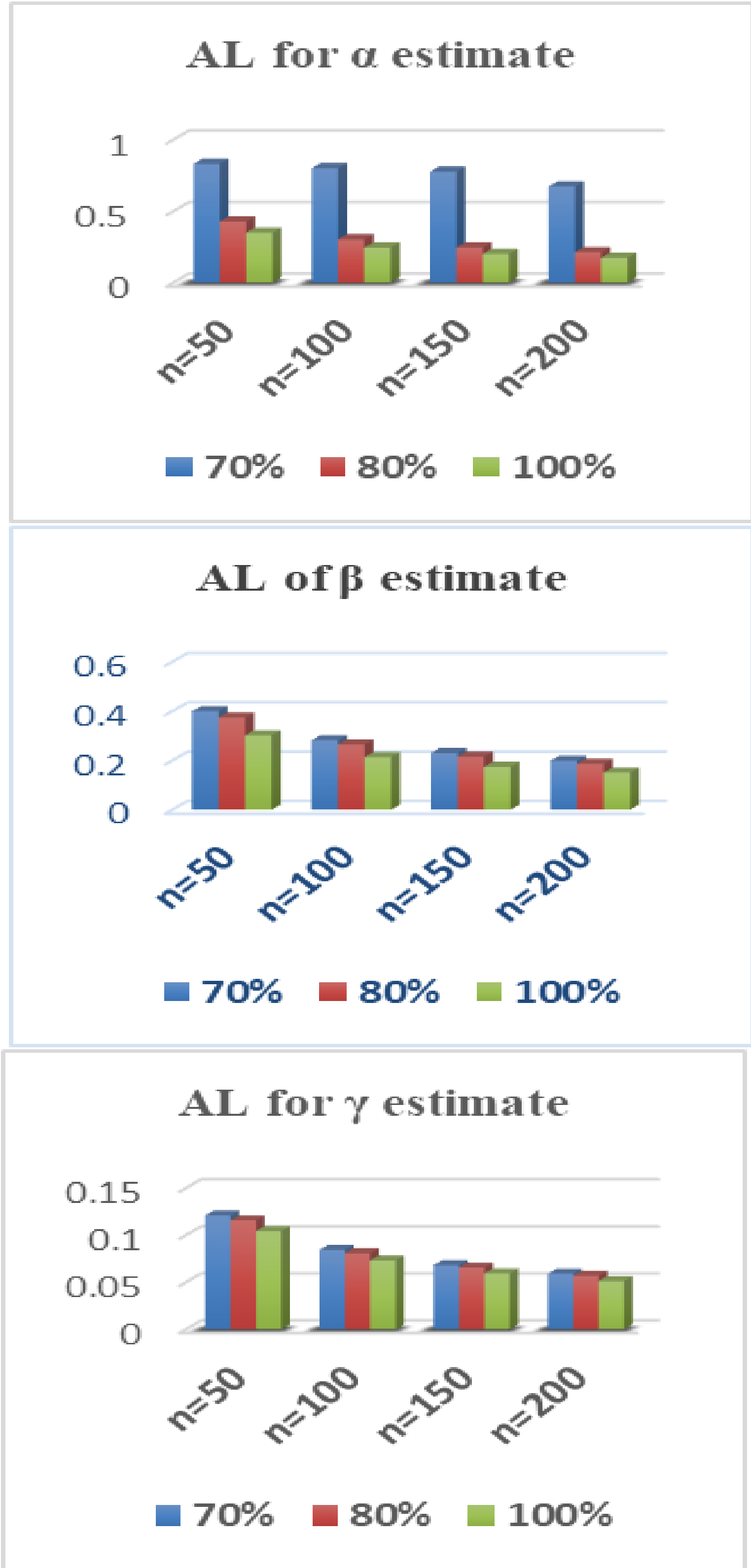

Figure 4. AL of the estimates at the true value of Set2.

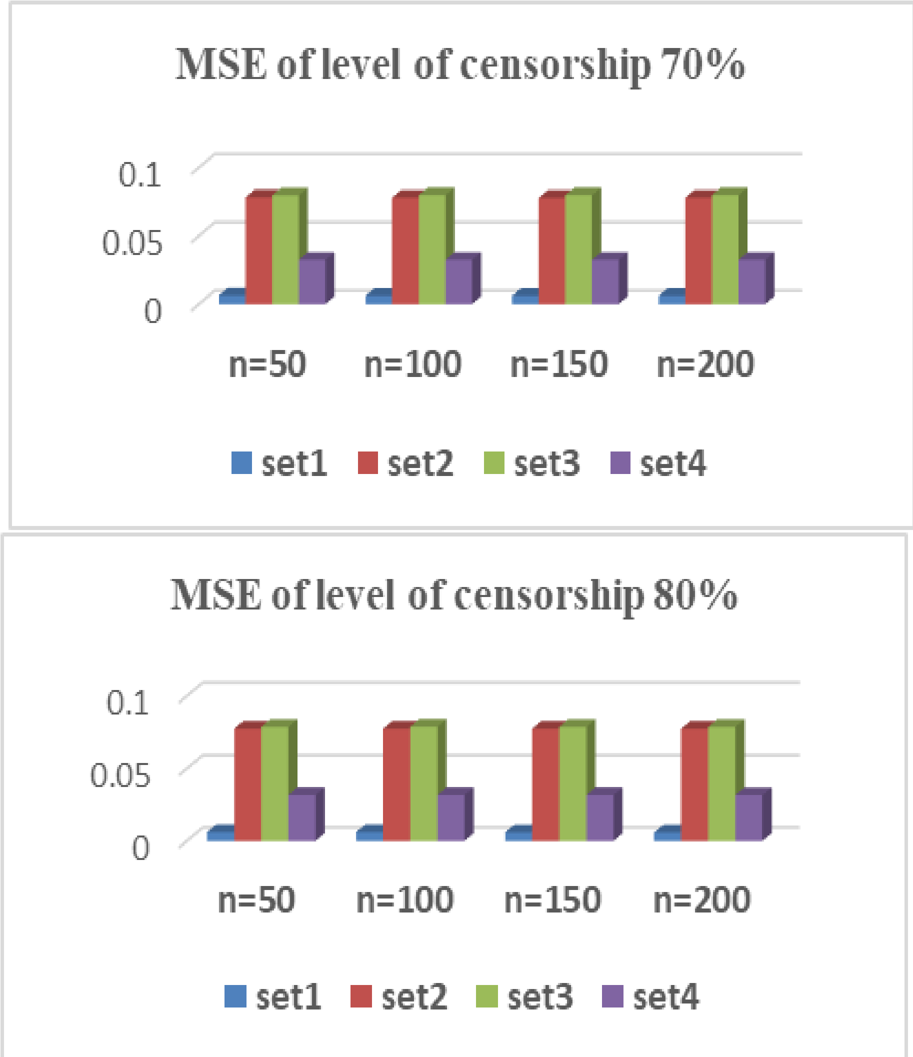

Figure 5. MSE of $\hat{\alpha}$ for all sets.

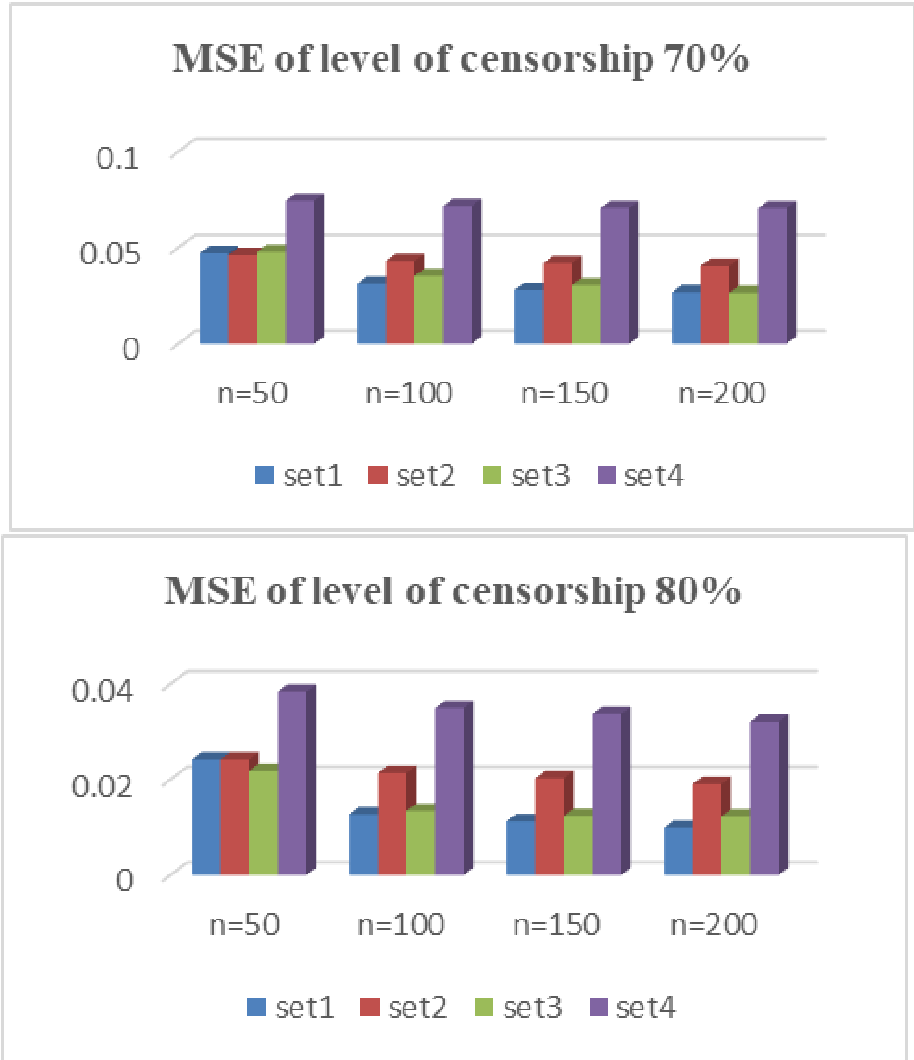

Figure 6. MSE of $\hat{\beta}$ for all sets.

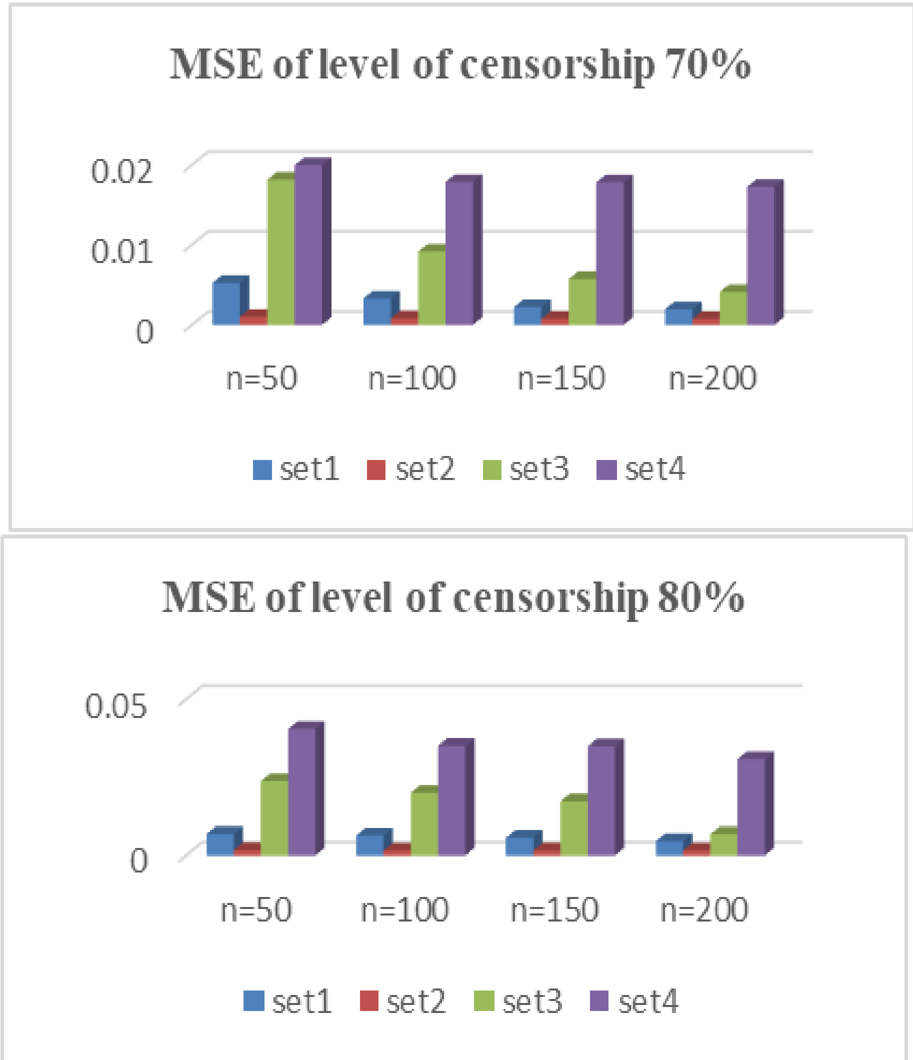

Figure 7. MSE of $\hat{\gamma}$ for all sets.

Table 7. The percent Global Reserves Natural Gas of the Countries (2020).

Rank	Country	% Global Reserves	Rank	Country	% Global Reserves
1	Russia	19.9	23	Ukraine	0.6
2	Iran	17.1	24	Malaysia	0.5
3	Qatar	13.1	25	Uzbekistan	0.4
4	Turkmenistan	7.2	26	Oman	0.4
5	United States	6.7	27	Vietnam	0.3
6	China	4.5	28	Israel	0.3
7	Venezuela	3.3	29	Argentina	0.2
8	Saudi Arabia	3.2	30	Pakistan	0.2
9	United Arab Emirates	3.2	31	Trinidad	0.2
10	Nigeria	2.9	32	Brazil	0.2
11	Iraq	1.9	33	Myanmar	0.2
12	Canada	1.3	34	United Kingdom	0.1
13	Australia	1.3	35	Thailand	0.1
14	Azerbaijan	1.3	36	Mexico	0.1
15	Algeria	1.2	37	Bangladesh	0.1
16	Kazakhstan	1.2	38	Netherlands	0.1
17	Egypt	1.1	39	Bolivia	0.1
18	Kuwait	0.9	40	Brunei	0.1
19	Norway	0.8	41	Peru	0.1
20	Libya	0.8	42	Syria	0.1
21	Indonesia	0.7	43	Yemen	0.1
22	India	0.7	44	Papua New Guinea	0.1

Table 8. Top 20 Countries with the Largest Oil Reserves (in thousand million barrels).

Rank	Country	reserves2020	Rank	Country	reserves2020
1	Venezuela	303.8	11	Nigeria	36.9
2	Saudi Arabia	297.5	12	Kazakhstan	30
3	Canada	168.1	13	China	26
4	Iran	157.8	14	Qatar	25.2
5	Iraq	145	15	Algeria	12.2
6	Russia	107.8	16	Brazil	11.9
7	Kuwait	101.5	17	Norway	7.9
8	United Arab Emirates	97.8	18	Angola	7.8
9	United States	68.8	19	Azerbaijan	7
10	Libya	48.4	20	Mexico	6.1

Table 9. Top 100 central bank owned the largest gold Reserves (in thousand tons).

Rank	Country	Reserves of Gold	Rank	Country	Reserves of Gold	Rank	Country	Reserves of Gold
1	USA	8.1335	35	LBY	0.1166	68	CYP	0.0139
2	DEU	3.3585	36	GRC	0.1141	69	CUW	0.0131
3	IMF	2.814	37	ROK	0.1045	70	MUS	0.0124
4	ITA	2.4518	38	ROU	0.1036	71	IRL	0.012
5	FRA	2.4365	39	BIS	0.102	72	CZE	0.0109
6	RUS	2.2985	40	IRQ	0.0964	73	KGZ	0.0102
7	CHN	1.9483	41	HUN	0.0945	74	GHA	0.0087
8	CHE	1.04	42	AUS	0.0798	75	PRY	0.0082
9	JPN	0.846	43	KWT	0.079	76	NPL	0.008
10	IND	0.7604	44	IDN	0.0786	77	MNG	0.0076
11	NLD	0.6125	45	DNK	0.0666	78	MMR	0.0073
12	ECB	0.5048	46	PAK	0.0647	79	GTM	0.0069
13	TUR	0.4311	47	ARG	0.0617	80	MKD	0.0069
14	TAI	0.4236	48	ARE	0.0553	81	TUN	0.0068
15	PRT	0.3826	49	BLR	0.0535	82	LVA	0.0067
16	KAZ	0.3681	50	QAT	0.0513	83	LTU	0.0058
17	UZB	0.3375	51	KHM	0.0504	84	COL	0.0047
18	SAU	0.3231	52	FIN	0.049	85	BHR	0.0047
19	GBR	0.3103	53	JOR	0.0435	86	BRN	0.0046
20	LBN	0.2868	54	BOL	0.0425	87	GIN	0.0042
21	ESP	0.2816	55	BGR	0.0408	88	MOZ	0.0039
22	AUT	0.28	56	MYS	0.0389	89	SVN	0.0032
23	THA	0.2442	57	SRB	0.0378	90	ABW	0.0031
24	POL	0.2287	58	WAEMU	0.0365	91	BIH	0.003
25	BEL	0.2274	59	PER	0.0347	92	ALB	0.0028
26	DZA	0.1736	60	SVK	0.0317	93	LUX	0.0022
27	VEN	0.1612	61	UKR	0.0271	94	HKG	0.0021
28	PHL	0.1563	62	SYR	0.0258	95	ISL	0.002
29	SGP	0.1537	63	MAR	0.0221	96	TTO	0.0019
30	BRA	0.1297	64	ECU	0.0219	97	HTI	0.0018
31	SWE	0.1257	65	AFG	0.0219	98	YEM	0.0016
32	ZAF	0.1254	66	NGA	0.0215	99	SUR	0.0015
33	EGY	0.125	67	BGD	0.014	100	SLV	0.0014
34	MEX	0.1199						

Table 10. Some descriptive analysis of all data sets.

	n	Mean	Median	Skewness	Kurtosis	Range	Min	Max	Sum
Data1	20	1.900	1.700	1.860	4.185	3.000	1.100	4.100	38.000
Data2	44	2.248	0.650	2.990	8.864	19.800	0.100	19.900	98.900
Data3	20	83.375	42.650	1.430	1.420	297.700	6.100	303.800	1667.500
Data4	100	0.347	0.050	5.590	38.257	8.130	0.001	8.133	34.676

Table 11. MLEs and SEs for the first data set.

Distributions	MLE and SE				
	α	β	γ	λ	θ
LBTLoW	8.648	3.074	0.042		
	(3.545)	(0.474)	(0.025)		
ETGR	0.103	0.692	23.539	-0.342	
	(0.436)	(0.086)	(105.137)	(1.971)	
BW	0.831	0.613	29.947	11.632	
	(0.954)	(0.340)	(40.414)	(21.900)	
T-Li	0.665	0.359			
	(0.332)	(0.048)			
McLL	0.881	2.070	1.926	19.225	32.033
	(0.109)	(3.693)	(5.165)	(22.341)	(43.081)
NMW	0.121	2.784	2.787	0.003	0.008
	(0.056)	(20.370)	(0.428)	(0.025)	(0.002)
W	0.122	2.787			
	(0.056)	(0.427)			

Table 12. Measures of fitting for the first data set.

Distributions	A_{IC}	C_{AIC}	B_{IC}	H_{QIC}	K_S	P_V
LBTLoW	40.140	41.640	38.040	40.720	0.146	0.790
ETGR	44.860	47.520	42.060	45.630	0.190	0.465
BW	42.400	45.060	39.600	43.170	0.160	0.683
T-Li	65.730	66.440	64.330	66.120	0.380	0.006
McLL	43.850	48.140	40.360	44.830	0.147	0.734
NMW	51.170	55.460	47.680	52.150	0.190	0.501
W	45.170	45.880	43.780	45.560	0.180	0.509

Table 13. MLEs and SEs for the global reserves natural gas data set.

Distributions	MLE and SE				
	α	β	γ	λ	θ
LBTLoW	6.268	0.623	0.484		
	(2.631)	(0.066)	(0.210)		
ETGR	0.055	0.071	8.773	0.947	
	(0.027)	(0.029)	(7.043)	(0.081)	
TCWG	34.076	0.802	0.005	1.12	
	(81.023)	(0.021)	(0.013)	(0.285)	
EKW	0.221	400.298	5.215	1	3.823
	(0.038)	(718.99)	(0.649)	(0.004)	(3.036)
TMW	0.851	1.159	-0.554	0.519	
	(0.163)	(1.026)	(0.985)	(0.379)	
BW	2.861	0.075	78.550	42.576	
	(69.095)	(0.090)	(167.320)	(187.300)	
T-Li	0.604	0.671			
	(0.155)	(0.074)			
McLL	0.181	1.565	1.286	21.234	28.124
	(0.193)	(9.254)	(5.432)	(34.701)	(45.757)
NMW	6.8×10^{-8}	0.680	0.223	0.015	0.806
	(0.623)	(0.110)	(617.48)	(0.015)	(0.418)
W	0.799	0.621			
	(0.136)	(0.068)			

Table 14. Measures of fitting for the global reserves natural gas data set.

Distributions	A_{IC}	C_{AIC}	B_{IC}	H_{QIC}	K_S	P_V
LBTLoW	132.210	132.810	131.140	134.200	0.130	0.425
ETGR	143.470	144.490	142.040	146.110	0.180	0.118
TCWG	137.690	138.710	136.260	140.330	0.150	0.251
EKW	133.890	135.470	132.110	140.330	0.140	0.355
TMW	140.900	142.480	139.120	144.210	0.150	0.276
BW	133.180	134.200	131.750	135.820	0.130	0.408
T-Li	174.360	174.660	173.650	175.690	0.200	0.057
McLL	134.830	136.410	133.040	138.130	0.130	0.419
NMW	143.780	145.360	142.000	147.090	0.160	0.243
W	138.650	138.940	137.940	139.970	0.170	0.139

Table 15. MLEs and SEs for the global oil reserves data set.

Distributions	MLE and SE				
	α	β	γ	λ	θ
LBTLoW	2.515	0.756	0.040		
	(3.877)	(0.166)	(0.048)		
WEIW	0.909	0.871	7.225		
	(106700)	(0.152)	(384700)		
TMW	0.998	0.459	-0.443	0.202	
	(0.081)	(18.537)	(18.537)	(0.769)	
T-Li	0.021	0.384			
	(0.345)	(0.004)			
McLL	0.208	93.978	1.279	24.759	32.815
	(0.499)	(1721)	(19.272)	(142.806)	(161.611)
NMW	10.7×10^{-8}	0.930	0.859	7.46×10^{-8}	0.017
	(0.001)	(0.250)	(1.216)	(0.002)	(0.017)
EKW	0.167	261.64	45.725	1.201	2.138
	(0.079)	(1709)	(219.725)	(0.741)	(7.209)

Table 16. Measures of fitting for the global oil reserves data set.

Distributions	A_{IC}	C_{AIC}	B_{IC}	H_{QIC}	K_S	P_V
LBTLoW	221.690	223.190	219.600	222.280	0.135	0.857
WEIW	223.400	224.900	221.300	223.980	0.157	0.708
TMW	226.410	230.690	222.910	227.380	0.153	0.734
T-Li	230.480	231.180	229.080	230.870	0.265	0.120
McLL	225.990	230.280	222.500	222.500	0.146	0.789
NMW	226.570	230.860	223.080	227.540	0.140	0.826
EKW	226.290	230.570	222.790	229.300	0.148	0.776

Table 17. MLEs and SEs for the global gold reserves data set.

Distributions	MLE and SE				
	α	β	γ	λ	θ
LBTLoW	6.498	0.482	1.490		
	(2.301)	(0.034)	(0.573)		
EKW	0.221	1096	4.424	1	1.717
	(0.030)	(1376)	(1.817)	(0.001)	(0.901)
TMW	0.596	2.612	0.588	-0.523	
	(0.057)	(0.689)	(0.256)	(0.346)	
BW	134.832	0.073	49.149	22.930	
	(956.622)	(0.060)	(74.497)	(46.500)	
WEIW	27.512	0.549	0.094		
	(3272000)	(0.042)	(856.967)		
W	2.648	0.489			
	(0.281)	(0.035)			

Table 18. Measures of fitting for the global gold reserves data set.

Distributions	A_{IC}	C_{AIC}	B_{IC}	H_{QIC}	K_S	P_V
LBTLoW	–170.510	–170.260	–170.510	–167.350	0.070	0.704
ETGR	–167.710	–167.070	–167.710	–157.710	0.078	0.584
BW	–158.180	–157.540	–158.180	–152.910	0.077	0.593
T-Li	–170.220	–169.800	–170.220	–166.010	0.071	0.703
McLL	–170.200	–169.950	–170.200	–167.040	0.091	0.383
W	–157.390	–157.260	–157.390	–155.280	0.100	0.269

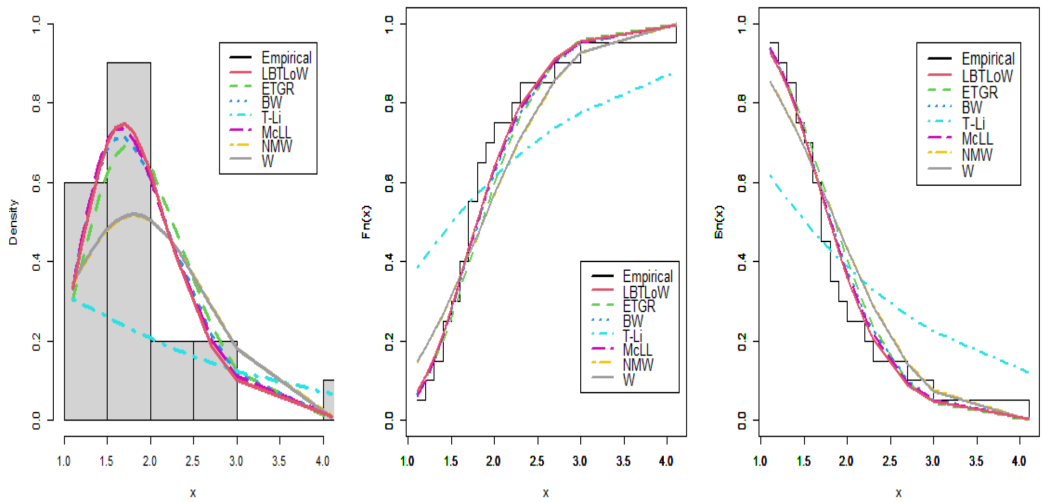

Figure 8. Estimated pdf, cdf and ccdf plots of the competitive models for the first data set.

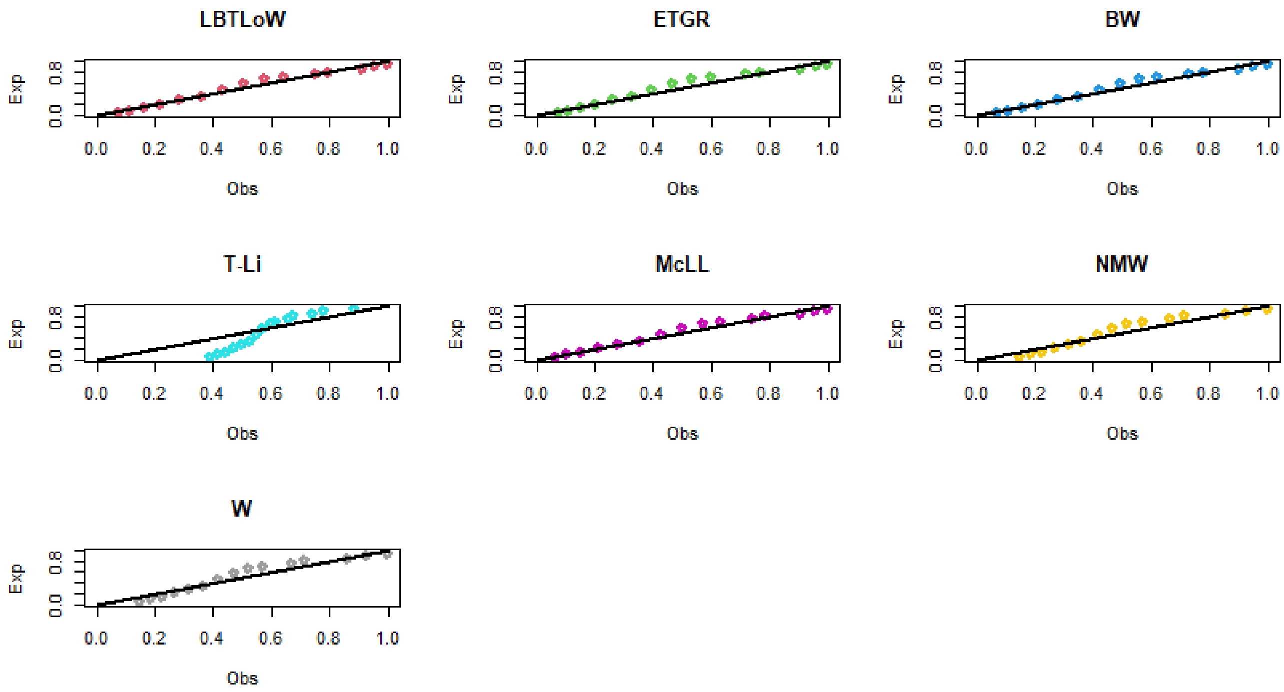

Figure 9. The PP plots of the fitted models for the first data set.

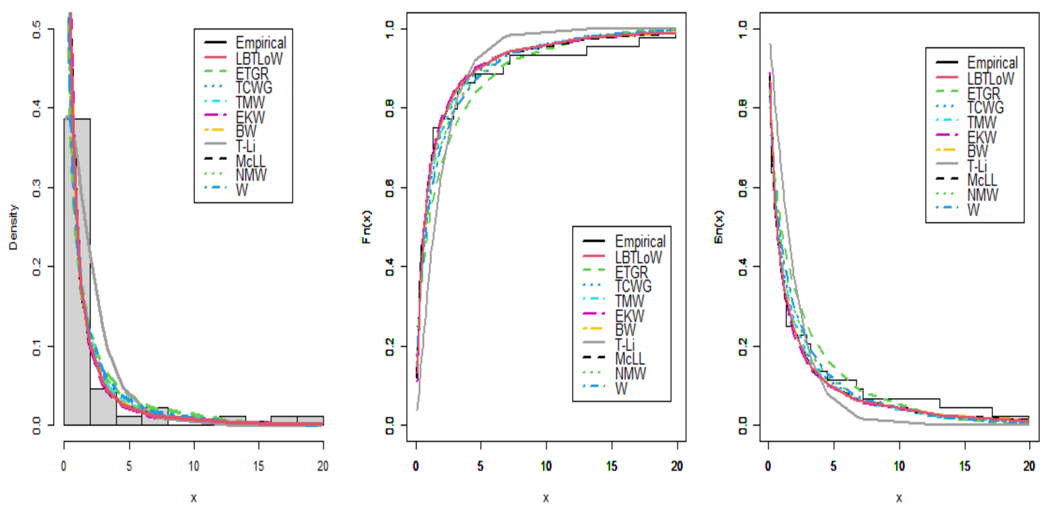

Figure 10. Estimated pdf, cdf and ccdf plots of the competitive models for global reserves natural gas data set.

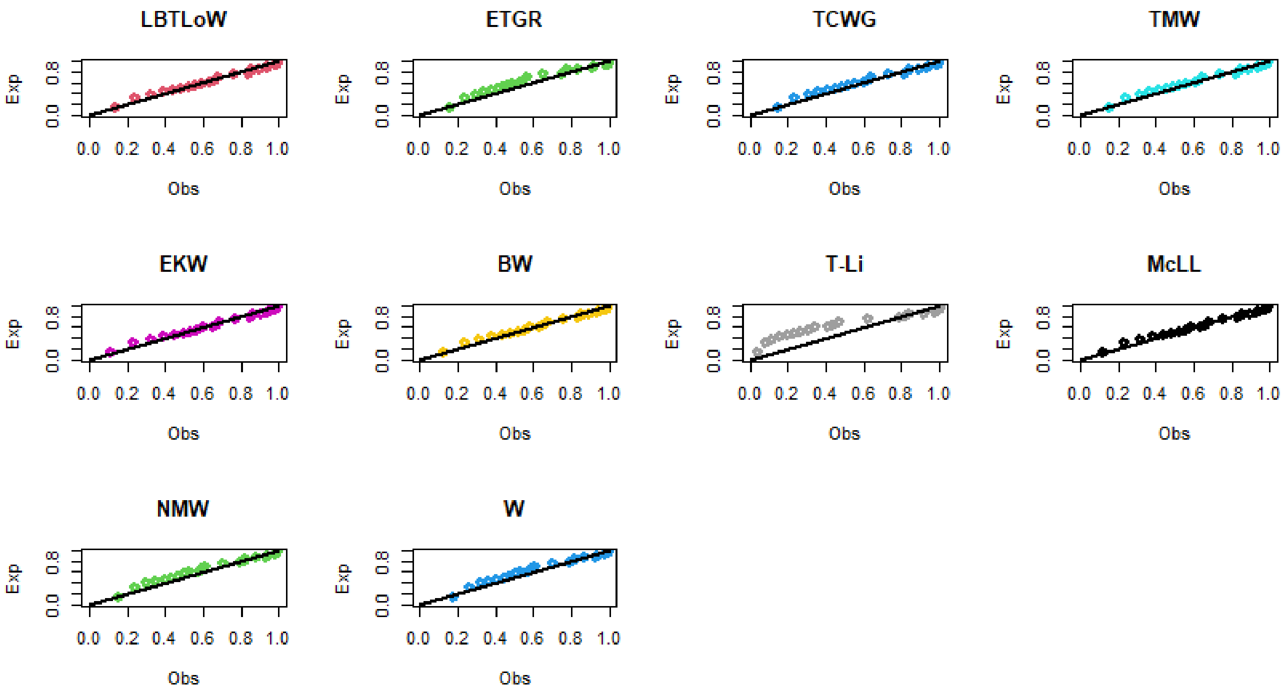

Figure 11. The PP plots of the fitted models for the global reserves natural gas data set.

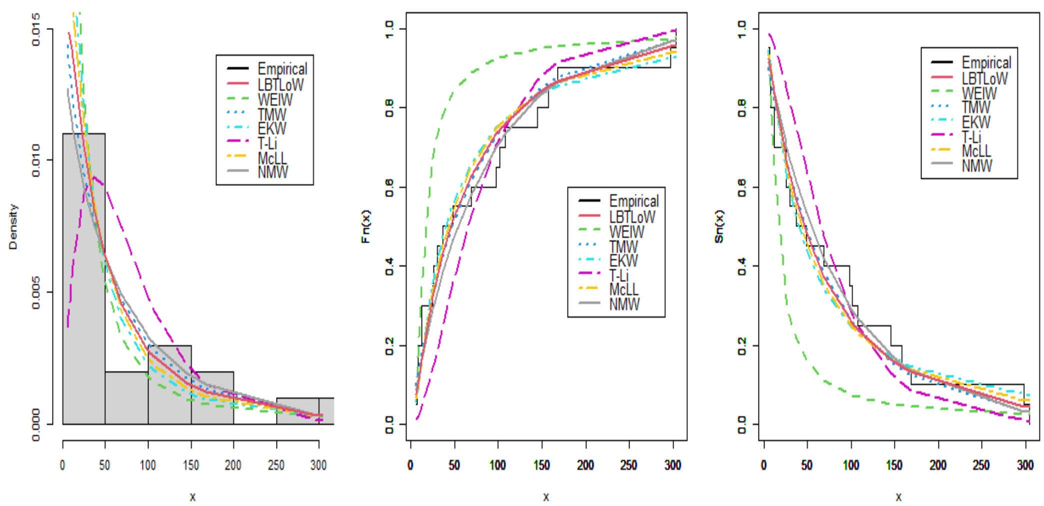

Figure 12. Estimated pdf, cdf and ccdf plots of the competitive models for global oil reserves data set.

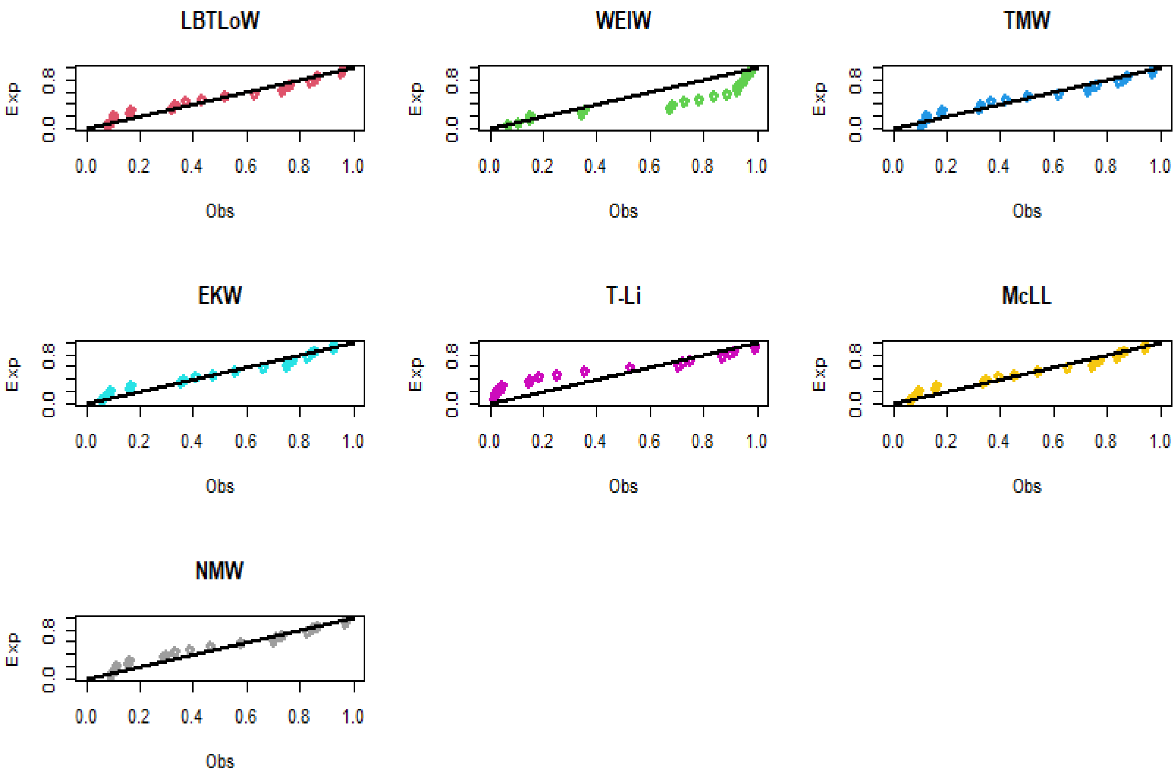

Figure 13. The PP plots of the fitted models for the global oil reserves data set.

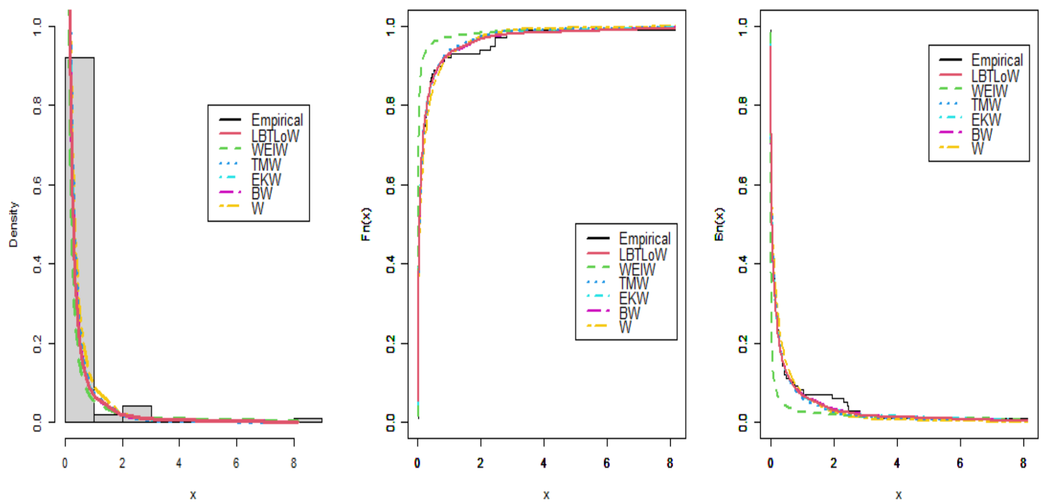

Figure 14. Estimated pdf, cdf and ccdf plots of the competitive models for global gold reserves data set.

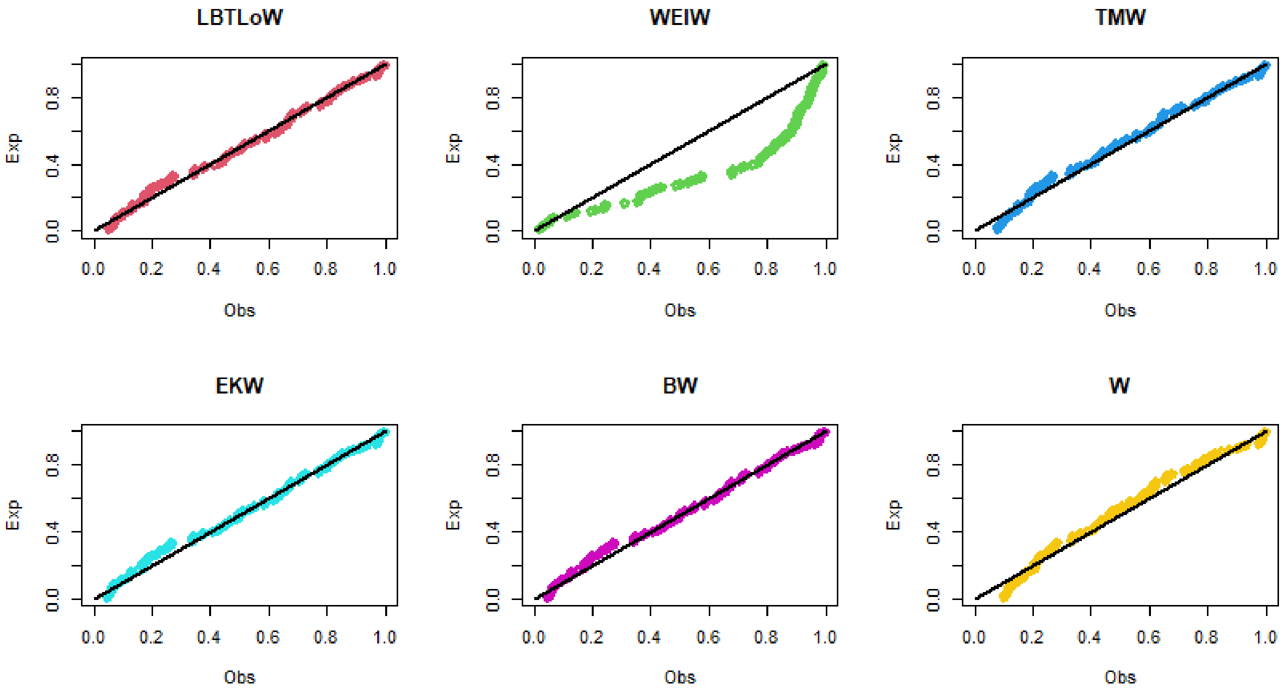

Figure 15. The PP plots of the fitted models for global gold reserves data set.

Use of AI tools declaration

The authors declare that they have not used artificial intelligence tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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