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*Research article*

## Sharp criterion of global existence and orbital stability of standing waves for the nonlinear Schrödinger equation with partial confinement

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**Abstract:** In this article, we consider the global existence and stability issues of the nonlinear Schrödinger equation with partial confinement. First, by establishing some new cross-invariant manifolds and variational problems, a new sharp criterion of global existence is derived in the  $L^2$ -critical and  $L^2$ -supercritical cases. Then, the existence of orbitally stable standing waves is obtained in the  $L^2$ -subcritical and  $L^2$ -critical cases by taking advantage of the profile decomposition technique. Our work extends and complements some earlier results.

**Keywords:** nonlinear Schrödinger equation; partial confinement; sharp criterion; global existence; orbital stability

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### 1. Introduction

In the current paper, we investigate the global existence and stability issues of the following nonlinear Schrödinger equation (NLS) with partial confinement

$$\begin{cases} iu_t = -\Delta u + V(x)u - |u|^{p-1}u, & (t, x) \in [0, T) \times \mathbb{R}^N, \\ u(0, x) = u_0, & x \in \mathbb{R}^N, \end{cases} \quad (1.1)$$

where  $N > 2$  represents the spatial dimension,  $0 < T \leq \infty$ ,  $u(t, x) : [0, T) \times \mathbb{R}^N \rightarrow \mathbb{C}$ ,  $1 < p < \frac{N+2}{N-2}$  and  $V(x) = \sum_{i=1}^k x_i^2$  ( $1 \leq k < N$ ) denotes the partial confinement.

It is well-known that the model (1.1) with partial confinement emerges in various kinds of physical environments, such as the propagation of a laser beam and plasma waves in the description of nonlinear waves. For  $p = 3$ , Eq (1.1) is also used to describe the Bose-Einstein condensate (BEC) [1–3]. In the experiment by BEC [4], the condensation phenomenon is observed due to the presence of a trapping potential, and the shape of external confining potential heavily influences the macroscopic behavior. For this consideration, the external confinement is usually chosen to be harmonic, i.e.,

$V(x) = \sum_{i=1}^N \omega_i^2 x_i^2$ ,  $\omega_i \in \mathbb{R}$ . In this manuscript, the model under consideration involves the simplified situation where  $\omega_i \equiv 1$  for  $1 \leq i \leq k$  and  $\omega_i \equiv 0$  for  $k+1 \leq i \leq N$ .

The energy space corresponding to Eq (1.1) is denoted by

$$\Sigma = \left\{ u \in H^1(\mathbb{R}^N), \int_{\mathbb{R}^N} \sum_{i=1}^k x_i^2 |u|^2 dx < \infty \right\} \quad (1.2)$$

with the norm

$$\|u\|_{\Sigma} = (\|\nabla u\|_{L^2(\mathbb{R}^N)}^2 + \|u\|_{L^2(\mathbb{R}^N)}^2 + \int_{\mathbb{R}^N} \sum_{i=1}^k x_i^2 |u|^2 dx)^{\frac{1}{2}}, \quad \text{for } 1 \leq k < N.$$

The energy functional associated to Eq (1.1) is written as

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + \sum_{i=1}^k x_i^2 |u|^2) dx - \frac{1}{p+1} \int_{\mathbb{R}^N} |u|^{p+1} dx, \quad u \in \Sigma. \quad (1.3)$$

Our main goal of this manuscript is to derive the criterion of blow-up or global existence as well as the orbital stability of standing waves to Eq (1.1).

We now review some earlier results on the above issues. Concerning the canonical NLS (i.e.,  $V(x) = 0$  in Eq (1.1)), Weinstein [5] and Zhang [6] derived several sharp thresholds of global and blow-up solutions for Eq (1.1) in the mass-critical and mass-supercritical cases using variational arguments, respectively. In addition, Berestycki and Cazenave [7] and Weinstein [5] addressed the instability issue of standing waves under the  $L^2$ -critical case  $p = 1 + \frac{4}{N}$ , while the first work done by Cazenave and Lions in [8] showed the orbital stability of normalized standing waves for the  $L^2$ -subcritical situation  $1 < p < 1 + \frac{4}{N}$  by utilizing concentration compactness theory. We refer the readers to [9–12] for more studies on Eq (1.1) which removes the confined potential, i.e.,  $V(x) = 0$ .

For NLS with complete harmonic confinement  $V(x) = \sum_{i=1}^N x_i^2$ , i.e., the case  $k = N$ , there is a large number of literatures concerning the corresponding Cauchy problems on blow-up and stability issues, see [13–20] for example. It is worth mentioning that Zhang [17] verified that the blow-up solutions exist for some special initial values and studied the sharp stability threshold for the  $L^2$ -critical NLS using scaling techniques and some compactness arguments related to compact embedding. Zhang [19], Shu and Zhang [20], Zhang and Ahmed [21] derived some sharp conditions for finite time blow-up and global existence to NLS with  $L^2$ -critical nonlinearity and  $L^2$ -supercritical nonlinearity or with  $L^2$ -supercritical nonlinearity respectively by constructing the cross-invariant sets and using variational methods.

It is worth noting that when  $V(x)$  represents a trapping potential confined on partial directions in the space, i.e.,  $V(x) = \sum_{i=1}^k x_i^2$  ( $1 \leq k < N$ ), it leads to the fact that the embedding from  $\Sigma$  (see (1.2)) to  $L^r$  with  $r \in [2, \frac{2N}{N-2})$  loses compactness, which makes a huge difference with the situation  $V(x) = |x|^2$  on the study of the stability and blow-up issues (see for example [13, 17]). Due to this reason, particular interest and increasing attention have been received for the study on the criterion of global existence versus blow-up and stability of normalized ground state to NLS or nonlinear Schrödinger system with a partial confinement, see for instance [22–27]. In particular, Zhang [23] studied the optimal condition of global existence to the  $L^2$ -critical NLS and showed that there exist solutions blowing up at finite

time for some special initial data via scaling approach. Based on the variational characterization of the ground state of a canonical elliptic equation without potential and the refined compactness argument established in [11], Pan and Zhang [28] derived a sharp threshold of blow-up versus global existence and researched the mass concentration phenomena for NLS with mass-critical nonlinearity and partial confinement in dimension  $N = 2$ . Recently, by employing the cross-constrained variational approach, Wang and Zhang [29] derived the sharp criterion of global existence and blow-up for the NLS with a special partial confinement  $V(x) = x_N^2$  for  $1 + \frac{4}{N-1} \leq p < \frac{N+2}{(N-2)^+}$  and  $N \geq 2$ . It's also worth to mention that the researchers in [30] proved a sharp criterion of global existence to Eq (1.1) in dimension  $N = 3$  by proposing some cross-invariant sets and using variational arguments. We are interested in extending the results of [30] to the case with space dimensions  $N > 2$  and deriving a new criterion for sharp global existence. With regard to the stability issues of the normalized standing waves, to overcome the loss of compactness, Bellazzini et al. [25] used the concentration compactness argument to investigate the existence of orbitally stable standing waves to Eq (1.1), including partial confinement in the  $L^2$ -supercritical case in dimension  $N = 3$ . Jia, Li and Luo [31] generalized the arguments in [25] to the cubic coupled Schrödinger system with a partial confinement in dimension  $N = 3$ . In [32], the concentration compactness principle was also applied to the study on the existence of stable standing waves for the Lee-Huang-Yang corrected dipolar NLS with partial confinement. More recently, for the NLS (1.1) with a partial confinement and inhomogeneous nonlinearity  $|x|^{-b}|u|^{p-1}u$  ( $0 < b < 2$ ), the authors in [33] showed the stability of normalized standing waves by utilizing the profile decomposition principle in the  $L^2$ -subcritical and  $L^2$ -critical cases and by applying concentration compactness principle in the  $L^2$ -supercritical case. It is shown in [34] that there exist normalized standing waves for the mixed dispersion NLS with a partial confinement and these solutions are orbitally stable, in which the main ingredients of the proofs are the profile decomposition principle and the concentration-compactness theory in  $H^2(\mathbb{R}^N) \cap \{u \in L^2(\mathbb{R}^N), \int_{\mathbb{R}^N} \sum_{i=1}^k x_i^2 |u|^2 dx < \infty\}$ .

As far as we know, the orbitally stable standing waves to Eq (1.1) with partial confinement in the  $L^2$ -subcritical and  $L^2$ -critical cases have not been investigated in the existing literatures. Inspired by the literatures mentioned above, our main contribution of this work is to derive the sharp criterion of global existence versus blow-up and the existence of stable standing waves to Eq (1.1) in the general  $N$ -dimensional space.

To these aims, the main difficulty stems from the presence of partial confinement  $V(x) = \sum_{i=1}^k x_i^2$ , which causes the loss of scale invariance and the lack of compactness. We first derive a new sharp criterion of global existence for Eq (1.1) with  $1 + \frac{4}{N} \leq p < \frac{N+2}{N-2}$  by establishing some new so-called cross-constrained manifolds and variational problems (see (1.7) and (1.8)), which are different from those in [30]. Then, the existence of orbitally stable standing waves is obtained in the  $L^2$ -subcritical and  $L^2$ -critical cases  $1 < p \leq 1 + \frac{4}{N}$ , by taking advantage of the profile decomposition technique to overcome the loss of compactness. Our work extends some earlier results of [17, 30] and complements partial arguments of [25, 33].

The first part of our paper is to consider the sharp criterion of global existence in the  $L^2$ -critical and  $L^2$ -supercritical cases  $1 + \frac{4}{N} \leq p < \frac{N+2}{N-2}$  by establishing some new so-called cross-invariant manifolds and proposing cross-constrained minimization problems. Before stating our results, for  $u \in \Sigma$ , we

define three important functionals as follows

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2 + \sum_{i=1}^k x_i^2 |u|^2) dx - \frac{1}{p+1} \int_{\mathbb{R}^N} |u|^{p+1} dx, \quad (1.4)$$

$$S(u) = \int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2 dx - \int_{\mathbb{R}^N} |u|^{p+1} dx, \quad (1.5)$$

$$P(u) = \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{N(p-1)}{2(p+1)} \int_{\mathbb{R}^N} |u|^{p+1} dx, \quad (1.6)$$

and set the following two minimizing problems by

$$d_M = \inf_{u \in M} I(u), \quad (1.7)$$

$$d_B = \inf_{u \in B} I(u), \quad (1.8)$$

where

$$M = \{u \in \Sigma \setminus \{0\}, P(u) = 0, S(u) < 0\},$$

$$B = \{u \in \Sigma \setminus \{0\}, S(u) = 0\}.$$

Let

$$d = \min\{d_M, d_B\}, \quad (1.9)$$

then from Lemmas 3.2 and 3.3, one can conclude that  $d > 0$ . Next define the following manifolds,

$$K = \{u \in \Sigma \setminus \{0\}, I(u) < d, S(u) < 0, P(u) < 0\},$$

$$K_+ = \{u \in \Sigma \setminus \{0\}, I(u) < d, S(u) < 0, P(u) > 0\},$$

$$R_+ = \{u \in \Sigma \setminus \{0\}, I(u) < d, S(u) > 0\},$$

$$R_- = \{u \in \Sigma \setminus \{0\}, I(u) < d, S(u) < 0\},$$

which will be proved as invariant sets in Section 3.

The following two assertions are about the existence of global solution and blow-up to Eq (1.1) for  $1 + \frac{4}{N} \leq p < \frac{N+2}{N-2}$ .

**Theorem 1.1.** Suppose  $1 + \frac{4}{N} \leq p < \frac{N+2}{N-2}$  and  $u_0 \in K_+ \cup R_+$ , then the solution  $u(t, x)$  to Eq (1.1) exists globally in time  $t \in [0, \infty)$ .

**Theorem 1.2.** Suppose  $1 + \frac{4}{N} \leq p < \frac{N+2}{N-2}$ , and assume  $u_0 \in K$  with  $|x|u_0 \in L^2(\mathbb{R}^N)$ , then the solution  $u(t, x)$  to Eq (1.1) blows up in finite time.

**Remark 1.3.** (i) From the definition of the invariant manifolds mentioned above, for  $1 + \frac{4}{N} \leq p < \frac{N+2}{N-2}$ , we see that

$$\{u \in \Sigma \setminus \{0\}, I(u) < d\} = K_+ \cup R_+ \cup K,$$

which indicates the conclusion of Theorem 1.1 is sharp if  $|x|u_0 \in L^2(\mathbb{R}^N)$ .

(ii) Notice that for  $1 < p < 1 + \frac{4}{N}$ , we can easily get the existence of global solution  $u(t, x)$  without any constrains. For  $p = 1 + \frac{4}{N}$  and  $u_0 \in \Sigma$ , the sharp threshold mass of blow-up versus global existence

is given in [23]. In the case  $N > 2$  and  $p = 1 + \frac{4}{N}$ , one can derive the mass-concentration property of blow-up solutions and the dynamical properties of  $L^2$ -minimal mass blow-up solutions, which have been discussed in [28] in the  $L^2$ -critical case with  $N = 2$  and  $p = 3$ , in terms of scaling techniques, a refined compactness argument and the variational characterization of the positive ground state solution  $Q(x)$  to the critical elliptic equation

$$-\Delta Q + Q - Q^{\frac{4}{N}}Q = 0, \quad x \in \mathbb{R}^N. \quad (1.10)$$

(iii) When  $N = 3$  and  $1 + \frac{4}{N} \leq p < \frac{N+2}{N-2}$ , Wang and Zhang [30] obtained a sharp criterion of global existence for Eq (1.1) by introducing some cross-invariant sets and variational problems. Our work, which is motivated by [20], derives a new sharp condition of global existence to Eq (1.1) in the case  $N > 2$  and  $1 + \frac{4}{N} \leq p < \frac{N+2}{N-2}$  by proposing some new cross-invariant manifolds and cross-constrained minimization problems, where the functionals in the constrained sets and variational problems we define (see (1.5)–(1.8)) are different from those in [30]. Moreover, we improve the corresponding results of [30] to space dimensions  $N > 2$ .

The second part of this work discusses the stability of normalized standing waves in the cases  $1 < p \leq 1 + \frac{4}{N}$  by taking advantage of the profile decomposition principle. Here, a solution  $u(t, x)$  to Eq (1.1), possessing the special form  $u(t, x) = e^{i\omega t}\varphi(x)$ , is said to be a standing wave, where  $\omega \in \mathbb{R}$  stands for a frequency and  $\varphi \in \Sigma \setminus \{0\}$  is a solution to the following elliptic equation

$$-\Delta\varphi + \omega\varphi + V(x)\varphi - |\varphi|^{p-1}\varphi = 0.$$

To research the orbital stability of normalized standing waves, applying the ideas of [8], we take into account the constrained minimization problem below

$$m(c) = \inf_{\varphi \in S(c)} E(\varphi), \quad (1.11)$$

where

$$S(c) = \{\varphi \in \Sigma : \|\varphi\|_{L^2(\mathbb{R}^N)} = c\}, \quad \text{for } c > 0.$$

For the  $L^2$ -subcritical case  $1 < p < 1 + \frac{4}{N}$ , or in the  $L^2$ -critical situation  $p = 1 + \frac{4}{N}$  and  $0 < c < \|Q\|_{L^2(\mathbb{R}^N)}$ , one can deduce from Gagliardo-Nirenberg inequality that  $E(\varphi)$  (see (1.3)) is bounded from below on  $S(c)$ , where  $Q(x)$  is the ground state solution to Eq (1.10). In addition, we know from Theorem 4.2 that the constrained variational problem (1.11) is attained. In what follows, we denote the set of whole minimizers to (1.11) by

$$M_c = \{\varphi \in S(c) : E(\varphi) = m(c)\}.$$

Let's now review the definition on orbital stability of standing waves.

**Definition 1.4.** The set  $A$  is said to be orbitally stable if for any given  $\epsilon > 0$ , there exists  $\delta > 0$  such that for any initial data  $u_0$  fulfilling

$$\inf_{\varphi \in A} \|u_0 - \varphi\|_{\Sigma} < \delta,$$

then the corresponding solution  $u(t, x)$  to Eq (1.1) satisfies

$$\inf_{\varphi \in A} \|u(t, x) - \varphi\|_{\Sigma} < \epsilon, \quad \text{for } \forall t > 0.$$

The last result is concerned with the orbital stability of normalized standing waves to Eq (1.1) in the  $L^2$ -subcritical and  $L^2$ -critical cases.

**Theorem 1.5.** *Suppose that  $c > 0$  if  $1 < p < 1 + \frac{4}{N}$  or  $0 < c < \|Q\|_{L^2(\mathbb{R}^N)}$  if  $p = 1 + \frac{4}{N}$ , where  $Q(x)$  is the ground state solution to Eq (1.10). Then,  $M_c \neq \emptyset$ , and is orbitally stable.*

**Remark 1.6.** (i) *To demonstrate the existence of orbitally stable standing waves for NLS with partial confinement  $V(x) = \sum_{i=1}^k x_i^2$  ( $1 \leq k < N$ ), the main challenge comes from the lack of compactness. With regard to NLS with complete harmonic potential  $V(x) = \sum_{i=1}^N x_i^2$ , Zhang [17] used the key fact that the embedding  $\Sigma \hookrightarrow L^r$  with  $r \in [2, \frac{2N}{N-2})$  is compact to give the sharp stability threshold of standing waves with prescribed mass for Eq (1.1) when  $p = 1 + \frac{4}{N}$ . Whereas, regarding the NLS with partial confinement, it's worth to note that the embedding  $\Sigma \hookrightarrow L^r$  with  $r \in [2, \frac{2N}{N-2})$  loses compactness, the method used by [17] is not suitable to obtain the stable standing waves to Eq (1.1). In [33], to overcome the main difficulty, Liu, He and Feng showed the orbital stability of normalized standing waves to the NLS with an inhomogeneous nonlinearity  $|x|^{-b}|u|^{p-1}u$  and partial confinement for  $0 < b < 2$  by taking advantage of the profile decomposition principle in the  $L^2$ -subcritical and  $L^2$ -critical cases, and by applying concentration compactness theory for the mass-supercritical case.*

(ii) *In this work, we survey only the existence of stable standing waves in the cases  $1 < p \leq 1 + \frac{4}{N}$  by profile decomposition theory, which is an improvement to [17] and a complement to [25, 33]. As far as the authors know, there are few literatures researching the stable standing waves to NLS with partial confinement in terms of the profile decomposition technique, except for [33, 34]. When  $N = 3$  and  $k = 2$  in Eq (1.1), Bellazzini et al. [25] obtained the orbital stability of normalized standing waves to Eq (1.1) in the  $L^2$ -supercritical and  $H^1$ -subcritical cases by utilizing concentration compactness principle and variational methods. In the general  $N$ -dimensional space, considering Eq (1.1) with  $L^2$ -supercritical nonlinearity, we can apply the ideas of [25, 33] to verify the stability of normalized standing waves.*

(iii) *The papers [5, 7] have addressed the instability of standing waves to Eq (1.1) without confinement for  $p = 1 + \frac{4}{N}$ . Our result (Theorem 1.5) reveals the stabilizing effect to the standing waves played by partial confinement  $V(x) = \sum_{i=1}^k x_i^2$  ( $1 \leq k < N$ ).*

Throughout this article, for the sake of convenience, we use the abbreviation  $\int \cdot dx$  to replace  $\int_{\mathbb{R}^N} \cdot dx$  and denote  $\|\cdot\|_{L^p(\mathbb{R}^N)}$  ( $1 < p < \frac{N+2}{N-2}$ ) by  $\|\cdot\|_p$ , and utilize  $C$  to represent a positive constant which may vary from line to line.

Our article is structured as follows. In Section 2, some preliminaries are presented, including several significant lemmas. In Section 3, the sharp criterion of global existence versus blow-up is established and the proofs to Theorems 1.1 and 1.2 are given. In Section 4, we address the orbital stability of normalized standing waves and prove Theorem 1.5. In the last section, the conclusions are given.

## 2. Preliminaries

In order to survey the global existence versus blow-up and the stability issues to standing waves, one requires the well-posedness to Eq (1.1). Based on [22] and [9], in the following we introduce the local well-posedness to problem (1.1).

**Proposition 2.1.** (*[9, 22]*) *Suppose  $u_0 \in \Sigma$  and  $1 < p < \frac{N+2}{N-2}$ . Then, there exist  $T = T(\|u_0\|_\Sigma)$  and a unique solution  $u(t, x) \in C([0, T], \Sigma)$  to Eq (1.1). Assume that the solution  $u(t, x)$  is well-defined on the*

maximal interval  $[0, T)$ . If  $T < \infty$ , then  $\lim_{t \rightarrow T} \|u(t, x)\|_{\Sigma} = \infty$  (blow-up). Moreover, for any  $t \in [0, T)$ , the following conservation laws of mass and energy hold

$$\|u(t, x)\|_2 = \|u_0\|_2, \quad E(u(t, x)) = E(u_0). \quad (2.1)$$

**Remark 2.2.** In the case  $1 < p < 1 + \frac{4}{N}$ , using Lemma 2.4 and Young's inequality, it is not hard to verify the existence of global solution for Eq (1.1). For the  $L^2$ -critical case  $p = 1 + \frac{4}{N}$ , the solution to Eq (1.1) with mass strictly less than  $\|Q\|_2$  is global. On the other hand, if the mass of initial data  $\|u_0\|_2 \geq \|Q\|_2$ , then finite time blow-up solutions exist, see [23]. Furthermore, in the case  $1 + \frac{4}{N} < p < \frac{N+2}{N-2}$ , using the local well-posedness theory to Eq (1.1), one can show the existence of global solution for initial data small enough, but for some large data, it is possible that the explosion of solutions happens at finite time.

Next, based on Cazenave [9], we give the virial identity for the Cauchy problem (1.1), which is of great importance in the analysis of blow-up behaviors to the solutions.

**Proposition 2.3.** Assume  $u_0 \in \Sigma$  and  $u(t, x)$  is the corresponding solution to problem (1.1) in  $C([0, T); \Sigma)$ . Let  $|x|u_0 \in L^2(\mathbb{R}^N)$  and take  $\Gamma(t) = \int |x|^2 |u(t, x)|^2 dx$ , then we get that

$$\Gamma''(t) = 8 \int (|\nabla u|^2 - \sum_{i=1}^k x_i^2 |u|^2) dx - \frac{4N(p-1)}{p+1} \int |u|^{p+1} dx.$$

Now, we recall some useful lemmas.

**Lemma 2.4.** ([5]) Let  $1 < p < \frac{N+2}{N-2}$ , then for all  $u \in H^1(\mathbb{R}^N)$ , we have the following sharp Gagliardo-Nirenberg inequality

$$\int |u|^{p+1} dx \leq C_{GN} \left( \int |\nabla u|^2 dx \right)^{\frac{N(p-1)}{4}} \left( \int |u|^2 dx \right)^{\frac{p+1}{2} - \frac{N(p-1)}{4}}.$$

In particular, in the mass-critical case  $p = 1 + \frac{4}{N}$ ,  $C_{GN} = \frac{N+2}{N} \|Q(x)\|_2^{-\frac{4}{N}}$ , where  $Q(x)$  is the positive ground state solution to Eq (1.10).

**Lemma 2.5.** ([25]) For  $1 \leq k < N$ , let

$$\Lambda_0 = \inf_{\int |w|^2 dx = 1} \left( \int |\nabla w|^2 dx + \int \sum_{i=1}^k x_i^2 |w|^2 dx \right)$$

and

$$\lambda_0 = \inf_{\int_{\mathbb{R}^k} |u|^2 dx_1 dx_2 \cdots dx_k = 1} \left( \int |\nabla u|^2 dx_1 dx_2 \cdots dx_k + \int_{\mathbb{R}^k} \sum_{i=1}^k x_i^2 |u|^2 dx_1 dx_2 \cdots dx_k \right).$$

Then, we have  $\Lambda_0 = \lambda_0$ .

To investigate the compactness of any minimizing sequence to (1.11), we introduce the corresponding profile decomposition of a bounded sequence in  $\Sigma$ , which is slightly different from [11].

**Lemma 2.6.** Suppose  $1 \leq k < N$ ,  $1 < p \leq 1 + \frac{4}{N}$  and let  $\{u_n\}$  be a bounded sequence in  $\Sigma$ . Then, there exist a subsequence of  $\{u_n\}$  (still denoted by  $\{u_n\}$ ), a family  $\{x_n^j\}_{n=1}^\infty$  of sequences in  $\mathbb{R}^{N-k}$  and a sequence  $\{U^j\}_{j=1}^\infty$  in  $\Sigma$  satisfying

(i) for each  $m \neq j$ ,  $|x_n^m - x_n^j| \rightarrow +\infty$ , as  $n \rightarrow \infty$ ;

(ii) for each  $l \geq 1$  and  $x \in \mathbb{R}^N$ , we have

$$u_n(x) = \sum_{j=1}^l \tau_{x_n^j} U^j(x) + r_n^l,$$

with  $\limsup_{n \rightarrow \infty} \|r_n^l\|_q \rightarrow 0$  as  $l \rightarrow \infty$  for any  $q \in [2, \frac{N+2}{N-2})$ , where  $\tau_y U^j(x) = U^j(x_1, \dots, x_k, x_{k+1} - y_1, \dots, x_N - y_{N-k})$  for  $x = (x_1, \dots, x_N) \in \mathbb{R}^N$  and  $y = (y_1, \dots, y_{N-k}) \in \mathbb{R}^{N-k}$ . In addition,

$$\|u_n\|_2^2 = \sum_{j=1}^l \|U^j\|_2^2 + \|r_n^l\|_2^2 + o(1), \quad (2.2)$$

$$\int V(x)|u_n|^2 dx = \sum_{j=1}^l \int V(x)|U^j|^2 dx + \int V(x)|r_n^l|^2 dx + o(1), \quad (2.3)$$

$$\|\nabla u_n\|_2^2 = \sum_{j=1}^l \|\nabla U^j\|_2^2 + \|\nabla r_n^l\|_2^2 + o(1), \quad (2.4)$$

$$\|u_n\|_{p+1}^{p+1} = \sum_{j=1}^l \int |\tau_{x_n^j} U^j|^{p+1} dx + \|r_n^l\|_{p+1}^{p+1} + o(1), \quad (2.5)$$

where  $o(1) = o_n(1) \rightarrow 0$  as  $n \rightarrow \infty$ .

### 3. Sharp criterion of global existence

In this section, the authors propose several new cross-constrained variational problems and invariant sets associated with problem (1.1) to discuss the sharp criterion of global existence.

**Proposition 3.1.** If  $1 + \frac{4}{N} \leq p < \frac{N+2}{N-2}$ , then  $M$  is not empty.

**Proof.** According to [35], there is  $u \in \Sigma \setminus \{0\}$  such that  $u$  is a nontrivial solution for Eq (1.10). Testing Eq (1.10) against  $u$  and integrating over  $\mathbb{R}^N$ , we see that  $S(u) = 0$ . Furthermore, multiplying Eq (1.10) by  $x \cdot \nabla u$ , one has the following Pohožaev identity

$$-\frac{1}{2}(N-2) \int |\nabla u|^2 dx + \frac{N}{p+1} \int |u|^{p+1} dx - \frac{N}{2} \int |u|^2 dx = 0. \quad (3.1)$$

Combining (3.1) with  $S(u) = 0$ , we have  $P(u) = 0$ . Put  $v = \nu u(t, x)$  for  $\nu > 1$ , then combining  $S(u) = 0$  and  $P(u) = 0$ , one can infer that  $S(v) < 0$  and  $P(v) < 0$ . Taking  $v^\lambda(t, x) = \lambda^{\frac{2}{p-1}} v(t, \lambda x)$  for  $\lambda > 0$ , from (1.5) and (1.6), we obtain

$$P(v^\lambda) = \lambda^{\frac{2+2p-Np+N}{p-1}} \int |\nabla v|^2 - \frac{N(p-1)}{2(p+1)} |v|^{p+1} dx,$$



$$S(v^\lambda) = \lambda^{\frac{2+2p-Np+N}{p-1}} \int |\nabla v|^2 - |v|^{p+1} dx + \lambda^{\frac{4-Np+N}{p-1}} \int |v|^2 dx.$$

Owing to  $P(v) < 0$ , we deduce that there is  $\lambda_0 > 1$  satisfying  $P(v^{\lambda_0}) = 0$ . Besides, according to the fact  $S(v) < 0$  and  $\lambda_0 > 1$ , we know that  $S(v^{\lambda_0}) < 0$ . Thus  $v^{\lambda_0} \in M$ , which means  $M \neq \emptyset$ .

**Lemma 3.2.** *Let  $1 + \frac{4}{N} \leq p < \frac{N+2}{N-2}$ , then  $d_M > 0$ .*

**Proof.** Take  $u \in M$ , it's clear that  $u \neq 0$ . Since  $P(u) = 0$ , one has

$$I(u) = \left(\frac{1}{2} - \frac{2}{N(p-1)}\right) \int |\nabla u|^2 dx + \frac{1}{2} \int |u|^2 dx + \frac{1}{2} \int \sum_{i=1}^k x_i^2 |u|^2 dx. \quad (3.2)$$

We prove the assertion in two situations: the mass-critical case  $p = 1 + \frac{4}{N}$  and the mass-supercritical case  $1 + \frac{4}{N} < p < \frac{N+2}{N-2}$ .

We first consider the case  $p = 1 + \frac{4}{N}$ . In the current situation, we claim that  $d_M > 0$ . Suppose  $d_M = 0$ , then we conclude from (1.7) that there exists a sequence  $\{u_n\}_{n=1}^\infty \in M$  satisfying  $I(u_n) \rightarrow 0$ ,  $S(u_n) < 0$  and  $P(u_n) = 0$  as  $n \rightarrow \infty$ . (3.2) leads to

$$\int V(x)|u_n|^2 dx \rightarrow 0, \quad \int |u_n|^2 dx \rightarrow 0, \quad \text{as } n \rightarrow \infty, \quad (3.3)$$

due to  $p = 1 + \frac{4}{N}$ . Applying Lemma 2.4, we obtain

$$\int |u_n|^{p+1} dx \leq C_{GN} \left( \int |\nabla u_n|^2 dx \right) \left( \int |u_n|^2 dx \right)^{\frac{2}{N}}.$$

This, together with  $S(u_n) < 0$ , implies

$$\int |\nabla u_n|^2 + |u_n|^2 dx < C_{GN} \left( \int |\nabla u_n|^2 dx \right) \left( \int |u_n|^2 dx \right)^{\frac{2}{N}}.$$

Nevertheless, when  $n$  is sufficiently large, from (3.3) we have

$$\int |\nabla u_n|^2 + |u_n|^2 dx > C_{GN} \left( \int |\nabla u_n|^2 dx \right) \left( \int |u_n|^2 dx \right)^{\frac{2}{N}},$$

which is a contradiction. Therefore,  $d_M > 0$  when  $p = 1 + \frac{4}{N}$ .

Now, let us handle the  $L^2$ -supercritical case  $1 + \frac{4}{N} < p < \frac{N+2}{N-2}$ . It follows from  $S(u) < 0$  and the continuous embedding  $H^1(\mathbb{R}^N) \hookrightarrow L^{p+1}(\mathbb{R}^N)$  that

$$\int |\nabla u|^2 + |u|^2 dx < \int |u|^{p+1} dx \leq C \left( \int |\nabla u|^2 + |u|^2 dx \right)^{\frac{p+1}{2}}.$$

Thus, we derive

$$\int |\nabla u|^2 + |u|^2 dx \geq C > 0. \quad (3.4)$$

Keeping in mind that  $1 + \frac{4}{N} < p < \frac{N+2}{N-2}$  and combining (3.2) with (3.4), one has

$$I(u) \geq C > 0, \quad \text{for any } u \in M,$$

which means  $d_M > 0$  for  $1 + \frac{4}{N} < p < \frac{N+2}{N-2}$ . Thus we claim that  $d_M > 0$  for  $1 + \frac{4}{N} \leq p < \frac{N+2}{N-2}$ .

**Lemma 3.3.** Suppose  $1 + \frac{4}{N} \leq p < \frac{N+2}{N-2}$ , then the set  $B$  is nonempty and  $d_B > 0$ .

**Proof.** According to [35], the set  $B$  is nonempty. By  $S(u) = 0$ , one can discover

$$I(u) = \left(\frac{1}{2} - \frac{1}{p+1}\right) \int |\nabla u|^2 + |u|^2 dx + \frac{1}{2} \int \sum_{i=1}^k x_i^2 |u|^2 dx. \quad (3.5)$$

Asserting Sobolev embedding inequality into  $S(u) = 0$ , we get

$$\int |\nabla u|^2 + |u|^2 dx \leq C \left( \int |\nabla u|^2 + |u|^2 dx \right)^{\frac{p+1}{2}}. \quad (3.6)$$

For  $1 + \frac{4}{N} \leq p < \frac{N+2}{N-2}$  and  $u \neq 0$ , one is able to infer from (3.5) that

$$\int (|\nabla u|^2 + |u|^2) dx \geq C > 0.$$

Thus, when  $1 + \frac{4}{N} \leq p < \frac{N+2}{N-2}$ , it follows from (1.8) and (3.5)–(3.6) that  $d_B > 0$ .

Next we shall show that  $K$ ,  $K_+$ ,  $R_+$  and  $R_-$  are all invariant sets related to Eq (1.1).

**Theorem 3.4.** Assume  $1 + \frac{4}{N} \leq p < \frac{N+2}{N-2}$ , then  $K$ ,  $K_+$ ,  $R_+$  and  $R_-$  are all invariant sets of Eq (1.1). That is, if  $u_0 \in K$ ,  $K_+$ ,  $R_+$  or  $R_-$ , then the solution  $u(t, x)$  to Eq (1.1) also fulfils  $u(t, x) \in K$ ,  $K_+$ ,  $R_+$  or  $R_-$  for  $\forall t \in [0, T)$ .

**Proof.** In the first, we demonstrate that the set  $K$  is an invariant manifold. Suppose  $u_0 \in K$  and  $u(t, x)$  is the unique solution to Eq (1.1). We infer from (2.1) that

$$I(u) = I(u_0), \text{ for arbitrary } t \in [0, T). \quad (3.7)$$

Owing to  $I(u_0) < d$ , we have  $I(u) < d$  for arbitrary  $t \in [0, T)$ .

Now we turn to show  $S(u) < 0$  for arbitrary  $t \in [0, T)$ . If otherwise, using the continuity of  $S(u)$  in  $t$ , one can find  $t_0 \in [0, T)$  satisfying  $S(u(t_0, \cdot)) = 0$ . From (3.7), we obtain  $u(t_0, \cdot) \neq 0$ . Combining (1.8) and (1.9), we know that  $I(u(t_0, \cdot)) \geq d$ . Obviously, it's contradictory to the fact  $I(u(t, \cdot)) < d$  for any  $t \in [0, T)$ . Thus  $S(u) < 0$  for all  $t \in [0, T)$ .

Subsequently, for  $t \in [0, T)$ , we claim that  $P(u) < 0$ . If  $P(u) < 0$  is false, since the functional  $S(u)$  is continuous, we could seek out  $t' \in [0, T)$  fulfilling  $P(u(t', \cdot)) = 0$ . Since we have demonstrated  $S(u(t', \cdot)) < 0$ , we conclude from  $P(u(t', \cdot)) = 0$  that  $u(t', \cdot) \in M$ . Thus, by utilizing (1.7) and (1.9), one has  $I(u(t', \cdot)) \geq d_M \geq d$ . This causes a contradiction because  $I(u(t', \cdot)) < d$  for every  $t \in [0, T)$ . Thus  $P(u) < 0$  when  $t \in [0, T)$ . Hence we have  $u(t, x) \in K$  for arbitrary  $t \in [0, T)$ .

By using similar method as the above process, one can also infer that the manifolds  $K_+$ ,  $R_+$  and  $R_-$  are all invariant sets.

Based on the conclusions we have proved, it is sufficient to show Theorems 1.1 and 1.2.

**Proof of Theorem 1.1.** We first study the case  $u_0 \in R_+$ . According to Proposition 2.1 and Theorem 3.4, the initial-value problem (1.1) possesses a unique solution  $u(t, x) \in R_+$  for arbitrary  $t \in [0, T)$ . Then, for all  $t \in [0, T)$ , we have  $I(u) < d$  and  $S(u) > 0$ . This implies

$$\int (|\nabla u|^2 + |u|^2 + \sum_{i=1}^k x_i^2 |u|^2) dx < \frac{2(p+1)}{p-1} d.$$

From Proposition 2.1, one can know that the solution  $u(t, x)$  is global in time  $t \in [0, \infty)$ .

Next we discuss the case  $u_0 \in K_+$ . In the light of Proposition 2.1 and Theorem 3.4, the unique solution  $u(t, x) \in K_+$  for  $t \in [0, T)$ . Hence one has  $I(u) < d$  and  $P(u) > 0$ , which yields

$$\left(\frac{1}{2} - \frac{2}{N(p-1)}\right) \int |\nabla u|^2 dx + \frac{1}{2} \int |u|^2 dx + \frac{1}{2} \int \sum_{i=1}^k x_i^2 |u|^2 dx < d. \quad (3.8)$$

In what follows, we shall give out the proof on the global existence of solution in two situations. One is the  $L^2$ -critical case, the other one is the  $L^2$ -supercritical case.

We first discuss the  $L^2$ -critical case  $p = 1 + \frac{4}{N}$ . By (3.8), we get

$$\frac{1}{2} \int |u|^2 dx + \frac{1}{2} \int \sum_{i=1}^k x_i^2 |u|^2 dx < d. \quad (3.9)$$

Set  $u^\lambda(t, x) = \lambda^{\frac{N}{p+1}} u(t, \lambda x)$ , then (1.6) gives us that

$$P(u^\lambda) = \lambda^{\frac{4}{N+2}} \int |\nabla u|^2 dx - \frac{N}{N+2} \int |u|^{p+1} dx.$$

Since  $P(u) > 0$ , then one can find  $0 < \lambda_* < 1$  satisfying  $P(u^{\lambda_*}) = 0$ . Putting (1.4) and (1.6) together, we obtain

$$I(u^{\lambda_*}) = \frac{1}{2} \int (\lambda_*^{-\frac{2N}{N+2}} |u|^2 + \lambda_*^{-\frac{4(N+1)}{N+2}} \sum_{i=1}^k x_i^2 |u|^2) dx.$$

Thus, from (3.9), one has that

$$I(u^{\lambda_*}) < \lambda_*^{-\frac{4(N+1)}{N+2}} d. \quad (3.10)$$

For  $S(u^{\lambda_*})$ , only two possibilities exist. One case is  $S(u^{\lambda_*}) < 0$ , and the remaining one is  $S(u^{\lambda_*}) \geq 0$ . For the case  $S(u^{\lambda_*}) < 0$ , since  $P(u^{\lambda_*}) = 0$ , together (1.7) with (1.9), one can show that

$$I(u^{\lambda_*}) \geq d_M \geq d > I(u).$$

Then we have

$$(1 - \lambda_*^{\frac{4}{N+2}}) \int |\nabla u|^2 dx + (1 - \lambda_*^{-\frac{4(N+1)}{N+2}}) \int \sum_{i=1}^k x_i^2 |u|^2 dx + (1 - \lambda_*^{-\frac{2N}{N+2}}) \int |u|^2 dx < 0. \quad (3.11)$$

Thus, from (3.9) and (3.11), one has

$$\int (|\nabla u|^2 + \sum_{i=1}^k x_i^2 |u|^2 + |u|^2) dx < C. \quad (3.12)$$

For  $S(u^{\lambda_*}) \geq 0$ , it follows from (3.10) that

$$I(u^{\lambda_*}) - \frac{1}{p+1} S(u^{\lambda_*}) < \lambda_*^{-\frac{4(N+1)}{N+2}} d,$$

which implies

$$\frac{p-1}{2(p+1)} \lambda_*^{-\frac{2N}{N+2}} \int \lambda_*^2 |\nabla u|^2 + |u|^2 dx + \frac{1}{2} \lambda_*^{-\frac{4(N+1)}{N+2}} \int \sum_{i=1}^k x_i^2 |u|^2 dx < \lambda_*^{-\frac{4(N+1)}{N+2}} d.$$

Therefore,

$$\int |\nabla u|^2 + |u|^2 + \sum_{i=1}^k x_i^2 |u|^2 dx < C. \quad (3.13)$$

Thus, for  $p = 1 + \frac{4}{N}$ , (3.12) and (3.13) imply that the solution  $u(t, x)$  is uniformly bounded in  $\Sigma$  for all  $t \in [0, T)$ . According to Proposition 2.1, we derive the global existence of  $u(t, x)$  in time  $t \in [0, \infty)$ .

We now argue the case  $1 + \frac{4}{N} < p < \frac{N+2}{N-2}$ . It is easy to see from (3.8) that

$$\int |\nabla u|^2 + \sum_{i=1}^k x_i^2 |u|^2 + |u|^2 dx < C.$$

Therefore, for  $1 + \frac{4}{N} \leq p < \frac{N+2}{N-2}$ , on account of Proposition 2.1, it suffices to show that the solution  $u(t, x)$  to Eq (1.1) exists globally for  $t \in [0, \infty)$ .

**Proof of Theorem 1.2.** Suppose  $u_0 \in K$  and  $|x|u_0 \in L^2(\mathbb{R}^N)$ , and assume  $u(t, x)$  is a solution to Eq (1.1). Then combining Proposition 2.1 with Theorem 3.4, one could derive that the solution  $u(t, x) \in K$  and  $|x|u(t, x) \in L^2(\mathbb{R}^N)$  when  $t \in [0, T)$ . It follows from the virial identity (see Proposition 2.3) and (1.6) that

$$\Gamma''(t) < 8P(u(t, \cdot)), \text{ for } t \in [0, T). \quad (3.14)$$

Thus, for  $0 \leq t < T$ ,  $u$  fulfils that  $S(u) < 0$  and  $P(u) < 0$ . For  $\mu > 0$ , we take  $u^\mu = \mu^{\frac{3}{p+1}} u(\mu x)$ , then

$$\begin{aligned} S(u^\mu) &= \mu^{\frac{5-p}{p+1}} \int |\nabla u|^2 dx + \mu^{\frac{3(1-p)}{p+1}} \int |u|^2 dx - \int |u|^{p+1} dx, \\ P(u^\mu) &= \mu^{\frac{5-p}{p+1}} \int |\nabla u|^2 dx - \frac{N(p-1)}{2(p+1)} \int |u|^{p+1} dx. \end{aligned}$$

Owing to  $1 + \frac{4}{N} \leq p < \frac{N+2}{N-2}$  and  $P(u) < 0$ , then there must exist  $\mu_* > 1$  fulfilling  $P(u^{\mu_*}) = 0$ , and  $P(u^\mu) < 0$  for  $1 \leq \mu < \mu_*$ . When  $1 \leq \mu < \mu_*$ , due to  $S(u) < 0$ ,  $S(u^\mu)$  may have the following two cases:

- (i)  $S(u^\mu) < 0$  for  $1 \leq \mu < \mu_*$ ;
- (ii) There is  $1 < \theta \leq \mu_*$  satisfying  $S(u^\theta) = 0$ .

Concerning the first situation (i), we have  $P(u^{\mu_*}) = 0$  and  $S(u^{\mu_*}) < 0$ , then  $u^{\mu_*} \in M$ . Based on (1.7) and (1.9), we discover

$$I(u^{\mu_*}) \geq d_M > d > I(u).$$

Furthermore, one has

$$\begin{aligned} I(u) - I(u^{\mu_*}) &= \frac{1}{2} (1 - \mu_*^{\frac{5-p}{p+1}}) \int |\nabla u|^2 dx + \frac{1}{2} (1 - \mu_*^{\frac{1-5p}{p+1}}) \int \sum_{i=1}^k x_i^2 |u|^2 dx \\ &\quad + \frac{1}{2} (1 - \mu_*^{-\frac{3(p-1)}{p+1}}) \int |u|^2 dx, \\ P(u) - P(u^{\mu_*}) &= (1 - \mu_*^{\frac{5-p}{p+1}}) \int |\nabla u|^2 dx. \end{aligned}$$

Due to the fact  $\mu_* > 1$  and  $1 + \frac{4}{N} \leq p < \frac{N+2}{N-2}$ , we derive

$$I(u) - I(u^{\mu_*}) \geq \frac{1}{2}[P(u) - P(u^{\mu_*})] = \frac{1}{2}P(u). \quad (3.15)$$

For the case (ii), we have  $u^\theta \in B$ . Thus, we deduce from (3.15) and (1.9) that

$$I(u^\theta) \geq d_B \geq d.$$

And so, we could get

$$I(u) - I(u^\theta) \geq \frac{1}{2}[P(u) - P(u^\theta)] \geq \frac{1}{2}P(u). \quad (3.16)$$

Since  $I(u^{\mu_*}) \geq d$ ,  $I(u^\theta) \geq d$ , combining (3.15) with (3.16), we conclude

$$P(u) < 2[I(u) - d]. \quad (3.17)$$

From (3.14), (3.17), (2.1) and  $u_0 \in K$ , one has the following estimate

$$\Gamma''(t) < 8P(u) \leq 16[I(u_0) - d] < 0.$$

Hence there exists  $0 < T < \infty$  satisfying  $\Gamma(T) = 0$ . Then using Lemma 4.2 in [17], one obtains

$$\lim_{t \rightarrow T} \|u\|_\Sigma = \infty,$$

which indicates the solution  $u(t, x)$  to Eq (1.1) must blow up in finite time.

#### 4. Orbital stability of standing waves

This part is concerned with the orbital stability of normalized standing waves of Eq (1.1) in the  $L^2$ -subcritical and  $L^2$ -critical cases, in which the proof to Theorem 1.5 is given. To go further, let us first introduce the non-vanishing conclusion as below.

**Lemma 4.1.** *Let  $1 \leq k < N$  and  $1 < p \leq 1 + \frac{4}{N}$ . Assume  $u_n$  is a minimizing sequence of (1.11), then there must exist  $\delta > 0$  meeting*

$$\liminf_{n \rightarrow \infty} \int |u_n|^{p+1} dx > \delta. \quad (4.1)$$

**Proof.** Assume by contradiction that there is a subsequence  $u_{n_j}$  fulfilling

$$\lim_{j \rightarrow \infty} \int |u_{n_j}(x)|^{p+1} dx = 0.$$

This, together with the definition of  $m(c)$ , deduces that

$$\begin{aligned} m(c) &= \lim_{j \rightarrow \infty} E(u_{n_j}) \\ &= \lim_{j \rightarrow \infty} \left[ \frac{1}{2} \int (|\nabla u_{n_j}|^2 + \sum_{i=1}^k x_i^2 |u_{n_j}|^2) dx - \frac{1}{p+1} \int |u_{n_j}|^{p+1} dx \right] \\ &= \lim_{j \rightarrow \infty} \frac{1}{2} \int (|\nabla u_{n_j}|^2 + \sum_{i=1}^k x_i^2 |u_{n_j}|^2) dx \\ &\geq \lim_{j \rightarrow \infty} \inf_{\int |u_{n_j}|^2 dx = c^2} \frac{1}{2} \int (|\nabla u_{n_j}|^2 + \sum_{i=1}^k x_i^2 |u_{n_j}|^2) dx. \end{aligned} \quad (4.2)$$

Taking  $v_{n_j} = \frac{u_{n_j}}{c}$ , then (4.2) can be rewritten as

$$\begin{aligned} m(c) = \lim_{j \rightarrow \infty} E(u_{n_j}) &\geq \lim_{j \rightarrow \infty} \inf_{\int |v_{n_j}|^2 dx = 1} \frac{c^2}{2} \int (|\nabla v_{n_j}|^2 + \sum_{i=1}^k x_i^2 |v_{n_j}|^2) dx \\ &\geq \frac{\Lambda_0}{2} c^2, \end{aligned} \quad (4.3)$$

where the last inequality according to Lemma 2.5. Furthermore, in view of the argument that the embedding  $H = \{v \in H^1(\mathbb{R}^k), \int \sum_{i=1}^k x_i^2 |v|^2 dx < \infty\} \hookrightarrow L^2(\mathbb{R}^k)$  is compact by Lemma 2.5, then there exists some  $\tau \in H^1(\mathbb{R}^k)$  with  $\int_{\mathbb{R}^k} |\tau|^2 dx = 1$  such that  $\lambda_0$  is achieved. Let  $\psi \in H^1(\mathbb{R}^{N-k})$  satisfy  $\int_{\mathbb{R}^{N-k}} |\psi|^2 dx = c^2$  and set

$$u_\lambda(x) = \tau(x_1, \dots, x_k) \psi_\lambda(x_{k+1}, \dots, x_N),$$

where  $\psi_\lambda(x_{k+1}, \dots, x_N) = \lambda^{\frac{N-k}{2}} \psi(\lambda x_{k+1}, \dots, \lambda x_N)$ . Then for any  $\lambda > 0$ , we can deduce  $u_\lambda \in S(c)$ , combining this fact and Lemma 2.5, we see that

$$\begin{aligned} E(u_\lambda) &= \frac{1}{2} \left( \int |\nabla u_\lambda|^2 + \sum_{i=1}^k x_i^2 |u_\lambda|^2 dx \right) - \frac{1}{p+1} \int |u_\lambda|^{p+1} dx \\ &= I_1 + I_2 - \frac{1}{p+1} \int |u_\lambda|^{p+1} dx, \end{aligned} \quad (4.4)$$

where

$$\begin{aligned} I_1 &= \frac{1}{2} \int |u_\lambda|^2 dx \\ &= \frac{1}{2} (c^2 \int_{\mathbb{R}^k} |\nabla_{x_1 \dots x_k} \tau(x_1, \dots, x_k)|^2 dx_1 \dots dx_k \\ &\quad + \lambda^2 \int_{\mathbb{R}^{N-k}} |\nabla_{x_{k+1} \dots x_N} \psi_\lambda|^2 dx_{k+1} \dots dx_N), \end{aligned}$$

and

$$\begin{aligned} I_2 &= \frac{1}{2} \int \sum_{i=1}^k x_i^2 |u_\lambda|^2 dx \\ &= \frac{c^2}{2} \int_{\mathbb{R}^k} \left| \sum_{i=1}^k x_i^2 \tau(x_1, x_2, \dots, x_k) \right|^2 dx_1 \dots dx_k, \end{aligned}$$

which implies that (4.4) can be written as

$$\begin{aligned}
E(u_\lambda) &= \frac{1}{2}(c^2 \int_{\mathbb{R}^k} |\nabla_{x_1 \dots x_k} \tau(x_1, \dots, x_k)|^2 dx_1 \dots dx_k + \lambda^2 \int_{\mathbb{R}^{N-k}} |\nabla_{x_{k+1} \dots x_N} \psi_\lambda|^2 dx_{k+1} \dots dx_N) \\
&\quad + \frac{c^2}{2} \int_{\mathbb{R}^k} \left| \sum_{i=1}^k x_i^2 \tau(x_1, x_2, \dots, x_k) \right|^2 dx_1 \dots dx_k \\
&\quad - \frac{1}{p+1} \int_{\mathbb{R}^N} |\tau(x_1, \dots, x_k)|^{p+1} |\psi_\lambda(x_{k+1}, \dots, x_N)|^{p+1} dx \\
&= \int_{\mathbb{R}^{N-k}} |\nabla_{x_{k+1} \dots x_N} \psi_\lambda| dx_{k+1} \dots dx_N + \frac{\Lambda_0}{2} c^2 \\
&\quad - \frac{1}{p+1} \int_{\mathbb{R}^N} |\tau(x_1, \dots, x_k)|^{p+1} |\psi_\lambda(x_{k+1}, \dots, x_N)|^{p+1} dx \\
&= \frac{\lambda^2}{2} \int_{\mathbb{R}^{N-k}} |\nabla_{x_{k+1} \dots x_N} \psi| dx_{k+1} \dots dx_N + \frac{\Lambda_0}{2} c^2 \\
&\quad - \frac{\lambda^{\frac{(N-k)(p-1)}{2}}}{p+1} \int_{\mathbb{R}^N} |\tau(x_1, \dots, x_k)|^{p+1} |\psi(x_{k+1}, \dots, x_N)|^{p+1} dx \\
&< \frac{\Lambda_0}{2} c^2,
\end{aligned}$$

where the last inequality bases on the fact  $1 < p \leq 1 + \frac{4}{N} < 1 + \frac{4}{N-k}$  when taking  $\lambda > 0$  sufficiently small. Moreover, since  $u_\lambda \in S(c)$  for  $\lambda > 0$  sufficiently small, one has

$$m(c) \leq E(u_\lambda) < \frac{\Lambda_0}{2} c^2,$$

which contradicts with (4.3). Thus (4.1) holds.

Then, we deal with problem (1.11) by utilizing the profile decomposition theory of a bounded sequence in  $\Sigma$  (see Lemma 2.6).

**Theorem 4.2.** *Let  $c > 0$  if  $1 < p < 1 + \frac{4}{N}$  or  $0 < c < \|Q\|_2$  if  $p = 1 + \frac{4}{N}$ , where  $Q(x)$  is the ground state solution to Eq (1.10). Then there must exist  $u \in S(c)$  satisfying  $m(c) = E(u)$ .*

**Proof.** We first demonstrate that the variational problem (1.11) is well-defined, and any minimizing sequence for (1.11) is bounded in  $\Sigma$ . By Lemma 2.4 and (1.3), one has the following estimate

$$\begin{aligned}
E(u) &= \frac{1}{2} \|\nabla u\|_2^2 + \frac{1}{2} \int \sum_{i=1}^k x_i^2 |u|^2 dx - \frac{1}{p+1} \int |u|^{p+1} dx \\
&\geq \frac{1}{2} \|u\|_\Sigma^2 - C_{GN} \left( \int |\nabla u|^2 dx \right)^{\frac{N(p-1)}{4}} \left( \int |u|^2 dx \right)^{\frac{2(p+1)-N(p-1)}{4}},
\end{aligned}$$

where  $\|u\|_\Sigma^2 = \|\nabla u\|_2^2 + \int \sum_{i=1}^k x_i^2 |u|^2 dx$ . For the case  $1 < p < 1 + \frac{4}{N}$ , using Young's inequality, one has that for any  $0 < \varepsilon < \frac{1}{2}$ , there is a positive constant  $C(\varepsilon, C_{GN}, c)$  fulfilling

$$C_{GN} \|u\|_\Sigma^{\frac{N(p-1)}{2}} \left( \int |u|^2 dx \right)^{\frac{p+1}{2} - \frac{N(p-1)}{4}} \leq \varepsilon \|u\|_\Sigma^2 + C(\varepsilon, C_{GN}, c),$$

which implies

$$E(u) + C(\varepsilon, C_{GN}, c) \geq \left(\frac{1}{2} - \varepsilon\right) \|u\|_{\Sigma}^2. \quad (4.5)$$

For  $p = 1 + \frac{4}{N}$  and  $0 < c < \|Q\|_2$ , applying Lemma 2.4 again, one derives from (1.3) that

$$\begin{aligned} E(u) &\geq \frac{1}{2} \|u\|_{\Sigma}^2 - \frac{1}{2} \frac{\|\nabla u\|_2^2 \|u\|_2^{\frac{4}{N}}}{\|Q\|_2^{\frac{4}{N}}} \\ &> \frac{\|Q\|_2^{\frac{4}{N}} - c^{\frac{4}{N}}}{2\|Q\|_2^{\frac{4}{N}}} \|u\|_{\Sigma}^2 > 0. \end{aligned} \quad (4.6)$$

Thus, the energy functional  $E(u)$  possesses a finite lower bound and the constrained minimization problem (1.11) is well-defined. In addition, it is clear that each minimizing sequence to (1.11) is bounded in  $\Sigma$  from (4.5) and (4.6).

Second, we argue that there only exists one term  $U^{j_0} \neq 0$  in (4.7) with the aid of profile decomposition technique in  $\Sigma$ . Let  $\{u_n\}_{n=1}^{\infty}$  be a minimizing sequence, using Lemma 2.6, then one gets

$$u_n(x) = \sum_{j=1}^l \tau_{x_n^j} U^j(x) + r_n^l, \quad (4.7)$$

with  $\limsup_{n \rightarrow \infty} \|r_n^l\|_q \rightarrow 0$  as  $l \rightarrow \infty$  when  $q \in [2, \frac{N+2}{N-2})$ . It follows from (4.7) and (2.2)–(2.5) that

$$E(u_n) = \sum_{j=1}^l E(\tau_{x_n^j} U^j) + E(r_n^l) + o(1), \text{ as } n \rightarrow \infty \text{ and } l \rightarrow \infty. \quad (4.8)$$

Let  $\tau_{x_n^j} U_{\lambda_j}^j(x) = \lambda_j \tau_{x_n^j} U^j(x)$  with  $\lambda_j = \frac{c}{\|U^j\|_2}$ , for every  $\tau_{x_n^j} U^j$  ( $1 \leq j \leq l$ ), we deduce

$$\|\tau_{x_n^j} U_{\lambda_j}^j\|_2 = c,$$

and

$$\begin{aligned} E(\tau_{x_n^j} U_{\lambda_j}^j) &= \frac{1}{2} \|\nabla \tau_{x_n^j} U_{\lambda_j}^j\|_2^2 + \frac{1}{2} \int V(x) |\tau_{x_n^j} U_{\lambda_j}^j|^2 dx - \frac{1}{p+1} \int |\tau_{x_n^j} U_{\lambda_j}^j|^{p+1} dx \\ &= \lambda_j^2 E(\tau_{x_n^j} U^j) - \frac{\lambda_j^2 (\lambda_j^{p-1} - 1)}{p+1} \int |\tau_{x_n^j} U^j|^{p+1} dx, \end{aligned}$$

which means that

$$E(\tau_{x_n^j} U^j) = \frac{E(\tau_{x_n^j} U_{\lambda_j}^j)}{\lambda_j^2} + \frac{\lambda_j^{p-1} - 1}{p+1} \int |\tau_{x_n^j} U^j|^{p+1} dx. \quad (4.9)$$

Similarly, we get the estimate of  $E(r_n^l)$  as below

$$\begin{aligned} E(r_n^l) &= \frac{\|r_n^l\|_2^2}{c^2} E\left(\frac{c}{\|r_n^l\|_2} r_n^l\right) + \frac{\left(\frac{c}{\|r_n^l\|_2}\right)^{p-1} - 1}{p+1} \int |r_n^l|^{p+1} dx + o(1) \\ &\geq \frac{\|r_n^l\|_2^2}{c^2} E\left(\frac{c}{\|r_n^l\|_2} r_n^l\right) + o(1). \end{aligned} \quad (4.10)$$



Due to  $\|\tau_{x_n^j} U_{\lambda_j}^j\|_2 = \|\frac{c}{\|r_n^l\|_2} r_n^l\|_2 = c$ , one has

$$E(\tau_{x_n^j} U_{\lambda_j}^j) \geq m(c), \text{ and } E(\frac{c}{\|r_n^l\|_2} r_n^l) \geq m(c).$$

It follows from (4.8)–(4.10) that

$$\begin{aligned} E(u_n) &\geq \sum_{j=1}^l \left( \frac{E(\tau_{x_n^j} U_{\lambda_j}^j)}{\lambda_j^2} + \frac{\lambda_j^{p-1} - 1}{p+1} \int |\tau_{x_n^j} U^j|^{p+1} dx \right) \\ &\quad + \frac{\|r_n^l\|_2^2}{c^2} E(\frac{c}{\|r_n^l\|_2} r_n^l) + o(1) \\ &\geq \frac{m(c)}{c^2} \sum_{j=1}^l \|U^j\|_2^2 + \inf_{j \geq 1} \frac{\lambda_j^{p-1} - 1}{p+1} \left( \sum_{j=1}^l \int |\tau_{x_n^j} U^j|^{p+1} dx \right) \\ &\quad + \frac{\|r_n^l\|_2^2}{c^2} m(c) + o(1). \end{aligned} \tag{4.11}$$

By the convergence of  $\sum_{j=1}^{\infty} \|U^j\|_2^2$ , there must exist  $j_0 \geq 1$  such that

$$\|U^{j_0}\|_2^2 = \sup\{\|U^j\|_2^2, j \geq 1\} \text{ and } \inf_{j \geq 1} \lambda_j = \lambda_{j_0} = \frac{c}{\|U^{j_0}\|_2}.$$

Let  $n \rightarrow \infty$  and  $l \rightarrow \infty$  in (4.11), applying Lemma 4.1, then one gets

$$m(c) \geq m(c) + \delta \left( \left( \frac{c}{\|U^{j_0}\|_2} \right)^{p-1} - 1 \right),$$

which yields

$$\|U^{j_0}\|_2 \geq c.$$

Thus, combining (2.2), we have  $\|U^{j_0}\|_2 = c$ , and there only exists one term  $U^{j_0} \neq 0$  in (4.7). Therefore, (4.7) can be rewritten as

$$u_n(x) = \tau_{x_n^{j_0}} U^{j_0}(x) + r_n(x).$$

Moreover, note that  $\|u_n\|_2 = \|U^{j_0}\|_2 + \|r_n\|_2 + o_n(1)$ , and  $\|u_n\|_2 = \|U^{j_0}\|_2 = c$ , one has  $\lim_{n \rightarrow \infty} \|r_n\|_2 = 0$ , which means  $r_n \rightarrow 0$  in  $L^2(\mathbb{R}^N)$ . This, together with Lemma 2.4, deduces that  $\lim_{n \rightarrow \infty} \|r_n\|_{q+1}^{q+1} = 0$  for  $q \in (1, \frac{N+2}{N-2})$ . Then we get

$$\int |r_n|^{p+1} dx \rightarrow 0.$$

By the lower semi-continuity, one has

$$\liminf_{n \rightarrow \infty} E(r_n) \geq 0,$$

and

$$\begin{aligned}
\liminf_{n \rightarrow \infty} E(\tau_{x_n^{j_0}} U^{j_0}) &\leq \liminf_{n \rightarrow \infty} E(\tau_{x_n^{j_0}} U^{j_0}) + \liminf_{n \rightarrow \infty} E(r_n) \\
&\leq \liminf_{n \rightarrow \infty} (E(\tau_{x_n^{j_0}} U^{j_0}) + E(r_n)) \\
&= \liminf_{n \rightarrow \infty} E(u_n) = m(c).
\end{aligned}$$

Besides, for  $n \geq 1$ , we infer from  $\|\tau_{x_n^{j_0}} U^{j_0}\|_2 = \|U^{j_0}\|_2 = c$  that  $E(\tau_{x_n^{j_0}} U^{j_0}) \geq m(c)$ . Thus,

$$\liminf_{n \rightarrow \infty} E(\tau_{x_n^{j_0}} U^{j_0}) = m(c).$$

Next, we claim that the sequence  $\{x_n^{j_0}\}$  is bounded. Let us argue by contradiction and suppose that up to a subsequence,  $|x_n^{j_0}| \rightarrow \infty$  as  $n \rightarrow \infty$ . For convenience, let  $U^{j_0}$  be continuous and compactly supported. Thus, one has

$$\int |\tau_{x_n^{j_0}} U^{j_0}|^{p+1} dx \rightarrow 0, \text{ as } n \rightarrow \infty.$$

This implies that

$$\liminf_{n \rightarrow \infty} E(\tau_{x_n^{j_0}} U^{j_0}) = \frac{1}{2} \|U^{j_0}\|_{\Sigma}^2 = m(c).$$

Furthermore, by the definition of  $E(U^{j_0})$  we infer that

$$E(U^{j_0}) + \frac{1}{p+1} \int |U^{j_0}|^{p+1} dx = m(c),$$

which means  $E(U^{j_0}) < m(c)$ . This contradicts to  $E(U^{j_0}) \geq m(c)$  since  $\|U^{j_0}\|_2^2 = c$ . Thus, the boundedness of the sequence  $\{x_n^{j_0}\} \subseteq \mathbb{R}^{N-k}$  is proved, and we could suppose that, up to a subsequence,  $x_n^{j_0} \rightarrow x^{j_0}$  in  $\mathbb{R}^{N-k}$  as  $n \rightarrow \infty$ .

Up to now, we can rewrite (4.7) as

$$u_n(x) = \widetilde{U}^{j_0}(x) + \widetilde{r}_n(x),$$

where  $\widetilde{U}^{j_0}(x) = \tau_{x_n^{j_0}} U^{j_0}(x)$  and  $\widetilde{r}_n(x) = \tau_{x_n^{j_0}} U^{j_0}(x) - \tau_{x_n^{j_0}} U^{j_0}(x) + r_n(x)$ . Since  $\|u_n\|_2 = \|U^{j_0}\|_2 = c$ , then

$$\widetilde{r}_n \rightharpoonup 0 \text{ in } \Sigma \text{ and } \widetilde{r}_n \rightharpoonup 0 \text{ in } L^2(\mathbb{R}^N).$$

Thus, we have

$$E(u_n) = E(\widetilde{U}^{j_0}) + E(\widetilde{r}_n) + o_n(1).$$

Applying the lower semi-continuity to norm, together with  $\lim_{n \rightarrow \infty} \int |\widetilde{r}_n|^{p+1} dx = 0$ , we know  $\liminf_{n \rightarrow \infty} E(\widetilde{r}_n) \geq 0$ . Thus, it follows from  $\|\widetilde{U}^{j_0}\|_2^2 = c$  that

$$\begin{aligned}
m(c) = \liminf_{n \rightarrow \infty} E(u_n) &\geq \liminf_{n \rightarrow \infty} (E(\widetilde{U}^{j_0}) + E(\widetilde{r}_n)) \\
&\geq E(\widetilde{U}^{j_0}) + \liminf_{n \rightarrow \infty} E(\widetilde{r}_n) \\
&\geq E(\widetilde{U}^{j_0}) \geq m(c),
\end{aligned}$$

which indicates that  $E(\widetilde{U}^{j_0}) = m(c)$ . Thus the proof is completed.

We now show that the standing waves to Eq (1.1) are orbitally stable with the help of Theorem 4.2. **Proof of Theorem 1.5.** According to Remark 2.2, we are aware the existence of unique global solution  $u(t, x)$  to Eq (1.1) under the assumptions. We prove this conclusion by contradiction. Suppose that there exists a sequence  $\{u_{0,n}\}_{n=1}^{\infty}$  satisfying

$$\inf_{\varphi \in M_c} \|u_{0,n} - \varphi\|_{\Sigma} < \frac{1}{n}, \quad (4.12)$$

and assume there exist a time sequence  $\{t_n\}_{n=1}^{\infty}$  and a positive constant  $\varepsilon_0$  such that the solution sequence  $\{u_n(t_n)\}_{n=1}^{\infty}$  to Eq (1.1) fulfils

$$\inf_{\varphi \in M_c} \|u_n(t_n) - \varphi\|_{\Sigma} \geq \varepsilon_0. \quad (4.13)$$

Next we show that there is  $v \in M_c$  satisfying

$$\lim_{n \rightarrow \infty} \|u_{0,n} - v\|_{\Sigma} = 0. \quad (4.14)$$

In fact, from (4.12), we can find a sequence  $\{v_n\}_{n=1}^{\infty} \subset M_c$  such that

$$\|u_{0,n} - v_n\|_{\Sigma} < \frac{2}{n}. \quad (4.15)$$

Since  $\{v_n\}_{n=1}^{\infty} \subset M_c$ , then  $\{v_n\}$  is a minimizing sequence to (1.11). In addition, applying the assertion in Theorem 4.2, one can conclude that there exists  $v \in M_c$  fulfilling

$$\lim_{n \rightarrow \infty} \|v_n - v\|_{\Sigma} = 0. \quad (4.16)$$

Thus, (4.14) follows immediately from (4.15) and (4.16). Then we have

$$\lim_{n \rightarrow \infty} \|u_{0,n}\|_2^2 = \|v\|_2^2 = c^2, \quad \lim_{n \rightarrow \infty} E(u_{0,n}) = E(v) = m(c).$$

In addition, we deduce from (2.1) that

$$\lim_{n \rightarrow \infty} \|u_n(t_n)\|_2^2 = c^2, \quad \lim_{n \rightarrow \infty} E(u_n(t_n)) = E(v) = m(c).$$

Moreover, thanks to Theorem 4.2, one knows that  $\{u_n(t_n)\}_{n=1}^{\infty}$  is bounded in  $\Sigma$ . Taking  $\tilde{u}_n = \frac{cu_n(t_n)}{\|u_n(t_n)\|_2}$ , one has  $\|\tilde{u}_n\|_2 = c$  and

$$\begin{aligned} E(\tilde{u}_n) &= \frac{c^2}{2\|u_n(t_n)\|_2^2} \|u_n(t_n)\|_{\Sigma}^2 - \frac{1}{p+1} \frac{c^{p+1}}{\|u_n(t_n)\|_2^{p+1}} \int |u_n(t_n)|^{p+1} dx \\ &= \frac{c^2}{\|u_n(t_n)\|_2^2} E(u_n(t_n)) + \frac{1}{p+1} \left( \frac{c^2}{\|u_n(t_n)\|_2^2} - \frac{c^{p+1}}{\|u_n(t_n)\|_2^{p+1}} \right) \int |u_n(t_n)|^{p+1} dx, \end{aligned}$$

which yields that

$$\lim_{n \rightarrow \infty} E(\tilde{u}_n) = E(u_n(t_n)) = m(c).$$

Therefore,  $\tilde{u}_n$  also becomes a minimizing sequence to (1.11). Then by Theorem 4.2, one can find an element  $\tilde{v} \in M_c$  such that

$$\tilde{u}_n \rightarrow \tilde{v} \text{ in } \Sigma.$$

Then, one has

$$\tilde{u}_n - u_n(t_n) \rightarrow 0 \text{ in } \Sigma.$$

It is clear that

$$u_n(t_n) \rightarrow \tilde{v} \text{ in } \Sigma,$$

which is contradictory to (4.13). Thus the conclusion holds true.

## 5. Conclusions

In this paper, we investigate the sharp global existence of solutions and the stability of standing waves for the NLS with partial confinement. More precisely, for  $1 + \frac{4}{N} \leq p < \frac{N+2}{N-2}$ , via constructing some novel cross-invariant manifolds and variational problems, we derive a novel sharp criterion for global existence. That is, the solution  $u(t, x)$  for Eq (1.1) exists globally in time  $t \in [0, \infty)$  if the initial data  $u_0 \in K_+ \cup R_+$ , while the solution  $u(t, x)$  blows up in finite time if  $u_0 \in K$  and  $|x|u_0 \in L^2(\mathbb{R}^N)$ . In addition, we utilize the profile decomposition technique to overcome the lack of compactness and show the existence and stability of normalized standing waves for  $1 < p < 1 + \frac{4}{N}$  or  $p = 1 + \frac{4}{N}$  with  $\|u_0\|_2 < \|Q\|_2$ , where  $Q(x)$  is the ground state to the critical elliptic equation (1.10).

### Use of AI tools declaration

The author declares she has not used Artificial Intelligence (AI) tools in the creation of this article.

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### Conflict of interest

The authors declare there is no conflict of interest.

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