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Research article

Flocking dynamics and pattern motion for the Cucker-Smale system with distributed delays

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Abstract: In this paper, a new class of Cucker-Smale systems with distributed delays are developed from the measurement perspective. By combining dissipative differential inequalities with a continuity argument, some new sufficient criteria for the flocking dynamics of the proposed model with general communication rate, especially the non-normalized rate, are established. In order to achieve the prescribed pattern motion, the driving force term is incorporated into the delayed collective system. Lastly, some examples and simulations are provided to illustrate the validity of the theoretical results.

Keywords: Cucker-Smale model; flocking behavior; distributed delay; pattern motion

1. Introduction

In the last years, self-organization systems have been getting a great deal of attention from researchers around the world, and have conducted a lot of research in many fields such as artificial intelligence, physics, biology and social sciences. The famous Cucker-Smale model [1, 2] offered a frame to describe aggregation behaviors, for instance, the flocking of birds, reaching a consensus. Considering that the interaction intensity depends on the number of agents, the Cucker-Smale model can not well describe the flocking behavior of non-uniform multi-particle swarm optimization. Motsch and Tadmor generalized the flocking system to the case of asymmetric interactions [3]. Recently, the classical Cucker-Smale model has been generalized and modified to several cases, such as various forms of stochastic noise, cone-vision constraints, the presence of leadership and more general interaction potentials [4–15, 22–28].

In practical applications, time-delay often leads to system instability, and its impact can not be ignored [29–31]. It has become a broad consensus that mathematical models with time delay always have greater practicability[23–25, 32–35]. The authors took into account heterogeneous delays in [23]. The velocity asymptotic alignment of the delayed Cucker-Smale model was investigated in the presence or absence of noise [24]. Very recently, the authors in [25] proposed the following improved model

$$\begin{cases} \dot{p}_i = c_i, \\ \dot{c}_i = \frac{1}{h(t)} \sum\limits_{m=1}^{N} \int_{t-T(t)}^{t} \beta(t-s)\Phi\left(p_m(s), p_i(t)\right)(c_m(s) \\ \qquad - c_i(t))\mathrm{d}s, \quad i = 1, 2, \ldots, N, \end{cases} \tag{1.1}$$

where $p_i(t)$ and $c_i(t)$ denote the position and velocity of agent i at time t, $\beta : [0, T_0] \to [0, \infty)$ is a weight function which requires

$$\int_0^{\hat{T}} \beta(s)\mathrm{d}s > 0 \quad \text{and} \quad h(k) := \int_0^{T(k)} \beta(s)\mathrm{d}s, \quad k \geq 0. \tag{1.2}$$

$\Phi(p_m(s), p_i(t))$ is the normalized communication weights provided by

$$\Phi\left(p_m(s), p_i(t)\right) = \begin{cases} \frac{\psi(|p_m(s)-p_i(t)|)}{\sum\limits_{j\neq i} \psi\left(\left|p_j(s)-p_i(t)\right|\right)}, & \text{if } m \neq i, \\ 0, & \text{if } m = i, \end{cases} \tag{1.3}$$

with ψ: [0,∞) → $(0, \infty)$, is positive, bounded, nonincremental and Lipschitz continuous on $[0, \infty)$, and $\psi(0) = 1$. Besides, there has $\hat{T} > 0$ and $T_0 > 0$ obeying

$$T(k) \geq \hat{T}, \;\; T'(k) \leq 0, \quad \text{and} \quad \hat{T} \leq T(k) \leq T_0 \quad \text{for} \quad k \geq 0. \tag{1.4}$$

On the one hand, assumption (1.4) is reasonable requirement for time-varying delays in specific background [21, 27]. As we all know, measurement of velocity is much more sensitive than measurement of position from the perspective of time delay and the measurement delay in this system mainly responds to velocity, not position. Furthermore, the introduction of distributed delay can effectively characterize the fact that the position and velocity of agents are not only affected by the behavior of other individuals in a certain period of time, but also affected by the behavior of other individuals at a varying time period [37]. And few researchers have considered the following distributed delay of Cucker-Smale model which governed by

$$\begin{cases} \dot{p}_j = c_j, \\ \dot{c}_j = \frac{1}{Nh(k)} \sum\limits_{m=1, m\neq j}^{N} \int_{k-T(k)}^{k} \beta(k-s)\Phi_{m,j}(k)(c_m(s) - c_j(k))\mathrm{d}s, \quad j = 1, 2, \ldots, N, \end{cases} \tag{1.5}$$

where $\Phi_{m,j}(k) = \Phi(p_m(k), p_j(k))$, $(\phi_j, \psi_j) \in \mathcal{S}^2 = \mathcal{S} \times \mathcal{S}$ and $\mathcal{S} := S([-T, 0], \mathbb{R}^d)$ is the Banach space of all continuous functions. Moreover, $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ requires $\phi(k) \leq 1$ for each $k \geq 0$. $\mathbb{R}^+$ denotes the set of positive real number.

$$p_j(s) =: \Phi_j(s), \quad c_j(s) =: \psi_j(s), \quad j = 1, \cdots, N, \quad s \in [-T_0, 0], \tag{1.6}$$

On the other hand, the pattern motion to self-organized systems come out so unaffectedly at lots of physical and biological scenes, which is important for us to further study the mechanism of swarm intelligence systems. However, there are few more practical research on flocking formation behavior based on Cucker-Smale model.

Illuminated by the aforementioned arguments, the main objective of this article is to establish the flocking behavior of the system (1.5). Specifically speaking, the focus of this article is as follows.

(i) In this article, we propose a new class of Cucker-Smale model incorporating distributed delays from the measurement perspective, which is different from the existing models, see, e.g., [23–25].
(ii) Under certain assumptions, by exploiting dissipative differential inequality, some new sufficient criteria for the flocking behavior of (1.5) and (1.6) with general communication rate (especially the non-normalized rate) are gained for the first time.
(iii) Numerical simulations are arranged to verify the effectiveness of the main theoretical analysis results.

In the rest of this paper, we give the definition and several useful lemmas in Section 2. The flocking result and motion pattern of the system (1.5) with a driving force are presented in Section 3. In Section 4, we also give the numeric calculations which are very good agreement with theoretical results. Lastly, we draw a brief conclusion in Section 5.

2. Preliminaries

In this section, for purpose of obtaining the main results of this paper, we firstly require the following definition and lemmas. Define the following quantities:

$$d_P(t) := \max_{1\le j,i\le N} \|p_j(t) - p_i(t)\|,$$
$$d_C(t) := \max_{1\le j,i\le N} \|c_j(t) - c_i(t)\|.$$

Definition 2.1 We say that a solution $\{p_j(t), c_j(t)\}, j \in \{1, ..., N\}$ of the systems (1.5) and (1.6) converges to flocking while the conditions as follow are satisfied.

$$\sup_{t\ge 0} d_P(k) < \infty, \quad \text{and} \quad \lim_{k\to\infty} d_C(k) = 0.$$

Due to the fact which the functions d_P and d_C usually are not at C^1 smooth, we do with the upper Dini derivative. For a function $G(t)$, G's upper Dini derivative at k is defined by

$$D^+G(k) = \limsup_{m\to 0^+} \frac{G(k+m) - G(k)}{m}.$$

In particular, the Dini derivative is the same as the usual derivative when G is differential at k. In this paper, we assume that the function Φ is bounded, positive, nonincremental and at Lipschitz continuous in $\mathbb{R}^+$, with $\Phi(0) = 1$.

Remark 2.1 We notice that the theoretical analysis in [21] the assumption of the function Φ. that it have the strictly positive lower bound . However, it isn't needed in our framework.

Lemma 2.2 Let $\left\{(p_j, c_j)\right\}_{j=1}^N$ be a solution to (1.5) and (1.6) and $F = \max_{k\in[-T,0]} \max_{1\le j\le N} |\Psi_j(k)| > 0$. Assume the premier velocity $\Psi_j (j = 1, 2, ..., N)$ is continuous on $[-T, 0]$. Then the solution satisfies

$$\max_{1\le j\le N} |c_j(t)| \le F \quad \text{for} \quad k \ge -T.$$

Proof Choose any $\epsilon > 0$ and set

$$Q^\epsilon := \{k > 0 : \max_{1\le i\le N} |c_i(t)| < F + \varepsilon, \quad \forall t \in [0, k)\}.$$

According to the assumption, $Q^\varepsilon \neq \emptyset$. Denote $R^\varepsilon := \sup Q^\varepsilon > 0$. We will prove that $R^\varepsilon = +\infty$. For contradiction, suppose $R^\varepsilon < +\infty$. This gives, by continuity, $\max_{1\le j\le N} |c_j(R^\varepsilon)| = F + \epsilon$.

On another scale, from (1.5) and (1.6), for $k < R^\varepsilon$ and $j = 1, \cdots, N$, we have

$$\begin{aligned}
\tfrac{1}{2}\mathrm{D}^+|c_j(k)|^2 &\le \langle c_j(k), \tfrac{dc_j(k)}{dk}\rangle \\
&= \langle c_j(k), \tfrac{1}{Nh(k)} \sum_{m=1,m\neq j}^{N} \int_{k-T(k)}^{k} \beta(k-s)\Phi_{m,i}(k)(c_m(s) - c_j(k))ds\rangle \\
&= \tfrac{1}{Nh(k)} \sum_{m=1,m\neq j}^{N} \int_{k-T(k)}^{k} \beta(k-s)\Phi_{m,i}(k) \left\langle c_j(k), c_m(s) - c_j(k)\right\rangle ds \\
&= \tfrac{1}{Nh(k)} \sum_{m=1,m\neq j}^{N} \int_{k-T(k)}^{k} \beta(k-s)\Phi_{m,i}(k)(\left\langle c_j(k), c_m(s)\right\rangle - |c_j(k)|^2)ds \\
&\le \tfrac{1}{Nh(k)} \sum_{m=1,m\neq j}^{N} \int_{k-T(k)}^{k} \beta(k-s)\Phi_{m,i}(k)|c_j(k)|(|c_m(s) - |c_j(k)|)ds.
\end{aligned}$$

Note that $\frac{1}{N} \sum_{m=1,m\neq j}^{N} \Phi_{m,i}(k) \le 1$ and $\max_{1\le j\le N} |c_m(s)| < F + \epsilon$ for $k < R^\epsilon$. Thus we receive

$$\begin{aligned}
\tfrac{1}{2}\mathrm{D}^+|c_j(k)|^2 &\le \tfrac{1}{h(t)} \int_{k-T(k)}^{k} \beta(k-s)ds|c_j(k)|(F + \epsilon - |c_j(k)|) \\
&= |c_j(k)|(F + \epsilon - |c_j(k)|),
\end{aligned}$$

which yields

$$\mathrm{D}^+|c_j(k)| \le (F + \epsilon) - |c_j(k)|.$$

With the help of Gronwall inequality, we have

$$|c_j(k)| \le e^{-k}(c_j(0) - F - \epsilon) + F + \epsilon < F + \epsilon.$$

Hence,

$$\lim_{t\to R^{\epsilon-}} \max_{1\le j\le N} |c_j(k)| < F + \epsilon,$$

which is in contradiction with hypothesis. Therefore, $R^\epsilon = +\infty$. Moreover, since ϵ is arbitrary, the lemma is proved.

Lemma 2.3. Let $\left\{(p_j, c_j)\right\}_{j=1}^N$ be the solution to (1.5) and (1.6). Afterwards, the diameters functions $d_P(k)$ and $d_C(k)$ require

$$\begin{aligned}
&D^+ d_P(k) \le d_C(k), \\
&D^+ d_C(k) \le -\Phi(d_P(k))d_C(k) + 2\Delta_N^T(k),
\end{aligned} \tag{2.1}$$

for all $k > 0$, where $\Delta_N^T(k)$ is given by

$$\Delta_N^T(k) = \frac{1}{Nh(k)} \max_{1\le j\le N} \sum_{m=1,m\neq j}^{N} \int_{k-T(k)}^{k} \beta(k-s)\Phi_{m,i}(k)|c_m(k) - c_m(s)|ds, \tag{2.2}$$

and satisfies

$$\Delta_N^T(k) \le \int_{k-T(k)}^{k} [\Delta_N^T(s) + d_C(s)]ds. \tag{2.3}$$

Proof We firstly obtain from (1.5) that

$$D^+ d_P(k) \le d_C(k).$$

On account of the continuity of $c_j(k)(j \in \{1, ..., N\})$, there is a times sequence $\{k_m\}_{m\in\mathbb{N}}$ such that

$$\bigcup_{m\in\mathbb{N}} [k_m, k_{m+1}) = [0, +\infty).$$

And for each $k \in \mathbb{N}$, there has $i, j \in \{1, ..., N\}$ so that $d_C(k) = |c_j(k) - c_i(k)|$ for $k \in [k_m, k_{m+1})$.

Consequently, one can get

$$\begin{aligned} \frac{1}{2}D^+ d_C^2(k) &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}k}|c_j(k) - c_i(k)|^2 \\ &= \langle c_j(k) - c_i(k), \dot{c}_j(k) - \dot{c}_i(k)\rangle \\ &= \langle c_j(k) - c_i(k), \frac{1}{Nh(k)} \sum_{m=1,m\neq j}^{N} \int_{k-T(k)}^{k} \beta(k-s)\Phi_{m,i}(k)[c_m(s) - c_j(k)]ds\rangle \\ &\quad -\langle c_j(k) - c_i(k), \frac{1}{Nh(k)} \sum_{m=1,m\neq i}^{N} \int_{k-T(k)}^{k} \beta(k-s)\Phi_{m,i}(k)[c_m(s) - c_i(k)]ds\rangle \\ &=: W_1(k) + W_2(k). \end{aligned} \tag{2.4}$$

For any $j \in \{1, ..., N\}$ and the function Φ is non-increasing, one can obtain

$$\Phi_{m,i}(k) \ge \Phi(d_P(k)), \tag{2.5}$$

and

$$\langle c_j(k) - c_i(k), c_m(k) - c_j(k)\rangle \le 0, \quad \forall k \ge 0,\ m \in \{1, ..., N\}. \tag{2.6}$$

According to the definitions of $h(k)$, and with $\Phi \le 1$, $\alpha(k)$, (2.5) and (2.6), we obtain as follows:

$$\begin{aligned} W_1(k) &= \langle c_j(k) - c_i(k), \frac{1}{Nh(k)} \sum_{m=1,m\neq j}^{N} \int_{k-T(k)}^{k} \beta(k-s)\Phi_{m,i}(k)[c_m(s) - c_j(k)]ds\rangle \\ &= \langle c_j(k) - c_i(k), \frac{1}{Nh(k)} \sum_{m=1,m\neq j}^{N} \int_{k-T(k)}^{k} \beta(k-s)\Phi_{m,i}(k)[c_m(k) - c_j(k)]ds\rangle \\ &\quad +\langle c_j(k) - c_i(k), \frac{1}{Nh(k)} \sum_{m=1,m\neq j}^{N} \int_{k-T(k)}^{k} \beta(k-s)\Phi_{m,i}(k)[c_m(s) - c_m(k)]ds\rangle \\ &\le \frac{\Phi(d_P(k))}{N} \sum_{m=1,m\neq j}^{N} \langle c_j(k) - c_i(k), c_m(k) - c_j(k)\rangle \\ &\quad + \frac{d_C(k)}{Nh(k)} \sum_{m=1,m\neq j}^{N} \int_{k-T(k)}^{k} \beta(k-s)\Phi_{m,i}(k)|c_m(k) - c_m(s)|ds. \end{aligned} \tag{2.7}$$

Similarly

$$
\begin{aligned}
W_2(k) &= \langle c_j(k)-c_i(k), \tfrac{-1}{Nh(k)} \sum_{m=1,m\neq i}^{N} \int_{k-T(k)}^{k} \beta(k-s)\Phi_{k,j}(k)[c_m(k)-c_i(k)]ds\rangle \\
&\quad +\langle c_j(k)-c_i(k), \tfrac{-1}{Nh(k)} \sum_{m=1,m\neq i}^{N} \int_{k-T(k)}^{k} \beta(k-s)\Phi_{k,j}(k)[c_m(s)-c_m(k)]ds\rangle \\
&\leq \tfrac{-\Phi(d_P(k))}{N} \sum_{m=1,m\neq i}^{N} \langle c_j(k)-c_i(k), c_m(k)-c_i(k)\rangle \\
&\quad + \tfrac{d_C(k)}{Nh(k)} \sum_{m=1,m\neq i}^{N} \int_{k-T(k)}^{k} \beta(k-s)\Phi_{m,i}(k)|c_m(k)-c_m(s)|ds.
\end{aligned} \tag{2.8}
$$

From (2.4), (2.7) and (2.8), for $k \geq 0$ we obtain that

$$
\begin{aligned}
\tfrac{1}{2}D^+d_C(k)^2 &\leq -\Phi(d_P(k))d_C(k)^2 \\
&\quad + \tfrac{2d_C(k)}{Nh(k)} \max_{1\leq ji\leq N} \sum_{m=1,m\neq i}^{N} \int_{k-T(k)}^{k} \beta(k-s)\Phi_{m,i}(k)|c_m(k)-c_m(s)|ds.
\end{aligned}
$$

Thereby,

$$
D^+d_C(k) \leq -\Phi(d_P(k))d_C(k) + 2\Delta_N^T(k).
$$

We next estimate the term $\Delta_N^T(k)$. One can get

$$
\begin{aligned}
\Delta_N^T(k) &= \tfrac{1}{Nh(k)} \max_{1\leq i\leq N} \sum_{m=1,m\neq i}^{N} \int_{k-T(k)}^{k} \beta(k-s)\Phi_{m,i}(s)|c_m(k)-c_m(s)|ds \\
&\leq \tfrac{1}{Nh(k)} \sum_{m=1}^{N} \int_{k-T(k)}^{k} \beta(k-s) \int_s^k |\dot{c}_m(\theta)|d\theta ds \\
&\leq \tfrac{1}{N} \sum_{m=1}^{N} \int_{k-T(k)}^{k} |\dot{c}_m(s)|ds \\
&\leq \int_{k-T(k)}^{k} |\dot{c}_m(s)|ds.
\end{aligned} \tag{2.9}
$$

Furthermore, it obeys from (2.7), that

$$
\begin{aligned}
|\dot{c}_m(s)| &= |\tfrac{1}{Nh(s)} \sum_{l=1,l\neq m}^{N} \int_{s-T(s)}^{s} \beta(s-\theta)\Phi_{m,l}(s)|c_l(\theta)-c_m(s)|d\theta| \\
&\leq \tfrac{1}{Nh(s)} \sum_{l=1}^{N} \int_{s-T(s)}^{s} \beta(s-\theta)\Phi_{m,l}(s)(|c_l(s)-c_m(s)| + |c_l(\theta)-c_l(s)|)d\theta \\
&\leq \Delta_N^T(s) + d_C(s).
\end{aligned} \tag{2.10}
$$

Hence, combining with (2.9) and (2.10), we obtain

$$
\Delta_N^T(k) \leq \int_{k-T(k)}^{k} [\Delta_N^T(s) + d_C(s)]ds, \tag{2.11}
$$

which proves the Lemma 2.3.

Remark 2.2 In the perspective of Lemma 2.3, we achieve

$$
|\dot{c}_m(s)| \leq |\tfrac{1}{Nh(s)} \sum_{l=1}^{N} \int_{s-T(s)}^{s} \beta(s-\theta)\Phi_{m,l}(s)|c_l(\theta)-c_m(s)|d\theta| \leq 2F.
$$

Estimation are given as follow:

$$
\Delta_N^T(k) \leq 2FT(k) \quad \text{for} \quad k \geq 0.
$$

3. Main results

3.1. Asymptotic flocking for the Cucker-Smale system

Theorem 3.1.1 Make $\left\{(p_i, c_j)\right\}_{i=1}^{N}$ as the solution to (1.5) and (1.6). Assume that there have some constants $\kappa, \delta > 0$ satisfying

$$0 < \kappa < \Phi(\delta), \quad d_P(0) + \tfrac{C_2}{\kappa} < \delta, \tag{3.1}$$

where $C_2 := \frac{2C_1}{\Phi(\delta)-\kappa}$ with $C_1 > \max\left\{\frac{d_C(0)}{2}[\Phi(\delta) - \kappa], 2GT_0\right\}$. Then, if

$$\tfrac{\Phi(\delta)-\kappa+2}{\kappa[\Phi(\delta)-\kappa]}(e^{\kappa T_0} - 1) < 1, \tag{3.2}$$

we have

$$d_P(k) < \delta \quad \text{and} \quad d_C(k) \leq C_2 e^{-\kappa k}, \quad \forall k \geq 0.$$

Aiming at proving Theorem 3.1.1, we give Lemma 3.1.2 as follow.

Lemma 3.1.2 Make $\left\{(p_i, c_j)\right\}_{i=1}^{N}$ as a global solution to the model (1.5) and (1.6) satisfying a priori assumption on the relative position:

$$\sup_{0 \leq k < +\infty} d_P(k) \leq \delta. \tag{3.3}$$

Then, we can gain

$$\Delta_N^T(k) < C_1 e^{-\kappa k} \quad \text{and} \quad d_C(k) < C_2 e^{-\kappa k} \quad \forall k > 0, \tag{3.4}$$

where $\kappa > 0$ and $C_1, C_2 > 0$ have been given in Theorem 3.1.1.

Proof Firstly if $\Delta_N^T(k) < C_1 e^{-\kappa k}$ for all $k \in [0, H]$ with fixing $H > 0$, we prove that $d_C(k) < C_2 e^{-\kappa k}$ for all $k \in [0, T]$. Actually, according to Lemma 2.3, one can easily obtain

$$\mathrm{D}^+ d_C(k) \leq -\Phi(d_P(k)) d_C(k) + 2C_1 e^{-\kappa k},$$

for $k \in [0, H]$. Using Growall's inequality and the fact that $d_C(0) < \frac{2C_1}{\Phi(\delta)-\kappa}$ yield

$$\begin{aligned} d_C(k) &\leq d_C(0) e^{-\Phi(\delta)k} + \tfrac{2C_1}{\Phi(\delta)-\kappa}\left[e^{-\kappa k} - e^{-\Phi(\delta)t}\right] \\ &= \left[d_C(0) - \tfrac{2C_1}{\Phi(\delta)-\kappa}\right] e^{-\Phi(\delta)k} + \tfrac{2C_1}{\Phi(\delta)-\kappa} e^{-\kappa k} \\ &< \tfrac{2C_1}{\Phi(\delta)-\kappa} e^{-\kappa k}. \end{aligned}$$

Set

$$M := \left\{H > 0 : \ \Delta_N^T(k) < C_1 e^{-\kappa k} \quad \text{and} \quad d_C(k) < C_2 e^{-\kappa k}, \ \ k \in [0, H]\right\}.$$

Which obeys Lemma 2.3 and $2GT_0 < C_1$ that $0 \in M$. Thus, $M \neq \emptyset$. We will prove $\sup M = \infty$. Suppose that $\overline{H} = \sup M < \infty$. Therefore by continuity of functions $d_C(k)$ and $\Delta_N^T(k)$, we have

$$\Delta_N^T(\overline{H}) = C_1 e^{-\kappa \overline{H}} \quad \text{or} \quad d_C(\overline{H}) = C_2 e^{-\kappa \overline{H}}, \tag{3.5}$$

Afterwards $\overline{H} \notin M$. Whatsmore, with using $\Delta_N^T(k)$ and M yields definitions.

$$\begin{array}{rcl}\Delta_N^T(\overline{H}) & = & \lim\limits_{k\to\overline{H}^-} \Delta_N^T(k) \\ & \leq & \lim\limits_{k\to\overline{H}^-} \int_{k-T(k)}^{k} [\Delta_N^T(s) + d_C(s)]ds \\ & \leq & \lim\limits_{k\to\overline{H}^-} (C_1 + C_2) \int_{k-T(k)}^{t} e^{-\kappa s} ds \\ & \leq & \frac{C_1+C_2}{\kappa} e^{-\kappa\overline{H}}(e^{\kappa T_0} - 1) \\ & < & C_1 e^{-\kappa\overline{H}}. \end{array} \tag{3.6}$$

Thus, from the assertion of the proof of the lemma, we can get $d_C(\overline{H}) < C_2 e^{-\kappa\overline{H}}$. Consequently, (3.5) doesn't hold, and we have $\overline{H} = \infty$. This completes the proof.

Next, we prove Theorem 3.1.1.

We prove that a priori assumption (3.3) is effective for given δ in Theorem 3.1.1. Indeed, label

$$S := \{H > 0 : \quad d_P(k) < \delta, \quad k \in [0, H]\}.$$

In addition, through (3.3) and the function $d_P(t)$ continuity, one can conclude that $S \neq \emptyset$. We are now ready to deduce that $\sup S = \infty$. Suppose that $\overline{\overline{H}} := \sup S < \infty$, then we get $d_P(\overline{\overline{H}}) < \delta$. On the other hand, applying Lemma 3.1.2 yields that $d_C(k) \leq C_2 e^{-\kappa k}$ for $k \in [0, \overline{\overline{H}})$. So, by defining $d_C(k)$, one can obtain that for $i, j \in \{1, ..., N\}$

$$\begin{array}{rcl}|p_i(\overline{\overline{H}}) - p_j(\overline{\overline{H}})| & \leq & |p_i(0) - p_j(0)| + \int_0^{\overline{\overline{H}}} |c_j(s) - c_i(s)| ds \\ & \leq & d_P(0) + \int_0^{\overline{\overline{H}}} |d_C(s)| ds \\ & \leq & d_P(0) + \int_0^{\overline{\overline{H}}} C_2 e^{-\kappa s} ds \\ & < & d_P(0) + \frac{C_2}{\kappa}. \end{array} \tag{3.7}$$

It can deduce $d_P(\overline{\overline{H}}) < d_P(0) + \frac{C_2}{\kappa}$. And we have

$$\delta = d_P(\overline{\overline{H}}) < d_P(0) + \frac{C_2}{\kappa} < \delta,$$

which conflicts. Thus, the priori assumption (3.3) is valid for δ. Combining with Lemma 3.1.2, we finish the proof of Theorem 3.1.1.

3.2. Future work

In practical application, we hope that agents can form formation, so we establish a distributed time-delay flocking model with controller. The driving force term is taken into account, precisely, a suitable F is introduced into the system (1.5), so that all the agents converge to flocking and achieve the prescribed pattern motion. The modified Cucker-Smale is given as follows:

$$\begin{cases} \dot{p}_i = c_j, \\ \dot{v}_i = \frac{1}{Nh(k)} \sum\limits_{m=1, m\neq i}^{N} \int_{k-T(k)}^{k} \beta(k-s)\Phi_{m,i}(k)(c_m(s) - c_j(k))\mathrm{d}s + F(p_i), \quad i = 1, 2, \ldots, N. \end{cases} \tag{3.8}$$

Inspired by the work of [36], the function is taken as

$$F(p_i(k)) = \gamma(\sin(\langle p-a, c\rangle) - \langle p-a, w\rangle)(c\cos(\langle p-a, v\rangle) - w), \tag{3.9}$$

where $a = \frac{1}{N}\sum_{i=1}^{N} p_i(k)$, v, w are two given vectors, γ is a positive force strength measured constant.

For the sake of brevity, we only list a framework model. Some simulations for special experiments are conducted in the next section. The range of control parameters for formation of flock, how to select the control design parameters effectively, and strict theoretical analysis will be our future study.

4. Numerical simulations

In this section, we provide several numerical simulations, which confirm the delay can affect the dynamic position and velocity of system (4.1).

$$\begin{cases} \dot{p}_j = c_j, \\ \dot{c}_j = \frac{1}{20h(k)} \sum_{m=1, m\neq j}^{20} \int_{k-T(k)}^{k} \phi_{m,i}(k)(c_m(s) - c_j(k))\mathrm{d}s, \quad j = 1, 2, \ldots, 20, \end{cases} \tag{4.1}$$

with the initial functions as below

$$p_j(s) =: \Phi_j(s), \quad c_j(s) =: \psi_j(s), \quad j = 1, \cdots, N, \quad s \in [-T_0, 0],$$

where $\phi_{m,i}(k) = \phi(p_m(k), p_i(k))$, $(\Phi_j, \psi_j) \in \mathcal{S}^2 = \mathcal{S} \times \mathcal{S}$ and $\mathcal{S} := S([-T, 0], \mathbb{R}^d)$ is the Banach space of all continuous functions. Moreover, $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ requires $\phi(k) \leq 1$ for each $k \geq 0$. $\mathbb{R}^+$ denote the set of positive real number.

The effectiveness and validity of the analytical results are demonstrated by the following examples.

Example 4.1 For system (4.1), we choose

$$\phi(r) = \frac{1}{(1+r^2)^{0.2}}, \quad \Phi_i(s) = (0.01, i-1), \quad \psi_i(s) = (0.01, i-1), i = 1, \cdots, 20, s \in [-T_0, 0]. \tag{4.2}$$

Taking $\delta = 70$, $C_1 = 0.84$, $T(k) \equiv T_0 = 0.04$, by some simple calculations, it is easy to see that all conditions of Theorem 3.1.1 are satisfied. Therefore, system (4.1) under condition (4.2) will asymptotically converge to a flock. This conclusion can be verified by the following numerical simulations in Figure 1.

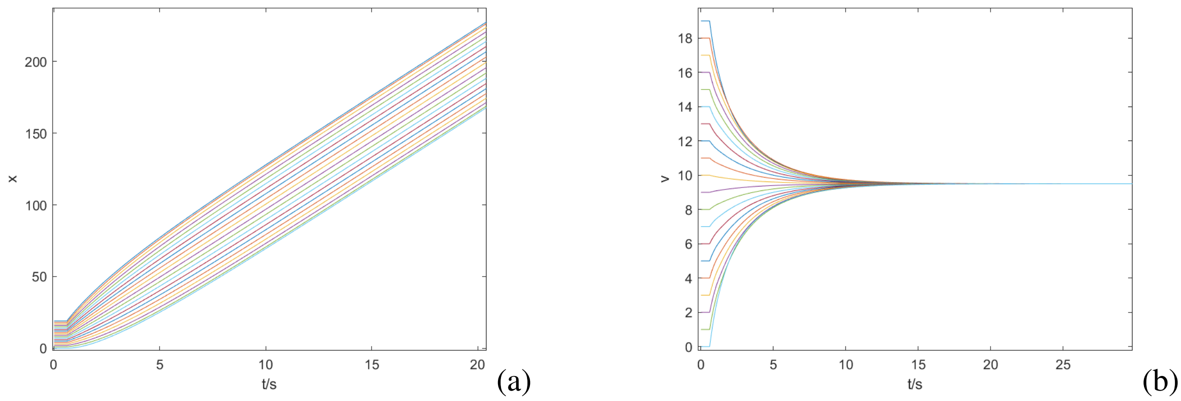


Figure 1. N=20, $T = 0.04$, the system (4.1) asymptotically converges to a flock.

Example 4.2 With the purpose of demonstrating the dynamic process of agents' flock convergence, for system (4.1), we choose

$$\phi(r) = \frac{1}{(1+r^2)^{0.4}}, \Phi_i(s) = (0.01, 2i-1), \psi_i(s) = (0.01, 2i-1), i = 1, \cdots, 20, s \in [-T_0, 0]. \tag{4.3}$$

Taking $\delta = 70$, $C_1 = 0.84$ and $T(k) \equiv T_0 = 0.01$, by some simple calculations, it is easy to see that all conditions of Theorem 3.1.1 are satisfied. Figure 2 shows the process of agents' flock convergence, which demonstrates that the dynamic graph of agents connected and changing until system (4.1) under condition (4.3) flocking achieved.

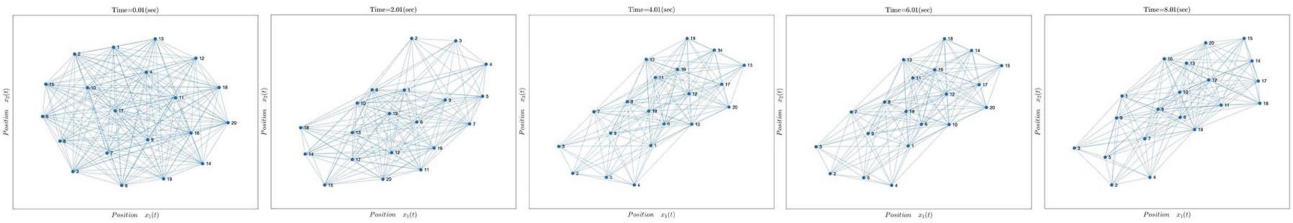


Figure 2. The dynamic process of agents' flock convergence.

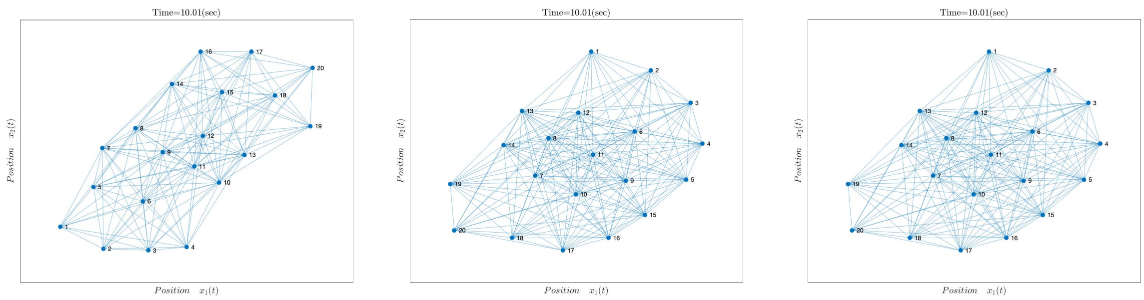
Example 4.3 Aiming at demonstrating that agents starting at different initial position and initial velocity can converging to varieties of different stable structure eventually. For system (4.1), we take $\delta = 70$, $C_1 = 0.84$ and $T(k) \equiv T_0 = 0.01$. It is easy to see that the below three situations satisfy Theorem 3.1.1, and therefore, the system (4.1) can work normally.

(i) $\phi(r) = e^{-r^{0.8}}$, $\Phi_i(s) = (0.01, i-1), \psi_i(s) = (0.01, i-1), i = 1, \cdots, 20, s \in [-T_0, 0]$. The simulation results are shown at (a) in Figure 3

(ii) $\phi(r) = (\sin r^2)^{0.8}$, $\Phi_i(s) = (0.01, 2i-1), \psi_i(s) = (0.01, 2i-1), i = 1, \cdots, 20, s \in [-T_0, 0]$. The simulation results are shown at (b) in Figure 3.

(iii) $\phi(r) = (\cos r^2)^{0.8}$, $\Phi_i(s) = (0.01, 3i-1), \psi_i(s) = (0.01, 3i-1), i = 1, \cdots, 20, s \in [-T_0, 0]$. The simulation results are shown at (c) in Figure 3.

Figure 3 displays a variety of different stable formation of 20 agents which start at random initial positions with different velocities.



(a) (b) (c)

Figure 3. Agents start at different position with different velocity, shape various different stable formation at last.

Example 4.4 For system (4.1), we choose

$$\Phi(r) = e^{-r^{0.6}}, \Phi_i(s) = (0.01, i-1), \psi_i(s) = (0.01, i-1), i = 1, \cdots, 20, s \in [-T_0, 0]. \tag{4.4}$$

Taking $T(k) \equiv T_0 = 3.0$, by some simple calculations, then the condition of Theorem 3.1.1 fails, which is perfectly verified in Figure 4.

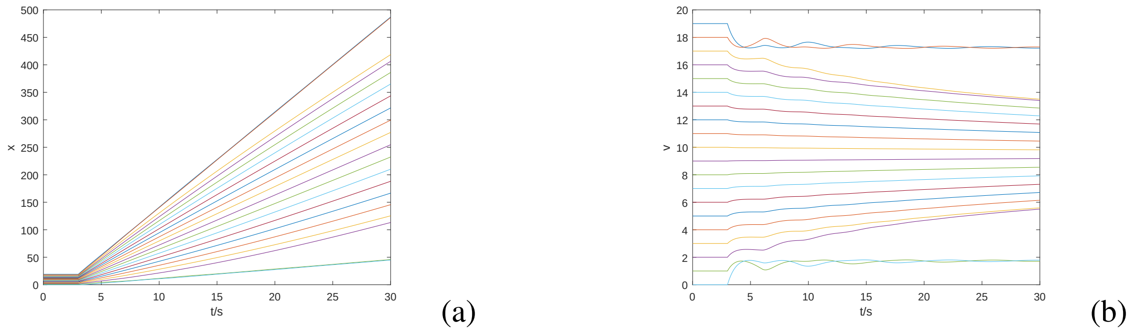


Figure 4. N=20, $T = 1.0$, system (4.1) with (4.4) cannot asymptotically converges to a flock.

Example 4.5 Consider the system (4.1), where $\phi(r) = \frac{1}{(1+r^2)^{\frac{1}{6}}}$, and take $\gamma = 1$ as the communication rate function and $T(k) \equiv T_0 = 0.8$. Besides, we let $p_i(k)$ and $c_i(k)$ $(i = 1, 2, \ldots, N)$ for $k \in [-T_0, 0]$ generated randomly and diverse in the region $[0, 10] \times [0, 1]$. Then our simulation verifies that the solution of the system (4.1) can converge to a flock with the prescribed motion pattern, which is shown in Figure 5.

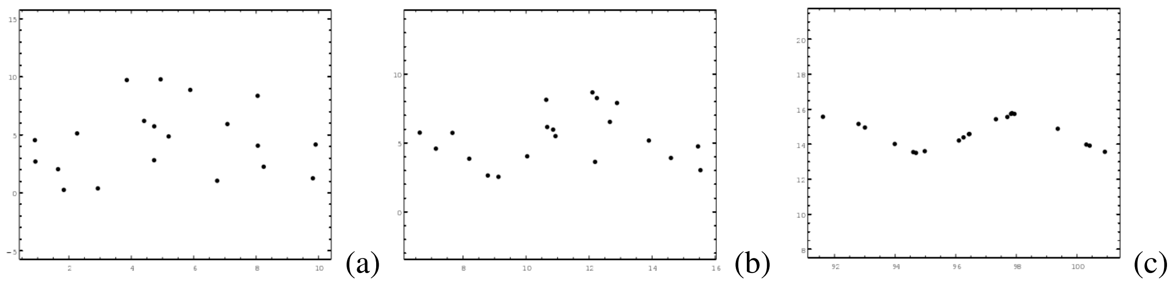


Figure 5. The position distribution of population at t = 0 s, t = 20 s and t = 100 s, respectively. And the values of each parameter are given as $w = (0, 1)$ and $v = (1, 0)$.

5. Conclusions

It is practical to understand how autonomous agents organize orderly movements based on finite information about the environment and monotonous rules. By utilizing dissipative differential inequality with a continuity argument, abundant conditions are the key insurance to the existence of flocking for the Cucker-Smale system with distributed delays from a measurement perspective. Several numerical simulations, it is a confirmation of that distributing delays can affect the flocking behavior. Meanwhile, the driving force term is added to the delay collective system to realize the specified pattern motion through numerical experiments.

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Conflict of interest

The authors declare that they have no conflict of interest.

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