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Research article

Threshold behaviour of a stochastic SIRS Lévy jump model with saturated incidence and vaccination

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Abstract: A stochastic SIRS system with Lévy process is formulated in this paper, and the model incorporates the saturated incidence and vaccination strategies. Due to the introduction of Lévy jump, the jump stochastic integral process is a discontinuous martingale. Then the Kunita's inequality is used to estimate the asymptotic pathwise of the solution for the proposed model, instead of Burkholder-Davis-Gundy inequality which is suitable for continuous martingales. The basic reproduction number R_0^s of the system is also derived, and the sufficient conditions are provided for the persistence and extinction of SIRS disease. In addition, the numerical simulations are carried out to illustrate the theoretical results. Theoretical and numerical results both show that Lévy process can suppress the outbreak of the disease.

Keywords: stochastic SIRS model; Lévy process; Kunita's inequality; extinction

1. Introduction

The method of mathematical modelling has been an important way to study the spreading and controlling of infectious disease. Since the pioneer work of Kermack and McKendrick [1], many deterministic mathematical models have been proposed to help us understand the spreading and controlling of infectious disease [2–5]. In [6], Lahrouz et al. put forward and analyzed a deterministic SIRS model with general incidence and vaccination. The authors derived the basic reproduction number and provided sufficient conditions for the persistence and extinction of disease. However, the model does not consider the environmental fluctuations. Many researchers have demonstrated that environmental fluctuation has largely affected the spreading of infectious disease [7, 8]. For human disease, due to the nondeterminacy of person-to-person contacts, the growth and spreading of infectious disease are inherently random.

There are several ways to model the influence of environmental fluctuations by stochastic differential equations [9, 10]. One of the most important way is to consider system perturbation which arises from the approach in [11]. On the basis of the work of Lahrouz et al. [6], the SIRS model incorporated by system perturbation can be described by

$$\begin{cases} \mathrm{d}S(t) &= \Big((1-\nu)\Lambda - \mu S - \frac{\beta S(t)I(t)}{1+aI(t)} + \delta R(t)\Big)\mathrm{d}t + S(t^-)\mathrm{d}Z_1(t), \\ \mathrm{d}I(t) &= \Big(\frac{\beta S(t)I(t)}{1+aI(t)} - (\mu + c + \xi)I(t)\Big)\mathrm{d}t + I(t^-)\mathrm{d}Z_2(t), \\ \mathrm{d}R(t) &= \Big(\nu\Lambda - (\mu+\delta)R(t) + \xi I(t)\Big)\mathrm{d}t + R(t^-)\mathrm{d}Z_3(t), \end{cases} \tag{1.1}$$

where the numbers of the susceptible, infectious and recovered individuals at time t are represented by $S(t)$, $I(t)$ and $R(t)$, respectively. $S(t^-)$, $I(t^-)$ and $R(t^-)$ are the left limits of $S(t)$, $I(t)$ and $R(t)$, respectively. $Z(t) = (Z_1(t), Z_2(t), Z_2(t))$ is a 3-dimensional stochastic process which models the random perturbation of the system. The parameters are all assumed to be positive. The biological meanings of the parameters are summarised in the following Table 1.

Table 1. The biological meanings of each parameter in system (1).

Notation	Biological meanings
ν	The proportion of population that is vaccinated
Λ	The population influx into the susceptible component
μ	The death rates of S, I, and R
β	The transmission coefficient between S and I
δ	The rate of the recovered individuals losing immunity
c	The death rate due to the disease
ξ	The recovery rate of I
a	The parameter measuring the inhibitory or psychological effect

When $Z_i(t) = 0$, system (1.1) is reduced into a deterministic model irrespective of the effects of environmental fluctuations. In fact, the spreading of disease is inevitably disturbed by environmental fluctuations. Usually, white noise [12, 13] is used to model the impact of environmental noise. However, when there exist large occasionally environmental shocks, such as earthquakes and floods, a stochastic model only with white noise cannot explain these discontinuous perturbations. Since Bao et al. [14] firstly proposed that Lévy process should be suitable to describe large occasionally fluctuations, some researchers use non-Gaussian Lévy jump noise to model these discontinuous phenomena [15,16]. For the sake of modelling the influence of environmental fluctuations, the form of $Z_i(t)$ is represented by

$$Z_i(t) = \sigma_i B_i(t) + \int_0^t \int_{\mathbb{Y}} \eta_i(z)\widetilde{N}(\mathrm{d}s, \mathrm{d}z), \quad i = 1, 2, 3, \tag{1.2}$$

and $Z_i(t)$ is a Lévy process and has the following Lévy-Khintchine representation

$$\mathbb{E}[e^{iu_1 Z_1(t) + iu_2 Z_2(t) + iu_3 Z_3(t)}] = \exp(-\frac{t}{2}(u, Au) + t\int_{\mathbb{R}\backslash 0}(e^{i(u,\eta(z))} - i(u, \eta(z)) - 1)\nu(\mathrm{d}z)),$$

where $u = (u_1, u_2, u_3) \in \mathbb{R}^3$ and A is a 3×3 diagonal matrix and its diagonal elements are $\sqrt{\sigma_i}$.

In (1.2), defined on a complete probability space $(\Omega, \mathcal{F}, P)$, $B_i(t)$ $(i = 1, 2, 3)$ are standard Brownian motions are independent with $B_i(0) = 0$. The intensities of the white noise is denoted by σ_i $(i = 1, 2, 3)$. $\widetilde{N}(\mathrm{d}t, \mathrm{d}z) := N(\mathrm{d}t, \mathrm{d}z) - \nu(\mathrm{d}z)\mathrm{d}t$ represents the compensated Poisson random measure, where $N(\mathrm{d}t, \mathrm{d}z)$ is the Poisson random measure and its characteristic measure is $\nu(\mathrm{d}z)$ which is finite Lévy measure. The Lévy measure satisfies $\int_{\mathbb{Y}} \min(|\eta_i(z)|^2, 1)\nu(\mathrm{d}z) < \infty \quad (i = 1, 2, 3)$, see Theorem 1.2.14 in [16]. The effects of jumps is represented by $\eta_i : \mathbb{Y} \to \mathbb{R}$ $(i = 1, 2, 3)$ which is supposed to be continuously differentiable and bounded.

On the basis of the above discussion, the model (1.1) incorporated by linear system perturbation can be described by

$$\begin{cases} \mathrm{d}S(t) = & \Big((1-\nu)\Lambda - \mu S - \frac{\beta S(t)I(t)}{1+aI(t)} + \delta R(t)\Big)\mathrm{d}t + \sigma_1 S(t)\mathrm{d}B_1(t) + \int_{\mathbb{Y}} \eta_1(z)S(t^-)\widetilde{N}(\mathrm{d}t, \mathrm{d}z), \\ \mathrm{d}I(t) = & \Big(\frac{\beta S(t)I(t)}{1+aI(t)} - (\mu + c + \xi)I(t)\Big)\mathrm{d}t + \sigma_2 I(t)\mathrm{d}B_2(t) + \int_{\mathbb{Y}} \eta_2(z)I(t^-)\widetilde{N}(\mathrm{d}t, \mathrm{d}z), \\ \mathrm{d}R(t) = & \Big(\nu\Lambda - (\mu + \delta)R(t) + \xi I(t)\Big)\mathrm{d}t + \sigma_3 R(t)\mathrm{d}B_3(t) + \int_{\mathbb{Y}} \eta_3(z)R(t^-)\widetilde{N}(\mathrm{d}t, \mathrm{d}z). \end{cases} \tag{1.3}$$

The spreading and extinction of infectious disease are two important and interesting topics in the control of infectious disease owing to their theoretical and practical meanings. The aim of the paper is to study the extinction and persistence of stochastic model (1.3). In order to investigate the extinction and persistence of stochastic model (1.3), it is required to estimate the solution of stochastic system (1.3). Usually, for continuous martingales, Zhou and Zhang [17] have used Burkholder-Davis-Gundy (BDG) inequality to prove the asymptotic pathwise estimation (see Lemmas 2.1–2.2 in [17]) of the solution. However, in this paper, due to the introduction of Lévy jump, the jump stochastic integral process is a discontinuous martingale. Therefore, the Kunita's inequality which is suitable for discontinuous martingales is used to estimate the asymptotic pathwise of the solution for the proposed model, instead of BDG inequality.

This paper is organized as follow: In Section 2, the asymptotic pathwise of the solution is estimated for stochastic model (1.3). In Section 3, sufficient conditions are provided for persistence and extinction of stochastic model (1.3). In Section 4, the paper ends with some discussions and numerical simulations.

2. Existence and uniqueness of the positive solution

For the jump diffusion coefficients of stochastic model (1.3), we suppose that

($\mathbf{H_1}$) For each $n > 0$, there exists a constant $L_n > 0$ such that $\int_{\mathbb{Y}} |H_i(x, z) - H_i(y, z)|^2 \nu(\mathrm{d}z) \le L_n|x - y|^2$ $(i = 1, 2, 3)$, where $H_1(x, z) = \eta_1(z)S(t^-)$, $H_2(x, z) = \eta_2(z)I(t^-)$, $H_3(x, z) = \eta_3(z)R(t^-)$ with $|x| \vee |y| \le n$;

($\mathbf{H_2}$) $\int_{\mathbb{Y}} |\eta_i(z) - \ln(1 + \eta_i(z))|\nu(\mathrm{d}z) < \infty$ for $\eta_i(z) > -1$ $(i = 1, 2, 3)$.

Theorem 2.1. Assume ($\mathbf{H_1}$) and ($\mathbf{H_2}$) hold. Then the stochastic system (1.3) has a unique global solution $(S(t), I(t), R(t)) \in \mathbb{R}^3_+$ for all $t \ge 0$ and any given initial value $(S(0), I(0), R(0)) \in \mathbb{R}^3_+$ a.s..

Proof. According to the assumption ($\mathbf{H_1}$), the coefficients of the model (1.3) satisfy local Lipschitz conditions. By Theorem 2.1 of [14], for any $t \in [0, \tau_e)$, there is a unique local solution $(S(t), I(t), R(t)) \in \mathbb{R}^3_+$ and τ_e is the explosion time. To illustrate the solution is global, it is necessary to show that $\tau_e = +\infty$ a.s.. Define the stopping time as

$$\tau_m = \inf\Big\{t \in [0, \tau_e) : \min\{S(t), I(t), R(t)\} \le \frac{1}{m} \quad or \quad \max\{S(t), I(t), R(t)\} \ge m\Big\}.$$

Set $\tau_\infty = \lim_{m\to\infty} \tau_m \leq \tau_e$ a.s.. It is sufficient to prove $\tau_\infty = +\infty$ a.s.. If it does not hold, then there exist constants $T > 0$ and $\epsilon \in (0, 1)$ such that

$$P(\tau_\infty \leq T) \geq \epsilon.$$

Consequently, for all $m \geq m_1$, there is an integer m_1 such that $P(\tau_m \leq T) \geq \epsilon$. Define

$$V(S(t), I(t), R(t)) = (S - b - b\ln\frac{S}{b}) + (I - 1 - \ln I) + (R - 1 - \ln R),$$

where $0 < b < \frac{\mu+c}{\beta}$. By Itô's formula with Lévy jump process, we have

$$\begin{aligned}
\mathrm{d}V(S(t), I(t), R(t)) =& LV\mathrm{d}t + \sigma_1(S(t) - b)\mathrm{d}B_1(t) + \sigma_2(I(t) - 1)\mathrm{d}B_2(t)\\
&+ \sigma_3(R(t) - 1)\mathrm{d}B_3(t) + \int_{\mathbb{Y}} [\eta_1(z)S(t^-)\\
&- b\ln(1 + \eta_1(z)) + \eta_2(z)I(t^-) - \ln(1 + \eta_2(z))\\
&+ \eta_3(z)R(t^-) - \ln(1 + \eta_3(z))]\widetilde{N}(\mathrm{d}t, \mathrm{d}z),
\end{aligned}$$

where LV is the generating operator of stochastic system (1.3):

$$\begin{aligned}
LV =&(1 - \frac{b}{S})[(1 - \nu)\Lambda - \mu S - \frac{\beta SI}{1 + aI} + \delta R] + (1 - \frac{1}{I})[-(\mu + c + \xi)I + \frac{\beta SI}{1 + aI}]\\
&+ (1 - \frac{1}{R})[\nu\Lambda - (\mu + \delta)R + \xi I] + \frac{b}{2}\sigma_1^2 + \frac{1}{2}\sigma_2^2 + \frac{1}{2}\sigma_3^2\\
&+ \int_{\mathbb{Y}} [\eta_1(z) - b\ln(1 + \eta_1(z)) + \eta_2(z) - \ln(1 + \eta_2(z)) + \eta_3(z) - \ln(1 + \eta_3(z))]\nu(\mathrm{d}z)\\
=&\Lambda + \mu b + 2\mu + c + \xi + \delta - \mu S - \frac{b(1 - \nu)\Lambda}{S} - \frac{b\delta R}{S} - \frac{\beta S}{1 + aI} - \frac{\nu\Lambda}{R}\\
&- \frac{\xi I}{R} - (\mu + c - \frac{b\beta}{1 + aI})I + \frac{b}{2}\sigma_1^2 + \frac{1}{2}\sigma_2^2 + \frac{1}{2}\sigma_3^2\\
&+ \int_{\mathbb{Y}} [\eta_1(z) - b\ln(1 + \eta_1(z)) + \eta_2(z) - \ln(1 + \eta_2(z)) + \eta_3(z) - \ln(1 + \eta_3(z))]\nu(\mathrm{d}z).
\end{aligned}$$

Applying the inequality $\eta_i(z) - \ln(1 + \eta_i(z)) \geq 0$ for $\eta_i(z) > -1$ and assumption ($\mathbf{H_2}$), we obtain

$$\begin{aligned}
LV \leq&\Lambda + \mu b + 2\mu + c + \xi + \delta + \frac{b}{2}\sigma_1^2 + \frac{1}{2}\sigma_2^2 + \frac{1}{2}\sigma_3^2 + \int_{\mathbb{Y}} [\eta_1(z) - b\ln(1 + \eta_1(z))\\
&+ \eta_2(z) - \ln(1 + \eta_2(z)) + \eta_3(z) - \ln(1 + \eta_3(z))]\nu(\mathrm{d}z)\\
:=&K,
\end{aligned}$$

where K is a positive constant.

The remaining steps are similar to [8], so they are omitted. □

3. Asymptotic pathwise estimation of the stochastic solution

Firstly, we provide some preliminary results for the asymptotic pathwise estimation of the solution of stochastic system (1.3).

For continuous martingales, Zhou and Zhang [17] used BDG inequality to prove asymptotic pathwise estimation of the solution. However, in this paper, for discontinuous martingales, Kunita's inequality is needed to prove asymptotic pathwise estimation of the solution and it is different from the BDG inequality for continuous martingales used by Zhou and Zhang [17].

Let us first state the Kunita inequality (Theorem 4.4.23 of [16]). Consider the stochastic integral for jump process

$$Y(t) = \int_0^t \int_{\mathbb{Y}} H(s,z)(N(\mathrm{d}s,\mathrm{d}z) - \nu(\mathrm{d}z)\mathrm{d}s), \quad t \in \mathbb{R}_+ \tag{3.1}$$

of the predictable integrand $(H(s,z))_{(s,z)\in\mathbb{R}_+\times\mathbb{Y}}$.

Lemma 3.1. *Assume that $Y(t)$ is a semi-martingale represented by* (3.1) *[16]. Then there exists a constant $C_p > 0$ such that for any $p \geq 2$,*

$$E[\sup_{0<s\leq t}|Y(s)|^p] \leq C_p\{E[(\int_0^t\int_{\mathbb{Y}}|H(s,z)|^2\mathrm{d}s\nu(\mathrm{d}z))^{p/2}] + E[\int_0^t\int_{\mathbb{Y}}|H(s,z)|^p\mathrm{d}s\nu(\mathrm{d}z)]\}$$

For convenience, denote

$$a \vee b = \max\{a,b\}; \quad a \wedge b = \min\{a,b\};$$
$$\sigma^2 = \sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2; \quad \lambda = \int_{\mathbb{Y}}[(1+\eta_1(z)\vee\eta_2(z)\vee\eta_3(z))^p - 1]\nu(\mathrm{d}z).$$

Now, let us state the results on the asymptotic pathwise estimation.

Theorem 3.1. *Suppose* $(\mathbf{H_1})$–$(\mathbf{H_2})$ *hold. If $\mu - \frac{p-1}{2}\sigma^2 - \frac{1}{p}\lambda > 0$ for some $p > 1$, then*

$$\lim_{t\to\infty}\frac{S(t)+I(t)+R(t)}{t} = 0 \quad a.s..$$

Moreover,

$$\lim_{t\to\infty}\frac{S(t)}{t} = 0, \quad \lim_{t\to\infty}\frac{I(t)}{t} = 0, \quad \lim_{t\to\infty}\frac{R(t)}{t} = 0, \quad a.s..$$

Proof. The proof is enlightened by [18]. Let $X(t) = S(t) + I(t) + R(t)$. Define

$$V(X) = (1+X)^p,$$

where $p > 0$ is a constant to be determined later. It follows that

$$\begin{aligned} \mathrm{d}V(X(t)) =& LV\mathrm{d}t + p(1+X(t))^{p-1}[\sigma_1 S(t)\mathrm{d}B_1(t) + \sigma_2 I(t)\mathrm{d}B_2(t) + \sigma_3 R(t)\mathrm{d}B_3(t)] \\ &+ \int_{\mathbb{Y}}(1+X(t^-)+\eta_1(z)S(t^-)+\eta_2(z)I(t^-)+\eta_3(z)R(t^-))^p - (1+X(t^-))^p\widetilde{N}(\mathrm{d}t,\mathrm{d}z), \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} LV(X) \leq& p(1+X)^{p-1}[\Lambda - \mu S - (\mu+c)I - \mu R] + \frac{p(p-1)}{2}(1+X)^{p-2}(\sigma_1^2S^2 + \sigma_2^2I^2 + \sigma_3^2R^2) \\ &+ \int_{\mathbb{Y}}(1+X(t))^p[(1+\eta_1(z)\vee\eta_2(z)\vee\eta_3(z))^p - 1]\nu(\mathrm{d}z) \\ \leq& p(1+X)^{p-2}\{-[\mu - \frac{p-1}{2}\sigma^2 - \frac{1}{p}\lambda]X^2 + (\Lambda - \mu + \frac{2}{p}\lambda)X + \Lambda + \frac{\lambda}{p}\}. \end{aligned}$$

Set $\rho=\mu-\frac{p-1}{2}\sigma^2-\frac{1}{p}\lambda$. It follows from $\rho=\mu-\frac{p-1}{2}\sigma^2-\frac{1}{p}\lambda>0$ that

$$LV(X)\leq p(1+X)^{p-2}\{-\rho X^2+(\Lambda+\frac{2}{p}\lambda-\mu)X+\Lambda+\frac{\lambda}{p}\}. \tag{3.3}$$

For any $\kappa\in\mathbb{R}$, direct calculation yields that

$$\begin{aligned}\mathrm{d}e^{\kappa t}V(X(t))=&L[e^{\kappa t}V(X(t))]\mathrm{d}t+e^{\kappa t}p(1+X(t))^{p-1}[\sigma_1S(t)\mathrm{d}B_1(t)+\sigma_2I(t)\mathrm{d}B_2(t)\\&+\sigma_3R(t)\mathrm{d}B_3(t)]+e^{\kappa t}\int_{\mathbb{Y}}\{(1+X(t^-)+\eta_1(z)S(t^-)+\eta_2(z)I(t^-)\\&+\eta_3(z)R(t^-))^p-(1+X(t^-))^p\}\widetilde{N}(\mathrm{d}t,\mathrm{d}z).\end{aligned} \tag{3.4}$$

Integrating from 0 to t, we have

$$\begin{aligned}e^{\kappa t}(1+X(t))^p=&(1+X(0))^p+\int_0^t[\kappa e^{\kappa s}(1+X(s))^p+e^{\kappa s}LV(X(s))]\mathrm{d}s+\int_0^t e^{\kappa s}p(1+X(s))^{p-1}\\&[\sigma_1S(s)\mathrm{d}B_1(s)+\sigma_2I(s)\mathrm{d}B_2(s)+\sigma_3R(s)\mathrm{d}B_3(s)]+\int_0^t e^{\kappa s}\int_{\mathbb{Y}}\{(1+X(s^-)\\&+\eta_1(z)S(s^-)+\eta_2(z)I(s^-)+\eta_3(z)R(s^-))^p-(1+X(s^-))^p\}\widetilde{N}(\mathrm{d}s,\mathrm{d}z).\end{aligned} \tag{3.5}$$

Taking expectations of (3.5) leads to

$$e^{\kappa t}E[(1+X(t))^p]=(1+X(0))^p+E\{\int_0^t[\kappa e^{\kappa s}(1+X(s))^p+e^{\kappa s}LV(X(s))]\mathrm{d}s\}. \tag{3.6}$$

According to the inequality (3.3), for any $\kappa<\rho p$, we have

$$\begin{aligned}&\kappa e^{\kappa t}(1+X(t))^p+e^{\kappa t}LV(X(t))\\\leq&\kappa e^{\kappa t}(1+X(t))^p+pe^{\kappa t}(1+X(t))^{p-2}[-\rho X^2(t)+(\Lambda+\frac{2}{p}\lambda-\mu)X(t)+\Lambda+\frac{\lambda}{p}]\\=&pe^{\kappa t}(1+X(t))^{p-2}[-(\rho-\frac{\kappa}{p})X^2(t)+(\Lambda+\frac{2}{p}\lambda-\mu+\frac{2\kappa}{p})X(t)+\Lambda+\frac{\lambda+\kappa}{p}]\\\leq&pe^{\kappa t}H,\end{aligned} \tag{3.7}$$

where

$$0<H:=1+\sup_{X\in\mathbb{R}_+}(1+X)^{p-2}[-(\rho-\frac{\kappa}{p})X^2+(\Lambda+\frac{2}{p}\lambda-\mu+\frac{2\kappa}{p})X+\Lambda+\frac{\lambda+\kappa}{p}]<\infty.$$

Therefore, for any $\kappa\in(0,\rho p)$, according to (3.6) and (3.7), we obtain

$$\begin{aligned}E[e^{\kappa t}(1+X(t))^p]\leq&(1+X(0))^p+pH\int_0^t e^{\kappa s}\mathrm{d}s\\=&(1+X(0))^p+\frac{pH}{\kappa}e^{\kappa t}.\end{aligned}$$

Consequently,

$$\limsup_{t\to\infty}E[(1+X(t))^p]\leq\frac{pH}{\kappa},$$

which indicates that we have $M > 0$ such that

$$E[(1 + X(t))^p] \leq M, \quad t \geq 0. \tag{3.8}$$

For $k = 1, 2, \ldots$ and sufficiently small $\theta > 0$, by (3.2) and (3.3) that

$$\begin{aligned}(1 + X(t))^p - (1 + X(k\theta))^p \leq & p\int_{k\theta}^{t}(1 + X(s))^{p-2}[-\rho X^2(s) + (\Lambda + \frac{2}{p}\lambda - \mu)X(s) + \Lambda + \frac{\lambda}{p}]\mathrm{d}s \\ & + p\int_{k\theta}^{t}(1 + X(s))^{p-1}(\sigma_1 S(s)\mathrm{d}B_1(s) + \sigma_2 I(s)\mathrm{d}B_2(s) + \sigma_3 R(s)\mathrm{d}B_3(s)) \\ & + \int_{k\theta}^{t}\int_{\mathbb{Y}}[(1 + X(s^-) + \eta_1(z)S(s^-) + \eta_2(z)I(s^-) \\ & + \eta_3(z)R(s^-))^p - (1 + X(s^-))^p]\widetilde{N}(\mathrm{d}s, \mathrm{d}z), \quad t \geq k\theta.\end{aligned}$$

Direct calculation yields that

$$\begin{aligned}\sup_{k\theta\leq t\leq(k+1)\theta}(1 + X(t))^p \leq & (1 + X(k\theta))^p + p\sup_{k\theta\leq t\leq(k+1)\theta}|\int_{k\theta}^{t}(1 + X(s))^{p-2}[-\rho X^2(s) + (\Lambda + \frac{2}{p}\lambda - \mu)X(s) \\ & + \Lambda + \frac{\lambda}{p}]\mathrm{d}s| + p\sup_{k\theta\leq t\leq(k+1)\theta}|\int_{k\theta}^{t}(1 + X(s))^{p-1}(\sigma_1 S(s)\mathrm{d}B_1(s) + \sigma_2 I(s)\mathrm{d}B_2(s) \\ & + \sigma_3 R(s)\mathrm{d}B_3(s))\,| + \sup_{k\theta\leq t\leq(k+1)\theta}|\int_{k\theta}^{t}\int_{\mathbb{Y}}[(1 + X(s^-) + \eta_1(z)S(s^-) + \eta_2(z)I(s^-) \\ & + \eta_3(z)R(s^-))^p - (1 + X(s^-))^p]\widetilde{N}(\mathrm{d}s, \mathrm{d}z)\,|\,.\end{aligned}$$

Taking expectations of the above inequality, it follows that

$$\begin{aligned}E[\sup_{k\theta\leq t\leq(k+1)\theta}(1 + X(t))^p] &\leq E[(1 + X(k\theta))^p] + I_1 + I_2 + I_3 \\ &\leq M + I_1 + I_2 + I_3,\end{aligned}$$

where

$$\begin{aligned}I_1 &= pE\{\sup_{k\theta\leq t\leq(k+1)\theta}|\int_{k\theta}^{t}(1 + X)^{p-2}[-\rho X^2 + (\Lambda + \frac{2}{p}\lambda - \mu)X + \Lambda + \frac{\lambda}{p}]\mathrm{d}s|\}; \\ I_2 &= pE\{\sup_{k\theta\leq t\leq(k+1)\theta}|\int_{k\theta}^{t}(1 + X(s))^{p-1}(\sigma_1 S(s)\mathrm{d}B_1(s) + \sigma_2 I(s)\mathrm{d}B_2(s) + \sigma_3 R(s)\mathrm{d}B_3(s))\,|\,\}; \\ I_3 &= E\{\sup_{k\theta\leq t\leq(k+1)\theta}|\int_{k\theta}^{t}\int_{\mathbb{Y}}[(1 + X(s^-) + \eta_1(z)S(s^-) + \eta_2(z)I(s^-) + \eta_3(z)R(s^-))^p - (1 + X(s^-))^p]\widetilde{N}(\mathrm{d}s, \mathrm{d}z)\,|\,\}.\end{aligned}$$

Now let us estimate I_1, I_2 and I_3. It is easy to see that there exists $c_{11} > 0$ and $c_{12} > 0$ such that

$$\begin{aligned}&-\rho(1 + X)^{p-2}X^2 \leq 0; \\ &(\Lambda + \frac{2}{p}\lambda - \mu)(1 + X)^{p-2}X \leq c_{11}(1 + X)^p; \\ &(\Lambda + \frac{\lambda}{p})(1 + X)^{p-2} \leq c_{12}(1 + X)^p,\end{aligned}$$

then we have

$$
\begin{aligned}
I_1 &\leq c_1 E\{\sup_{k\theta \leq t \leq (k+1)\theta} | \int_{k\theta}^{t} (1+X)^p \mathrm{d}s|\} \\
&\leq c_1 E\{\int_{k\theta}^{(k+1)\theta} (1+X)^p \mathrm{d}s\} \\
&\leq c_1 \theta E\{\sup_{k\theta \leq t \leq (k+1)\theta} (1+X)^p\},
\end{aligned}
$$

where $c_1 = p(c_{11} + c_{12})$ is a positive constant. By using BDG inequality, it follows that

$$
\begin{aligned}
I_2 &\leq \sqrt{32} p E\{(\int_{k\theta}^{(k+1)\theta} (1+X)^{2(p-1)} (\sigma_1^2 S^2 + \sigma_2^2 I^2 + \sigma_3^2 R^2) \mathrm{d}s)^{\frac{1}{2}}\} \\
&\leq c_2 \theta^{\frac{1}{2}} p\sigma E\{(\sup_{k\theta \leq t \leq (k+1)\theta} (1+X)^{2p})^{\frac{1}{2}}\} \\
&= c_2 \theta^{\frac{1}{2}} p\sigma E\{\sup_{k\theta \leq t \leq (k+1)\theta} (1+X)^p\},
\end{aligned}
$$

where c_2 is positive constant. Since $\widetilde{N}(\mathrm{d}t, \mathrm{d}z) := N(\mathrm{d}t, \mathrm{d}z) - \nu(\mathrm{d}z)\mathrm{d}t$, then,

$$
\begin{aligned}
I_3 \leq & E\{ | \int_{k\theta}^{(k+1)\theta} \int_{\mathbb{Y}} [(1 + X(s^-) + \eta_1(z)S(s^-) + \eta_2(z)I(s^-) + \eta_3(z)R(s^-))^p - (1 + X(s^-))^p] N(\mathrm{d}s, \mathrm{d}z) | \} \\
& + E\{ | \int_{k\theta}^{(k+1)\theta} \int_{\mathbb{Y}} [(1 + X(s^-) + \eta_1(z)S(s^-) + \eta_2(z)I(s^-) + \eta_3(z)R(s^-))^p - (1 + X(s^-))^p] \mathrm{d}s\nu(\mathrm{d}z) | \} \\
= & 2E\{ | \int_{k\theta}^{(k+1)\theta} \int_{\mathbb{Y}} [(1 + X(s^-) + \eta_1(z)S(s^-) + \eta_2(z)I(s^-) + \eta_3(z)R(s^-))^p - (1 + X(s^-))^p] \mathrm{d}s\nu(\mathrm{d}z) | \} \\
\leq & 2\theta E[\sup_{k\theta \leq t \leq (k+1)\theta} (1+X)^p] \int_{\mathbb{Y}} | (1 + \eta_1(z) \vee \eta_2(z) \vee \eta_3(z))^p - 1 | \nu(\mathrm{d}z).
\end{aligned}
$$

Therefore, it follows that

$$
\begin{aligned}
E[\sup_{k\theta \leq t \leq (k+1)\theta} (1 + X(t))^p] \leq & E[(1 + X(k\theta))^p] + \{c_1\theta + c_2\theta^{\frac{1}{2}} p\sigma + 2\theta \int_{\mathbb{Y}} | (1 + \eta_1(z) \vee \eta_2(z) \vee \eta_3(z))^p \\
& - 1 | \nu(\mathrm{d}z)\} E[\sup_{k\theta \leq t \leq (k+1)\delta} (1 + X(t))^p].
\end{aligned} \tag{3.9}
$$

Moreover, it is easy to see that we can choose $\theta > 0$ small enough such that

$$
c_1\theta + c_2\theta^{\frac{1}{2}} p\sigma + 2\theta \int_{\mathbb{Y}} | (1 + \eta_1(z) \vee \eta_2(z) \vee \eta_3(z))^p - 1 | \nu(\mathrm{d}z) < \frac{1}{2}.
$$

It follows from (3.8) and (3.9) that

$$
E[\sup_{k\theta \leq t \leq (k+1)\theta} (1 + X(t))^p] \leq 2E[(1 + X(k\theta))^p] \leq 2M. \tag{3.10}
$$

Set arbitrary $\epsilon > 0$. For all $k \geq 1$, applying Chebyshev's inequality yields

$$P\{\sup_{k\theta \leq t \leq (k+1)\theta} (1 + X(t))^p > (k\theta)^{1+\epsilon}\} \leq \frac{E[\sup_{k\theta \leq t \leq (k+1)\theta} (1 + X(t))^p]}{(k\theta)^{1+\epsilon}}$$
$$\leq \frac{2M}{(k\theta)^{1+\epsilon}}$$

By Borel-Cantelli Lemma ([19]),

$$\sup_{k\theta \leq t \leq (k+1)\theta} (1 + X(t))^p \leq (k\theta)^{1+\epsilon} \tag{3.11}$$

holds for all but finitely many k. Then, there exists $k_0(\omega)$ such that whenever $k \geq k_0$, then

$$\frac{\ln(1 + X(t))^p}{\ln t} \leq \frac{(1 + \epsilon_X)\ln k\theta}{\ln(k\theta)} = 1 + \epsilon, \quad \epsilon > 0, \quad k\theta \leq t \leq (k+1)\theta.$$

Therefore, we have

$$\limsup_{t\to\infty} \frac{\ln(1 + X(t))^p}{\ln t} \leq 1 + \epsilon, \quad a.s..$$

Letting $\epsilon \to 0$ yields

$$\limsup_{t\to\infty} \frac{\ln(1 + X(t))^p}{\ln t} \leq 1, \quad a.s..$$

For $p > 1$,

$$\limsup_{t\to\infty} \frac{\ln X(t)}{\ln t} \leq \limsup_{t\to\infty} \frac{\ln(1 + X(t))}{\ln t} \leq \frac{1}{p}, \quad a.s.,$$

i.e., for any $b \in (0, 1 - \frac{1}{p})$, we can find a finite random time $T(\omega)$ such that

$$\ln X(t) \leq (\frac{1}{p} + b)\ln t, \quad t \geq T(\omega).$$

It implies

$$\limsup_{t\to\infty} \frac{X(t)}{t} \leq \limsup_{t\to\infty} \frac{t^{\frac{1}{p}+b}}{t} = 0.$$

Thus, we get

$$\lim_{t\to\infty} \frac{X(t)}{t} = \lim_{t\to\infty} \frac{S(t) + I(t) + R(t)}{t} = 0. \tag{3.12}$$

The positivity of the solution and the equality (3.12) imply

$$\lim_{t\to\infty} \frac{S(t)}{t} = 0, \quad \lim_{t\to\infty} \frac{I(t)}{t} = 0, \quad \lim_{t\to\infty} \frac{R(t)}{t} = 0, \quad a.s..$$

□

In the following, Kunita's inequality is used to estimate the asymptotic pathwise of the jump stochastic integral process instead of BDG inequality.

Theorem 3.2. *Assume* ($\mathbf{H_1}$)–($\mathbf{H_2}$) *hold. If there exists some* $p > 2$ *such that* $\mu - \frac{p-1}{2}\sigma^2 - \frac{1}{p}\lambda > 0$, *then*

$$\lim_{t\to\infty}\frac{\int_0^t\int_{\mathbb{Y}}\eta_1(z)S(s^-)\widetilde{N}(\mathrm{d}s,\mathrm{d}z)}{t}=0,\quad \lim_{t\to\infty}\frac{\int_0^t\int_{\mathbb{Y}}\eta_2(z)I(s^-)\widetilde{N}(\mathrm{d}s,\mathrm{d}z)}{t}=0,$$
$$\lim_{t\to\infty}\frac{\int_0^t\int_{\mathbb{Y}}\eta_3(z)R(s^-)\widetilde{N}(\mathrm{d}s,\mathrm{d}z)}{t}=0,\quad a.s..$$

Proof. Denote

$$X_1(t)=\int_0^t\int_{\mathbb{Y}}\eta_1(z)S(s^-)\widetilde{N}(\mathrm{d}s,\mathrm{d}z),\quad X_2(t)=\int_0^t\int_{\mathbb{Y}}\eta_2(z)I(s^-)\widetilde{N}(\mathrm{d}s,\mathrm{d}z),$$
$$X_3(t)=\int_0^t\int_{\mathbb{Y}}\eta_3(z)R(s^-)\widetilde{N}(\mathrm{d}s,\mathrm{d}z).$$

By Kunita's inequality, there exists $C_p > 0$ for any $p \geq 2$ such that

$$\begin{aligned}E[\sup_{0\le s\le t}|X_1(s)|^p]\le&C_pE[(\int_0^t\int_{\mathbb{Y}}|\eta_1(z)S(s^-)|^2\nu(\mathrm{d}z)\mathrm{d}s)^{\frac{p}{2}}]+C_pE[\int_0^t\int_{\mathbb{Y}}|\eta_1(z)S(s^-)|^p\nu(\mathrm{d}z)\mathrm{d}s]\\=&C_p(\int_{\mathbb{Y}}\eta_1^2(z)\nu(\mathrm{d}z))^{\frac{p}{2}}E[(\int_0^t|S(s)|^2\mathrm{d}s)^{\frac{p}{2}}]+C_p(\int_{\mathbb{Y}}\eta_1^p(z)\nu(\mathrm{d}z))E[\int_0^t|S(s)|^p\mathrm{d}s]\\\le&C_pt^{\frac{p}{2}}(\int_{\mathbb{Y}}\eta_1^2(z)\nu(\mathrm{d}z))^{\frac{p}{2}}E[\sup_{0\le s\le t}|S(s)|^p]+C_ptM\int_{\mathbb{Y}}\eta_1^p(z)\nu(\mathrm{d}z),\end{aligned}$$

where the inequality (3.8) has been used in the last inequality. By the above inequality and (3.10), then

$$E[\sup_{k\theta\le t\le(k+1)\theta}|X_1(t)|^p]\le C_p2M((k+1)\theta)^{\frac{p}{2}}(\int_{\mathbb{Y}}\eta_1^2(z)\nu(\mathrm{d}z))^{\frac{p}{2}}+C_pM(k+1)\theta\int_{\mathbb{Y}}\eta_1^p(z)\nu(\mathrm{d}z).$$

Let $\epsilon > 0$ be arbitrary. Then, by Doob's martingale inequality,

$$\begin{aligned}&P\{\omega:\sup_{k\theta\le t\le(k+1)\theta}|X_1(t)|^p>(k\theta)^{1+\epsilon+\frac{p}{2}}\}\\&\le\frac{E[\sup_{k\theta\le t\le(k+1)\theta}|X_1(t)|^p]}{(k\theta)^{1+\epsilon+\frac{p}{2}}}\\&\le\frac{2MC_p((k+1)\theta)^{\frac{p}{2}}}{(k\theta)^{1+\epsilon+\frac{p}{2}}}(\int_{\mathbb{Y}}\eta_1^2(z)\nu(\mathrm{d}z))^{\frac{p}{2}}+\frac{MC_p(k+1)\theta}{(k\theta)^{1+\epsilon+\frac{p}{2}}}\int_{\mathbb{Y}}\eta_1^p(u)\nu(\mathrm{d}z).\end{aligned}$$

By Borel-Cantelli Lemma ([19]),

$$\sup_{k\theta\le t\le(k+1)\theta}|X_1(t)|^p\le(k\theta)^{1+\epsilon+\frac{p}{2}}\quad a.s.$$

holds for all but finitely many k. Therefore, there exists a positive $k_0(\omega)$ such that for all $k > k_0$, we have

$$\frac{\ln|X_1(t)|^p}{\ln t} \leq \frac{(1+\epsilon+\frac{p}{2})\ln k\theta}{\ln(k\theta)} = 1+\epsilon+\frac{p}{2}, \quad \epsilon > 0, \quad k\theta \leq t \leq (k+1)\theta.$$

Therefore, we obtain

$$\limsup_{t\to\infty} \frac{\ln|X_1|}{\ln t} \leq \frac{1}{2} + \frac{1+\epsilon}{p}.$$

Letting $\epsilon \to 0$, we have

$$\limsup_{t\to\infty} \frac{\ln|X_1(t)|}{\ln t} \leq \frac{1}{2} + \frac{1}{p}, \quad p > 2.$$

Similar to the proof of Theorem 3.1, it follows that

$$\limsup_{t\to\infty} \frac{X_1(t)}{t} \leq \limsup_{t\to\infty} \frac{t^{\frac{1}{2}+\frac{1}{p}}}{t} = 0.$$

Together with $\liminf\limits_{t\to\infty} \frac{|X_1(t)|}{t} \geq 0$, this yields

$$\lim_{t\to\infty} \frac{|X_1(t)|}{t} = 0, \quad a.s..$$

We also obtain

$$\lim_{t\to\infty} \frac{X_2(t)}{t} = 0, \quad \lim_{t\to\infty} \frac{X_3(t)}{t} = 0, \quad a.s..$$

□

Theorem 3.3. *Assume* ($\mathbf{H_1}$)–($\mathbf{H_2}$) *hold. If there exists some* $p > 1$ *such that* $\mu - \frac{p-1}{2}\sigma^2 - \frac{1}{p}\lambda > 0$, *then*

$$\lim_{t\to\infty} \frac{\int_0^t S(s)\mathrm{d}B_1(s)}{t} = 0, \quad \lim_{t\to\infty} \frac{\int_0^t I(s)\mathrm{d}B_2(s)}{t} = 0, \quad \lim_{t\to\infty} \frac{\int_0^t R(s)\mathrm{d}B_3(s)}{t} = 0, \quad a.s..$$

The proof is omitted here because it is similar to that of Lemma 2.2 in [18].

4. Threshold behaviour

When $Z_i(t) = 0$ ($i = 1, 2, 3$), the stochastic system (1.3) is reduced into a deterministic system investigated by Lahrouz et al. [6]. In [6], the basic reproduction number R_0 is derived represented by

$$R_0 = \frac{\beta\Lambda((1-\nu)\mu+\delta)}{\mu(\mu+\delta)(\mu+c+\xi)}.$$

The purpose of this paper is to provide sufficient conditions for the persistence and extinction of disease for stochastic model (1.3). Define

$$R_0^s = R_0 - \frac{1}{\mu + c + \xi}(\frac{\sigma_2^2}{2} + \int_{\mathbb{Y}} [\eta_2(z) - \ln(1 + \eta_2(z))]\nu(\mathrm{d}z)).$$

In the following, we will show that R_0^s completely determines the persistence and extinction of the disease. Throughout this section, always assume

$(\mathbf{H_3})$ $\int_{\mathbb{Y}} [\ln(1 + \eta_i(z))]^2 \nu(\mathrm{d}z) < \infty.$

For convenience, denote $\langle x(t) \rangle = \frac{1}{t}\int_0^t x(s)\mathrm{d}s$.

Theorem 4.1. *Suppose* $(\mathbf{H_1})$–$(\mathbf{H_3})$ *hold. If* $R_0^s < 1$ *and there exists some* $p > 2$ *such that* $\mu - \frac{p-1}{2}\sigma^2 - \frac{1}{p}\lambda > 0$, *then*

$$\lim_{t\to\infty}\langle S(t)\rangle = \frac{\Lambda(\mu + \delta + \mu\nu)}{\mu(\mu + \delta)}, \quad \lim_{t\to\infty}\langle I(t)\rangle = 0, \quad \lim_{t\to\infty}\langle R(t)\rangle = \frac{\nu\Lambda}{\mu + \delta}, \quad a.s.,$$

i.e., the disease will die out a.s..

Proof. From stochastic model (1.3),

$$\begin{aligned}
&\mathrm{d}(S(t) + I(t)) + \frac{\delta}{\mu + \delta}\mathrm{d}R(t) \\
=&(1 - \nu)\Lambda + \frac{\nu\Lambda\delta}{\mu + \delta} - \mu S(t) - (\frac{\xi\mu}{\mu + \delta} + \mu + c)I(t) + \sigma_1 S(t)\mathrm{d}B_1(t) + \sigma_2 I(t)\mathrm{d}B_2(t) \\
&+ \frac{\sigma_3\delta}{\mu + \delta}R(t)\mathrm{d}B_3(t) + \int_{\mathbb{Y}} \eta_1(z)S(t^-)\widetilde{N}(\mathrm{d}t, \mathrm{d}z) + \int_{\mathbb{Y}} \eta_2(z)I(t^-)\widetilde{N}(\mathrm{d}t, \mathrm{d}z) \\
&+ \frac{\delta}{\mu + \delta}\int_{\mathbb{Y}} \eta_3(z)R(t^-)\widetilde{N}(\mathrm{d}t, \mathrm{d}z).
\end{aligned}$$

Integrating the above equality from 0 to t and dividing both sides by t,

$$(1 - \nu)\Lambda + \frac{\nu\Lambda\delta}{\mu + \delta} - \mu\langle S(t)\rangle - (\frac{\xi\mu}{\mu + \delta} + \mu + c)\langle I(t)\rangle = \Psi_1(t), \tag{4.1}$$

where

$$\begin{aligned}
\Psi_1(t) =&\frac{S(t) - S(0)}{t} + \frac{I(t) - I(0)}{t} + \frac{\delta}{\mu + \delta}\frac{R(t) - R(0)}{t} - \frac{\sigma_1}{t}\int_0^t S(s)\mathrm{d}B_1(s) - \frac{\sigma_2}{t}\int_0^t I(s)\mathrm{d}B_2(s) \\
&- \frac{\sigma_3}{t}\frac{\delta}{\mu + \delta}\int_0^t R(s)\mathrm{d}B_3(s) - \frac{1}{t}\int_0^t\int_{\mathbb{Y}} \eta_1(z)S(s^-)\widetilde{N}(\mathrm{d}s, \mathrm{d}z) - \frac{1}{t}\int_0^t\int_{\mathbb{Y}} \eta_2(z)I(s^-)\widetilde{N}(\mathrm{d}s, \mathrm{d}z) \\
&- \frac{1}{t}\frac{\delta}{\mu + \delta}\int_0^t\int_{\mathbb{Y}} \eta_3(z)R(s^-)\widetilde{N}(\mathrm{d}s, \mathrm{d}z).
\end{aligned}$$

Similarly, for the first equation of system (1.3),

$$\begin{aligned}&= (1-\nu)\Lambda - \mu\langle S(t)\rangle - \beta\langle \frac{S(t)(1+aI(t)) - S(t)}{a(1+aI(t))}\rangle + \delta\langle R(t)\rangle \\ &= (1-\nu)\Lambda - (\mu + \frac{\beta}{a})\langle S(t)\rangle + \frac{1}{a}\langle \frac{\beta S}{1+aI}\rangle + \delta\langle R(t)\rangle \\ &= \Psi_2(t),\end{aligned} \tag{4.2}$$

where

$$\Psi_2(t) = \frac{S(t)-S(0)}{t} - \frac{\sigma_1}{t}\int_0^t S(s)\mathrm{d}B_1(s) - \frac{1}{t}\int_0^t\int_{\mathbb{Y}} \eta_1(z)S(s^-)\widetilde{N}(\mathrm{d}s,\mathrm{d}z).$$

For the third equation in system (1.3), we obtain

$$\nu\Lambda - (\mu+\delta)\langle R(t)\rangle + \xi\langle I(t)\rangle = \Psi_3(t), \tag{4.3}$$

where

$$\Psi_3(t) = \frac{R(t)-R(0)}{t} - \frac{\sigma_3}{t}\int_0^t R(s)\mathrm{d}B_3(s) - \frac{1}{t}\int_0^t\int_{\mathbb{Y}} \eta_3(z)R(s^-)\widetilde{N}(\mathrm{d}s,\mathrm{d}z).$$

According to Theorem 3.1–3.3, then

$$\lim_{t\to\infty}\Psi_1(t) = 0, \quad \lim_{t\to\infty}\Psi_2(t) = 0, \quad \lim_{t\to\infty}\Psi_3(t) = 0. \tag{4.4}$$

By Itô's formula with jump process,

$$\begin{aligned}\mathrm{d}\ln I(t) =& \{\frac{\beta S}{1+aI} - (\mu+c+\xi) - \frac{\sigma_2^2}{2} - \int_{\mathbb{Y}}[\eta_2(z) - \ln(1+\eta_2(z))]\nu(\mathrm{d}z)\}\mathrm{d}t \\ &+ \sigma_2\mathrm{d}B_2(t) + \int_{\mathbb{Y}}\ln(1+\eta_2(z))\widetilde{N}(\mathrm{d}t,\mathrm{d}z).\end{aligned}$$

It follows that

$$\begin{aligned}\frac{\ln I(t) - \ln I(0)}{t} =& \langle\frac{\beta S}{1+aI}\rangle - (\mu+c+\xi) - \frac{\sigma_2^2}{2} - \int_{\mathbb{Y}}[\eta_2(z) - \ln(1+\eta_2(z))]\nu(\mathrm{d}z) \\ &+ \sigma_2\frac{B_2(t)}{t} + \frac{1}{t}\int_0^t\int_{\mathbb{Y}}\ln(1+\eta_2(z))\widetilde{N}(\mathrm{d}s,\mathrm{d}z).\end{aligned} \tag{4.5}$$

Substituting (4.1)–(4.3) into (4.5) leads to

$$\begin{aligned}\frac{\ln I(t)}{t} =& \frac{\ln I(0)}{t} + \frac{\beta\Lambda((1-\nu)\mu+\delta)}{\mu(\mu+\delta)} - (\mu+c+\xi) - a[(\mu+\frac{\beta}{a})(\frac{\xi}{\mu+\delta} + \frac{\mu+c}{\mu}) + \frac{\xi\delta}{\mu+\delta}]\langle I(t)\rangle \\ &+ a\Psi_2(t) - \frac{a\mu+\beta}{\mu}\Psi_1(t) + \frac{a\delta}{\mu+\delta}\Psi_3(t) - \frac{\sigma_2^2}{2} - \int_{\mathbb{Y}}[\eta_2(z) - \ln(1+\eta_2(z))]\nu(\mathrm{d}z) \\ &+ \sigma_2\frac{B_2(t)}{t} + \frac{1}{t}\int_0^t\int_{\mathbb{Y}}\ln(1+\eta_2(z))\widetilde{N}(\mathrm{d}s,\mathrm{d}z) \\ \le& (\mu+c+\xi)(R_0^s - 1) + a\Psi_2(t) - \frac{a\mu+\beta}{\mu}\Psi_1(t) + \frac{a\delta}{\mu+\delta}\Psi_3(t) + \sigma_2\frac{B_2(t)}{t} \\ &+ \frac{1}{t}\int_0^t\int_{\mathbb{Y}}\ln(1+\eta_2(z))\widetilde{N}(\mathrm{d}s,\mathrm{d}z) + \frac{\ln I(0)}{t}.\end{aligned} \tag{4.6}$$

Besides, define

$$M_1(t) := \int_0^t \int_{\mathbb{Y}} \ln(1+\eta_2(z))\widetilde{N}(\mathrm{d}s, \mathrm{d}z).$$

By assumption ($\mathbf{H_3}$), we obtain

$$\int_0^t \frac{\mathrm{d}\langle M_1, M_1\rangle(s)}{(1+s)^2}\mathrm{d}s = \frac{t}{1+t}\int_{\mathbb{Y}} (\ln(1+\eta_2(z)))^2\nu(\mathrm{d}z) < +\infty.$$

Law of large numbers (Theorem 1 of [20]) yields that

$$\lim_{t\to\infty} \frac{M_1(t)}{t} = 0, \quad a.s.. \tag{4.7}$$

Employing the law of large number yields $\lim_{t\to\infty} \frac{B_2(t)}{t} = 0$, a.s.. This with (4.4) and (4.7) yields

$$\limsup_{t\to\infty} \frac{\ln I(t)}{t} \leq (\mu + c + \xi)(R_0^s - 1), \quad a.s..$$

Then

$$\lim_{t\to\infty} I(t) = 0, \quad a.s.. \tag{4.8}$$

Moreover, from (4.1), (4.3) and (4.4) we obtain

$$\lim_{t\to\infty}\langle S(t)\rangle = \frac{\Lambda(\mu+\delta+\mu\nu)}{\mu(\mu+\delta)}, \quad \lim_{t\to\infty}\langle R(t)\rangle = \frac{\nu\Lambda}{\mu+\delta}, \quad a.s..$$

□

Now, sufficient condition for the persistence of the disease is provided.

Theorem 4.2. *Suppose* ($\mathbf{H_1}$)–($\mathbf{H_3}$) *hold. If* $R_0^s > 1$ *and there exists some* $p > 2$ *such that* $\mu - \frac{p-1}{2}\sigma^2 - \frac{1}{p}\lambda > 0$*, then*

$$\begin{aligned}
\lim_{t\to\infty}\langle S(t)\rangle &= \frac{1}{\mu}((1-\nu)\Lambda + \frac{\nu\Lambda\delta}{\mu+\delta}) - \frac{1}{\mu}(\frac{\xi\mu}{\mu+\delta} + \mu + c)\frac{(\mu+c+\xi)(R_0^s - 1)}{a[(\mu+\frac{\beta}{a})(\frac{\xi}{\mu+\delta} + \frac{\mu+c}{\mu}) + \frac{\xi\delta}{\mu+\delta}]};\\
\lim_{t\to\infty}\langle I(t)\rangle &= \frac{(\mu+c+\xi)(R_0^s - 1)}{a[(\mu+\frac{\beta}{a})(\frac{\xi}{\mu+\delta} + \frac{\mu+c}{\mu}) + \frac{\xi\delta}{\mu+\delta}]};\\
\lim_{t\to\infty}\langle R(t)\rangle &= \frac{\nu\Lambda}{\mu+\delta} + \frac{\xi}{\mu+\delta}\frac{(\mu+c+\xi)(R_0^s - 1)}{a[(\mu+\frac{\beta}{a})(\frac{\xi}{\mu+\delta} + \frac{\mu+c}{\mu}) + \frac{\xi\delta}{\mu+\delta}]}, \quad a.s.,
\end{aligned}$$

i.e., the disease will persist a.s..

Proof. According to (4.6), we have

$$\frac{\ln I(t)}{t} = (\mu+c+\xi)(R_0^s - 1) - a[(\mu+\frac{\beta}{a})(\frac{\xi}{\mu+\delta} + \frac{\mu+c}{\mu}) + \frac{\xi\delta}{\mu+\delta}]\langle I(t)\rangle + \Psi(t),$$

where

$$\Psi(t) := a\Psi_2(t) - \frac{a\mu+\beta}{\mu}\Psi_1(t) + \frac{a\delta}{\mu+\delta}\Psi_3(t) + \sigma_2\frac{B_2(t)}{t} + \frac{1}{t}\int_0^t\int_{\mathbb{Y}}\ln(1+\eta_2(z))\widetilde{N}(\mathrm{d}s,\mathrm{d}z) + \frac{\ln I(0)}{t}.$$

Theorem 3.1–3.3, the law of large number, (4.4) and (4.7) yield that $\lim_{t\to\infty}\Psi(t) = 0$. It follows from Lemma 2 of Liu and Wang [21] that

$$\lim_{t\to\infty}\langle I(t)\rangle = \frac{(\mu+c+\xi)(R_0^s-1)}{a[(\mu+\frac{\beta}{a})(\frac{\xi}{\mu+\delta}+\frac{\mu+c}{\mu})+\frac{\xi\delta}{\mu+\delta}]}.$$

Meanwhile, from (4.1) and (4.4) we obtain

$$\lim_{t\to\infty}\langle S(t)\rangle = \frac{1}{\mu}((1-\nu)\Lambda + \frac{\nu\Lambda\delta}{\mu+\delta}) - \frac{1}{\mu}(\frac{\xi\mu}{\mu+\delta}+\mu+c)\lim_{t\to\infty}\langle I(t)\rangle.$$

Consequently, it follows that

$$\lim_{t\to\infty}\langle S(t)\rangle = \frac{1}{\mu}((1-\nu)\Lambda + \frac{\nu\Lambda\delta}{\mu+\delta}) - \frac{1}{\mu}(\frac{\xi\mu}{\mu+\delta}+\mu+c)\frac{(\mu+c+\xi)(R_0^s-1)}{a[(\mu+\frac{\beta}{a})(\frac{\xi}{\mu+\delta}+\frac{\mu+c}{\mu})+\frac{\xi\delta}{\mu+\delta}]}.$$

By virtue of (4.3), we have

$$\lim_{t\to\infty}\langle R(t)\rangle = \frac{\nu\Lambda}{\mu+\delta} + \frac{\xi}{\mu+\delta}\frac{(\mu+c+\xi)(R_0^s-1)}{a[(\mu+\frac{\beta}{a})(\frac{\xi}{\mu+\delta}+\frac{\mu+c}{\mu})+\frac{\xi\delta}{\mu+\delta}]}.$$

□

5. Conclusions

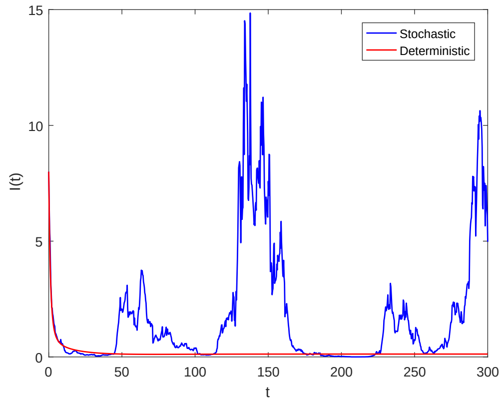
In the paper, we analyze the asymptotic behaviour of a stochastic SIRS system with Lévy process. Due to the introduction of Lévy jump, the jump stochastic integral process is a discontinuous martingale. Thus the Kunita's inequality is used to estimate the asymptotic pathwise of solution for stochastic system (1.3). On this basis, the basic reproduction number for stochastic system (1.3) is defined:

$$R_0^s = \frac{\beta\Lambda((1-\nu)\mu+\delta)}{\mu(\mu+\delta)(\mu+c+\xi)} - \frac{1}{\mu+c+\xi}(\frac{\sigma_2^2}{2} + \int_{\mathbb{Y}}[\eta_2(z) - \ln(1+\eta_2(z))]\nu(\mathrm{d}z)).$$

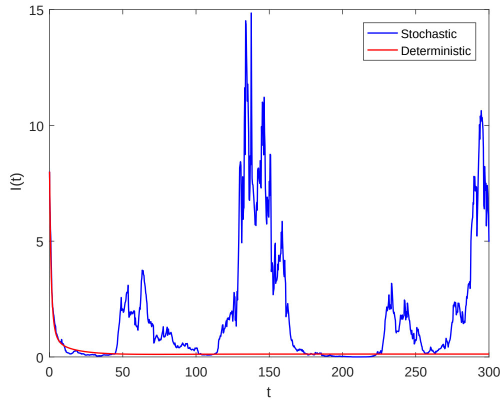
If $R_0^s < 1$ and some other conditions hold, the disease will die out. If $R_0^s > 1$ and some other conditions hold, the disease will persist. By comparing $R_0 = \frac{\beta\Lambda((1-\nu)\mu+\delta)}{\mu(\mu+\delta)(\mu+c+\xi)}$, it is easy to see that $R_0^s < R_0$. This indicates that Lévy noise can suppress the outbreak of infectious disease.

Next, let us make numerical simulations to verify the theoretical results. The initial value is given by $(S(0), I(0), R(0)) = (80, 8, 5)$. The parameters ν, Λ, μ, β, a, δ, c and ξ are chosen as 0.7346, 6, 0.04, 0.02, 1, 0.005, 0.01 and 0.8, respectively. By calculation, we can obtain $R_0 = 1.1023$. This implies that the disease will persist when the system does not have environmental perturbation.

To consider the effect of Lévy process in the spreading of disease, the values of σ_2 are both taken as 0.4. We first choose η_2 as 0.08. By calculating the value of R_0^s, we get the value is 1.0046. It follows from Theorem 4.2 that the disease will persist. The simulated result is shown in Figure 1(a). Then we choose η_2 as 0.2. The value of R_0^s is 0.9874 in this case. By Theorem 4.1, the disease will die out. The simulation result is presented in Figure 1(b). From Figure 1, it can be seen that the larger jump noise induces the extinction of disease, i.e., the Lévy noise suppress the outbreak of infectious disease while the corresponding deterministic model is persist.



(a) Persistence of I(t) with $\eta_2 = 0.08$



(b) Extinction of I(t) with $\eta_2 = 0.2$

Figure 1. The paths of the infected for two different values of η_2.

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Conflict of interest

The authors declare that there is no conflict of interest.

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