Mathematical Biosciences and Engineering

http://www.aimspress.com/journal/mbe

MBE, 20(1): 437–455.
DOI: 10.3934/mbe.2023020
Received: 01 August 2022
Revised: 09 September 2022
Accepted: 25 September 2022
Published: 10 October 2022

Research article

Stability and bifurcation control for a fractional-order chemostat model with time delays and incommensurate orders

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Abstract: In this paper, a delayed fractional Lotka-Volterra food chain chemostat model with incommensurate orders is proposed, and the effect on system stability and bifurcation of this model are discussed. First, for the system with no controller, the stability and Hopf bifurcation with respect to time delay are investigated. Taking the time delay as the bifurcation parameter, the relevant characteristic equations are analyzed, and the conditions for Hopf bifurcation are proposed. The results show that the controller can fundamentally affect the stability of the system, and that they both have an important impact on the generation of bifurcation at the same time. Finally, numerical simulation is carried out to support the theoretical data.

Keywords: fractional system; time delay; stability; control; chemostat model; Hopf bifurcation; incommensurate orders

1. Introduction

In the last 100 years, with the continuous development of mathematical biology, food chain models [1] have received much attention from scientists. In this field, mathematical models are established by scientific methods and reasonable assumptions, then, the specific problems are explained, predicted and controlled. In the 1920s, the Lotka-Volterra model explained the fluctuations in the number of fish and shark populations. The chemostat is a simple and well-adopted laboratory apparatus used to culture microorganisms. It can be used to investigate microbial growth and it has the advantage that the parameters are easily measurable. A sterile growth medium enters the chemostat at a constant rate; the volume within the chemostat is preserved by allowing excess medium to flow out through a siphon. We inoculate this chemostat with a heterotrophic bacterium that finds, in the medium, all of the necessary nutrients but one. This last nutrient is the limiting substrate. Based on previous works, the Lotka-Volterra food chain chemostat model has attracted a lot of attention. It has long been recognized that there is a time lag in the growth response of the population to environmental change. The growth of predators is based on the number of prey over a period of time. Therefore, it is feasible to add distributed delay to the food chain chemostat model. In [2], a model of the chemostat involving two species of microorganisms competing for two perfectly complementary, growth-limiting nutrients is considered.

With the increasing complexity of the actual biological mathematical system and the increasing requirements for control, the control technology of integer-order calculus theory has been unable to achieve satisfactory results. However, using the model of an integer-order system can-not well describe some systems' dynamic processes; fractional calculus not only provides a new mathematical tool for the model of biological mathematics, but it also can solve some problems in real life; fractional calculus theory [3] can solve this problem. The order of integrals and differentials in fractional calculus can be changed at will. To a certain extent, it better expands the description ability of integral calculus. According to the historical information of the system, it is necessary and urgent to analyze the impact of time delay on the dynamics of a biological model in order to accurately describe the dynamics of the food chain model in theory and practice. In [4], a new integrated pest management predator-prey model is presented, and the existence and stability of the order-1 periodic orbit of the proposed model is discussed. Whether in population dynamics or epidemic dynamics [5–7], the relevant research of mathematical biology has been ongoing. In the field, fractional calculus has attracted the attention of engineers and scientists. It has been successfully applied in various fields, such as medicine, industry, finance, physics, security communication, system biology [8–10] and so on. With the rapid development of fractional calculus, the Hopf bifurcation of fractional-order models [11, 12] has attracted more and more attention. In [13], the author mainly uses fractional-order differential equations to describe the dynamic behavior of the chemostat system. The integer-order chemostat model in the form of the ordinary differential equation was extended to the fractional-order differential equations. The stability and bifurcation analyses of the fractional-order chemostat model have been investigated using the Adams-type predictor-corrector method. In [14], the fractional-order form of a three dimensional chemostat model with variable yields is introduced. The stability analysis of this fractional system is discussed in detail. In order to study the dynamic behaviors of the mentioned fractional system, the well known the non-standard finite difference scheme was implemented.

The time delay is inevitable in most practical dynamical networks, including biological models, neural networks [15] and evolutionary dynamics. For delayed fractional-order systems, the bifurcation problem has attracted more and more attention. Fractional calculus can be used as a mathematical analysis tool to study arbitrary order integrals and derivatives. It can describe many systems in the real world. However, due to these remarkable results, the influence of a time delay on bifurcation is ignored. The chemostat is a simple and well-adopted laboratory apparatus used to culture microorganisms. It can be used to investigate microbial growth and it has the advantage that the parameters are easily measurable. A sterile growth medium enters the chemostat at a constant rate; the volume within the chemostat is preserved by allowing excess medium to flow out through a siphon. We inoculate this chemostat with a heterotrophic bacterium that finds, in the medium, all of the necessary nutrients but one. This last nutrient is the limiting substrate.

In [16], we study the following n-dimensional linear fractional differential system with multiple time delays

$$\begin{cases}\dfrac{d^{q_1}x_1(t)}{d^{q_1}t} = a_{11}x_1(t-\tau_{11}) + a_{12}x_2(t-\tau_{12}) + \cdots + a_{1n}x_n(t-\tau_{1n}),\\ \dfrac{d^{q_2}x_2(t)}{d^{q_2}t} = a_{21}x_1(t-\tau_{21}) + a_{22}x_2(t-\tau_{22}) + \cdots + a_{2n}x_n(t-\tau_{2n}),\\ \qquad\qquad\cdots\cdots\\ \dfrac{d^{q_n}x_n(t)}{d^{q_n}t} = a_{n1}x_1(t-\tau_{n1}) + a_{n2}x_2(t-\tau_{n2}) + \cdots + a_{nn}x_n(t-\tau_{nn}).\end{cases} \tag{1.1}$$

In [17], the authors considered the chaotic control of integer-orders and fractional-orders of a chaotic Burke-Shaw system by using time delayed feedback control

$$\begin{cases}D^{\alpha}x(t) = -S(x(t)+y(t)),\\ D^{\alpha}y(t) = -y(t) - Sx(t)z(t) + K[y(t) - y(t-\tau)],\\ D^{\alpha}z(t) = V + Sx(t)y(t).\end{cases}$$

They investigated the control of a chaotic Burke-Shaw system using the Pyragas method. This system is derived from a Lorenz system which has several applications in physics and engineering. The linear stability and the existence of Hopf bifurcation of this system were investigated.

In [18], the authors promote and consider a Lotka-Volterra food chain chemostat model that incorporates both distributed delay and stochastic perturbations. In this paper, our main work is to consider a fractional-order model with time delays

$$\begin{cases}\dfrac{dS(t)}{dt} = d(a-S) - \dfrac{m_1SX}{\varepsilon},\\ \dfrac{dX(t)}{dt} = m_1SX - dX - \dfrac{m_2XY}{\eta},\\ \dfrac{dY(t)}{dt} = m_2XY - dY.\end{cases} \tag{1.2}$$

Next, in order to better study the control of System (1.2), an extended feedback controller can be added; the controller is represented as follows

$$\mu(t) = h[X(t) - X(t-\nu)];$$

clearly, if $h = 0$ or $\nu = 0$, it is obtained that the controller is meaningless. In this case, it will not change the final result of the equilibrium point of System (1.2).

In this paper, the controller is added to an incommensurate order a delayed fractional-order model, shown as the following system

$$\begin{cases}D_t^{\alpha_1}S(t) = d(a-S) - \dfrac{m_1SX}{\varepsilon},\\ D_t^{\alpha_2}X(t) = m_1SX - dX - \dfrac{m_2XY}{\eta} + \mu(t),\\ D_t^{\alpha_3}Y(t) = m_2X(t-\tau)Y(t-\tau) - dY,\end{cases} \tag{1.3}$$

where $\alpha_i \in (0,1] (i = 1,2,3)$, and the initial values are as follows:

$$S(t) = \phi_1(t),\ X(t) = \phi_2(t),\ Y(t) = \phi_3(t), \phi_1(t) \geq 0, \phi_2(t) \geq 0, \phi_3(t) \geq 0,\ t \in [-\max(\tau,\nu), 0];$$

the model is based on the Caputo derivative. The meanings of various parameters in the system are shown in the following figure.

Table 1. Definitions of parameters.

Parameter	Description
$S(t)$	The concentration of a nutrient at time t
$X(t)$	The concentration of the prey at time t
$Y(t)$	The concentration of the predator at time t
m_1	The per capita growth rate of the prey
m_2	The per capita growth rate of the predator
a	The concentration of the growth limiting nutrient in the feed vessel
d	The dilution rate
ε	The yield constant for prey growth on a nutrient
η	The yield constant for predator growth on the prey
h	The negative feedback gain
ν	The feedback control delay

Motivated by the works mentioned above, this paper introduces a controller to a delayed fractional Lotka-Volterra food chain chemostat model with incommensurate orders. The architecture of our current paper is as follows. In the second section, some preliminary preparations are made. In the third section, some properties of the system, bifurcation control strategy and the stability of the system under the influence of the controller are studied. In the fourth section, the numerical simulation is described according to the theoretical knowledge of the previous sections. Finally, the corresponding conclusions are given.

2. Preliminaries

In this paper, all results are based on the Caputo derivative definition. The definition of the Caputo derivative can form the initial conditions of the fractional equation expressed in the form of integer derivative. With this advantage, some practical problems can be better solved. For the convenience of the reader, we present some necessary fractional definitions. The definitions can be found in recent literature.

Definition 2.1. *([19]) The Caputo fractional derivative of order α of a function $f(t)$ is defined as*

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau,$$

where n is the positive integer and $n-1 < \alpha < n$.

Definition 2.2. *([20]) Let $f(x)$ denote a function which vanishes for negative values of x. Its Laplace's transform $L_\alpha\{f(x)\}$ of order α (or α-th fractional Laplace transform) is defined by the following expression when it is finite:*

$$L_\alpha\{f(x)\} :=: F_\alpha(s) = \int_0^{\infty} E_\alpha(-s^\alpha x^\alpha) f(x) (dx)^\alpha,$$

$$:= \lim_{M\uparrow\infty}\int_0^M E_\alpha(-s^\alpha x^\alpha)f(x)(dx)^\alpha,$$

where $s \in C$ and $E_\alpha(u)$ is the Mittag-Leffler function $\sum u^k(k\alpha)!$. The following operational formulae can be easily obtained:

$$\begin{aligned}
&L_\alpha\{x^\alpha f(x)\} = -D_s^\alpha L\{f(x)\},\\
&L_\alpha\{f(ax)\}_s = (1/a)^\alpha L_\alpha\{f(x)\}_{x/a},\\
&L_\alpha\{f(x-b)\} = E_\alpha(-s^\alpha b^\alpha)L_\alpha\{f(x)\},\\
&L_\alpha\{E_\alpha(-c^\alpha x^\alpha)f(x)\}_s = L_\alpha\{f(x)\}_{s+c},\\
&L_\alpha\{-x^\alpha f(x)\}_= D_s^\alpha L_\alpha\{f(x)\},\\
&L_\alpha\left\{\int_0^x f(u)(du)^\alpha\right\} = \Gamma^{-1}(1+\alpha)s^{-\alpha}L_\alpha\{f(x)\}.
\end{aligned}$$

Furthermore, using the properties of the Mittag-Leffler function and integration by parts, we find that

$$L_\alpha\{f^{(\alpha)}(x)\} = s^\alpha L_\alpha\{f(x)\} - \Gamma(1+\alpha)f(0).$$

According to the relevant conclusions of [15], the system (1.1) is extended to a more general linear system, System (2.1)

$$\begin{cases}
\dfrac{d^{q_1}x_1(t)}{d^{q_1}t} = a_{11}x_1(t-\tau_{11}) + b_{11}x_1(t) + a_{12}x_2(t-\tau_{12}) + b_{12}x_2(t)\cdots + a_{1n}x_n(t-\tau_{1n}) + b_{1n}x_n(t),\\
\dfrac{d^{q_2}x_2(t)}{d^{q_2}t} = a_{21}x_1(t-\tau_{21}) + b_{21}x_1(t) + a_{22}x_2(t-\tau_{22}) + b_{22}x_2(t)\cdots + a_{2n}x_n(t-\tau_{2n}) + b_{2n}x_n(t),\\
\qquad\qquad\cdots\cdots\\
\dfrac{d^{q_n}x_n(t)}{d^{q_n}t} = a_{n1}x_1(t-\tau_{n1}) + b_{n1}x_1(t) + a_{n2}x_2(t-\tau_{n2}) + b_{n2}x_2(t)\cdots + a_{nn}x_n(t-\tau_{nn}) + b_{nn}x_n(t);
\end{cases} \tag{2.1}$$

the characteristic matrix of System (2.1) is as follows:

$$\Delta(s) = \begin{pmatrix}
s^{q_1} - a_{11}e^{-s\tau_{11}} - b_{11} & -a_{12}e^{-s\tau_{12}} - b_{12} & \cdots & -a_{1n}e^{-s\tau_{1n}} - b_{1n}\\
-a_{21}e^{-s\tau_{21}} - b_{21} & s^{q_2} - a_{22}e^{-s\tau_{22}} - b_{22} & \cdots & -a_{2n}e^{-s\tau_{2n}} - b_{2n}\\
\vdots & \vdots & \ddots & \vdots\\
-a_{n1}e^{-s\tau_{n1}} - b_{n1} & -a_{n2}e^{-s\tau_{n2}} - b_{n2} & \cdots & s^{q_n} - a_{nn}e^{-s\tau_{nn}} - b_{nn}
\end{pmatrix}$$

$$= \begin{pmatrix}
s^{q_1} & 0 & \cdots & 0\\
0 & s^{q_2} & \cdots & 0\\
\vdots & \vdots & \ddots & \vdots\\
0 & 0 & \cdots & s^{q_n}
\end{pmatrix} - \begin{pmatrix}
a_{11}e^{-s\tau_{11}} & a_{12}e^{-s\tau_{12}} & \cdots & a_{1n}e^{-s\tau_{1n}}\\
a_{21}e^{-s\tau_{21}} & a_{22}e^{-s\tau_{22}} & \cdots & a_{2n}e^{-s\tau_{2n}}\\
\vdots & \vdots & \ddots & \vdots\\
a_{n1}e^{-s\tau_{n1}} & a_{n2}e^{-s\tau_{n2}} & \cdots & a_{nn}e^{-s\tau_{nn}}
\end{pmatrix} - \begin{pmatrix}
b_{11} & b_{12} & \cdots & b_{1n}\\
b_{21} & b_{22} & \cdots & b_{2n}\\
\vdots & \vdots & \ddots & \vdots\\
b_{n1} & b_{n2} & \cdots & b_{nn}
\end{pmatrix};$$

then the following conclusion can be established.

Theorem 2.1. *If all roots of the characteristic equation det*$(\Delta(s)) = 0$ *have negative real parts, then the zero solution of System* (2.1) *is locally asymptotically stable.*

According to the solution method for the equilibrium point and the relevant definitions of the stability of the equilibrium point, we calculated that the positive equilibrium $E_*(S_*, I_*, Y_*)$ of System (1.3) is shown as

$$S_* = \frac{am_2\varepsilon}{m_1 + m_2\varepsilon}, \quad X_* = \frac{d}{m_2}, \quad Y_* = \left(\frac{am_1\varepsilon}{m_1 + m_2\varepsilon} - \frac{d}{m_2}\right)\eta,$$

if $am_1m_2\varepsilon - d(m_1 + m_2\varepsilon) > 0$; then, the system (1.3) has a positive equilibrium $E_*(S_*, I_*, Y_*)$.

The linearized system of System (1.3) at the positive equilibrium $E_*(S_*, I_*, Y_*)$ is

$$\begin{cases} D_t^{\alpha_1} S(t) = \left(-d - \frac{m_1X_*}{\varepsilon}\right)S(t) - \frac{m_1S_*}{\varepsilon}X(t), \\ D_t^{\alpha_2} X(t) = (m_1X_*)S(t) + \left(m_1S_* - d - \frac{m_2Y_*}{\eta} + h\right)X(t) - hX(t-v) - \frac{m_2X_*}{\eta}Y(t), \\ D_t^{\alpha_3} Y(t) = m_2Y_*X(t-\tau) + m_2X_*Y(t-\tau) - dY(t). \end{cases} \tag{2.2}$$

By a Laplace transform, we have

$$\begin{cases} s^{\alpha_1}L[S(t)] - s^{\alpha_1-1}\phi_1(0) = \left(-d - \frac{m_1X_*}{\varepsilon}\right)L[S(t)] - \frac{m_1S_*}{\varepsilon}L[X(t)], \\ s^{\alpha_2}L[X(t)] - s^{\alpha_2-1}\phi_2(0) = (m_1X_*)L[S(t)] + \left(m_1S_* - d - \frac{m_2Y_*}{\eta} + h\right)L[X(t)] \\ \qquad - he^{-sv}\left(L[X(t)] + \int_{-v}^{0} e^{-st}\phi_2(t)dt\right) - \frac{m_2X_*}{\eta}L[Y(t)], \\ s^{\alpha_3}L[Y(t)] - s^{\alpha_3-1}\phi_3(0) = m_2Y_*e^{-s\tau}\left(L[X(t)] + \int_{-\tau}^{0} e^{-st}\phi_2(t)dt\right) \\ \qquad + m_2X_*e^{-s\tau}\left(L[Y(t)] + \int_{-\tau}^{0} e^{-st}\phi_3(t)dt\right) - dL[Y(t)], \end{cases} \tag{2.3}$$

where $L[F(t)]$ represents the Laplace transform of $F(t)$. Let $\Delta(s)$ represents the characteristic matrix of System (2.3); then, System (2.3) can be rewritten as follows:

$$\Delta(s) \cdot \begin{pmatrix} L[S(t)] \\ L[X(t)] \\ L[Y(t)] \end{pmatrix} = \begin{pmatrix} b_1(s) \\ b_2(s) \\ b_3(s) \end{pmatrix} \tag{2.4}$$

$$\begin{cases} b_1(s) = s^{\alpha_1-1}\phi_1(0), \\ b_2(s) = s^{\alpha_2-1}\phi_2(0) - he^{sv}\int_{-v}^{0} e^{-st}\phi_2(t)dt, \\ b_3(s) = s^{\alpha_3-1}\phi_3(0) + m_2Y^*e^{-s\tau}\int_{-\tau}^{0} e^{-st}\phi_2(t)dt + m_2X^*e^{-s\tau}\int_{-\tau}^{0} e^{-st}\phi_3(t)dt, \end{cases} \tag{2.5}$$

and

$$\Delta(s) = \begin{pmatrix} s^{\alpha_1} - a_{11} & a_{12} & 0 \\ -a_{21} & s^{\alpha_2} - a_{22} - h + he^{-sv} & a_{23} \\ 0 & -a_{32}e^{-s\tau} & s^{\alpha_3} - a_{33}e^{-s\tau} + d \end{pmatrix}, \tag{2.6}$$

where

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} -d - \frac{m_1X^*}{\varepsilon} & \frac{m_1S^*}{\varepsilon} & 0 \\ m_1X^* & m_1S^* - d - \frac{m_2Y^*}{\eta} & \frac{m_2X^*}{\eta} \\ 0 & m_2Y^* & m_2X^* \end{pmatrix}. \tag{2.7}$$

3. Main results

3.1. Bifurcation analysis without the effects of control

In this subsection, the stability and bifurcation of the positive equilibrium E_* are discussed with $\mu(t) \equiv 0$.

Theorem 3.1. *Suppose that $\mu(t) = 0, \tau = 0$ and $\alpha_i \in (0,1](i = 1,2,3)$, and the following condition holds:*
(T_1) : $am_1m_2\varepsilon - d(m_1 + m_2\varepsilon) > 0$.
Then, the positive equilibrium E_* of System (1.3) is locally asymptotically stable.

Proof. When $\mu(t) = 0$,

$$\Delta(s) = \begin{pmatrix} s^{\alpha_1} - a_{11} & a_{12} & 0 \\ -a_{21} & s^{\alpha_2} - a_{22} & a_{23} \\ 0 & -a_{32}e^{-s\tau} & s^{\alpha_3} - a_{33}e^{-s\tau} + d \end{pmatrix}; \tag{3.1}$$

the characteristic polynomial is written as

$$a_1(s) + a_2(s)e^{-s\tau} = 0, \tag{3.2}$$

where

$$\begin{aligned} a_1(s) =& s^{\alpha_1+\alpha_2+\alpha_3} + ds^{\alpha_1+\alpha_2} - a_{22}s^{\alpha_1+\alpha_3} - a_{11}s^{\alpha_2+\alpha_3} - a_{22}ds^{\alpha_1} \\ & - a_{11}ds^{\alpha_2} + (a_{11}a_{22} + a_{12}a_{21})s^{\alpha_3} + a_{11}a_{22}d + a_{12}a_{21}d, \\ a_2(s) =& - a_{33}s^{\alpha_1+\alpha_2} + (a_{22}a_{33} + a_{23}a_{32})s^{\alpha_1} + a_{11}a_{33}s^{\alpha_2} - a_{11}a_{22}a_{33} \\ & - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}; \end{aligned} \tag{3.3}$$

assuming that $\tau = 0$ and $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$, (3.2) becomes

$$\lambda^3 + H_1\lambda^2 + H_2\lambda + H_3 = 0, \tag{3.4}$$

where

$$\begin{aligned} \lambda =& s^{\alpha}, \\ H_1 =& d - a_{22} - a_{11} - a_{33}, \\ H_2 =& a_{22}a_{33} + a_{23}a_{32} + a_{11}a_{33} + a_{11}a_{22} + a_{12}a_{21} - a_{22}d - a_{11}d, \\ H_3 =& a_{11}a_{22}d + a_{12}a_{21}d - a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}. \end{aligned} \tag{3.5}$$

When $\tau = 0$ and $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$, (3.4) can be written as

$$(\lambda - z_1)(\lambda - z_2)(\lambda - z_3) = 0,$$

where z_1, z_2 and z_3 are the roots of (3.4). If (T_1) is satisfied, all roots of (3.2) have negative real parts, i.e., all eigenvalues λ_i of the Jacobian matrix evaluated at the equilibrium points satisfy $|\arg(\lambda_i)| > \frac{\alpha\pi}{2} (i = 1, 2, 3)$; then, the positive equilibrium E_* is locally asymptotically stable.

When $\tau = 0$, for all $\alpha_i \in (0, 1]$ $(i = 1, 2, 3)$, let $s^{\alpha_1} = \lambda_1$, $s^{\alpha_2} = \lambda_2$ and $s^{\alpha_3} = \lambda_3$. Without loss of generality, (3.2) can be written as

$$(\lambda_1 - z_1)(\lambda_2 - z_2)(\lambda_3 - z_3) = 0. \tag{3.6}$$

According to relevant calculations, the characteristic equation satisfies the Hurwitz criterion, and if (T_1) is satisfied, all roots of (3.6) have negative real parts, i.e., all eigenvalues λ_i of the Jacobian matrix evaluated at the equilibrium points satisfy $|\arg(\lambda_i)| > \frac{\alpha\pi}{2} (i = 1, 2, 3)$; then, the positive equilibrium E_* is locally asymptotically stable. This completes the proof. □

Theorem 3.2. *Assume $\mu(t) = 0, \tau > 0$ and the condition (T_1) holds. If the following condition holds: $(T_3) : \varpi_1\vartheta_1 + \varpi_2\vartheta_2 > 0$, where*

$$\begin{aligned}
\varpi_1 &= (Re[a_1{}'(i\varphi_0)] + Re[a_2{}'(i\varphi_0)]) \cos\varphi_0\tau_0 - (Im[a_1{}'(i\varphi_0)] + Im[a_2{}'(i\varphi_0)]) \sin\varphi_0\tau_0,\\
\varpi_2 &= (Re[a_1{}'(i\varphi_0)] + Re[a_2{}'(i\varphi_0)]) \sin\varphi_0\tau_0 + (Im[a_1{}'(i\varphi_0)] + Im[a_2{}'(i\varphi_0)]) \cos\varphi_0\tau_0,\\
\vartheta_1 &= -\varphi_0 Im[a_2(i\varphi_0)], \quad \vartheta_2 = \varphi_0 Re[a_2(i\varphi_0)];
\end{aligned}$$

then

1) *The positive equilibrium of System (1.3) is locally asymptotically stable for $\tau \in [0, \tau_0)$.*
2) *There exists a constant $\tau_0 > 0$ such that System (1.3) has a Hopf bifurcation.*

Proof. Let $s = i\varphi = \varphi e^{\frac{\pi i}{2}}$ and substituting it into (3.2), it can be obtained that

$$(\cos\varphi\tau - i\sin\varphi\tau) = -\frac{a_1(i\varphi)}{a_2(i\varphi)}, \qquad a_2(i\varphi) \neq 0; \tag{3.7}$$

separating the real and imaginary parts, one gets

$$\begin{cases}
\cos\varphi\tau = -\dfrac{Re[a_1(i\varphi)]Re[a_2(i\varphi)] + Im[a_1(i\varphi)]Im[a_2(i\varphi)]}{Re^2[a_2(i\varphi)] + Im^2[a_2(i\varphi)]},\\
\sin\varphi\tau = \dfrac{Im[a_1(i\varphi)]Re[a_2(i\varphi)] - Re[a_1(i\varphi)]Im[a_2(i\varphi)]}{Re^2[a_2(i\varphi)] + Im^2[a_2(i\varphi)]},
\end{cases}$$

where

$$\begin{aligned}
Re[a_1(i\varphi)] =& \varphi^{\alpha_1+\alpha_2+\alpha_3} \cos\frac{(\alpha_1 + \alpha_2 + \alpha_3)\pi}{2} + d\varphi^{\alpha_1+\alpha_2} \cos\frac{(\alpha_1 + \alpha_2)\pi}{2} - a_{22}\varphi^{\alpha_1+\alpha_3} \cos\frac{(\alpha_1 + \alpha_3)\pi}{2}\\
&- a_{11}\varphi^{\alpha_2+\alpha_3} \cos\frac{(\alpha_2 + \alpha_3)\pi}{2} - a_{22}d\varphi^{\alpha_1} \cos\frac{\alpha_1\pi}{2} - a_{11}d\varphi^{\alpha_2} \cos\frac{\alpha_2\pi}{2}\\
&+ (a_{11}a_{22} + a_{12}a_{21})\varphi^{\alpha_3} \cos\frac{\alpha_3\pi}{2} + a_{11}a_{22}d + a_{12}a_{21}d,\\
Im[a_1(i\varphi)] =& \varphi^{\alpha_1+\alpha_2+\alpha_3} \sin\frac{(\alpha_1 + \alpha_2 + \alpha_3)\pi}{2} + d\varphi^{\alpha_1+\alpha_2} \sin\frac{(\alpha_1 + \alpha_2)\pi}{2} - a_{22}\varphi^{\alpha_1+\alpha_3} \sin\frac{(\alpha_1 + \alpha_3)\pi}{2}\\
&- a_{11}\varphi^{\alpha_2+\alpha_3} \sin\frac{(\alpha_2 + \alpha_3)\pi}{2} - a_{22}d\varphi^{\alpha_1} \sin\frac{\alpha_1\pi}{2} - a_{11}d\varphi^{\alpha_2} \sin\frac{\alpha_2\pi}{2}
\end{aligned}$$

$$
\begin{aligned}
&+ (a_{11}a_{22} + a_{12}a_{21})\varphi^{\alpha_3} \sin\frac{\alpha_3\pi}{2},\\
Re[a_2(i\varphi)] =& - a_{33}\varphi^{\alpha_1+\alpha_2}\cos\frac{(\alpha_1+\alpha_2)\pi}{2} + (a_{22}a_{33} + a_{23}a_{32})\varphi^{\alpha_1}\cos\frac{\alpha_1\pi}{2} + a_{11}a_{33}\varphi^{\alpha_2}\cos\frac{\alpha_2\pi}{2}\\
&- a_{12}a_{21}a_{33} - a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32},\\
Im[a_2(i\varphi)] =& - a_{33}\varphi^{\alpha_1+\alpha_2}\sin\frac{(\alpha_1+\alpha_2)\pi}{2} + (a_{22}a_{33} + a_{23}a_{32})\varphi^{\alpha_1}\sin\frac{\alpha_1\pi}{2} + a_{11}a_{33}\varphi^{\alpha_2}\sin\frac{\alpha_2\pi}{2}.
\end{aligned}
$$

Then, it can be obtained that

$$
\Psi_1{}^2(\varphi) + \Psi_2{}^2(\varphi) = 1, \tag{3.8}
$$

where

$$
\begin{cases}
\Psi_1(\varphi) = \cos\varphi\tau = -\dfrac{Re[a_1(i\varphi)]Re[a_2(i\varphi)] + Im[a_1(i\varphi)]Im[a_2(i\varphi)]}{Re^2[a_2(i\varphi)] + Im^2[a_2(i\varphi)]},\\
\Psi_2(\varphi) = \sin\varphi\tau = \dfrac{Im[a_1(i\varphi)]Re[a_2(i\varphi)] - Re[a_1(i\varphi)]Im[a_2(i\varphi)]}{Re^2[a_2(i\varphi)] + Im^2[a_2(i\varphi)]};
\end{cases}
$$

suppose that (3.8) has at least a positive real root φ_0. Then, it is obtained that

$$
\begin{cases}
\tau_0^{(k)} = \dfrac{1}{\varphi_0}[\arccos\Psi_1(\varphi_0) + 2k\pi], \qquad k = 0, 1, 2\dots\\
\tau_0^{(k)} = \dfrac{1}{\varphi_0}[\arcsin\Psi_2(\varphi_0) + 2k\pi], \qquad k = 0, 1, 2\dots
\end{cases} \tag{3.9}
$$

here, the bifurcation point is defined as $\tau_0 = \min\{\tau_0^{(k)}\}$. Then, we get the positive equilibrium of system (1.3) is locally asymptotically stable.

Next, differentiating (3.2) with respect to τ, we obtain

$$
\left(a_1{}'(s) + a_2{}'(s) - \tau a_2(s)e^{-s\tau}\right)\frac{ds}{d\tau} - sa_2(s)e^{-s\tau} = 0;
$$

then,

$$
\left(\frac{ds}{d\tau}\right)^{-1} = \frac{a_1{}'(s)e^{s\tau} + a_2{}'(s)}{sa_2(s)} - \frac{\tau}{s}; \tag{3.10}
$$

substituting $s = i\varphi_0$ and $\tau = \tau_0$ into (3.10), it gives

$$
\left(\frac{ds}{d\tau}\right)^{-1}\Bigg|_{s=i\varphi_0} = \frac{\varpi_1 + i\varpi_2}{\vartheta_1 + i\vartheta_2} - \frac{\tau_0}{i\varphi_0};
$$

thus, one obtains

$$
Re\left[\left(\frac{ds}{d\tau}\right)^{-1}\right] = \frac{\varpi_1\vartheta_1 + \varpi_2\vartheta_2}{\vartheta_1{}^2 + \vartheta_2{}^2}.
$$

If $\varpi_1\vartheta_1 + \varpi_2\vartheta_2 > 0$, we have that $Re\left[\left(\frac{ds}{d\tau}\right)^{-1}\right]_{\tau=\tau_0} > 0$. According to Theorem 2.1, this section demonstrates completion. □

3.2. Effects of the control on the bifurcation

In this subsection, consider the case with the controller below. First, under the influence of $\mu(t)$, the bifurcation and stability of E_* with respect to time delay are studied. Given the controller, the characteristic equation is written as follows

$$\Lambda_1(s) + \Lambda_2(s)e^{-s\tau} = 0, \tag{3.11}$$

where

$$\begin{aligned}\Lambda_1(s) =& a_1(s) + he^{-sv}s^{\alpha_1+\alpha_3} + hde^{-sv}s^{\alpha_1} - a_{11}he^{-sv}s^{\alpha_3} - a_{11}hde^{-sv}\\ &- hs^{\alpha_1+\alpha_3} - dhs^{\alpha_1} + a_{11}hs^{\alpha_3} + a_1hd,\\ \Lambda_2(s) =& a_2(s) - ha_{33}e^{-sv}s^{\alpha_1} + a_{11}a_{33}he^{sv} + ha_{33}s^{\alpha_1} - a_{11}a_{33}h.\end{aligned} \tag{3.12}$$

Theorem 3.3. *Assume* $\tau > 0$ *and the condition* (T_1) *hold. If the following condition holds:* $(T_4) : \kappa_1\iota_1 + \kappa_2\iota_2 > 0$*, where*

$$\begin{aligned}\kappa_1 &= (Re[\Lambda_1'(i\varphi_1)] + Re[\Lambda_2'(i\varphi_1)])\cos\varphi_1\tau_1 - (Im[\Lambda_1'(i\varphi_1)] + Im[\Lambda_2'(i\varphi_1)])\sin\varphi_1\tau_1,\\ \kappa_2 &= (Re[\Lambda_1'(i\varphi_1)] + Re[\Lambda_2'(i\varphi_1)])\sin\varphi_1\tau_1 + (Im[\Lambda_1'(i\varphi_1)] + Im[\Lambda_2'(i\varphi_1)])\cos\varphi_1\tau_1,\\ \iota_1 &= -\varphi_1 Im[\Lambda_2(i\varphi_1)], \quad \iota_2 = \varphi_1 Re[\Lambda_2(i\varphi_1)];\end{aligned}$$

then

1) *The positive equilibrium of System* (1.3) *is locally asymptotically stable for* $\tau \in [0, \tau_1^0)$.
2) *There exists a constant* $\tau_1^0 > 0$ *such that System* (1.3) *has a Hopf bifurcation.*

Proof. Let $s = i\varphi = \varphi e^{\frac{\pi i}{2}}$ and substituting it into (3.2), it can be obtained that

$$(\cos\varphi\tau - i\sin\varphi\tau) = -\frac{\Lambda_1(i\varphi)}{\Lambda_2(i\varphi)}, \qquad \Lambda_2(i\varphi) \neq 0; \tag{3.13}$$

separating the real and imaginary parts, one gets

$$\begin{cases}\cos\varphi\tau = -\dfrac{Re[\Lambda_1(i\varphi)]Re[\Lambda_2(i\varphi)] + Im[\Lambda_1(i\varphi)]Im[\Lambda_2(i\varphi)]}{Re^2[\Lambda_2(i\varphi)] + Im^2[\Lambda_2(i\varphi)]},\\ \sin\varphi\tau = \dfrac{Im[\Lambda_1(i\varphi)]Re[\Lambda_2(i\varphi)] - Re[\Lambda_1(i\varphi)]Im[\Lambda_2(i\varphi)]}{Re^2[\Lambda_2(i\varphi)] + Im^2[\Lambda_2(i\varphi)]},\end{cases}$$

where

$$\begin{aligned}Re[\Lambda_1(i\varphi)] =& Re[a_1(i\varphi)] + h\varphi^{(\alpha_1+\alpha_3)}\Big(\cos\varphi v\cos\frac{(\alpha_1+\alpha_3)\pi}{2} + \sin\varphi v\sin\frac{(\alpha_1+\alpha_3)\pi}{2}\\ &- \cos\frac{(\alpha_1+\alpha_3)\pi}{2}\Big) + dh\varphi^{\alpha_1}\Big(\cos\varphi v cos\frac{\alpha_1\pi}{2} + \sin\varphi v\sin\frac{\alpha_1\pi}{2} - \cos\frac{\alpha_1\pi}{2}\Big)\\ &- a_{11}h\varphi^{\alpha_3}\Big(\cos\varphi v\cos\frac{\alpha_3\pi}{2} + \sin\varphi v\sin\frac{\alpha_3\pi}{2} - \cos\frac{\alpha_3\pi}{2}\Big)\\ &- a_{11}hd\cos\varphi v + a_{11}hd,\\ Im[\Lambda_1(i\varphi)] =& Im[a_1(i\varphi)] + h\varphi^{(\alpha_1+\alpha_3)}\Big(\cos\varphi v\sin\frac{(\alpha_1+\alpha_3)\pi}{2} + \sin\varphi v\cos\frac{(\alpha_1+\alpha_3)\pi}{2}\end{aligned}$$

$$- \sin\frac{(\alpha_1+\alpha_3)\pi}{2}\Big) + dh\varphi^{\alpha_1}\Big(\cos\varphi v \sin\frac{\alpha_1\pi}{2} + \sin\varphi v\cos\frac{\alpha_1\pi}{2} - \sin\frac{\alpha_1\pi}{2}\Big)$$
$$- a_{11}h\varphi^{\alpha_3}\Big(\cos\varphi v\sin\frac{\alpha_3\pi}{2} + \sin\varphi v\cos\frac{\alpha_3\pi}{2} - \sin\frac{\alpha_3\pi}{2}\Big) - a_{11}hd\sin\varphi v,$$
$$\begin{aligned}Re[\Lambda_2(i\varphi)] =&Re[a_2(i\varphi)] - a_{33}h\varphi^{\alpha_1}\cos\varphi v\cos\frac{\alpha_1\pi}{2} - a_{33}h\varphi^{\alpha_1}\sin\varphi v\sin\frac{\alpha_1\pi}{2}\\ &+ a_{11}a_{33}h\cos\varphi v + a_{33}h\varphi^{\alpha_1}\cos\frac{\alpha_1\pi}{2} - a_{11}a_{33}h,\\ Im[\Lambda_2(i\varphi)] =&Im[a_2(i\varphi)] - a_{33}h\varphi^{\alpha_1}\cos\varphi v sin\frac{\alpha_1\pi}{2} + a_{33}h\varphi^{\alpha_1}\sin\varphi v\cos\frac{\alpha_1\pi}{2}\\ &+ a_{11}a_33h\sin\varphi v + a_{33}h\varphi^{\alpha_1}\sin\frac{\alpha_1\pi}{2}.\end{aligned}$$

Then, it can be obtained that

$$j_1{}^2(\varphi) + j_2{}^2(\varphi) = 1, \tag{3.14}$$

where

$$\begin{cases} j_1(\varphi) = \cos\varphi\tau = -\dfrac{Re[\Lambda_1(i\varphi)]Re[\Lambda_2(i\varphi)] + Im[\Lambda_1(i\varphi)]Im[\Lambda_2(i\varphi)]}{Re^2[\Lambda_2(i\varphi)] + Im^2[\Lambda_2(i\varphi)]},\\ j_2(\varphi) = \sin\varphi\tau = \dfrac{Im[\Lambda_1(i\varphi)]Re[\Lambda_2(i\varphi)] - Re[\Lambda_1(i\varphi)]Im[\Lambda_2(i\varphi)]}{Re^2[\Lambda_2(i\varphi)] + Im^2[\Lambda_2(i\varphi)]};\end{cases}$$

suppose that (3.14) has at least a positive real root φ_1. Then, it is obtained that

$$\begin{cases}\tau_1^{(k)} = \dfrac{1}{\varphi_1}[\arccos j_1(\varphi_1) + 2k\pi], \qquad k = 0, 1, 2\ldots\\ \tau_1^{(k)} = \dfrac{1}{\varphi_1}[\arcsin j_2(\varphi_1) + 2k\pi], \qquad k = 0, 1, 2\ldots\end{cases} \tag{3.15}$$

the bifurcation point is defined as $\tau_1 = \min\{\tau_1^{(k)}\}$.

Next, differentiating (3.11) with respect to τ gives

$$\Big(\Lambda_1{}'(s) + \Lambda_2{}'(s) - \tau\Lambda_2(s)e^{-s\tau}\Big)\frac{ds}{d\tau} - s\Lambda_2(s)e^{-s\tau} = 0;$$

then,

$$\left(\frac{ds}{d\tau}\right)^{-1} = \frac{\Lambda_1{}'(s)e^{s\tau} + \Lambda_2{}'(s)}{s\Lambda_2(s)} - \frac{\tau}{s}; \tag{3.16}$$

substituting $s = i\varphi_1$ and $\tau = \tau_1$ into (3.16), it gives

$$\left(\frac{ds}{d\tau}\right)^{-1}\Bigg|_{s=i\varphi_1} = \frac{\kappa_1 + i\kappa_2}{\iota_1 + i\iota_2} - \frac{\tau_1}{i\varphi_1};$$

thus, one obtains

$$Re\left[\left(\frac{ds}{d\tau}\right)^{-1}\right] = \frac{\kappa_1\iota_1 + \kappa_2\iota_2}{\iota_1{}^2 + \iota_2{}^2}.$$

If $\kappa_1\iota_1 + \kappa_2\iota_2 > 0$, we have that $Re\left[\left(\frac{ds}{d\tau}\right)^{-1}\right]_{\tau=\tau_1} > 0$. According to Theorem 2.1, this section demonstrates completion. □

Theorem 3.4 mainly provides a method to find a given $\mu(t)$ bifurcation point. However, the most important point should be to find the appropriate control parameters for the time delay. And the characteristic equation is written as follows

$$W_1(s) + W_2(s)e^{-s\upsilon} = 0, \tag{3.17}$$

where

$$\begin{aligned} W_1(s) =& a_1(s) + \mathcal{A}_2(s)e^{-s\tau} - W_2(s), \\ W_2(s) =& h[(a_{11}a_{33} - a_{33}s^{\alpha_1})e^{-s\tau} + ds^{\alpha_1} + s^{\alpha_1+\alpha_3} - a_{11}d - a_{11}s^{\alpha_3}]. \end{aligned} \tag{3.18}$$

Theorem 3.4. *Assume the condition* (T_1) *holds. If the following condition holds:*
$(T_5) : P_1Q_1 + P_2Q_2 > 0$, *where*

$$\begin{aligned} P_1 &= (Re[W_1{}'(i\varphi_2)] + Re[W_2{}'(i\varphi_2)])\cos\varphi_2\upsilon_0 - (Im[W_1{}'(i\varphi_2)] + Im[W_2{}'(i\varphi_2)])\sin\varphi_2\upsilon_0, \\ P_2 &= (Re[W_1{}'(i\varphi_2)] + Re[W_2{}'(i\varphi_2)])\sin\varphi_2\upsilon_0 + (Im[W_1{}'(i\varphi_2)] + Im[W_2{}'(i\varphi_2)])\cos\varphi_2\upsilon_0, \\ Q_1 &= -\varphi_2 Im[W_2(i\varphi_2)], \quad Q_2 = \varphi_2 Re[W_2(i\varphi_2)]; \end{aligned}$$

then

(1) *The positive equilibrium of System* (1.3) *is locally asymptotically stable for* $\upsilon \in [0, \upsilon_0)$.
(2) *There exists a constant* $\upsilon = \upsilon_0 > 0$ *such that System* (1.3) *has a Hopf bifurcation.*

Proof. Let $s = i\varphi = \varphi e^{\frac{\pi i}{2}}$ and substituting it into (3.17), it can be obtained that

$$(\cos\varphi\upsilon - i\sin\varphi\upsilon) = -\frac{W_1(i\varphi)}{W_2(i\varphi)}, \qquad W_2(i\varphi) \neq 0; \tag{3.19}$$

separating the real and imaginary parts, one gets

$$\begin{cases} \cos\varphi\upsilon = -\dfrac{Re[W_1(i\varphi)]Re[W_2(i\varphi)] + Im[W_1(i\varphi)]Im[W_2(i\varphi)]}{Re^2[W_2(i\varphi)] + Im^2[W_2(i\varphi)]}, \\ \sin\varphi\upsilon = \dfrac{Im[W_1(i\varphi)]Re[W_2(i\varphi)] - Re[W_1(i\varphi)]Im[W_2(i\varphi)]}{Re^2[W_2(i\varphi)] + Im^2[W_2(i\varphi)]}, \end{cases}$$

where

$$\begin{aligned} Re[W_1(i\varphi)] =& Re[a_1(i\varphi)] + Re[a_2(i\varphi)]\cos\varphi\tau + Im[a_2(i\varphi)]\sin\varphi\tau - Re[W_2(i\varphi)], \\ Im[W_1(i\varphi)] =& Im[a_1(i\varphi)] + Im[a_2(i\varphi)]\cos\varphi\tau - Re[a_2(i\varphi)]\sin\varphi\tau - Im[W_2(i\varphi)], \\ Re[W_2(i\varphi)] =& ha_{11}a_{33}\cos\varphi v - ha_{33}\varphi^{\alpha_1}\Big(\cos\varphi v\cos\frac{\alpha_1\pi}{2} + \sin\frac{\alpha_1\pi}{2}\sin\varphi v\Big) + hd\varphi^{\alpha_1}\cos\frac{\alpha_1\pi}{2}, \\ & + h\varphi^{\alpha_1+\alpha_3}\cos\frac{(\alpha_1+\alpha_3)\pi}{2} - a_{11}dh - a_{11}h\varphi^{\alpha_3}\cos\frac{\alpha_3\pi}{2} \\ Im[W_2(i\varphi)] =& -ha_{11}a_{33}\sin\varphi v - ha_{33}\varphi^{\alpha_1}\Big(-\cos\varphi v\sin\frac{\alpha_1\pi}{2} + \sin\frac{\alpha_1\pi}{2}\cos\varphi v\Big) + hd\varphi^{\alpha_1}\sin\frac{\alpha_1\pi}{2} \\ & + h\varphi^{\alpha_1+\alpha_3}\sin\frac{(\alpha_1+\alpha_3)\pi}{2} - a_{11}dh - a_{11}h\varphi^{\alpha_3}\sin\frac{\alpha_3\pi}{2}. \end{aligned}$$

Then, it can be obtained that

$$ {r_1}^2(\varphi) + {r_2}^2(\varphi) = 1, \tag{3.20} $$

where

$$ \begin{cases} r_1(\varphi) = \cos\varphi\upsilon = -\dfrac{Re[W_1(i\varphi)]Re[W_2(i\varphi)] + Im[W_1(i\varphi)]Im[W_2(i\varphi)]}{Re^2[W_2(i\varphi)] + Im^2[W_2(i\varphi)]}, \\ r_2(\varphi) = \sin\varphi\upsilon = \dfrac{Im[W_1(i\varphi)]Re[W_2(i\varphi)] - Re[W_1(i\varphi)]Im[W_2(i\varphi)]}{Re^2[W_2(i\varphi)] + Im^2[W_2(i\varphi)]}; \end{cases} $$

suppose that (3.20) has at least a positive real root φ_2. Then, it is obtained that

$$ \begin{cases} \upsilon_0^{(k)} = \dfrac{1}{\varphi_2}[\arccos r_1(\varphi_1) + 2k\pi], \qquad k = 0, 1, 2 \ldots \\ \upsilon_0^{(k)} = \dfrac{1}{\varphi_2}[\arcsin r_2(\varphi_1) + 2k\pi], \qquad k = 0, 1, 2 \ldots \end{cases} \tag{3.21} $$

the bifurcation point is defined as $\upsilon_0 = \min\{\upsilon_0^{(k)}\}$.

Next, differentiating (3.17) with respect to υ gives

$$ \left(W_1'(s) + W_2'(s) - \tau W_2(s)e^{-s\upsilon}\right)\frac{ds}{d\upsilon} - sW_2(s)e^{-s\upsilon} = 0; $$

then, we have

$$ \left(\frac{ds}{d\upsilon}\right)^{-1} = \frac{W_1'(s)e^{s\upsilon} + W_2'(s)}{sW_2(s)} - \frac{\upsilon}{s}; \tag{3.22} $$

substituting $s = i\varphi_2$ and $\upsilon = \upsilon_0$ into (3.22), it gives

$$ \left.\left(\frac{ds}{d\upsilon}\right)^{-1}\right|_{s=i\varphi_2} = \frac{P_1 + iP_2}{Q_1 + iQ_2} - \frac{\upsilon_0}{i\varphi_2}; $$

thus, one obtains

$$ Re\left[\left(\frac{ds}{d\upsilon}\right)^{-1}\right] = \frac{P_1Q_1 + P_2Q_2}{{Q_1}^2 + {Q_2}^2}. $$

If $P_1Q_1 + P_2Q_2 > 0$, we have that $Re\left[\left(\frac{ds}{d\upsilon}\right)^{-1}\right]_{\upsilon=\upsilon_0} > 0$. According to Theorem 2.1, this section demonstrates completion. □

4. Numerical simulations

In this section, some concrete examples are given to illustrate the theory presented in the previous section. First, the system unaffected by the controller is considered; the system is shown as

Example 4.1.

$$ \begin{cases} D_t^{\alpha_1}S(t) = \dfrac{7}{20}\left(\dfrac{3}{2} - S\right) - \dfrac{15SX}{7}, S(0) = 0.7, \\ D_t^{\alpha_2}X(t) = \dfrac{3}{2}SX - \dfrac{7}{20}X - \dfrac{75XY}{8}, X(0) = 0.1, \\ D_t^{\alpha_3}Y(t) = 3X(t-\tau)Y(t-\tau) - \dfrac{7}{20}Y, Y(0) = 0.08. \end{cases} \tag{4.1} $$

According to the above system, the equilibrium point can be calculated as $E^* = (0.8750, 0.1167, 0.1027)$, and all of the conditions in Theorem 3.1 can be satisfied. Thus, if $\tau = 0$, the positive equilibrium of System (4.1) is locally asymptotically stable, in which $\alpha_1 = 0.98$, $\alpha_2 = 0.96$ and $\alpha_3 = 0.97$, as shown in Figure 1.

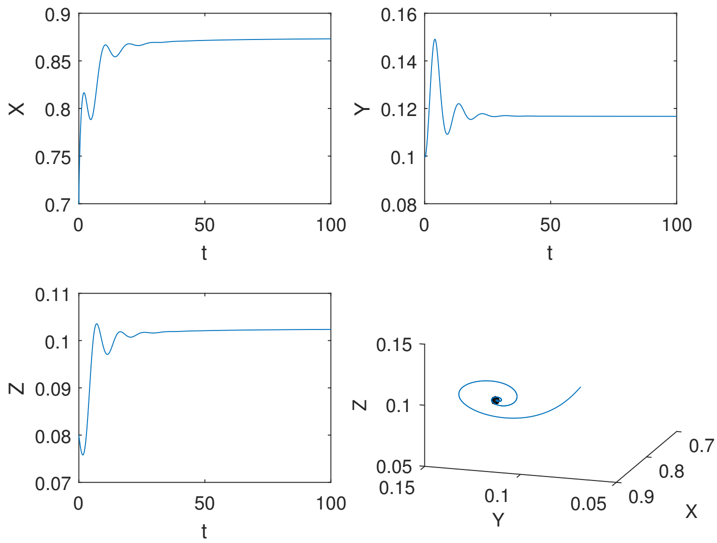

Figure 1. Local asymptotic stability of the positive equilibrium E_* when $\tau = 0$.

Then, according to Theorem 3.2, when $\alpha_1 = 0.8$, $\alpha_2 = 0.95$ and $\alpha_3 = 0.97$, it can be calculated that $\tau_0 = 3.7286$. As a consequence, Hopf bifurcation occurs at $\tau_0 = 3.7286$. As shown in Figure 2, when $\tau = 3 < \tau_0$, the positive equilibrium point is locally asymptotically stable.

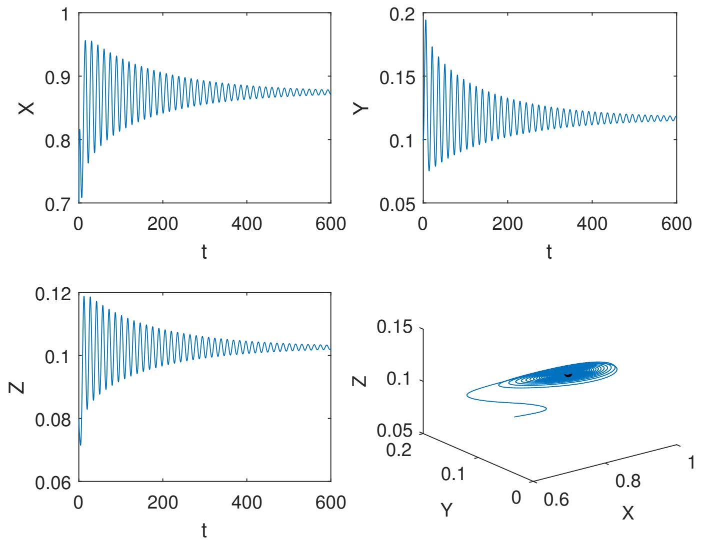

Figure 2. Effects of τ on E_* when $\tau = 3$.

When $\tau = 3.8 > \tau_0$, it becomes unstable, as shown in Figure 3.

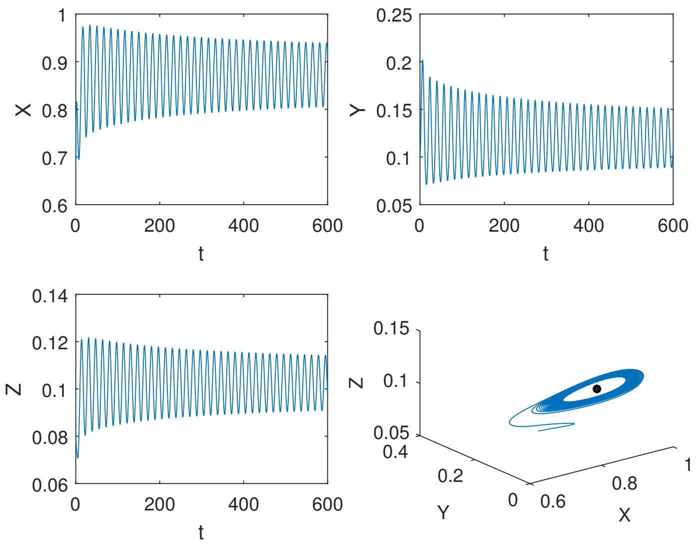

Figure 3. Effects of τ on E_* when $\tau = 3.8$.

In order to show the effects of incommensurate orders on Hopf bifurcation, let $\alpha_1 = 0.8$, $\alpha_2 = 0.95$ and $\tau = 3.8$; when $\alpha_3 = 0.92$, we obtain $\tau_0 = 4.236 > 3.8$; when $\alpha_3 = 0.97$, we obtain $\tau_0 = 3.6285 < 3.8$. As Figure 4 shows, when α_3 changes from 0.92 to 0.97, the equilibrium point becomes stable. It can be seen that the change of fractional-order will also affect the change of system stability.

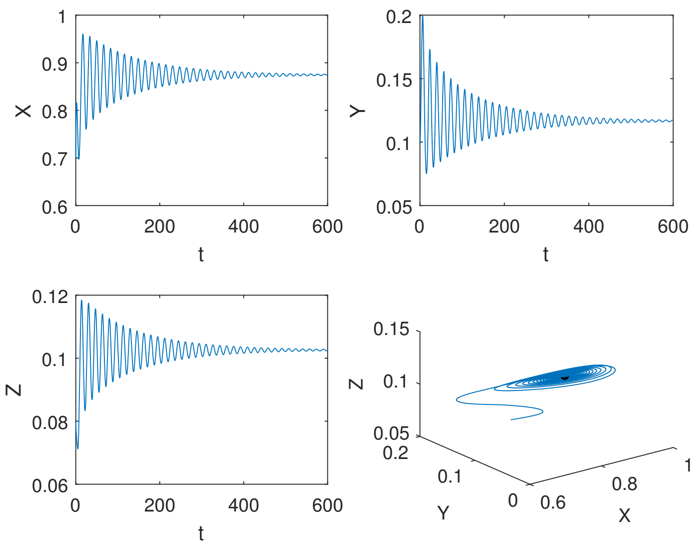

Figure 4. Effects on E_* when $\tau = 3.8, \alpha_3 = 0.92$.

Next, consider a system that adds an extended feedback controller:

Example 4.2.

$$
\begin{cases}
D_t^{\alpha_1} S(t) = \frac{7}{20}\left(\frac{3}{2} - S\right) - \frac{15SX}{7}, S(0) = 0.7, \\
D_t^{\alpha_2} X(t) = \frac{3}{2} SX - \frac{7}{20} X - \frac{75XY}{8} + h[X(t) - X(t-\nu)], X(0) = 0.1, \\
D_t^{\alpha_3} Y(t) = 3X(t-\tau)Y(t-\tau) - \frac{7}{20} Y, Y(0) = 0.08.
\end{cases}
\tag{4.2}
$$

According to Theorem 3.4, when $h = 0.5$, it is calculated that $\nu_0 = 3.3769$. As can be seen from Figures 5 and 6, when $\nu = 3 < \nu_0$, the positive equilibrium point of System (4.2) is locally asymptotically stable; when $\nu = 4 > \nu_0$, the system is unstable.

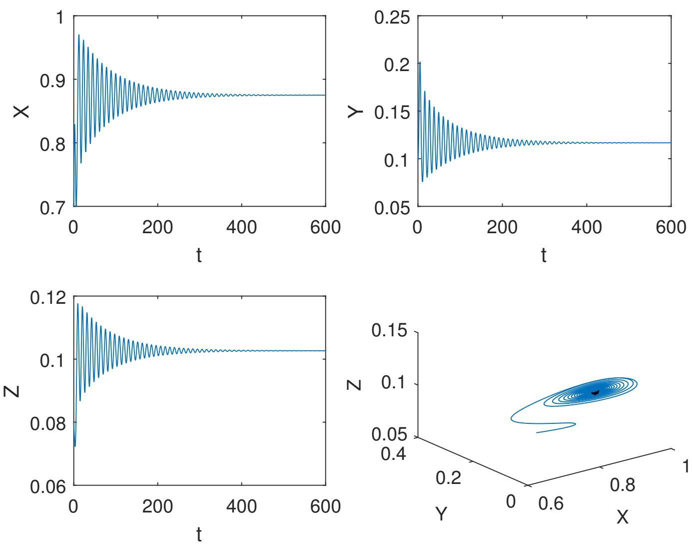

Figure 5. Effects of υ on E_* when $\upsilon = 3$.

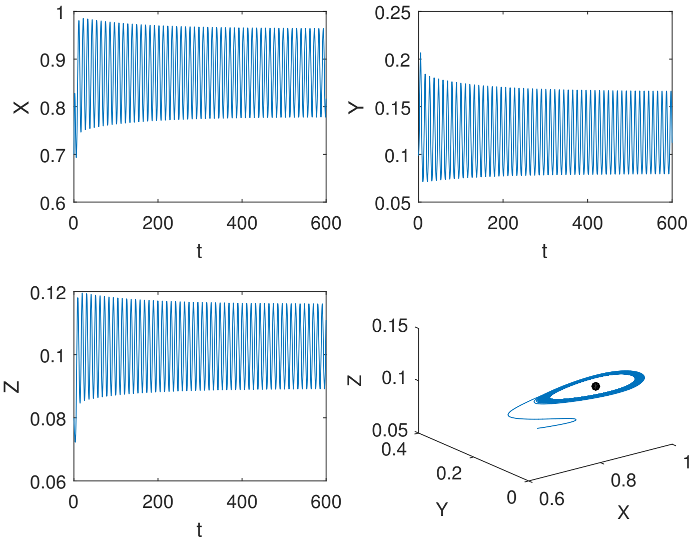

Figure 6. Effects of υ on E_* when $\upsilon = 4$.

5. Conclusions

In this work, a controller was added to a fractional-order chemostat model with incommensurate delay to study its effect on system stability and bifurcation. We first considered some basic results for the positive equilibrium E_* in the absence of a controller. Then the influence of the controller on the bifurcation and stability of the system was analyzed in detail. Considering the case of incommensurate order, given certain conditions, the corresponding control bifurcation parameters were obtained accurately. Finally, in order to support the theoretical analysis results, numerical simulations were carried out. The results show that the stability and bifurcation of the system were effectively controlled.

Acknowledgments

This research was funded by NSFC grant number 11571207, SDNSF grant number ZR2021MA064 and the Taishan Scholar project.

Conflict of interest

The authors declare that there is no conflict of interest.

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