



Research article

Periodic functions related to the Gompertz difference equation

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Abstract: We investigate periodicity of functions related to the Gompertz difference equation. In particular, we derive difference equations that must be satisfied to guarantee periodicity of the solution.

Keywords: difference equations; periodic solution; Gompertz; population model

1. Introduction

We study the Gompertz difference equation

$$y^\Delta(t) = (\ominus r)(t)y(t) \left(K(t) + a + \int_0^t \frac{y^\Delta(\tau)}{y(\tau)} \Delta\tau \right), \quad y(0) = y_0, \quad (1.1)$$

as well as periodic functions that arise from it. This is to say when $\omega \in \{1, 2, \dots\}$, $f: \mathbb{N}_0 \rightarrow \mathbb{R}$ is ω -periodic if $f(t + \omega) = f(t)$ for all $t \in \mathbb{N}_0$. Here, $(\ominus r)(t) = \frac{-r(t)}{1 + r(t)}$ is the time scales analogue of the growth rate while $K(t)$ is the analogue of the carrying capacity at time t from the traditional continuous Gompertz model. Throughout, we will use notation inspired from time scales calculus for the time scale $\mathbb{T} = \mathbb{N}_0$, including $\sigma(t) = t + 1$, $y^\Delta(t) = y(\sigma(t)) - y(t)$, and the integration symbol representing summation, i.e. $\int_a^b f(t)\Delta t = \sum_{k=a}^{b-1} f(k)$. See the monograph [1] for the usual introduction to dynamic equations on time scales and see the recent texts on first and second order boundary value problems on time scales [2] and its companion book on third, fourth, and higher-order boundary value problems on time scales [3] for more recent books.

In [4], the model (1.1) as well as a second model without the \ominus was introduced, solved, and bounds of its solutions were established. Three discrete fractional analogues of (1.1) were explored in [5] by changing the difference to a fractional difference and exploring defining the logarithm with a fractional

integral. These three models were compared to another existing fractional Gompertz difference equation [6], which was built around using the classical logarithm instead of a time scales logarithm. The solution of (1.1) can be normalized to create a probability distribution which was studied in [7] where bounds on the expected value were derived and a connection between the classical continuous Gompertz distribution with the q -geometric distribution of the second kind was established. An alternative approach to Gompertz equations on time scales appears in [8] which uses the \odot operation to define a Gompertz dynamic equation.

Gompertz models have been used to study a number of applications in both discrete and continuous settings. This includes studying the growth rate of tumors [9, 10], modeling growth of prey in predator-prey dynamics [11], as well as study the change in cost in adopting new technologies [12, 13], effect of seasonality for Gompertz models using time series [14], and the spread of COVID-19 [15, 16].

2. Preliminaries and definitions

Before introducing our main results, some preliminary definitions and results are in order. Equation (1.1) has the unique solution $y(t) = y_0 e_p(t, 0)$, where

$$p(t) = (\ominus r)(t) \left(a e_{\ominus r}(t, 0) - \int_0^t (\ominus r)(s) e_{\ominus r}(t, \sigma(s)) K(s) \Delta s - K(t) \right). \quad (2.1)$$

Here, $e_f: \mathbb{N}_0 \times \mathbb{N}_0 \rightarrow \mathbb{R}$, called the discrete exponential, is the unique solution of the initial value problem $y^\Delta = fy, y(0) = 1$. We often make use of the so-called “simple useful formula,”

$$e_p(\sigma(t), 0) = (1 + p(t)) e_p(t, 0). \quad (2.2)$$

when rewriting exponentials.

Time scales integration by parts is given by

$$\int_a^b f(\tau) g^\Delta(\tau) \Delta\tau = f(t)g(t) \Big|_a^b - \int_a^b f^\Delta(\tau)g(\sigma(\tau))\Delta\tau. \quad (2.3)$$

A function $f: \mathbb{N}_0 \rightarrow \mathbb{C}$ is said to be of exponential order α [17, Definition 4.1] if there is an $\alpha \in \mathbb{R}$ with $1 + \alpha > 0$ and a $M > 0$ such that $|f(t)| \leq M e_\alpha(t, s)$ for all $t \in \mathbb{N}_0$. In particular, [17, Lemma 4.4] shows that if f is of exponential order α and $|z + 1| > 1 + \alpha$, then $\lim_{t \rightarrow \infty} f(t) e_{\ominus z}(t, 0) = 0$.

The time scales Laplace transform is given by [1, Section 3.10]

$$\mathcal{L}\{f\}(z) = \int_0^\infty f(\tau) e_{\ominus z}(\sigma(\tau), 0) \Delta\tau.$$

which for $\mathbb{T} = \mathbb{N}_0$ is a scaled and shifted \mathcal{Z} -transform. It's known [18, Theorem 3.2] that if w is a regressive constant and $\mathbb{T} = \mathbb{N}_0$, then

$$\mathcal{L}\{f e_w^\sigma(\cdot, s)\}(z) = \mathcal{L}\{f\}(z \ominus w), \quad (2.4)$$

and if $X(t) = \int_0^t x(\tau) \Delta\tau$, then [18, Theorem 6.4]

$$\mathcal{L}\{X\}(z) = \frac{1}{z} \mathcal{L}\{x\}(z). \quad (2.5)$$

A well-known identity for the $\mathbb{T} = \mathbb{N}_0$ delta derivative operator is

$$f(k + \omega) = \sum_{j=0}^{\omega} \binom{\omega}{j} f^{\Delta^j}(k). \quad (2.6)$$

The Laplace transform for differences of f is given by

$$\mathcal{L}\{f^{\Delta^j}\}(z) = z^j \mathcal{L}\{f\}(z) - \sum_{\ell=0}^{j-1} z^\ell f^{\Delta^{j-\ell-1}}(0). \quad (2.7)$$

The Laplace transform of the shifted argument is useful for the sequel.

Lemma 1. *If f is of exponential order α , then*

$$\mathcal{L}\{f(\cdot + \omega)\}(z) = (z + 1)^\omega \mathcal{L}\{f\}(z) - \sum_{j=0}^{\omega} \binom{\omega}{j} \sum_{\ell=0}^{j-1} z^\ell f^{\Delta^{j-\ell-1}}(0). \quad (2.8)$$

Proof. Applying the Laplace transform to (2.6) and using (2.7), we have

$$\begin{aligned} \mathcal{L}\{f(\cdot + \omega)\}(z) &= \sum_{j=0}^{\omega} \binom{\omega}{j} \mathcal{L}\{f^{\Delta^j}\}(z) \\ &= \sum_{j=0}^{\omega} \binom{\omega}{j} \left[z^j \mathcal{L}\{f\}(z) - \sum_{\ell=0}^{j-1} z^\ell f^{\Delta^{j-\ell-1}}(0) \right] \\ &= \mathcal{L}\{f\}(z) \left(\sum_{j=0}^{\omega} \binom{\omega}{j} z^j \right) - \sum_{j=0}^{\omega} \binom{\omega}{j} \sum_{\ell=0}^{j-1} z^\ell f^{\Delta^{j-\ell-1}}(0). \end{aligned}$$

An application of the binomial theorem to the first summation completes the proof. \square

Now we calculate the discrete Laplace transform of a certain time-dependent delta integral.

Lemma 2. *If f is of exponential order α and $X(t) = \int_t^{t+\omega} f(\tau) \Delta\tau$, then for all $z \in \mathbb{C}$ with $|1 + z| > 1 + \alpha$,*

$$\mathcal{L}\{X\}(z) = \frac{1}{z} \sum_{k=0}^{\omega-1} f(k) + \frac{(z + 1)^\omega - 1}{z} \mathcal{L}\{f\}(z) - \frac{1}{z} \sum_{j=0}^{\omega} \binom{\omega}{j} \sum_{\ell=0}^{j-1} z^\ell f^{\Delta^{j-\ell-1}}(0). \quad (2.9)$$

Proof. First use (2.2) to see

$$e_{\ominus z}(k + 1, 0) = (1 + (\ominus z)) e_{\ominus z}(k, 0) = \frac{1}{1 + z} e_{\ominus z}(k, 0).$$

Calculate, where k is understood to be the variable,

$$\left[\sum_{\ell=k}^{k+\omega-1} f(\ell) \right]^\Delta = \sum_{\ell=k+1}^{k+\omega} f(\ell) - \sum_{\ell=k}^{k+\omega-1} f(\ell) = f(k + \omega) - f(k).$$

Now since $X(t) = \sum_{\ell=t}^{t+\omega-1} f(\ell)$, (2.2) reveals

$$\mathcal{L}\{X\}(z) = \int_0^\infty \left(\sum_{\ell=\tau}^{\tau+\omega-1} f(\ell) \right) e_{\Theta z}(\sigma(\tau), 0) \Delta\tau = \frac{1}{1+z} \int_0^\infty \left(\sum_{\ell=\tau}^{\tau+\omega-1} f(\ell) \right) e_{\Theta z}(\tau, 0) \Delta\tau.$$

Since (Θz) is constant and $e_{\Theta z}^\Delta = (\Theta z)e_{\Theta z}$, we observe

$$\mathcal{L}\{X\}(z) = \frac{1}{1+z} \frac{1}{(\Theta z)} \int_0^\infty \left(\sum_{\ell=\tau}^{\tau+\omega-1} f(\ell) \right) e_{\Theta z}^\Delta(\tau, 0) \Delta\tau = -\frac{1}{z} \int_0^\infty \left(\sum_{\ell=\tau}^{\tau+\omega-1} f(\ell) \right) e_{\Theta z}^\Delta(\tau, 0) \Delta\tau.$$

Apply (2.3) to obtain

$$\mathcal{L}\{X\}(z) = -\frac{1}{z} \left(\sum_{\ell=\tau}^{\tau+\omega-1} f(\ell) \right) e_{\Theta z}(\tau, 0) \Big|_{\tau=0}^{\tau=\infty} + \frac{1}{z} \int_0^\infty (f(\tau+\omega) - f(\tau)) e_{\Theta z}(\sigma(\tau), 0) \Delta\tau.$$

Thus we have

$$\mathcal{L}\{X\}(z) = \frac{1}{z} \sum_{k=0}^{\omega-1} f(k) + \frac{1}{z} \mathcal{L}\{f(\cdot + \omega)\}(z) - \frac{1}{z} \mathcal{L}\{f\}(z).$$

Applying (2.8) to the middle term of the right-hand side completes the proof. \square

3. Periodicity of p

First we establish which functions r yield $e_{\Theta r}$ to be ω -periodic.

Lemma 3. *The discrete exponential is periodic, meaning*

$$e_{\Theta r}(t + \omega, 0) = e_{\Theta r}(t, 0) \tag{3.1}$$

if and only if

$$r(t + \omega - 1) = -1 + \prod_{k=t}^{t+\omega-2} \frac{1}{1+r(k)}. \tag{3.2}$$

Proof. First calculate

$$1 + (\Theta r)(t) = 1 - \frac{r(t)}{1+r(t)} = \frac{1}{1+r(t)}.$$

Now, (3.1) becomes

$$\prod_{k=0}^{t+\omega-1} \frac{1}{1+r(k)} = \prod_{k=0}^{t-1} \frac{1}{1+r(k)},$$

hence $1 = \prod_{k=t}^{t+\omega-1} \frac{1}{1+r(k)}$, which is equivalent to $r(t + \omega - 1) = -1 + \prod_{k=t}^{t+\omega-2} \frac{1}{1+r(k)}$. Since all steps are reversible, the proof is complete. \square

Since the ω -periodicity of $e_{\ominus r}$ is equivalent to r satisfying the difference equation (3.2), solving it is of importance.

Lemma 4. *If $r(0), \dots, r(\omega - 2)$ are known, then the unique solution of (3.2) is ω -periodic.*

Proof. Use (3.2) with $t = 0$ to generate the ω th value

$$r(\omega - 1) = -1 + \frac{1}{(1 + r(0))(1 + r(1)) \dots (1 + r(\omega - 2))}.$$

We claim that the function r is ω -periodic. From (3.2), we obtain

$$r(t + \omega) = -1 + \prod_{k=t+1}^{t+\omega-1} \frac{1}{1 + r(k)} = -1 + \frac{1}{(1 + r(t+1))(1 + r(t+2)) \dots (1 + r(t + \omega - 1))}.$$

But also by (3.2),

$$1 + r(t + \omega - 1) = \prod_{k=t}^{t+\omega-2} \frac{1}{1 + r(k)} = \frac{1}{(1 + r(t))(1 + r(t+1)) \dots (1 + r(t + \omega - 2))}.$$

Therefore,

$$\begin{aligned} r(t + \omega) &= -1 + \frac{1}{(1 + r(t+1))(1 + r(t+2)) \dots (1 + r(t + \omega - 2)) \left[\frac{1}{(1 + r(t)) \dots (1 + r(t + \omega - 2))} \right]} \\ &= -1 + \frac{1}{\frac{1}{1 + r(t)}} = r(t), \end{aligned}$$

completing the proof. □

By (2.1), when p is ω -periodic, $p(t + \omega) = p(t)$ expands to

$$\begin{aligned} (\ominus r)(t + \omega) \left(ae_{\ominus r}(t + \omega, 0) - \int_0^{t+\omega} (\ominus r)(s) e_{\ominus r}(t + \omega, \sigma(s)) K(s) \Delta s - K(t + \omega) \right) \\ = (\ominus r)(t) \left(ae_{\ominus r}(t, 0) - \int_0^t (\ominus r)(s) e_{\ominus r}(t, \sigma(s)) K(s) \Delta s - K(t) \right). \quad (3.3) \end{aligned}$$

Theorem 5. *If r solves (3.2), then $p(t + \omega) = p(t)$ if and only if*

$$K(t + \omega) = K(t) - \int_t^{t+\omega} (\ominus r)(s) e_{\ominus r}(t + \omega, \sigma(s)) K(s) \Delta s.$$

Proof. By Lemma 3, $e_{\ominus r}(\cdot, 0)$ is ω -periodic. By Lemma 4, r is ω -periodic, hence $(\ominus r)$ is also ω -periodic. Using (3.3), we divide by $(\ominus r)(t + \omega)$ and subtract $ae_{\ominus r}(t + \omega, 0)$ to obtain

$$- \int_0^{t+\omega} (\ominus r)(s) e_{\ominus r}(t + \omega, \sigma(s)) K(s) \Delta s - K(t + \omega) = - \int_0^t (\ominus r)(s) e_{\ominus r}(t, \sigma(s)) K(s) \Delta s - K(t).$$

Hence

$$0 = K(t + \omega) - K(t) + \int_0^t (\Theta r)(s) [e_{\Theta r}(t + \omega, \sigma(s)) - e_{\Theta r}(t, \sigma(s))] K(s) \Delta s + \int_t^{t+\omega} (\Theta r)(s) e_{\Theta r}(t + \omega, \sigma(s)) \Delta s. \quad (3.4)$$

By the semigroup property and the periodicity of $e_{\Theta r}(\cdot, 0)$,

$$e_{\Theta r}(t + \omega, \sigma(s)) - e_{\Theta r}(t, \sigma(s)) = [e_{\Theta r}(t + \omega, 0) - e_{\Theta r}(t, 0)] e_{\Theta r}(0, \sigma(s)) = 0, \quad (3.5)$$

and applying (3.5) to (3.4) completes the proof. \square

Theorem 6. *If $r(t) = r$ is constant and K is ω -periodic, then $p(t + \omega) = p(t)$ if and only if*

$$K(t + \omega - 1) = \frac{a}{r(1+r)^t} \left[1 - \frac{1}{(1+r)^\omega} \right] + \frac{1}{1+r} \int_0^t \frac{K(s)}{(1+r)^{t-s-1}} \Delta s - \frac{1}{1+r} \int_0^{t+\omega-1} \frac{K(s)}{(1+r)^{t+\omega-s-1}} \Delta s.$$

Proof. From (3.3), since r is constant, so is (Θr) , hence both $(\Theta r)(t + \omega)$ and $(\Theta r)(t)$ can be divided off. Similarly, since $K(t + \omega) = K(t)$, those terms also vanish in (3.3). What remains is

$$ae_{\Theta r}(t + \omega, 0) - (\Theta r) \int_0^{t+\omega} e_{\Theta r}(t + \omega, \sigma(s)) K(s) \Delta s = ae_{\Theta r}(t, 0) - (\Theta r) \int_0^t e_{\Theta r}(t, \sigma(s)) K(s) \Delta s$$

Thus,

$$\frac{a}{(1+r)^{t+\omega}} + \frac{r}{1+r} \int_0^{t+\omega} \frac{K(s)}{(1+r)^{t+\omega-s-1}} \Delta s = \frac{a}{(1+r)^t} + \frac{r}{1+r} \int_0^t \frac{K(s)}{(1+r)^{t-\sigma(s)}} \Delta s$$

Now

$$\frac{a}{(1+r)^{t+\omega}} + \frac{r}{1+r} \int_0^{t+\omega-1} \frac{K(s)}{(1+r)^{t+\omega-s-1}} \Delta s + rK(t + \omega - 1) = \frac{a}{(1+r)^t} + \frac{r}{1+r} \int_0^t \frac{K(s)}{(1+r)^{t-s-1}} \Delta s,$$

and solving for $K(t + \omega - 1)$ completes the proof. \square

4. Periodicity of e_p

Define $\alpha(t, s) := (\Theta r)(s) e_{\Theta r}(t, \sigma(s)) K(s)$ and

$$\beta(t) := \frac{1}{(\Theta r)(t + \omega - 1)} \left[1 - \frac{1}{e_p(t + \omega - 1, t)} \right] + ae_{\Theta r}(t + \omega - 1, 0).$$

Theorem 7. *If $r: \mathbb{N}_0 \rightarrow \mathbb{R}$, then the function $t \mapsto e_p(t, 0)$ is ω -periodic if and only if*

$$K(t + \omega - 1) = \beta(t) - \int_0^{t+\omega-1} \alpha(t + \omega - 1, s) \Delta s. \quad (4.1)$$

Proof. If e_p is ω -periodic, then using the semigroup property of e_p , we obtain

$$p(t + \omega - 1) = -1 + \frac{1}{e_p(t + \omega - 1, t)}.$$

By (2.1), this becomes

$$(\ominus r)(t + \omega - 1) \left[ae_{\ominus r}(t + \omega - 1, 0) - \int_0^{t+\omega-1} \alpha(t + \omega - 1, s) \Delta s - K(t + \omega - 1) \right] = -1 + \frac{1}{e_p(t + \omega - 1, t)}, \quad (4.2)$$

which we rearrange to obtain (4.1). All steps are reversible so the converse is also true, completing the proof. \square

We provide a numerical example of Theorem 7 in Figure 1. It is difficult in general to solve (4.1) in closed form, but if r is a constant function, then it may be solved with Laplace transform techniques.

Theorem 8. *If $r \in \mathcal{R}_c(\mathbb{N}_0, \mathbb{R})$ and K is of exponential order α , then for all $|z + 1| > 1 + \alpha$, the Laplace transform of (4.1) is*

$$\begin{aligned} \mathcal{L}\{K\}(z) = & \frac{1}{(z + 1)^{\omega-1} - \frac{r((z \oplus r) + 1)^{\omega-1}}{(1 + r)^{\omega-1}(z \oplus r)}} \times \left[\mathcal{L}\{\beta\}(z) + \frac{r}{(1 + r)^{\omega-1}(z \oplus r)} \sum_{k=0}^{\omega-2} e_r(\sigma(k), 0)K(k) \right. \\ & \left. + \sum_{j=0}^{\omega-1} \binom{\omega-1}{j} \sum_{\ell=0}^{j-1} z^\ell \left[K^{\Delta^{k-\ell-1}}(0) - \frac{r}{(1 + r)^{\omega-1}(z \oplus r)} [e_r(\sigma(\cdot), 0)K(\cdot)]^{\Delta^{k-\ell-1}}(0) \right] \right]. \end{aligned}$$

Proof. By the semigroup and reciprocal properties for the discrete exponential, (4.1) becomes

$$K(t + \omega - 1) = \beta(t) - (\ominus r)e_{\ominus r}(t + \omega - 1, 0) \int_0^{t+\omega-1} e_r(\sigma(s), 0)K(s) \Delta s.$$

By (2.8), we know that

$$\mathcal{L}\{K(\cdot + \omega - 1)\}(z) = (z + 1)^{\omega-1} \mathcal{L}\{K\}(z) - \sum_{j=0}^{\omega-1} \binom{\omega-1}{j} \sum_{\ell=0}^{j-1} z^\ell K^{\Delta^{j-\ell-1}}(0)$$

Using (2.2), compute

$$e_{\ominus r}(t + \omega - 1, 0) = e_{\ominus r}(\sigma^{\omega-1}(t), 0) = (1 + (\ominus r))^{\omega-2} e_{\ominus r}(t + 1, 0) = \frac{e_{\ominus r}(t + 1, 0)}{(1 + r)^{\omega-2}}. \quad (4.3)$$

Let

$$\begin{aligned} g(t) &= (\ominus r)e_{\ominus r}(t + \omega - 1, 0) \int_0^t e_r(\sigma(s), 0)K(s) \Delta s \\ &= \frac{-r}{1 + r} e_{\ominus r}(t + \omega - 1, 0) \int_0^t e_r(\sigma(s), 0)K(s) \Delta s \\ &= \frac{-r}{(1 + r)^{\omega-1}} e_{\ominus r}(t + 1, 0) \int_0^t e_r(\sigma(s), 0)K(s) \Delta s \end{aligned}$$

Using (2.4), (2.5), and (4.3), we compute

$$\begin{aligned}\mathcal{L}\{g\}(z) &= \frac{-r}{(1+r)^{\omega-1}} \mathcal{L} \left\{ \int_0^{\cdot} e_r(\sigma(s), 0)K(s)\Delta s \right\} (z \oplus r) \\ &= \frac{-r}{(1+r)^{\omega-1}(z \oplus r)} \mathcal{L} \{e_r(\sigma(\cdot), 0)K(\cdot)\} (z \oplus r) \\ &= \frac{-r}{(1+r)^{\omega-1}(z \oplus r)} \mathcal{L} \{K\}(z).\end{aligned}$$

Now let $h(t) = \int_t^{t+\omega-1} (\ominus r)(s)e_{\ominus r}(t + \omega - 1, \sigma(s))K(s)\Delta s$. Using (2.9),

$$\begin{aligned}\mathcal{L}\{h\}(z) &= \frac{-r}{1+r} \mathcal{L} \left\{ \int_{\cdot}^{\cdot+\omega-1} e_{\ominus r}(\cdot + \omega - 1, \sigma(s))K(s)\Delta s \right\} (z) \\ &= \frac{-r}{(1+r)^{\omega-1}} \mathcal{L} \left\{ e_{\ominus r}(\sigma(\cdot), 0) \int_{\cdot}^{\cdot+\omega-1} e_r(\sigma(s), 0)K(s)\Delta s \right\} (z) \\ &= \frac{-r}{(1+r)^{\omega-1}} \mathcal{L} \left\{ \int_{\cdot}^{\cdot+\omega-1} e_r(\sigma(s), 0)K(s)\Delta s \right\} (z \oplus r) \\ &= \frac{-r}{(1+r)^{\omega-1}} \left[\frac{1}{z \oplus r} \sum_{k=0}^{\omega-2} e_r(\sigma(k), 0)K(k) + \frac{\left((z \oplus r) + 1\right)^{\omega-1} - 1}{z \oplus r} \mathcal{L} \{e_r(\sigma(\cdot), 0)K(\cdot)\} (z \oplus r) \right. \\ &\quad \left. - \frac{1}{z \oplus r} \sum_{j=0}^{\omega-1} \binom{\omega-1}{j} \sum_{\ell=0}^{j-1} z^{\ell} \left[e_r(\sigma(\cdot), 0)K(\cdot) \right]^{\Delta^{j-\ell-1}}(0) \right] \\ &= \frac{-r}{(1+r)^{\omega-1}(z \oplus r)} \left[\sum_{k=0}^{\omega-2} e_r(k+1, 0)K(k) + \left((z \oplus r) + 1\right)^{\omega-1} - 1 \right] \mathcal{L} \{e_r(\sigma(\cdot), 0)K(\cdot)\} (z \oplus r) \\ &\quad - \sum_{j=0}^{\omega-1} \binom{\omega-1}{j} \sum_{\ell=0}^{j-1} z^{\ell} \left[e_r(\sigma(\cdot), 0)K(\cdot) \right]^{\Delta^{j-\ell-1}}(0) \left. \right].\end{aligned}$$

One further step applying (2.8) on the second term yields

$$\begin{aligned}\mathcal{L}\{h\}(z) &= \frac{-r}{(1+r)^{\omega-1}(z \oplus r)} \left[\sum_{k=0}^{\omega-2} e_r(\sigma(k), 0)K(k) + \left(\left((z \oplus r) + 1\right)^{\omega-1} - 1\right) \mathcal{L} \{K\}(z) \right. \\ &\quad \left. - \sum_{j=0}^{\omega-1} \binom{\omega-1}{j} \sum_{\ell=0}^{j-1} z^{\ell} \left[e_r(\sigma(\cdot), 0)K(\cdot) \right]^{\Delta^{j-\ell-1}}(0) \right].\end{aligned}$$

Therefore we have shown that the Laplace transform of (4.1) is

$$\begin{aligned}(z+1)^{\omega-1} \mathcal{L}\{K\}(z) - \sum_{j=0}^{\omega-1} \binom{\omega-1}{j} \sum_{\ell=0}^{j-1} z^{\ell} \Delta^{j-\ell-1} K(0) &= \mathcal{L}\{\beta\}(z) \\ + \frac{r}{(1+r)^{\omega-1}(z \oplus r)} \left[\sum_{k=0}^{\omega-2} e_r(k+1, 0)K(k) + \left((z \oplus r) + 1\right)^{\omega-1} \mathcal{L} \{K\}(z) \right]\end{aligned}$$

$$- \sum_{j=0}^{\omega-1} \binom{\omega-1}{j} \sum_{\ell=0}^{j-1} z^\ell [e_r(\sigma(\cdot), 0)K(\cdot)]^{\Delta^{j-\ell-1}}(0) \Big].$$

Solving for $\mathcal{L}\{K\}(z)$ completes the proof. \square

Now we consider the reverse case of Theorem 7 where K is given and r must be solved for.

Theorem 9. *If $K: \mathbb{N}_0 \rightarrow \mathbb{R}$ is known, then the function $t \mapsto e_p(t, 0)$ is ω -periodic if and only if*

$$r(t + \omega - 1) = \ominus \left(\frac{-1 + \frac{1}{e_p(t + \omega - 1, t)}}{ae_{\ominus r}(t + \omega - 1, 0) - \int_0^{t+\omega-1} \alpha(t + \omega - 1, s)\Delta s - K(t + \omega - 1)} \right).$$

Proof. By solving (4.2) for $(\ominus r)(t + \omega - 1)$, we obtain

$$(\ominus r)(t + \omega - 1) = \frac{-1 + \frac{1}{e_p(t + \omega - 1, t)}}{ae_{\ominus r}(t + \omega - 1, 0) - \int_0^{t+\omega-1} \alpha(t + \omega - 1, s)\Delta s - K(t + \omega - 1)},$$

and so taking \ominus of both sides completes the proof, since all steps are algebraically reversible. \square

We provide a numerical example of Theorem 9 in Figure 2.

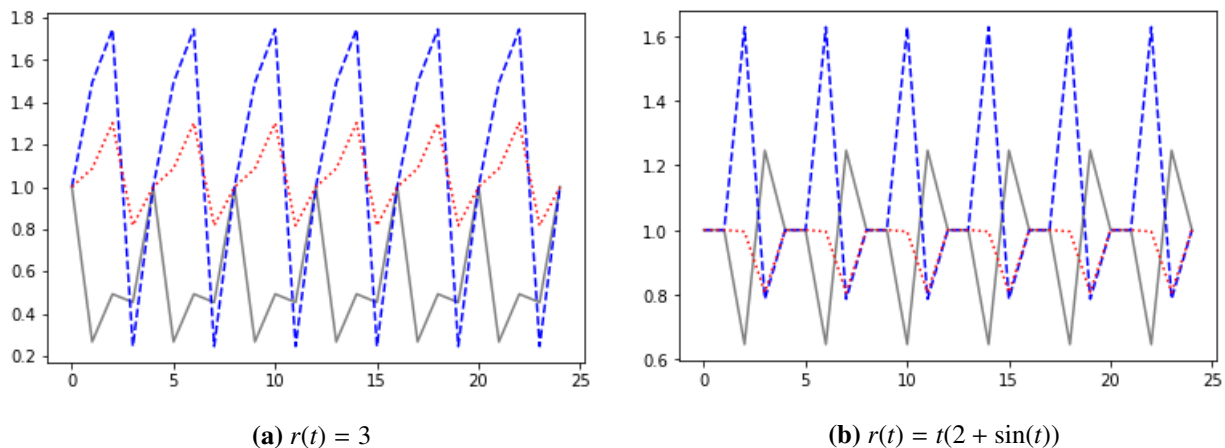


Figure 1. As an application of Theorem 7, three 4-periodic solutions of (1.1) with initial condition $y(0) = 1$ are plotted for given r and randomly selected initial values for $K(0)$, $K(1)$, and $K(2)$ chosen from the interval $(0, 2)$.

5. Conclusion

We have explored periodicity of functions related to the Gompertz difference equation (1.1). In Theorem 5, we found a difference equation that K must satisfy in order for p to be ω -periodic whenever

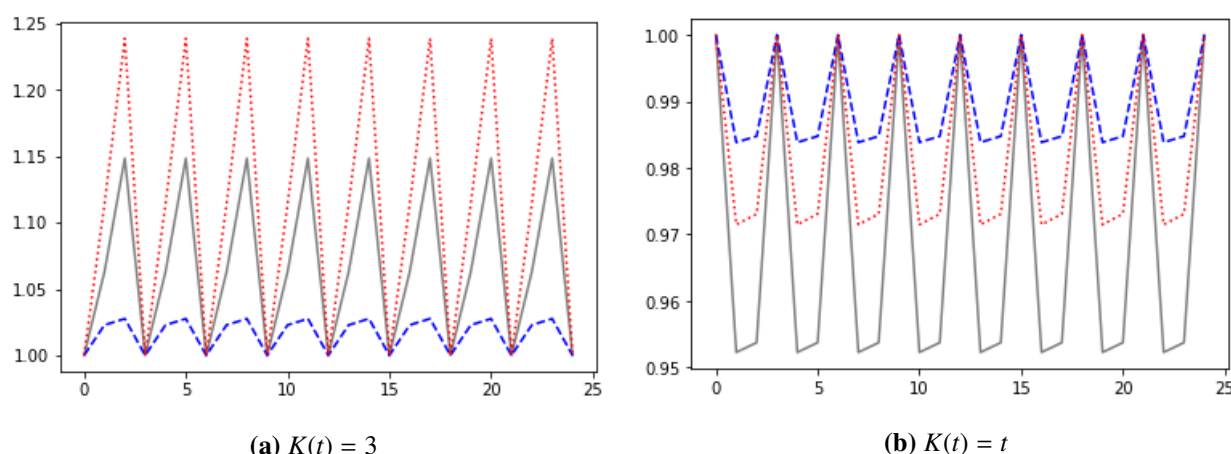


Figure 2. As an application of Theorem 9, three 4-periodic solutions of (1.1) with initial condition $y(0) = 1$ for given K and randomly selected initial values for $r(0)$, $r(1)$, and $r(2)$ chosen from the interval $(0, 0.1)$.

r is itself ω -periodic. Theorem 6 does the same thing, but when r is constant. In Theorem 7, we considered ω -periodicity of solutions of (1.1) and arrived at difference equations that K must solve in order to guarantee it. In Theorem 8 we solved that difference equation in the special case of a constant r using Laplace transform techniques. Finally, in Theorem 9, we instead found a difference equation that r must solve if K is known.

Future work in this area includes the extension of the results to ω -periodic functions on more general time scales as studied in [19,20]. Throughout, we have showcased the basic framework for these results on a more general time scale to aid in such a generalization. The connections between Volterra integral equations and generalizations of (1.1) are of interest, as well as interpreting the function K as a periodic control for population models.

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Conflict of interest

The authors declare that there is no conflict of interest.

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