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Research article

Statistical modelling for a new family of generalized distributions with real data applications

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Abstract: The modern trend in distribution theory is to propose hybrid generators and generalized families using existing algebraic generators along with some trigonometric functions to offer unique, more flexible, more efficient, and highly productive G-distributions to deal with new data sets emerging in different fields of applied research. This article aims to originate an odd sine generator of distributions and construct a new G-family called “The Odd Lomax Trigonometric Generalized Family of Distributions”. The new densities, useful functions, and significant characteristics are thoroughly determined. Several specific models are also presented, along with graphical analysis and detailed description. A new distribution, “The Lomax cosecant Weibull” (LocscW), is studied in detail. The versatility, robustness, and competency of the LocscW model are confirmed by applications on hydrological and survival data sets. The skewness and kurtosis present in this model are explained using modern graphical methods, while the estimation and statistical inference are explored using many estimation approaches.

Keywords: Lomax distribution; T-X transformation; odd trigonometric generator; stress-strength reliability; stochastic ordering; order statistics

1. Introduction

Since the last few decades, almost all research about the origination of G-families is just adopting the approaches of differential equations, compounding, weighting, etc., and thousands of statistical models have been added to the literature. No doubt, some very useful models are introduced using the above-described techniques. Still, a keen analysis reveals that mostly out of these models are internally correlated and maybe the replacement of one another in definite parametric conditions. Moreover, they are similar in mathematical appearance with only mild differences. The critical point is that almost all models are algebraic and non-trigonometric. For a brief study, we refer the reader to [1–5]. Also, for more information about machine learning, see [6–8].

Recently, the attention of statisticians has turned towards the directional data and disbursing the trigonometric functions in the existing classical models in order to construct generalized trigonometric families of distributions which open new and co-research horizons for Mathematicians and Statisticians. It is observed that the trigonometric functions enhance the flexibility prominently, keep relative balance and simplicity, show vast applicability in modeling different types of practical data sets and explore the skewness, kurtosis, and tail characteristics along with improving the goodness-of-fit (GoF). Table 1 presents chronological literature review on sine function based families and distributions.

Table 1. Sine generalized distributions and families in chronological order.

S.No.	Introducer(s)(year)	Sine based distributions and G-families.
1	Chakraborty et al. (2012)	explored the sin-skew logistic distribution.
2	Souza (2015)	suggested new trigonometric classes of probabilistic distributions.
3	Kharazmi and Saadatinik (2016)	presented hyperbolic sine-Weibull distribution.
4	Chesneau et al. (2018)	explored the cosine-sine distribution.
5	Mahmood et al. (2019)	presented a new sine-G family of distributions.
6	Chesneau et al. (2020)	deduced sine kumaraswamy-G family of distributions.
7	Al-Babtain et al. (2020)	introduced sine topp-leone-G family of distributions.
8	Nagarjuna et al. (2021)	worked on sine power lomax model.
9	Shrahili et al. (2021)	did the estimation of sine inverse exponential model.
10	Shrahili et al. (2021)	introduced sine half-logistic inverse rayleigh distribution.
11	Gang Shi et al. (2021)	presented sine entropy of uncertain random variables.

The development of trigonometric and algebraic functions mixed with a new generalized Lomax family of probability distributions is the basic motivation. The remainder motivations are five folded:

- To develop a new sine generator of distributions using the spirit of odd generator and combination of algebraic and trigonometric functions concurrently;
- To introduce a new G-family called ”The Odd Lomax Trigonometric Generalized Family of Distributions” (Locsc-G for short) in a trigonometric scenario;
- The proposed family is simple, free from non-identifiability and over parametrization issues;

- To investigate the injection of sine-cosecant functions in odd generator methods in classical distributions, leading to a novel, more versatile, and effective models;
- The new density adopts uni modal features or shapes as well (in almost all base models), and the hazard function adopts all monotone and non-monotone shapes.

The current study is conducted following the spirit of the odd generator presented by [9], the Weibull-G family developed by [10], the sine-G family introduced by [11] and the generalized odd Gamma-G family introduced by [12] collectively. In modern distribution theory, in our view, the trigonometric functions based on generalized families and distributions will prove a breakthrough for modeling the data of physical phenomena.

In Section 1, an introduction about trigonometric work with motivations are presented. New generator and family with special members are presented in Section 2, whereas the new family characteristics are derived in Section 3. In Section 5, the graphical behavior of the new family is observed using famous statistical models. The special member using Weibull as the baseline is investigated in Section 6 along with sub-models. Two data applications demonstrate the significance of the new family and model in Section 8 while final remarks and conclusions end the study in Section 9.

2. The new family

2.1. Genesis of odd sine/reciprocal cosecant generator

The odd generator $\left(\frac{G(x)}{1-G(x)}\right)$ and sine function $\left[\sin\left(\frac{\pi}{2}G(x)\right)\right]$ are used collectively to develop the new generator.

$$W[G(x)] = \frac{\sin\left(\frac{\pi}{2}G(x)\right)}{1-\sin\left(\frac{\pi}{2}G(x)\right)} = \left(\frac{1-\sin\left(\frac{\pi}{2}G(x)\right)}{\sin\left(\frac{\pi}{2}G(x)\right)}\right)^{-1} = \left(\csc(\frac{\pi}{2}G(x)) - 1\right)^{-1} \tag{2.1}$$

This generator $W[G(x)] : [0,1] \longrightarrow \mathbb{R}$ (a link function) satisfies all required conditions of T-X family of distributions.

2.2. Origination of the basic functions

Let $r(t) = \frac{\alpha}{\lambda}\left[1+(\frac{t}{\lambda})\right]^{-(\alpha+1)}$ is the lomax density where $0 < t < \infty$. Replace "t" by the new generator $W[G(x)] = \left[\csc\left(\frac{\pi}{2}G(x)\right)-1\right]^{-1}$ in lomax function, we arrived at the new family "Locsc-G" whose distribution function in Eq (2.2), and probability density function in Eq (2.3) and hazard rate function in Eq (2.4) , respectively, are given below.

$$F(x) = \int_0^{\left[\csc\left(\frac{\pi}{2}G(x)\right)-1\right]^{-1}} r(t)\,dt = 1 - \left\{1+\lambda^{-1}\left[\csc\left(\frac{\pi}{2}G(x)\right)-1\right]^{-1}\right\}^{-\alpha} \tag{2.2}$$

$$f(x) = \frac{\alpha\,\pi\,g(x)\,\csc\left(\frac{\pi}{2}G(x)\right)\cot\left(\frac{\pi}{2}G(x)\right)}{2\,\lambda\left[\csc\left(\frac{\pi}{2}G(x)\right)-1\right]^{-2}}\left\{1+\lambda^{-1}\left[\csc\left(\frac{\pi}{2}G(x)\right)-1\right]^{-1}\right\}^{-(\alpha+1)} \tag{2.3}$$

$$h(x) = \frac{\alpha\,\pi\,g(x)\,\csc\left(\frac{\pi}{2}G(x)\right)\cot\left(\frac{\pi}{2}G(x)\right)}{2\,\lambda\left[\csc\left(\frac{\pi}{2}G(x)\right)-1\right]^{-2}}\left\{1+\lambda^{-1}\left[\csc\left(\frac{\pi}{2}G(x)\right)-1\right]^{-1}\right\}^{-1} \tag{2.4}$$

In Table 2, for example, eight new recruits are added by employing the well-known statistical distributions on all feasible intervals.

Table 2. Specific models of the new family.

cdf $G(x)$	Support	New cdf $F(x)$	Parameters
Uniform	$(0,\theta)$	$1-\left[1+\left(\lambda^{-1}\left(\csc\left(\frac{\pi}{2}\left(\frac{x}{\theta}\right)\right)-1\right)^{-1}\right)\right]^{-\alpha}$	(λ,α,θ)
Exponential	$(0,\infty)$	$1-\left[1+\left(\lambda^{-1}\left(\csc\left(\frac{\pi}{2}\left(1-e^{-\beta x}\right)\right)-1\right)^{-1}\right)\right]^{-\alpha}$	(λ,α,β)
Weibull	$(0,\infty)$	$1-\left[1+\left(\lambda^{-1}\left(\csc\left(\frac{\pi}{2}\left(1-e^{-(\beta x)^{\sigma}}\right)\right)-1\right)^{-1}\right)\right]^{-\alpha}$	$(\lambda,\alpha,\beta,\sigma)$
Frechet	$(0,\infty)$	$1-\left[1+\left(\lambda^{-1}\left(\csc\left(\frac{\pi}{2}\left(e^{-(\beta x)^{\sigma}}\right)\right)-1\right)^{-1}\right)\right]^{-\alpha}$	$(\lambda,\alpha,\beta,\sigma)$
Burr XII	$(0,\infty)$	$1-\left[1+\left(\lambda^{-1}\left(\csc\left(\frac{\pi}{2}\left(1-[1+(x/s)^{c}]^{-k}\right)\right)-1\right)^{-1}\right)\right]^{-\alpha}$	(λ,α,c,k,s)
Logistic	$\mathbb{R}$	$1-\left[1+\left(\lambda^{-1}\left(\csc\left(\frac{\pi}{2}\left(\left[1+e^{-(x-\mu)/s}\right]^{-1}\right)\right)-1\right)^{-1}\right)\right]^{-\alpha}$	(λ,α,μ,s)
Gumbel	$\mathbb{R}$	$1-\left[1+\left(\lambda^{-1}\left(\csc\left(\frac{\pi}{2}\left(e^{-e^{-(x-\mu)/\sigma}}\right)\right)-1\right)^{-1}\right)\right]^{-\alpha}$	$(\lambda,\alpha,\mu,\sigma)$
Normal	$\mathbb{R}$	$1-\left[1+\left(\lambda^{-1}\left(\csc\left(\frac{\pi}{2}\Phi((x-\mu)/\sigma)\right)-1\right)^{-1}\right)\right]^{-\alpha}$	$(\lambda,\alpha,\mu,\sigma)$

3. Characteristics of the new family

3.1. Quantile function and quantile density function

We will discuss a characteristic of X called the quantile function (qf), which may be determined directly by inverting (2.2) as below

$$Q(u)=G^{-1}\left\{\frac{2}{\pi}\sin^{-1}\left[\left(\lambda\,(1-u)^{\frac{-1}{\alpha}}-1\right)^{-1}+1\right]^{-1}\right\} \tag{3.1}$$

Equation (3.1) possesses a lot of applications; some are given below: 1) To find median, quartiles, deciles and percentiles. 2) Replacing any standard model, this equation can be used to simulate density, histogram, and exact cdfs for these data can be accomplished. 3) The variability analysis related to skewness and kurtosis can be performed on the basis of quantile measures as the Bowley skewness (see [13]) and the Moors kurtosis (see [14]), respectively. A remarkable function related to Eq (3.1), having statistical significance discussed in [15], is the quantile density function denoted by $Q\prime(U)$ is:

$$Q\prime(U)=\frac{2\,\lambda\,(1-u)^{-(\frac{1+\alpha}{\alpha})}\left[\lambda\,(1-u)^{\left(\frac{-1}{\alpha}\right)}-1\right]^{-2}}{\pi\,\alpha\left[\left(\lambda\,(1-u)^{\left(\frac{-1}{\alpha}\right)}-1\right)^{-1}+1\right]^{2}\left[1-\left(\left(\lambda\,(1-u)^{\left(\frac{-1}{\alpha}\right)}-1\right)^{-1}+1\right)^{-2}\right]^{\frac{1}{2}}}$$

3.2. Basic reliability functions

The hazard rate function, which is an essential concept that plays a vital role in risk and survival analysis, is an example of an important function. There are some other important functions such as survival function $S(x)$, also another important one is reversed hazard rate $r(x)$, at last we must not forget the cumulative hazard rate $H(x)$, and the very interesting mills' ratio $m(x)$, elasticity $e(x)$ and finally the conditional reliability function $\bar{G}(G(x), \alpha, \beta\,|t)$, which are respectively, presented below.

$$S(x) = \left\{1 + \lambda^{-1}\left[\csc\left(\frac{\pi}{2}G(x)\right) - 1\right]^{-1}\right\}^{-\alpha}$$

$$h(x) = \frac{\alpha\,\pi\,g(x)\,\csc\left(\frac{\pi}{2}G(x)\right)\,\cot\left(\frac{\pi}{2}G(x)\right)}{2\,\lambda\left[\csc\left(\frac{\pi}{2}G(x)\right) - 1\right]^{-2}}\left\{1 + \lambda^{-1}\left[\csc\left(\frac{\pi}{2}G(x)\right) - 1\right]^{-1}\right\}^{-1}$$

$$r(x) = \frac{\frac{\alpha\,\pi\,g(x)\,\csc\left(\frac{\pi}{2}G(x)\right)\,\cot\left(\frac{\pi}{2}G(x)\right)}{2\,\lambda\left[\csc\left(\frac{\pi}{2}G(x)\right)-1\right]^{-2}}\left\{1 + \lambda^{-1}\left[\csc\left(\frac{\pi}{2}G(x)\right) - 1\right]^{-1}\right\}^{-(\alpha+1)}}{1 - \left\{1 + \lambda^{-1}\left[\csc\left(\frac{\pi}{2}G(x)\right) - 1\right]^{-1}\right\}^{-\alpha}}$$

$$m(x) = \frac{2\,\lambda\left[1 + \lambda^{-1}\left(\csc\left(\frac{\pi}{2}G(x)\right) - 1\right)^{-1}\right]}{\alpha\,\pi\,g(x)\,\csc\left(\frac{\pi}{2}G(x)\right)\,\cot\left(\frac{\pi}{2}G(x)\right)\left(\csc(\frac{\pi}{2}G(x)) - 1\right)^{2}}$$

$$H(x) = \log\left[\left\{1 + \lambda^{-1}\left[\csc\left(\frac{\pi}{2}G(x)\right) - 1\right]^{-1}\right\}^{-\alpha}\right]^{-1}$$

$$e(x) = \frac{\partial}{\partial(\ln(G(x)))}\ln\left\{1 - \left[1 + \left(\lambda^{-1}\left(\csc\left(\frac{\pi}{2}G(x)\right) - 1\right)^{-1}\right)\right]^{-\alpha}\right\}$$

$$\bar{G}(G(x), \alpha, \lambda\,|t) = \frac{\bar{G}(G(x+t), \alpha, \lambda)}{\bar{G}(G(t), \alpha, \lambda)} = \frac{\left[1 + \lambda^{-1}\left(\csc\left(\frac{\pi}{2}G(t)\right) - 1\right)^{-1}\right]^{\alpha}}{\left[1 + \lambda^{-1}\left(\csc\left(\frac{\pi}{2}G(x+t)\right) - 1\right)^{-1}\right]^{\alpha}}$$

3.3. Useful series expansions

We have the following linear representations for the new families CDF and pdf.

Proposition 3.1. *The new family's cdf and pdf have the following linear representations:*

$$F(x) = \sum_{k=0}^{\infty} w_{i,j}G(x)^{2k}, \quad f(x) = \sum_{k=0}^{\infty} w_{i,j}\left[2kg(x)G(x)^{2k-1}\right], \tag{3.2}$$

where

$$w_{i,j} = \sum_{i,j=0}^{\infty} \frac{(\frac{\pi}{2})^{2k}\,(-1)^{-i+j+1}}{(\lambda)^{i}}\binom{-\alpha}{i}\binom{-i}{j}a_k(j) \tag{3.3}$$

Proof. If the cdf and pdf of a random variable Y can be stated as $H_c(x) = G(x)^c$ and $h_c(x) = c\,G(x)^{c-1}\,g(x)$ then we say that this random variable has exp-G with power parameter $c > 0$, The cdf of the new family, required to be linearized, is

$$F(x) = 1 - \left\{1 + \lambda^{-1}\left[\csc\left(\frac{\pi}{2}G(x)\right) - 1\right]^{-1}\right\}^{-\alpha}$$

using the expansion $(1+x)^{-n} = \sum_{i=0}^{\infty}\binom{-n}{i}x^i$ and binomial expansion simultaneously, $F(x)$ becomes

$$F(x) = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}(-1)^{-i+j+1}(\lambda)^i\binom{-\alpha}{i}\binom{-i}{j}\left[\csc\left(\frac{\pi}{2}G(x)\right)\right]^j$$

using MATHEMATICA 11.1, $\left[\csc\left(\frac{\pi}{2}\,G(x)\right)\right]^j = \sum_{k=0}^{\infty} a_k(j)\,\left(\frac{\pi}{2}G(x)\right)^{2k}$
where $a_0(j) = 1$, $a_1(j) = j/6$, $a_2(j) = (j/180) + (j^2/72)$,etc.

For $F(x)$, the required linear representation is obtained. Moreover, just by simple differentiation, the linear representation of $f(x)$ can be obtained.

3.4. Moments and derivations

For the new family, the rth moment is (r be an integer and all sum and integrals are assumed to exist)

$$\mu_r' = \int_{-\infty}^{\infty} x^r f(x)\,dx = \int_{-\infty}^{\infty} x^r \sum_{k=0}^{\infty} w_{i,j}(2k)g(x)G(x)^{2k-1}\,dx = \sum_{k=0}^{\infty} w_{i,j}\,\mathbb{E}(X_k^r) \tag{3.4}$$

μ_r' is also expressed by consuming the quantile function (or changing the variable $x = Q(p)$) given by Eq (3.1), in this way

$$\mu_r' = \int_0^1 \,[Q(p)]^r\, dp. = \int_0^1 \left\{Q_G\left[\frac{2}{\pi}\,\sin^{-1}\left(\left(\lambda(1-u)^{(\frac{-1}{\alpha})} - 1\right)^{-1} + 1\right)^{-1}\right]\right\}. \tag{3.5}$$

The derived integral is computable using any modern mathematical software like Mathematica, R, Matlab, or Maple for given $G(x)$, α, and λ.

3.5. Probability weighted moments

For $r \geq 1$, $s \geq 0$, routinely, the (r, s)th probability weighted moment (PWM) is expressed as

$$\rho_{r,s} = E[X^r\,F(X)^s] = \int_0^{\infty} x^r\,F(x)^s\,f(x)\,dx. \tag{3.6}$$

Then, we have

$$F(x)^s = \left[1 - \left[1 + \lambda^{-1}\left(\csc\left(\frac{\pi}{2}G(x)\right) - 1\right)^{-1}\right]^{-\alpha}\right]^s. \tag{3.7}$$

After a bit modification using trigonometric relations, (3.7) can be written as:

$$F(x)^s = \left[1 - \left[1 + \lambda^{-1}\left(\frac{\sin\left(\frac{\pi}{2}G(x)\right)}{1-\sin\left(\frac{\pi}{2}G(x)\right)}\right)\right]^{-\alpha}\right]^s$$

and

$$f(x) = \frac{\alpha\,\pi\,g(x)\,\csc\left(\frac{\pi}{2}G(x)\right)\cot\left(\frac{\pi}{2}G(x)\right)}{2\,\lambda\left(\csc\left(\frac{\pi}{2}G(x)\right)-1\right)^{-2}}\left[1+\lambda^{-1}\left(\csc\left(\frac{\pi}{2}G(x)\right)-1\right)^{-1}\right]^{-(\alpha+1)}.$$

Similarly,

$$f(x) = \frac{\alpha\,\pi\,g(x)\,\cos\left(\frac{\pi}{2}G(x)\right)}{2\,\lambda\,\left(1-\sin\left(\frac{\pi}{2}G(x)\right)\right)^2}\left[1+\lambda^{-1}\left(\frac{\sin\left(\frac{\pi}{2}G(x)\right)}{\left(1-\sin\left(\frac{\pi}{2}G(x)\right)\right)}\right)\right]^{-(\alpha+1)}$$

in Eq (3.6), $\rho_{r,s}$ can be written as:

$$\rho_{r,s} = \sum_{l,m=0}^{\infty} w^*_{(i,j,k)} \int_{-\infty}^{\infty} x^r\,(2l+m+1)\,g(x)\,(G(x))^{(2l+m+1)-1}\,dx. \tag{3.8}$$

$$\rho_{r,s} = \sum_{l,m=0}^{\infty} w^*_{(i,j,k)} \int_{-\infty}^{\infty} x^r\,h_{(2l+m+1)(x)}dx. \tag{3.9}$$

$$\rho_{r,s} = \sum_{l,m=0}^{\infty} w^*_{(i,j,k)}\,\mathbb{E}(X^r_{(l,m)}) \tag{3.10}$$

where

$$\begin{aligned} w^*_{(i,j,k)} \quad = \quad & \sum_{i,j,k=0}^{\infty} \frac{\alpha}{(2l)!(2l+m+1)}\left(\frac{\pi}{2}\right)^{(2l+m+1)}(-1)^{(i+k+l)}\,(\lambda)^{(-\alpha(i+1)-2)} \\ & \binom{s}{i}\binom{(-\alpha(i+1)-1)}{j}\binom{(\alpha(i+1)-1)}{k}C_{m(-\alpha(i+1)+k-1)}. \end{aligned}$$

3.6. Moment generating function

Introducing mathematical properties is very important. The moment generating function is stated in the following mathematical format:

$$M(t) = \int_{-\infty}^{+\infty} e^{tx} f(x)dx = \sum_{r=0}^{+\infty}\frac{t^r}{r!}\mu'_r.$$

Otherwise, without utilizing the moments but consuming the linear representation presented in expression (3.2), $M(t)$ is expressed as

$$M(t) \quad = \quad \sum_{k=0}^{+\infty} w_{i,j}\int_{-\infty}^{+\infty} e^{tx}\,2\,k\,g(x)\,G(x)^{2k-1}\,dx = \sum_{k=0}^{+\infty} w_{i,j}(2k)\int_0^1 p^{(2k-1)}\,e^{tQ_G(p)}dp.$$

3.7. *Critical points of the density and hazard rate function*

By solving the following equation $\frac{\partial \log[f(x)]}{\partial x} = 0$ and $\frac{\partial \log[h(x)]}{\partial x} = 0$ we can provide the density and hazard rate function critical points respectively. For the Locsc-G density function, the nonlinear equation related to density is:

$$\frac{g'(x)}{g(x)} - \frac{\pi(\alpha+1)g(x)\cot\left(\frac{1}{2}\pi G(x)\right)\csc\left(\frac{1}{2}\pi G(x)\right)}{\left(\csc\left(\frac{1}{2}\pi G(x)\right)-1\right)^3\left(\frac{1}{\left(\csc\left(\frac{1}{2}\pi G(x)\right)-1\right)^2}+\lambda\right)}$$
$$-\frac{1}{2}\pi g(x)\cot\left(\frac{1}{2}\pi G(x)\right) - \pi g(x)\csc(\pi G(x)) + \frac{\pi g(x)\cot\left(\frac{1}{2}\pi G(x)\right)\csc\left(\frac{1}{2}\pi G(x)\right)}{\csc\left(\frac{1}{2}\pi G(x)\right)-1} = 0. \tag{3.11}$$

While the critical points of the hrf are obtained from the equation $\frac{\partial \log[h(x)]}{\partial x} = 0$

$$\frac{g'(x)}{g(x)} + \frac{1}{4}\pi g(x)\left(\frac{\sin(\pi G(x))\csc^3\left(\frac{1}{2}\pi G(x)\right)\left(2\lambda\csc\left(\frac{1}{2}\pi G(x)\right)-2\lambda+3\right)}{\left(\csc\left(\frac{1}{2}\pi G(x)\right)-1\right)\left(\lambda\csc\left(\frac{1}{2}\pi G(x)\right)\right)}\right)$$
$$-\left(\frac{1}{4}\pi g(x)\left((\lambda+1) - 2\tan\left(\frac{1}{2}\pi G(x)\right) - 4\csc(\pi G(x))\right)\right) = 0. \tag{3.12}$$

3.8. *Stochastic ordering*

A detailed description of stochastic ordering is available in [16], here, utilizing the family parameters α and λ, a proof is presented concerning the stochastic ordering.

Proposition 3.2. *let us suppose that we heave a random variable let us say it X came from a distribution with the density function $f_1(x)$ as defined in* (2.3) *with parameters α_1 and λ and let us suppose that we heave a random variable let us say it Y came from a distribution with the density function as defined in $f_2(x)$ as defined in* (2.3) *with parameters α_2 and λ. So, if $\alpha_1 \geqslant \alpha_2$, we have $X \geqslant_{lr} Y$, i.e., $\frac{f_1(x)}{f_2(x)}$ is decreasing.*

Proof. The density is

$$f(x) = \frac{\alpha\,\pi\, g(x)\,\csc\left(\frac{\pi}{2}G(x)\right)\,\cot\left(\frac{\pi}{2}G(x)\right)}{2\,\lambda\left(\csc\left(\frac{\pi}{2}G(x)\right)-1\right)^{-2}}\left[1+\lambda^{-1}\left(\csc\left(\frac{\pi}{2}G(x)\right)-1\right)^{-1}\right]^{-(\alpha+1)}.$$

Then

$$\frac{f_1(x)}{f_2(x)} = \left(\frac{\alpha_1}{\alpha_2}\right)\left[1+\lambda^{-1}\left(\csc\left(\frac{\pi}{2}G(x)\right)-1\right)^{-1}\right]^{-(\alpha_1-\alpha_2)}.$$

Since $\alpha_1 \geqslant \alpha_2$, then after differentiation with respect to x, we get

$$\frac{\partial}{\partial x}\frac{f_1(x)}{f_2(x)} = \left(\frac{-\alpha_1}{\alpha_2}\right)(\alpha_1-\alpha_2)\,\frac{\pi}{2\,\lambda}\,g(x)\,\csc\left(\frac{\pi}{2}G(x)\right)\cot\left(\frac{\pi}{2}G(x)\right)\left(\csc\left(\frac{\pi}{2}G(x)\right)-1\right)^{-2}$$
$$\left[1+\lambda^{-1}\left(\csc\left(\frac{\pi}{2}G(x)\right)-1\right)^{-1}\right]^{-(\alpha_1-\alpha_2)-1} \leq 0.$$

The proof of Proposition 3.2 ends with the conclusion that $X \geqslant_{lr} Y$.

3.9. Stress-strength reliability parameter

The reliability parameter is very important. See [17] for a detailed study on stress-strength reliability.

Let us suppose that we heave a random variable X came from a distribution with density function $f_1(x)$ given by (2.3) with parameters α_1 and λ_1 and another variable Y came from a distribution with distribution function $F_2(x)$ defined as (2.2) with parameters α_2 and λ_2. Then, the reliability parameter is defined by :

$$R = \mathbb{P}(Y < X) = \int_{-\infty}^{\infty} f_1(x)F_2(x)dx.$$

With $f_1(x)$ and $F_2(x)$ functions,

$$R = \int_{-\infty}^{\infty} \frac{\alpha_1\,\pi\,g(x)\,\cos\left(\frac{\pi}{2}G(x)\right)}{2\,\lambda_1\,\left(1-\sin\left(\frac{\pi}{2}G(x)\right)\right)^2}\left[1+\lambda_1{}^{-1}\left(\frac{\sin\left(\frac{\pi}{2}G(x)\right)}{1-\sin\left(\frac{\pi}{2}G(x)\right)}\right)\right]^{-(\alpha+1)}$$
$$\left(1-\left[1+\lambda_2{}^{-1}\left(\frac{\sin\left(\frac{\pi}{2}G(x)\right)}{1-\sin\left(\frac{\pi}{2}G(x)\right)}\right)\right]^{-\alpha_2}\right)dx.$$

After simplification, we get

$$R = \sum_{k,l,n=0}^{\infty} V_{(i,j,m)} \int_{-\infty}^{\infty} g(x)\,G(x)^{2(k+l+n)-1}\,dx.$$

Where

$$\begin{aligned}V_{(i,j,m)} &= \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{(k+m+1)}\left(\frac{\alpha_1\,\pi\,(\lambda_1)^{(j-1)}\,(\lambda_2)^i}{(2k)!2}\right)\binom{-\alpha_2}{i}\binom{-(\alpha_2+1)}{j}\\ &\quad\binom{(i+j+2)}{m}(\frac{\pi}{2})^{2(k+l+n)}\,d_{l(i+j)}\,e^{*}_{n(m)}\end{aligned}$$

where $d_0(i+j) = 1$, $d_1(i+j) = l/6$, $d_2(i+j) = (l/180)+((l^2)/72)$,etc. and similar for $e^{*}_{n(m)}$ .If $\alpha_1 = \alpha_2$ and $\lambda_1 = \lambda_2$ (corresponds to the case being distributed identically), at end, we obtained $R = \frac{1}{2(k+l+n)}$.

3.10. Order statistics

Order statistics invariably appear in a variety of applications requiring data related to survival testing. You will find all the information in the book [18].

Consider the ith order statistic $X_{i:n}$ and its density is to find. Let a random sample $X_1,\ldots,X_n$ is chosen from the new family then

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!}f(x)F(x)^{i-1}\left[1-F(x)\right]^{n-i}, \qquad x\in\mathbb{R}. \tag{3.13}$$

Equations (2.2) and (2.3) are substituted in the Eq (3.13), we get

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \frac{\alpha\,\pi\,g(x)\,\cos\left(\frac{\pi}{2}G(x)\right)}{2\,\lambda\left(1-\sin\left(\frac{\pi}{2}G(x)\right)\right)^2} \left[1+\lambda^{-1}\left(\frac{\sin\left(\frac{\pi}{2}G(x)\right)}{1-\sin\left(\frac{\pi}{2}G(x)\right)}\right)\right]^{-(\alpha+1)}$$
$$\left\{1-\left[1+\lambda^{-1}\left(\frac{\sin\left(\frac{\pi}{2}G(x)\right)}{1-\sin\left(\frac{\pi}{2}G(x)\right)}\right)\right]^{-\alpha}\right\}^{i-1} \left\{\left[1+\lambda^{-1}\left(\frac{\sin\left(\frac{\pi}{2}G(x)\right)}{1-\sin\left(\frac{\pi}{2}G(x)\right)}\right)\right]^{-\alpha}\right\}^{n-i}.$$

Notably, $f_{1:n}(x)$ and $f_{n:n}(x)$ are the densities of $X_{1:n} = \inf(X_1,\ldots,X_n)$ and $X_{n:n} = \sup(X_1,\ldots,X_n)$ respectively.

Proposition 3.3. *The$X_{i:n}$ pdf may be represented as a linear combination of pdfs from the exp-G distribution family.*

Proof. Firstly, consider the Eq (3.13) which displays the expression of $f_{i:n}(x)$. Applying the binomial series expansion and substituting the Eq (3.2) in Eq (3.13), we get

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}f(x)\,F(x)^{j+i-1}$$
$$= \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}$$
$$\frac{\alpha\,\pi\,g(x)\,\cos\left(\frac{\pi}{2}G(x)\right)}{2\,\lambda\left(1-\sin\left(\frac{\pi}{2}G(x)\right)\right)^2} \left[1+\lambda^{-1}\left(\frac{\sin\left(\frac{\pi}{2}G(x)\right)}{1-\sin\left(\frac{\pi}{2}G(x)\right)}\right)\right]^{-(\alpha+1)}$$
$$\left\{1-\left[1+\lambda^{-1}\left(\frac{\sin\left(\frac{\pi}{2}G(x)\right)}{1-\sin\left(\frac{\pi}{2}G(x)\right)}\right)\right]^{-\alpha}\right\}^{j+i-1}.$$

By virtue of generalized binomial expansion and relevant series on sine and cosine trigonometric functions, we have

$$f_{i:n}(x) = \sum_{p,q=0}^{\infty} V^*_{(j,l,m,s)}\,[2(p+q)+1]\,g(x)\,(G(x))^{[2(p+q)+1]-1}, \tag{3.14}$$

$$f_{i:n}(x) = \sum_{p,q=0}^{\infty} V^*_{(j,l,m,s)}\,h^*_{(2(p+q)+1)(x)}, \tag{3.15}$$

where

$$V^*_{(j,l,m,n)} = \sum_{j=0}^{n-i}\sum_{l,m,s=0}^{+\infty}\frac{n!}{(i-1)!(n-i)!}\frac{(-1)^{j+l+p+s}}{(2p)!\,(2(p+q)+1)}\,\alpha\,(\frac{\pi}{2})^{(2(p+q)+1)}\,(\lambda)^{(m-1)}\,d_q(m+n)$$
$$\binom{n-i}{j}\binom{j+i-1}{l}\binom{-\alpha(l+1)-1}{m}\binom{-(m+2)}{s}$$

Moreover, $h^*_{(2(p+q)+1)(x)}$ is a pdf of the exp-G family of distributions with parameter $(2(p+q)+1)$, the proposal evidence (3.3) is accomplished.

4. Estimation

In this section, we introduce different classical estimation methods for estimating the new family parameters α, λ, and ξ, which are obtained by maximization of minimization of the objective function, as we will see in this section. For more information about the introduced estimation methods, see [19–21].

The estimated parameters of our proposed family by the maximum likelihood estimation (MLE) method are obtained by maximizing the log-likelihood function of (2.3) which is defined in the following equation.

$$\ell(\alpha,\lambda,\xi) = n\left[\log(\pi\alpha) - (\alpha+1)\log\left(\frac{1}{\lambda\left(\csc\left(\frac{1}{2}\pi G(x_i,\xi)\right)-1\right)^2}+1\right) - \log(2\lambda)\right] + \sum_{i=1}^{n}\log\left[g(x_i,\xi)\right]$$
$$+\sum_{i=1}^{n}\log\left[\cot\left(\frac{1}{2}\pi G(x_i,\xi)\right)\right] - 2\sum_{i=1}^{n}\log\left[\csc\left(\frac{1}{2}\pi G(x_i,\xi)\right)-1\right] + \sum_{i=1}^{n}\log\left[\csc\left(\frac{1}{2}\pi G(x_i,\xi)\right)\right].$$

The estimated parameters of our proposed family by Anderson-Darling estimation (ADE) method is obtained by minimizing the following equation ($x_{(1)} \le x_{(2)} \le \ldots \le x_{(n)}$)

$$A(\alpha,\lambda,\xi) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log F(x_i) + \log S(x_i)\right] = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)$$
$$\times\left[\log\left(1-\left\{1+\lambda^{-1}\left[\csc\left(\frac{\pi}{2}G(x_i,\xi)\right)-1\right]^{-1}\right\}^{-\alpha}\right) + \log\left\{1+\lambda^{-1}\left[\csc\left(\frac{\pi}{2}G(x_i,\xi)\right)-1\right]^{-1}\right\}^{-\alpha}\right].$$

The estimated parameters of our proposed family by right-tail Anderson-Darling estimation (RADE) method is obtained by minimizing the following equation ($x_{(1)} \le x_{(2)} \le \ldots \le x_{(n)}$)

$$\begin{aligned}R(\alpha,\lambda,\xi) &= \frac{n}{2} - 2\sum_{i=1}^{n}F\left(x_{i:n}|\alpha,\lambda,\xi\right) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log S\left(x_{n+1-i:n}|\alpha,\lambda,\xi\right)\\ &= \frac{n}{2} - 2\sum_{i=1}^{n}\left(1-\left\{1+\lambda^{-1}\left[\csc\left(\frac{\pi}{2}G(x_i,\xi)\right)-1\right]^{-1}\right\}^{-\alpha}\right)\\ &\quad - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log\left\{1+\lambda^{-1}\left[\csc\left(\frac{\pi}{2}G(x_{n+1-i},\xi)\right)-1\right]^{-1}\right\}^{-\alpha}.\end{aligned}$$

The estimated parameters of our proposed family by Cramér-von Mises estimation (CVME) method is obtained by minimizing the following equation ($x_{(1)} \le x_{(2)} \le \ldots \le x_{(n)}$)

$$\begin{aligned}C(\alpha,\lambda,\xi) &= -\frac{1}{12n} + \sum_{i=1}^{n}\left[F(x_i|\alpha,\lambda,\xi) - \frac{2i-1}{2n}\right]^2\\ &= -\frac{1}{12n} + \sum_{i=1}^{n}\left[1-\left\{1+\lambda^{-1}\left[\csc\left(\frac{\pi}{2}G(x_i,\xi)\right)-1\right]^{-1}\right\}^{-\alpha} - \frac{2i-1}{2n}\right]^2.\end{aligned}$$

The estimated parameters of our proposed family by least-squares estimation (LSE) method is obtained by minimizing the following equation ($x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(n)}$)

$$V(\alpha, \lambda, \xi) = \sum_{i=1}^{n} \left[F(x_i|\alpha, \lambda, \xi) - \frac{i}{n+1}\right]^2 = \sum_{i=1}^{n} \left[1 - \left\{1 + \lambda^{-1}\left[\csc\left(\frac{\pi}{2}G\left(x_i, \xi\right)\right) - 1\right]^{-1}\right\}^{-\alpha} - \frac{i}{n+1}\right]^2.$$

The estimated parameters of our proposed family by weighted least-squares estimation (WLSE) method is obtained by minimizing the following equation ($x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(n)}$)

$$\begin{aligned} W(\alpha, \lambda, \xi) &= \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F(x_i|\alpha, \lambda, \xi) - \frac{i}{n+1}\right]^2 \\ &= \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[1 - \left\{1 + \lambda^{-1}\left[\csc\left(\frac{\pi}{2}G\left(x_i, \xi\right)\right) - 1\right]^{-1}\right\}^{-\alpha} - \frac{i}{n+1}\right]^2. \end{aligned}$$

The estimated parameters of our proposed family by maximum product of spacing estimation (MPSE) method is obtained by maximizing the following equation ($x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(n)}$)

$$H\left(\alpha, \lambda, \xi\right) = \frac{1}{n+1} \sum_{i=1}^{n+1} \log D_i(\alpha, \lambda, \xi),$$

where

$$\begin{aligned} D_i(\alpha, \lambda, \xi) &= F\left(x_{(i)}|\alpha, \lambda, \xi\right) - F\left(x_{(i-1)}|\alpha, \lambda, \xi\right) \\ &= \left\{1 + \lambda^{-1}\left[\csc\left(\frac{\pi}{2}G\left(x_{i-1}, \xi\right)\right) - 1\right]^{-1}\right\}^{-\alpha} + \left\{1 + \lambda^{-1}\left[\csc\left(\frac{\pi}{2}G\left(x_i, \xi\right)\right) - 1\right]^{-1}\right\}^{-\alpha}. \end{aligned}$$

5. Special Locsc-G distributions with graphical analysis

We presented a few special models of the new family using well-known statistical distributions as a baseline, developed main functions, and analyzed and described graphical flexibility.

5.1. The lomax cosecant exponentiated exponential (LocscEE) distribution

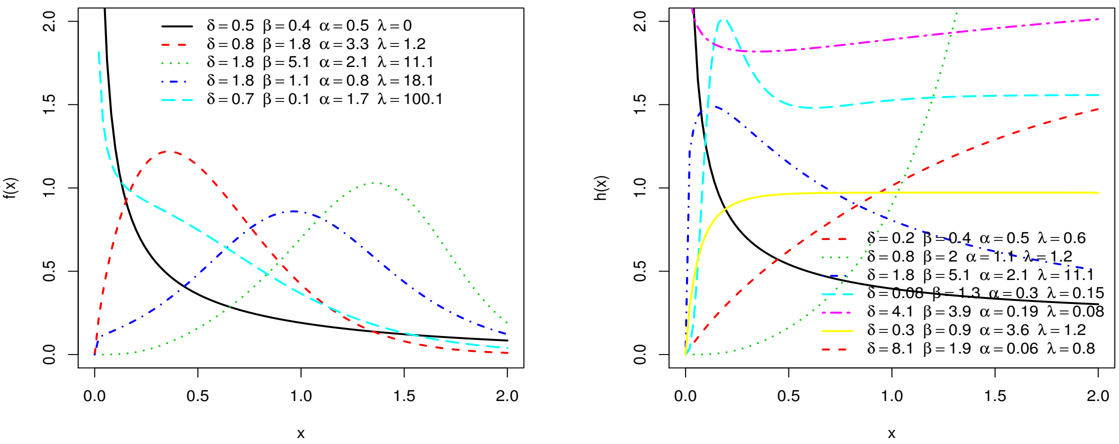
Let X be an exponentiated exponential random variable with cdf $G(x) = \left(1 - e^{-\delta x}\right)^{\beta}$ and density $g(x) = \delta\beta e^{-\delta x}\left(1 - e^{-\delta x}\right)^{(\beta-1)}$. Then the CDF, pdf, and hazard rate function of the LocscEE distribution, respectively, become as (for $x > 0$)

$$F(x) = 1 - \left[1 + \lambda^{-1}\left(\csc\left(\frac{\pi}{2}\left(\left(1 - e^{-\delta x}\right)^{\beta}\right)\right) - 1\right)^{-1}\right]^{-\alpha}$$

$$f(x) \quad = \quad \frac{\alpha\,\pi\,(\delta\beta e^{-\delta x}\left(1 - e^{-\delta x}\right)^{(\beta-1)})\csc\left(\frac{\pi}{2}\left(1 - e^{-\delta x}\right)^{\beta}\right)\cot\left(\frac{\pi}{2}\left(1 - e^{-\delta x}\right)^{\beta}\right)}{2\,\lambda\left(\csc(\frac{\pi}{2}\,(1 - e^{-\delta x})^{\beta}) - 1\right)^{-2}\left[1 + \lambda^{-1}\left(\csc(\frac{\pi}{2}\,(1 - e^{-\delta x})^{\beta}) - 1\right)^{-1}\right]^{(\alpha+1)}}$$

$$h(x) \;=\; \frac{\alpha\,\pi\,(\delta\,\beta\,\mathrm{e}^{-\delta\,x}\left(1-\mathrm{e}^{-\delta\,x}\right)^{(\beta-1)})\,\csc\left(\frac{\pi}{2}\left(1-\mathrm{e}^{-\delta\,x}\right)^{\beta}\right)\,\cot\left(\frac{\pi}{2}\left(1-\mathrm{e}^{-\delta\,x}\right)^{\beta}\right)}{2\,\lambda\left(\csc(\frac{\pi}{2}\,(1-\mathrm{e}^{-\delta\,x})^{\beta})-1\right)^{-2}\left[1+\lambda^{-1}\left(\csc(\frac{\pi}{2}\,(1-\mathrm{e}^{-\delta\,x})^{\beta})-1\right)^{-1}\right]}.$$

Figure 1 displays some plots of the density and hazard rate function of the LocscEE distribution for some parametric values. Figure 1(a) depicts that the LocscEE density exhibits reverse-j, approximately symmetrical, left-skewed and right-skewed shapes. Figure 1(b) reveals that the LocscEE hazard rate function has increasing, decreasing, increasing-decreasing-increasing, and upside-down bathtub shapes.



(a) (b)

Figure 1. Plots of the LocscEE (a) density (b) hazard rate for some parametric values.

5.2. *The lomax cosecant Weibull (LocscW) distribution*

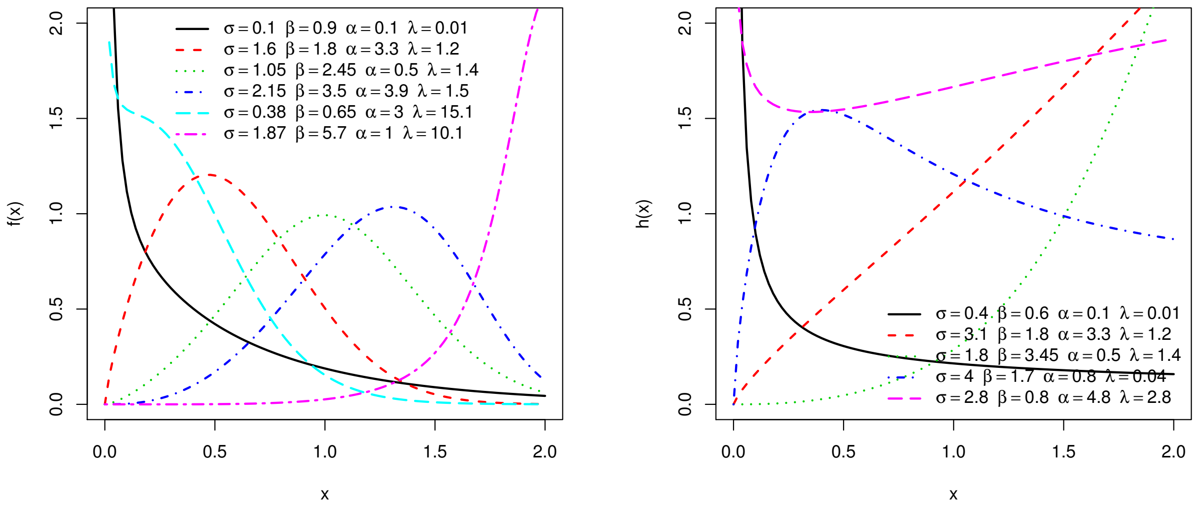
Taking $G(x)$ to be the Weibull cdf with scale parameter $\sigma > 0$ and shape parameter $\beta > 0$, say $G(x) = 1 - \mathrm{e}^{-\sigma\,x^{\beta}}$, and the weibull density $g(x) = \sigma\beta x^{\beta-1}\mathrm{e}^{-\sigma\,x^{\beta}}$, it follows the four-parameters LocscW having the following new cdf, pdf and hazard rate function (for $x > 0$)

$$F(x) = 1 - \left[1+\lambda^{-1}\left(\csc\left(\frac{\pi}{2}\left(1-\mathrm{e}^{-\sigma\,x^{\beta}}\right)\right)-1\right)^{-1}\right]^{-\alpha}$$

$$f(x) \;=\; \frac{\alpha\,\pi\,(\sigma\beta x^{\beta-1}\mathrm{e}^{-\sigma\,x^{\beta}})\,\csc\left(\frac{\pi}{2}(1-\mathrm{e}^{-\sigma\,x^{\beta}})\right)\,\cot\left(\frac{\pi}{2}(1-\mathrm{e}^{-\sigma\,x^{\beta}})\right)}{2\,\lambda\left(\csc(\frac{\pi}{2}(1-\mathrm{e}^{-\sigma\,x^{\beta}}))-1\right)^{-2}\left[1+\lambda^{-1}\left(\csc(\frac{\pi}{2}(1-\mathrm{e}^{-\sigma\,x^{\beta}}))-1\right)^{-1}\right]^{(\alpha+1)}}$$

$$h(x) \;=\; \frac{\alpha\,\pi\,(\sigma\beta x^{\beta-1}\mathrm{e}^{-\sigma(x)^{\beta}})\,\csc\left(\frac{\pi}{2}(1-\mathrm{e}^{-\sigma\,x^{\beta}})\right)\,\cot\left(\frac{\pi}{2}(1-\mathrm{e}^{-\sigma\,x^{\beta}})\right)}{2\,\lambda\left(\csc(\frac{\pi}{2}(1-\mathrm{e}^{-\sigma\,x^{\beta}}))-1\right)^{-2}\left[1+\lambda^{-1}\left(\csc(\frac{\pi}{2}(1-\mathrm{e}^{-\sigma\,x^{\beta}}))-1\right)^{-1}\right]}.$$

Figure 2 displays some plots of the density and hazard rate function of the LocscW distribution for some parametric values. Figure 2(a) depicts that the LocscW density have symmetrical, right-skewed, left-skewed, reversed-J and J shapes. Figure 2(b) reveals that the LocscW hazard rate function has decreased, increasing bathtub and upside-down bathtub shapes.



(a) (b)

Figure 2. Plots of the LocscW (a) density (b) hazard rate for some parametric values.

5.3. The lomax cosecant Burr(LocscB) distribution

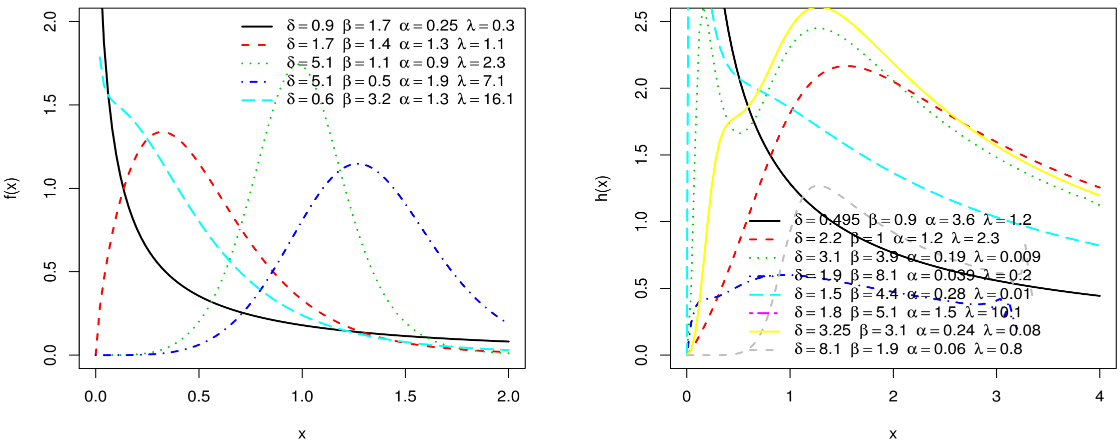
Let X be an burr random variable with pdf $g(x) = \delta\beta\,(x)^{\delta-1}(1+(x)^{\delta})^{-(\beta+1)}$ and cdf $G(x) = 1 - \left(1+x^{\delta}\right)^{-\beta}$, $\quad x > 0 \quad \delta, \beta > 0$. it follows the four-parameters LocscB having the following cdf, pdf and hazard function (for $x > 0$)

$$F(x) = 1 - \left[1 + \lambda^{-1}\left(\csc\left(\frac{\pi}{2}\left(1-\left(1+x^{\delta}\right)^{-\beta}\right)\right) - 1\right)^{-1}\right]^{-\alpha}$$

$$f(x) = \frac{\alpha\,\pi\,\delta\,\beta\,x^{\delta-1}\left(1+x^{\delta}\right)^{-(\beta+1)}\csc\left(\frac{\pi}{2}\left(1-\left(1+x^{\delta}\right)^{-\beta}\right)\right)\cot\left(\frac{\pi}{2}\left(1-\left(1+x^{\delta}\right)^{-\beta}\right)\right)}{2\,\lambda\left(\csc\left(\frac{\pi}{2}\left(1-(1+x^{\delta})^{-\beta}\right)\right)-1\right)^{-2}\left[1+\lambda^{-1}\left(\csc\left(\frac{\pi}{2}\left(1-(1+x^{\delta})^{-\beta}\right)\right)-1\right)^{-1}\right]^{(\alpha+1)}}$$

$$h(x) = \frac{\alpha\,\pi\,\delta\,\beta\,(x)^{\delta-1}(1+(x)^{\delta})^{-(\beta+1)}\csc\left(\frac{\pi}{2}\left(1-\left(1+x^{\delta}\right)^{-\beta}\right)\right)\cot\left(\frac{\pi}{2}\left(1-\left(1+x^{\delta}\right)^{-\beta}\right)\right)}{2\,\lambda\left(\csc\left(\frac{\pi}{2}\left(1-(1+x^{\delta})^{-\beta}\right)\right)-1\right)^{-2}\left[1+\lambda^{-1}\left(\csc\left(\frac{\pi}{2}\left(1-(1+x^{\delta})^{-\beta}\right)\right)-1\right)^{-1}\right]}.$$

Figure 3 displays some plots of the density and hazard rate function of the LocscB distribution for some parametric values. Figure 3(a) depicts that the LocscB density have symmetrical, right-skewed, left-skewed and reversed-J shapes. Figure 3(b) reveals that the LocscB hazard rate function have decreasing, increasing and upside down bathtub shapes.



(a) (b)

Figure 3. Plots of the LocscB (a) density (b) hazard rate for some parametric values.

6. The new distribution

6.1. Main properties

In this part, we will look at the unique member of the Locsc-G family of distributions that uses the Weibull distribution as a baseline, as well as its key features. As a result, by swapping the CDF $G(x) = 1 - \mathrm{e}^{-\sigma x^{\beta}}$, $x > 0$, into Eq (2.2), The new distribution's CDF can be written as below

$$F(x) = 1 - \left[1 + \lambda^{-1}\left(\csc\left(\frac{\pi}{2}\left(1 - \mathrm{e}^{-\sigma x^{\beta}}\right)\right) - 1\right)^{-1}\right]^{-\alpha} \qquad x > 0. \tag{6.1}$$

The corresponding pdf is

$$f(x) = \frac{\alpha\,\pi\,(\sigma\beta x^{\beta-1}\mathrm{e}^{-\sigma x^{\beta}})\,\csc\left(\frac{\pi}{2}(1-\mathrm{e}^{-\sigma x^{\beta}})\right)\,\cot\left(\frac{\pi}{2}(1-\mathrm{e}^{-\sigma x^{\beta}})\right)}{2\,\lambda\left(\csc(\frac{\pi}{2}(1-\mathrm{e}^{-\sigma x^{\beta}})) - 1\right)^{-2}\left[1+\lambda^{-1}\left(\csc(\frac{\pi}{2}(1-\mathrm{e}^{-\sigma x^{\beta}})) - 1\right)^{-1}\right]^{(\alpha+1)}}. \tag{6.2}$$

The corresponding hazard rate function is

$$h(x) = \frac{\alpha\,\pi\,(\sigma\beta x^{\beta-1}\mathrm{e}^{-\sigma x^{\beta}})\,\csc\left(\frac{\pi}{2}(1-\mathrm{e}^{-\sigma x^{\beta}})\right)\,\cot\left(\frac{\pi}{2}(1-\mathrm{e}^{-\sigma x^{\beta}})\right)}{2\,\lambda\left(\csc(\frac{\pi}{2}(1-\mathrm{e}^{-\sigma x^{\beta}})) - 1\right)^{-2}\left[1+\lambda^{-1}\left(\csc(\frac{\pi}{2}(1-\mathrm{e}^{-\sigma x^{\beta}})) - 1\right)^{-1}\right]}. \tag{6.3}$$

Figure 4 illustrates the suggested model's showing density forms.

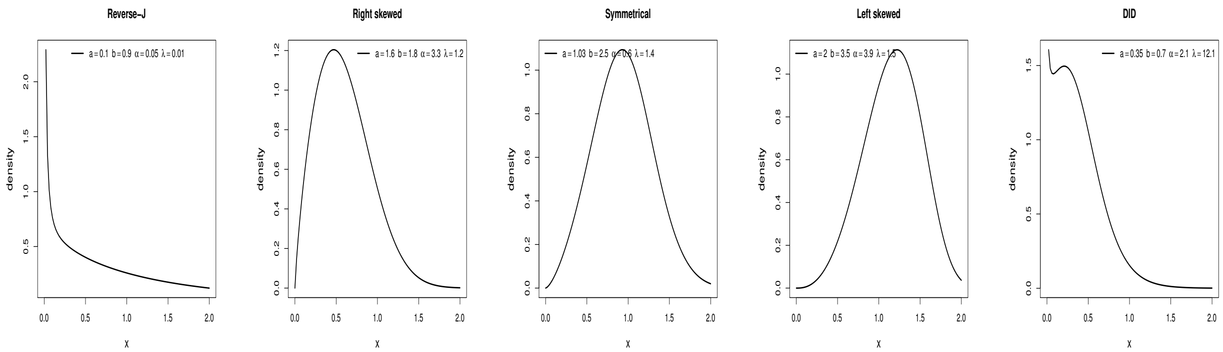


Figure 4. Different plots of the LocscW density.

Figure 5 illustrates the suggested model's displaying hazard rate function forms.

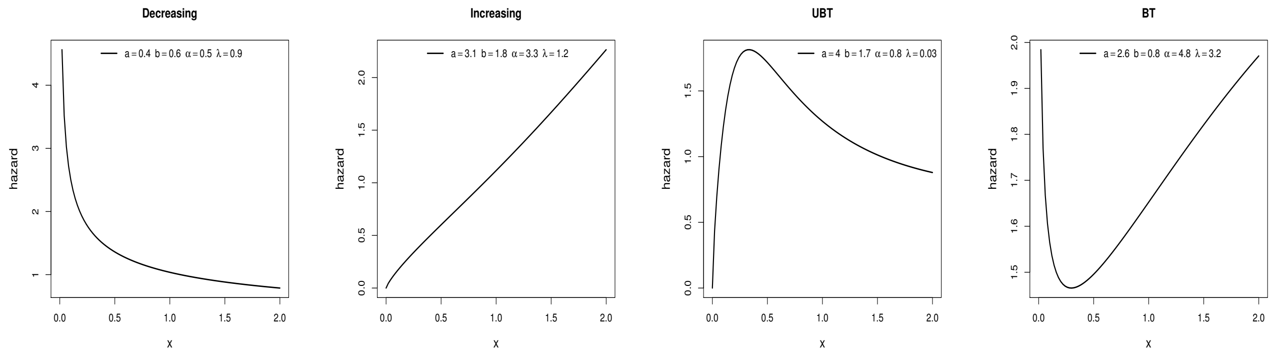


Figure 5. Different plots of the LocscW hazard rate function.

6.1.1. Reliability measures

The hazard rate function is a key notion and performs a central role in risk and survival analysis. There are some other important functions such survival function $S(x)$, also another important one is reversed hazard rate $r(x)$, at last we must not forget the cumulative hazard rate $H(x)$, and the very interesting mills' ratio $m(x)$, elasticity $e(x)$ and finally the conditional reliability function $\bar{G}(G(x), \alpha, \beta\,|t)$, which are respectively, presented below.

$$S(x) = \left[1 + \lambda^{-1}\left(\csc\left(\frac{\pi}{2}\left(1 - e^{-\sigma x^{\beta}}\right)\right) - 1\right)^{-1}\right]^{-\alpha}$$

$$h(x) = \frac{\alpha\,\pi\,\sigma\,\beta x^{\beta-1}\, e^{-\sigma x^{\beta}}\, \csc\left(\frac{\pi}{2}\left(1 - e^{-\sigma x^{\beta}}\right)\right)\cot\left(\frac{\pi}{2}\left(1 - e^{-\sigma x^{\beta}}\right)\right)}{2\,\lambda\left(\csc\left(\frac{\pi}{2}\left(1 - e^{-\sigma x^{\beta}}\right)\right) - 1\right)^{-2}\left[1 + \lambda^{-1}\left(\csc\left(\frac{\pi}{2}\left(1 - e^{-\sigma x^{\beta}}\right)\right) - 1\right)^{-1}\right]}$$

$$r(x) = \frac{\alpha\,\pi\,\sigma\,\beta x^{\beta-1}\,e^{-\sigma\,x^{\beta}}\ \csc\left(\frac{\pi}{2}\left(1-e^{-\sigma\,x^{\beta}}\right)\right)\cot\left(\frac{\pi}{2}\left(1-e^{-\sigma\,x^{\beta}}\right)\right)}{\left(1-\left[1+\lambda^{-1}\left(\csc\left(\frac{\pi}{2}\left(1-e^{-\sigma\,x^{\beta}}\right)\right)-1\right)^{-1}\right]^{-\alpha}\right)} * \frac{\left[1+\lambda^{-1}\left(\csc\left(\frac{\pi}{2}\left(1-e^{-\sigma\,x^{\beta}}\right)\right)-1\right)^{-1}\right]^{-(\alpha+1)}}{2\,\lambda\left(\csc\left(\frac{\pi}{2}\left(1-e^{-\sigma\,x^{\beta}}\right)\right)-1\right)^{-2}}$$

$$m(x) = \frac{2\,\lambda\left[1+\lambda^{-1}\left(\csc\left(\frac{\pi}{2}\left(1-e^{-\sigma\,x^{\beta}}\right)\right)-1\right)^{-1}\right]\left(\csc\left(\frac{\pi}{2}\left(1-e^{-\sigma\,x^{\beta}}\right)\right)-1\right)^{-2}}{\alpha\,\pi\,g(x)\ \csc\left(\frac{\pi}{2}\left(1-e^{-\sigma\,x^{\beta}}\right)\right)\cot\left(\frac{\pi}{2}\left(1-e^{-\sigma\,x^{\beta}}\right)\right)}$$

$$H(x) = -\log\left[1-F(x)\right] = \log\left\{\left[1+\lambda^{-1}\left(\csc\left(\frac{\pi}{2}\left(1-e^{-\sigma\,x^{\beta}}\right)\right)-1\right)^{-1}\right]^{-\alpha}\right\}^{-1}$$

$$e(x) = \frac{\partial}{\partial(\ln\left(1-e^{-\sigma\,x^{\beta}}\right))}\ \ln\left\{1-\left[1+\lambda^{-1}\left(\csc\left(\frac{\pi}{2}\left(1-e^{-\sigma\,x^{\beta}}\right)\right)-1\right)^{-1}\right]^{-\alpha}\right\}$$

$$\bar{G}(G(x),\alpha,\lambda\,|t) = \frac{\left[1+\lambda^{-1}\left(\csc\left(\frac{\pi}{2}\left(1-e^{-\sigma\,t^{\beta}}\right)\right)-1\right)^{-1}\right]^{\alpha}}{\left[1+\lambda^{-1}\left(\csc\left(\frac{\pi}{2}\left(1-e^{-\sigma\,(x+t)^{\beta}}\right)\right)-1\right)^{-1}\right]^{\alpha}}.$$

6.1.2. Residual and reverse residual life

The residual life has several uses in probability and statistics and risk assessment. The residual lifetime of LocscW random variable X denoted by $R_t(x)$ is

$$R_t(x) = \frac{\left[1+\lambda^{-1}\left(\csc\left(\frac{\pi}{2}\left(1-e^{-\sigma\,x^{\beta}}\right)\right)-1\right)^{-1}\right]^{\alpha}}{\left[1+\lambda^{-1}\left(\csc\left(\frac{\pi}{2}\left(1-e^{-\sigma\,(x+t)^{\beta}}\right)\right)-1\right)^{-1}\right]^{\alpha}}$$

Additionally, the reversed hazard rate function $\bar{R}_t(x)$ is

$$\bar{R}_t(x) = \frac{\left[1+\lambda^{-1}\left(\csc\left(\frac{\pi}{2}\left(1-e^{-\sigma\,x^{\beta}}\right)\right)-1\right)^{-1}\right]^{\alpha}}{\left[1+\lambda^{-1}\left(\csc\left(\frac{\pi}{2}\left(1-e^{-\sigma\,(x-t)^{\beta}}\right)\right)-1\right)^{-1}\right]^{\alpha}}.$$

6.1.3. Quantile function, quantile density function and median

The quantile function of LocscW is given by

$$Q_{LocscW}(p) = \left(\frac{\log\left(\frac{\pi}{2\sec^{-1}\left(\frac{1}{\lambda(1-p)^{-1/\alpha}-\lambda}+1\right)}\right)}{\sigma}\right)^{1/\beta}, \qquad p \in (0,1). \tag{6.4}$$

The quartiles and octiles, as well as skewness and kurtosis, can be calculated from this description, and the following distribution results are useful: For a random variable U with a uniform distribution on (0, 1), $Q_{LocscW(U)}$ has the LocscW distribution.

Furthermore, the quantile density function (qdf) for LocscW may indeed be calculated by getting the differentiation of $Q_{LocscW(U)}$ with respect to p. The median, in an instance, is provided as

$$Median = Q_F(0.5) = Q_G(x_*) = \left[\frac{-1}{\sigma}\log\left(1-\frac{2}{\pi}\csc^{-1}\left(1+\frac{\lambda}{(0.5)^{\frac{-1}{\alpha}}-1}\right)\right)\right]^{(\frac{1}{\beta})}.$$

6.2. MacGillivary's skewness

Obtaining skewness is very important for researchers , however MacGillivary (1986) created a technique to obtain it by the aid of the quantile function, such as below

$$\delta(p) = \frac{\delta_{(1)}(\alpha,\beta,\sigma,\lambda)}{\delta_{(2)}(\alpha,\beta,\sigma,\lambda)} = \frac{Q_{(1-p)} + Q_{(p)} - 2Q_{(1/2)}}{Q_{(1-p)} - Q_{(p)}} \tag{6.5}$$

where $p \epsilon (0, 1)$ and $Q(.)$ is the qf stated in Eq (6.4).

$$\begin{aligned}\delta_{(1)}(\alpha,\beta,\sigma,\lambda) &= \left[\frac{-1}{\sigma}\log\left(1-\frac{2}{\pi}\csc^{-1}\left(1+\frac{\lambda}{\left(p^{\frac{-1}{\alpha}}-1\right)}\right)\right)\right]^{(\frac{1}{\beta})}\\ &\quad+\left[\frac{-1}{\sigma}\log\left(1-\frac{2}{\pi}\csc^{-1}\left(1+\frac{\lambda}{(1-p)^{\frac{-1}{\alpha}}-1}\right)\right)\right]^{(\frac{1}{\beta})}\\ &\quad-2\left[\frac{-1}{\sigma}\log\left(1-\frac{2}{\pi}\csc^{-1}\left(1+\frac{\lambda}{(1/2)^{\frac{-1}{\alpha}}-1}\right)\right)\right]^{(\frac{1}{\beta})}\end{aligned}$$

$$\begin{aligned}\delta_{(2)}(\alpha,\beta,\sigma,\lambda) &= \left[\frac{-1}{\sigma}\log\left(1-\frac{2}{\pi}\csc^{-1}\left(1+\frac{\lambda}{\left(p^{\frac{-1}{\alpha}}-1\right)}\right)\right)\right]^{(\frac{1}{\beta})}\\ &\quad-\left[\frac{-1}{\sigma}\log\left(1-\frac{2}{\pi}\csc^{-1}\left(1+\frac{\lambda}{(1-p)^{\frac{-1}{\alpha}}-1}\right)\right)\right]^{(\frac{1}{\beta})}\end{aligned}$$

Because the MacGillivary skewness measure $\delta(p)$ is simply dependent on qf, it can efficiently characterize the influence of the parameters $(\alpha,\beta,\sigma,\lambda)$ just on the skewness of X. In Figure 6, the plots in Figure 6(left) describes keeping parameters $(\alpha = 1, \lambda = 0.1, \sigma = 0.5)$ as constant while the parameter β values are increased from 0.1 to 0.9, then $\delta(p) \to 0$ means skewness approaches to zero (or approaching to symmetry).

In Figure 6, the plots in Figure 6(middle) describes keeping parameters $(\alpha = 1.5, \lambda = 0.5, \sigma = 1.0)$ as constant (as compared to Figure 6(left),the values of $(\alpha, \lambda, \sigma)$ are increased by 0.5) while the parameter β values are also increased from 0.1 to 1.0 on regular spacing, then $\delta(p) \to 0.5$ means lightly skewness is observed.

In Figure 6, the plots in Figure 6(right) describes keeping parameters $(\alpha = 1.5, \lambda = 1.0, \sigma = 1.0)$ as constant (as compared to Figure 6(middle),the values of (α, σ) are not changes but λ in increased

0.5 only) while the parameter β values are increased from 0.05 to 0.5 on different spacing values, then $\delta(p) \to 1.0$ means significant skewness is produced.

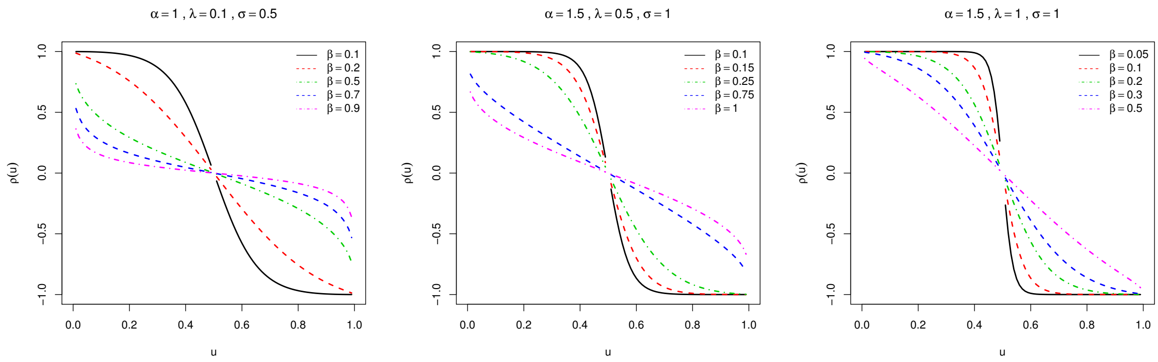


Figure 6. MacGillivary's skewness plots for selected values of the parameters.

In Figure 7, the plots in Figure 7(left) describes keeping parameters ($\alpha = 2.0, \beta = 1.5, \sigma = 0.1$) as constant while the parameter λ values are increased from 0.1 to 0.95, then the symmetry is loosed towards left (negative skewness is observed).

In Figure 7, the plots in Figure 7(middle) describes keeping parameters ($\beta = 1.0, \lambda = 0.1, \sigma = 0.5$) as constant while the parameter α values are increased from 0.1 to 1.05 on different spacing values, then $\delta(p)$ increases heavily means the highly skewness is produced on right side.

In Figure 7, the plots in Figure 7(right) describes keeping parameters ($\alpha = 1.5, \lambda = 0.5, \sigma = 0.5$) as constant while the parameter β values are increased from 1.05 to 3.5 on different spacing values, then $\delta(p)$ increases means the right skewness is produced.

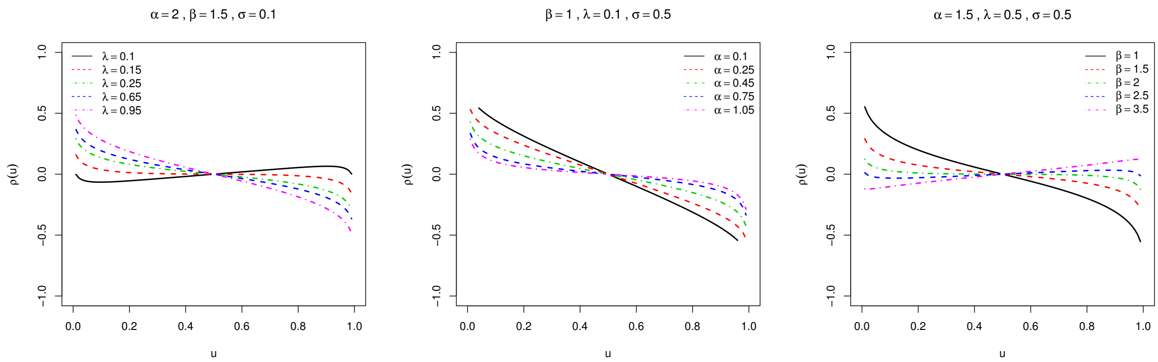
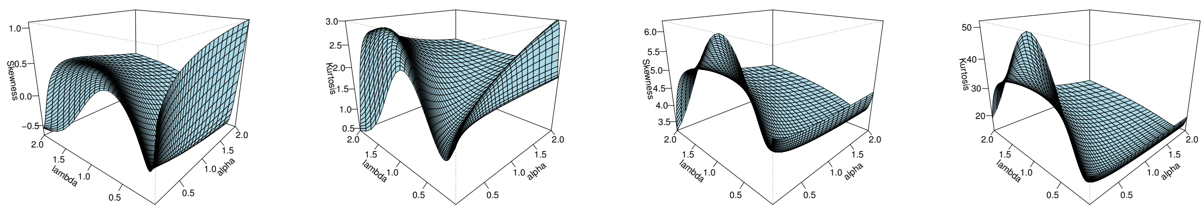


Figure 7. MacGillivary's skewness plots for selected values of the parameters.

6.3. *Skewness and kurtosis via 3D graphs*

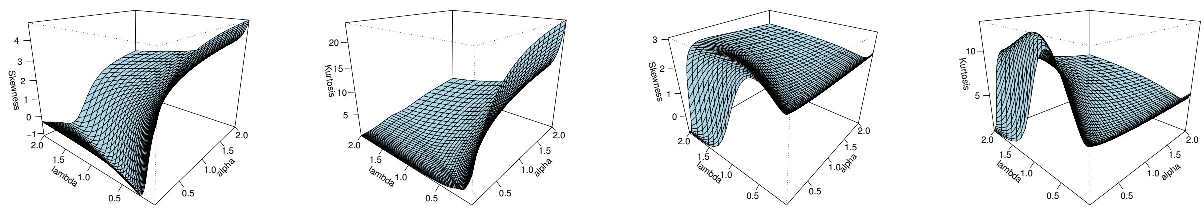
Recently, the tendency has shifted, and the graphical image is now more common and preferred than numerical and tabular representation. The 3D figures showed below vividly demonstrate the shift in skewness and kurtosis that occurs when the parental model parameters are changed. In Figure 8, the alternate curves 8(a) and 8(c) are for skewness while 8(b) and 8(d) are for kurtosis respectively. Both measures of skewness and kurtosis for the proposed model are highly dependent on the fixed values of α and λ.



(a) (b) (c) (d)

Figure 8. For chosen parametric values, curves for skewness and kurtosis.

In Figure 9, the curves 9(a) and 9(c) are for skewness while 9(b) and 9(d) are for kurtosis respectively.



(a) (b) (c) (d)

Figure 9. Curves for the skewness and the kurtosis for selected parametric values.

The baseline parameters $\sigma = 2.5$ and $\beta = 3.1$ are taken in Figure 9, it is observed that the skewness is decreased (symmetry is increased) in Figure 9(a) as well as the kurtosis is reduced (normality is increased) in Figure 9(b).

6.4. Reduced models of LocscW

In Table 3, three new reduced/sub-models of LocscW distribution are deduced here, just limiting the parametric values.

Table 3. Reduced models of LocscW distribution.

$S.No.$	σ	β	Reduced models of LocscW distribution	Comments
1	1	-	Lomax cosecant 1 parameter Weibull (LocscW(1P)) distribution	New
2	-	1	Lomax cosecant exponential (LocscE) distribution	New
3	-	2	Lomax cosecant rayleigh (LocscR) distribution	New

Table 4. Simulation values of BIAS, MSE and MRE for ($\sigma = 0.25,\ \beta = 0.5,\ \alpha = 0.75,\ \lambda = 0.5$).

n	Est.	Est. Par.	MLE	ADE	CVME	MPSE	LSE	RTADE	WLSE
50	BIAS	$\hat{\sigma}$	0.166439$^{\{7\}}$	0.15964$^{\{3\}}$	0.156342$^{\{1\}}$	0.156488$^{\{2\}}$	0.165037$^{\{6\}}$	0.161629$^{\{4\}}$	0.16181$^{\{5\}}$
		$\hat{\beta}$	0.203115$^{\{3\}}$	0.201461$^{\{2\}}$	0.302332$^{\{7\}}$	0.141869$^{\{1\}}$	0.256889$^{\{5\}}$	0.287445$^{\{6\}}$	0.220661$^{\{4\}}$
		$\hat{\alpha}$	0.3894$^{\{2\}}$	0.390991$^{\{3\}}$	0.41841$^{\{6\}}$	0.365869$^{\{1\}}$	0.416399$^{\{5\}}$	0.406432$^{\{4\}}$	0.44011$^{\{7\}}$
		$\hat{\lambda}$	0.356816$^{\{1\}}$	0.382947$^{\{3\}}$	0.413734$^{\{6\}}$	0.367721$^{\{2\}}$	0.417675$^{\{7\}}$	0.409102$^{\{5\}}$	0.39411$^{\{4\}}$
	MSE	$\hat{\theta}$	0.0465$^{\{7\}}$	0.034473$^{\{3\}}$	0.031461$^{\{1\}}$	0.037084$^{\{6\}}$	0.035224$^{\{4\}}$	0.032308$^{\{2\}}$	0.035791$^{\{5\}}$
		$\hat{\beta}$	0.065583$^{\{2\}}$	0.070206$^{\{3\}}$	0.158663$^{\{7\}}$	0.036064$^{\{1\}}$	0.125967$^{\{5\}}$	0.146189$^{\{6\}}$	0.087577$^{\{4\}}$
		$\hat{\alpha}$	0.209603$^{\{1\}}$	0.225049$^{\{2\}}$	0.244866$^{\{5\}}$	0.233893$^{\{3\}}$	0.263057$^{\{6\}}$	0.24107$^{\{4\}}$	0.278205$^{\{7\}}$
		$\hat{\lambda}$	0.144604$^{\{1\}}$	0.167324$^{\{3\}}$	0.185792$^{\{6\}}$	0.15515$^{\{2\}}$	0.188622$^{\{7\}}$	0.182576$^{\{5\}}$	0.174821$^{\{4\}}$
	MRE	$\hat{\theta}$	0.665755$^{\{7\}}$	0.638559$^{\{3\}}$	0.625369$^{\{1\}}$	0.625954$^{\{2\}}$	0.660147$^{\{6\}}$	0.646518$^{\{4\}}$	0.647238$^{\{5\}}$
		$\hat{\beta}$	0.40623$^{\{3\}}$	0.402921$^{\{2\}}$	0.604664$^{\{7\}}$	0.283738$^{\{1\}}$	0.513778$^{\{5\}}$	0.574889$^{\{6\}}$	0.441321$^{\{4\}}$
		$\hat{\alpha}$	0.5192$^{\{2\}}$	0.521321$^{\{3\}}$	0.557879$^{\{6\}}$	0.487826$^{\{1\}}$	0.555199$^{\{5\}}$	0.541909$^{\{4\}}$	0.586813$^{\{7\}}$
		$\hat{\lambda}$	0.713633$^{\{1\}}$	0.765894$^{\{3\}}$	0.827467$^{\{6\}}$	0.735443$^{\{2\}}$	0.83535$^{\{7\}}$	0.818203$^{\{5\}}$	0.788219$^{\{4\}}$
	$\sum Ranks$		37$^{\{3\}}$	33$^{\{2\}}$	59$^{\{5\}}$	24$^{\{1\}}$	68$^{\{7\}}$	55$^{\{4\}}$	60$^{\{6\}}$
100	BIAS	$\hat{\sigma}$	0.153797$^{\{7\}}$	0.147122$^{\{5\}}$	0.135032$^{\{3\}}$	0.129218$^{\{1\}}$	0.134948$^{\{2\}}$	0.143487$^{\{4\}}$	0.153338$^{\{6\}}$
		$\hat{\beta}$	0.120608$^{\{2\}}$	0.128477$^{\{3\}}$	0.186858$^{\{7\}}$	0.086896$^{\{1\}}$	0.164689$^{\{5\}}$	0.171597$^{\{6\}}$	0.136009$^{\{4\}}$
		$\hat{\alpha}$	0.377902$^{\{3\}}$	0.386078$^{\{4\}}$	0.403267$^{\{6\}}$	0.288504$^{\{1\}}$	0.392481$^{\{5\}}$	0.367004$^{\{2\}}$	0.407771$^{\{7\}}$
		$\hat{\lambda}$	0.29636$^{\{1\}}$	0.326781$^{\{3\}}$	0.372342$^{\{7\}}$	0.313049$^{\{2\}}$	0.362218$^{\{6\}}$	0.362159$^{\{5\}}$	0.344262$^{\{4\}}$
	MSE	$\hat{\theta}$	0.039643$^{\{7\}}$	0.034054$^{\{5\}}$	0.025299$^{\{2\}}$	0.025729$^{\{3\}}$	0.024872$^{\{1\}}$	0.026762$^{\{4\}}$	0.036139$^{\{6\}}$
		$\hat{\beta}$	0.024458$^{\{2\}}$	0.026304$^{\{3\}}$	0.056386$^{\{7\}}$	0.012241$^{\{1\}}$	0.045467$^{\{5\}}$	0.048396$^{\{6\}}$	0.031656$^{\{4\}}$
		$\hat{\alpha}$	0.214127$^{\{2\}}$	0.234593$^{\{4\}}$	0.250263$^{\{6\}}$	0.175237$^{\{1\}}$	0.240497$^{\{5\}}$	0.220399$^{\{3\}}$	0.254909$^{\{7\}}$
		$\hat{\lambda}$	0.107776$^{\{1\}}$	0.132118$^{\{3\}}$	0.157282$^{\{7\}}$	0.120781$^{\{2\}}$	0.150874$^{\{5\}}$	0.15114$^{\{6\}}$	0.143373$^{\{4\}}$
	MRE	$\hat{\theta}$	0.61519$^{\{7\}}$	0.588486$^{\{5\}}$	0.540129$^{\{3\}}$	0.516873$^{\{1\}}$	0.539792$^{\{2\}}$	0.573948$^{\{4\}}$	0.613352$^{\{6\}}$
		$\hat{\beta}$	0.241216$^{\{2\}}$	0.256955$^{\{3\}}$	0.373716$^{\{7\}}$	0.173793$^{\{1\}}$	0.329378$^{\{5\}}$	0.343194$^{\{6\}}$	0.272019$^{\{4\}}$
		$\hat{\alpha}$	0.503869$^{\{3\}}$	0.514771$^{\{4\}}$	0.537689$^{\{6\}}$	0.384672$^{\{1\}}$	0.523308$^{\{5\}}$	0.489339$^{\{2\}}$	0.543695$^{\{7\}}$
		$\hat{\lambda}$	0.59272$^{\{1\}}$	0.653562$^{\{3\}}$	0.744684$^{\{7\}}$	0.626099$^{\{2\}}$	0.724437$^{\{6\}}$	0.724317$^{\{5\}}$	0.688523$^{\{4\}}$
	$\sum Ranks$		38$^{\{2\}}$	45$^{\{3\}}$	68$^{\{7\}}$	17$^{\{1\}}$	52$^{\{4\}}$	53$^{\{5\}}$	63$^{\{6\}}$
200	BIAS	$\hat{\sigma}$	0.143248$^{\{6\}}$	0.139091$^{\{5\}}$	0.128434$^{\{3\}}$	0.10969$^{\{1\}}$	0.137393$^{\{4\}}$	0.120289$^{\{2\}}$	0.1452$^{\{7\}}$
		$\hat{\beta}$	0.081747$^{\{2\}}$	0.085726$^{\{3\}}$	0.113757$^{\{6\}}$	0.060472$^{\{1\}}$	0.108641$^{\{5\}}$	0.116098$^{\{7\}}$	0.089427$^{\{4\}}$
		$\hat{\alpha}$	0.343484$^{\{3\}}$	0.352998$^{\{4\}}$	0.353592$^{\{5\}}$	0.233319$^{\{1\}}$	0.369344$^{\{6\}}$	0.333755$^{\{2\}}$	0.38331$^{\{7\}}$
		$\hat{\lambda}$	0.257837$^{\{1\}}$	0.272929$^{\{3\}}$	0.306274$^{\{6\}}$	0.266796$^{\{2\}}$	0.313787$^{\{7\}}$	0.301849$^{\{5\}}$	0.287142$^{\{4\}}$
	MSE	$\hat{\theta}$	0.036058$^{\{7\}}$	0.033323$^{\{5\}}$	0.024352$^{\{3\}}$	0.02014$^{\{1\}}$	0.02754$^{\{4\}}$	0.020168$^{\{2\}}$	0.035779$^{\{6\}}$
		$\hat{\beta}$	0.011033$^{\{2\}}$	0.011812$^{\{3\}}$	0.021639$^{\{6\}}$	0.00578$^{\{1\}}$	0.019095$^{\{5\}}$	0.021922$^{\{7\}}$	0.01286$^{\{4\}}$
		$\hat{\alpha}$	0.182778$^{\{2\}}$	0.209958$^{\{5\}}$	0.20589$^{\{4\}}$	0.139641$^{\{1\}}$	0.23027$^{\{6\}}$	0.185814$^{\{3\}}$	0.231535$^{\{7\}}$
		$\hat{\lambda}$	0.091114$^{\{1\}}$	0.101621$^{\{3\}}$	0.115374$^{\{5\}}$	0.101162$^{\{2\}}$	0.121723$^{\{7\}}$	0.117581$^{\{6\}}$	0.112911$^{\{4\}}$
	MRE	$\hat{\theta}$	0.57299$^{\{6\}}$	0.556365$^{\{5\}}$	0.513737$^{\{3\}}$	0.438759$^{\{1\}}$	0.549572$^{\{4\}}$	0.481154$^{\{2\}}$	0.580799$^{\{7\}}$
		$\hat{\beta}$	0.163494$^{\{2\}}$	0.171453$^{\{3\}}$	0.227514$^{\{6\}}$	0.120944$^{\{1\}}$	0.217282$^{\{5\}}$	0.232195$^{\{7\}}$	0.178855$^{\{4\}}$
		$\hat{\alpha}$	0.457978$^{\{3\}}$	0.470533$^{\{4\}}$	0.471456$^{\{5\}}$	0.311092$^{\{1\}}$	0.492459$^{\{6\}}$	0.445007$^{\{2\}}$	0.51108$^{\{7\}}$
		$\hat{\lambda}$	0.515674$^{\{1\}}$	0.545859$^{\{3\}}$	0.612548$^{\{6\}}$	0.533592$^{\{2\}}$	0.627574$^{\{7\}}$	0.603699$^{\{5\}}$	0.574284$^{\{4\}}$
	$\sum Ranks$		36$^{\{2\}}$	46$^{\{3\}}$	58$^{\{5\}}$	15$^{\{1\}}$	66$^{\{7\}}$	50$^{\{4\}}$	65$^{\{6\}}$
300	BIAS	$\hat{\sigma}$	0.138561$^{\{5\}}$	0.138807$^{\{6\}}$	0.124072$^{\{3\}}$	0.090879$^{\{1\}}$	0.131573$^{\{4\}}$	0.118318$^{\{2\}}$	0.146928$^{\{7\}}$
		$\hat{\beta}$	0.058973$^{\{2\}}$	0.067624$^{\{3\}}$	0.090901$^{\{7\}}$	0.050863$^{\{1\}}$	0.084914$^{\{5\}}$	0.088902$^{\{6\}}$	0.069056$^{\{4\}}$
		$\hat{\alpha}$	0.337647$^{\{2\}}$	0.350945$^{\{4\}}$	0.360021$^{\{6\}}$	0.193548$^{\{1\}}$	0.359778$^{\{5\}}$	0.346452$^{\{3\}}$	0.369353$^{\{7\}}$
		$\hat{\lambda}$	0.219299$^{\{1\}}$	0.245087$^{\{3\}}$	0.279345$^{\{6\}}$	0.232013$^{\{2\}}$	0.279926$^{\{7\}}$	0.274285$^{\{5\}}$	0.247637$^{\{4\}}$
	MSE	$\hat{\theta}$	0.034373$^{\{6\}}$	0.033063$^{\{5\}}$	0.023835$^{\{3\}}$	0.013914$^{\{1\}}$	0.026133$^{\{4\}}$	0.020005$^{\{2\}}$	0.037861$^{\{7\}}$
		$\hat{\beta}$	0.005718$^{\{2\}}$	0.007306$^{\{3\}}$	0.013345$^{\{7\}}$	0.004054$^{\{1\}}$	0.011805$^{\{5\}}$	0.012591$^{\{6\}}$	0.007862$^{\{4\}}$
		$\hat{\alpha}$	0.191074$^{\{2\}}$	0.214125$^{\{4\}}$	0.222387$^{\{6\}}$	0.109753$^{\{1\}}$	0.225309$^{\{7\}}$	0.208967$^{\{3\}}$	0.220442$^{\{5\}}$
		$\hat{\lambda}$	0.072871$^{\{1\}}$	0.091013$^{\{3\}}$	0.104698$^{\{6\}}$	0.082986$^{\{2\}}$	0.106309$^{\{7\}}$	0.102139$^{\{5\}}$	0.095991$^{\{4\}}$
	MRE	$\hat{\theta}$	0.554242$^{\{5\}}$	0.556827$^{\{6\}}$	0.496287$^{\{3\}}$	0.363517$^{\{1\}}$	0.526293$^{\{4\}}$	0.473271$^{\{2\}}$	0.587712$^{\{7\}}$
		$\hat{\beta}$	0.117947$^{\{2\}}$	0.135248$^{\{3\}}$	0.181803$^{\{7\}}$	0.101726$^{\{1\}}$	0.169828$^{\{5\}}$	0.177805$^{\{6\}}$	0.138111$^{\{4\}}$
		$\hat{\alpha}$	0.450196$^{\{2\}}$	0.471926$^{\{4\}}$	0.480028$^{\{6\}}$	0.258064$^{\{1\}}$	0.479704$^{\{5\}}$	0.461936$^{\{3\}}$	0.492471$^{\{7\}}$
		$\hat{\lambda}$	0.438598$^{\{1\}}$	0.490174$^{\{3\}}$	0.55869$^{\{6\}}$	0.464026$^{\{2\}}$	0.559852$^{\{7\}}$	0.548569$^{\{5\}}$	0.495274$^{\{4\}}$
	$\sum Ranks$		31$^{\{2\}}$	47$^{\{3\}}$	66$^{\{7\}}$	15$^{\{1\}}$	65$^{\{6\}}$	48$^{\{4\}}$	64$^{\{5\}}$
500	BIAS	$\hat{\sigma}$	0.123165$^{\{3\}}$	0.13065$^{\{6\}}$	0.125115$^{\{4\}}$	0.080862$^{\{1\}}$	0.129143$^{\{5\}}$	0.110391$^{\{2\}}$	0.142231$^{\{7\}}$
		$\hat{\beta}$	0.04558$^{\{2\}}$	0.050097$^{\{3\}}$	0.068652$^{\{7\}}$	0.041848$^{\{1\}}$	0.065164$^{\{5\}}$	0.065489$^{\{6\}}$	0.051396$^{\{4\}}$
		$\hat{\alpha}$	0.304704$^{\{2\}}$	0.3462$^{\{4\}}$	0.346591$^{\{5\}}$	0.156206$^{\{1\}}$	0.36074$^{\{6\}}$	0.345446$^{\{3\}}$	0.372417$^{\{7\}}$
		$\hat{\lambda}$	0.189126$^{\{1\}}$	0.205681$^{\{3\}}$	0.240779$^{\{6\}}$	0.206657$^{\{4\}}$	0.251445$^{\{7\}}$	0.227929$^{\{5\}}$	0.199507$^{\{2\}}$
	MSE	$\hat{\theta}$	0.027603$^{\{5\}}$	0.032787$^{\{6\}}$	0.02505$^{\{3\}}$	0.011429$^{\{1\}}$	0.025908$^{\{4\}}$	0.018361$^{\{2\}}$	0.035584$^{\{7\}}$
		$\hat{\beta}$	0.003292$^{\{2\}}$	0.004$^{\{3\}}$	0.007501$^{\{7\}}$	0.002583$^{\{1\}}$	0.006854$^{\{5\}}$	0.006865$^{\{6\}}$	0.004201$^{\{4\}}$
		$\hat{\alpha}$	0.166447$^{\{2\}}$	0.213746$^{\{5\}}$	0.209538$^{\{4\}}$	0.083822$^{\{1\}}$	0.23977$^{\{6\}}$	0.208212$^{\{3\}}$	0.242504$^{\{7\}}$
		$\hat{\lambda}$	0.059636$^{\{1\}}$	0.070925$^{\{4\}}$	0.086588$^{\{6\}}$	0.07067$^{\{3\}}$	0.094284$^{\{7\}}$	0.07912$^{\{5\}}$	0.066433$^{\{2\}}$
	MRE	$\hat{\theta}$	0.492662$^{\{3\}}$	0.542602$^{\{6\}}$	0.500462$^{\{4\}}$	0.323447$^{\{1\}}$	0.516574$^{\{5\}}$	0.441562$^{\{2\}}$	0.568925$^{\{7\}}$
		$\hat{\beta}$	0.091159$^{\{2\}}$	0.100195$^{\{3\}}$	0.137304$^{\{7\}}$	0.083696$^{\{1\}}$	0.130329$^{\{5\}}$	0.130978$^{\{6\}}$	0.102792$^{\{4\}}$
		$\hat{\alpha}$	0.406272$^{\{2\}}$	0.4616$^{\{4\}}$	0.462121$^{\{5\}}$	0.208274$^{\{1\}}$	0.480986$^{\{6\}}$	0.460594$^{\{3\}}$	0.496556$^{\{7\}}$
		$\hat{\lambda}$	0.378253$^{\{1\}}$	0.411362$^{\{3\}}$	0.481558$^{\{6\}}$	0.413313$^{\{4\}}$	0.50289$^{\{7\}}$	0.455858$^{\{5\}}$	0.399014$^{\{2\}}$
	$\sum Ranks$		26$^{\{2\}}$	50$^{\{4\}}$	64$^{\{6\}}$	20$^{\{1\}}$	68$^{\{7\}}$	48$^{\{3\}}$	60$^{\{5\}}$

Table 5. Simulation values of BIAS, MSE and MRE for (σ = 1.5, β = 0.25, α = 1.5, λ = 0.75).

n	Est.	Est. Par.	MLE	ADE	CVME	MPSE	LSE	RTADE	WLSE
50	BIAS	$\hat{\sigma}$	2.368473[3]	2.686051[4]	4.986483[7]	0.924537[1]	4.967695[6]	2.141082[2]	3.185041[5]
		$\hat{\beta}$	0.097271[2]	0.100905[3]	0.136041[7]	0.05979[1]	0.126029[5]	0.129388[6]	0.108112[4]
		$\hat{\alpha}$	0.768542[2]	0.932529[6]	0.817627[4]	0.528023[1]	0.810726[3]	0.887373[5]	1.025206[7]
		$\hat{\lambda}$	0.519419[2]	0.597718[5]	0.560367[3]	0.496842[1]	0.578398[4]	0.622348[6]	0.635568[7]
	MSE	$\hat{\theta}$	20.346394[2]	23.821214[4]	322.421312[6]	4.01809[1]	501.265568[7]	22.158699[3]	40.016079[5]
		$\hat{\beta}$	0.017265[2]	0.017372[3]	0.039424[7]	0.007731[1]	0.0332[6]	0.031762[5]	0.022171[4]
		$\hat{\alpha}$	0.881246[2]	1.233622[6]	1.011954[4]	0.584107[1]	0.992128[3]	1.180926[5]	1.406141[7]
		$\hat{\lambda}$	0.341328[1]	0.411674[5]	0.382462[3]	0.345231[2]	0.39751[4]	0.439778[6]	0.45337[7]
	MRE	$\hat{\theta}$	1.578982[3]	1.790701[4]	3.324322[7]	0.616358[1]	3.311797[6]	1.427388[2]	2.123361[5]
		$\hat{\beta}$	0.389083[2]	0.403621[3]	0.544163[7]	0.239161[1]	0.504116[5]	0.517553[6]	0.432447[4]
		$\hat{\alpha}$	0.512361[2]	0.621686[6]	0.545085[4]	0.352015[1]	0.540484[3]	0.591582[5]	0.683471[7]
		$\hat{\lambda}$	0.692559[2]	0.796958[5]	0.747155[3]	0.662455[1]	0.771197[4]	0.829797[6]	0.847424[7]
	$\sum$ *Ranks*		24[2]	53[3]	60[6]	17[1]	56[4.5]	56[4.5]	70[7]
100	BIAS	$\hat{\sigma}$	1.503596[2]	2.181679[4]	2.409605[6]	0.503445[1]	2.327618[5]	1.52453[3]	2.41262[7]
		$\hat{\beta}$	0.056812[2]	0.065851[3]	0.085008[6]	0.03857[1]	0.078473[5]	0.085975[7]	0.069386[4]
		$\hat{\alpha}$	0.697288[2]	0.956184[6]	0.816886[3]	0.398195[1]	0.848384[4]	0.907708[5]	1.022525[7]
		$\hat{\lambda}$	0.483617[2]	0.561585[5]	0.526647[4]	0.421428[1]	0.52466[3]	0.581553[6]	0.583707[7]
	MSE	$\hat{\theta}$	7.260581[3]	11.729038[4]	18.781725[7]	1.165435[1]	18.232529[6]	5.523101[2]	15.078136[5]
		$\hat{\beta}$	0.005806[2]	0.00713[3]	0.013695[7]	0.002865[1]	0.011771[5]	0.0128[6]	0.008196[4]
		$\hat{\alpha}$	0.794167[2]	1.330844[6]	1.077834[3]	0.454167[1]	1.167044[4]	1.289307[5]	1.437883[7]
		$\hat{\lambda}$	0.320773[2]	0.393818[5]	0.348875[4]	0.274376[1]	0.347553[3]	0.406129[6]	0.4076[7]
	MRE	$\hat{\theta}$	1.002397[2]	1.454453[4]	1.606403[6]	0.33563[1]	1.551745[5]	1.016354[3]	1.608414[7]
		$\hat{\beta}$	0.227249[2]	0.263404[3]	0.340033[6]	0.154279[1]	0.31389[5]	0.343899[7]	0.277546[4]
		$\hat{\alpha}$	0.464859[2]	0.637456[6]	0.544591[3]	0.265463[1]	0.565589[4]	0.605139[5]	0.681683[7]
		$\hat{\lambda}$	0.644823[2]	0.74878[5]	0.702196[4]	0.561904[1]	0.699547[3]	0.775404[6]	0.778275[7]
	$\sum$ *Ranks*		29[2]	52[4]	57[5]	12[1]	50[3]	65[6]	71[7]
200	BIAS	$\hat{\sigma}$	1.037079[2]	1.792147[5]	1.793938[6]	0.22218[1]	1.722149[4]	1.18732[3]	2.132244[7]
		$\hat{\beta}$	0.036793[2]	0.043914[3]	0.053356[6]	0.024479[1]	0.050413[5]	0.053529[7]	0.047812[4]
		$\hat{\alpha}$	0.626596[2]	0.923064[5]	0.850873[3]	0.234961[1]	0.862918[4]	0.938707[6]	1.045254[7]
		$\hat{\lambda}$	0.424606[2]	0.509404[5]	0.495485[4]	0.337934[1]	0.470998[3]	0.51275[6]	0.518243[7]
	MSE	$\hat{\theta}$	3.318423[3]	6.496031[4]	9.106189[7]	0.224702[1]	7.988472[5]	2.720292[2]	9.025803[6]
		$\hat{\beta}$	0.002296[2]	0.003097[3]	0.005123[7]	0.001004[1]	0.004715[5]	0.004757[6]	0.003752[4]
		$\hat{\alpha}$	0.718876[2]	1.320814[5]	1.176936[3]	0.234086[1]	1.259898[4]	1.424838[6]	1.574473[7]
		$\hat{\lambda}$	0.271847[2]	0.352271[7]	0.336857[5]	0.205673[1]	0.300407[3]	0.336161[4]	0.349541[6]
	MRE	$\hat{\theta}$	0.691386[2]	1.194765[5]	1.195959[6]	0.14812[1]	1.148099[4]	0.791547[3]	1.421496[7]
		$\hat{\beta}$	0.147173[2]	0.175657[3]	0.213425[6]	0.097916[1]	0.201652[5]	0.214117[7]	0.191247[4]
		$\hat{\alpha}$	0.41773[2]	0.615376[5]	0.567249[3]	0.15664[1]	0.575279[4]	0.625805[6]	0.696836[7]
		$\hat{\lambda}$	0.566142[2]	0.679205[5]	0.660647[4]	0.450579[1]	0.627997[3]	0.683667[6]	0.69099[7]
	$\sum$ *Ranks*		25[2]	55[4]	60[5]	12[1]	49[3]	62[6]	73[7]
300	BIAS	$\hat{\sigma}$	0.934476[2]	1.652431[5]	1.698162[6]	0.163951[1]	1.648421[4]	1.131801[3]	1.790429[7]
		$\hat{\beta}$	0.029234[2]	0.036177[3]	0.044418[7]	0.021807[1]	0.042904[5]	0.044382[6]	0.036583[4]
		$\hat{\alpha}$	0.549292[2]	0.912171[5]	0.879001[3]	0.182861[1]	0.895503[4]	0.963101[6]	1.051865[7]
		$\hat{\lambda}$	0.376163[2]	0.480521[5]	0.450893[3]	0.320269[1]	0.464435[4]	0.485137[7]	0.483487[6]
	MSE	$\hat{\theta}$	2.944863[3]	5.125591[4]	6.998085[7]	0.104482[1]	6.214904[6]	2.282795[2]	5.749254[5]
		$\hat{\beta}$	0.001448[2]	0.001952[3]	0.003519[7]	0.000764[1]	0.003149[5]	0.00318[6]	0.002026[4]
		$\hat{\alpha}$	0.633235[2]	1.271483[3]	1.280931[4]	0.136351[1]	1.30316[5]	1.514906[6]	1.618684[7]
		$\hat{\lambda}$	0.235934[2]	0.32667[7]	0.288002[3]	0.202082[1]	0.302176[4]	0.315643[5]	0.320899[6]
	MRE	$\hat{\theta}$	0.622984[2]	1.101621[5]	1.132108[6]	0.1093[1]	1.098947[4]	0.754534[3]	1.193619[7]
		$\hat{\beta}$	0.116935[2]	0.144707[3]	0.177672[7]	0.087228[1]	0.171617[5]	0.177528[6]	0.146332[4]
		$\hat{\alpha}$	0.366195[2]	0.608114[5]	0.586[3]	0.121907[1]	0.597002[4]	0.642067[6]	0.701243[7]
		$\hat{\lambda}$	0.501551[2]	0.640695[5]	0.601191[3]	0.427026[1]	0.619247[4]	0.64685[7]	0.644649[6]
	$\sum$ *Ranks*		25[2]	53[3]	59[5]	12[1]	54[4]	63[6]	70[7]
500	BIAS	$\hat{\sigma}$	0.707979[2]	1.670229[6]	1.495733[5]	0.096022[1]	1.415746[4]	1.155813[3]	1.76218[7]
		$\hat{\beta}$	0.022165[2]	0.027766[3]	0.032408[6]	0.016335[1]	0.03191[5]	0.034972[7]	0.029233[4]
		$\hat{\alpha}$	0.468421[2]	0.955412[6]	0.916134[4]	0.105347[1]	0.891212[3]	0.948734[5]	1.03083[7]
		$\hat{\lambda}$	0.31123[2]	0.421891[4]	0.441501[6]	0.232734[1]	0.430158[5]	0.443041[7]	0.411027[3]
	MSE	$\hat{\theta}$	1.945207[2]	5.032233[6]	4.953797[5]	0.022692[1]	4.44341[4]	2.264562[3]	5.551723[7]
		$\hat{\beta}$	0.000806[2]	0.00117[3]	0.00175[6]	0.000427[1]	0.001659[5]	0.001978[7]	0.00128[4]
		$\hat{\alpha}$	0.520655[2]	1.411546[5]	1.382384[4]	0.031728[1]	1.346687[3]	1.49178[6]	1.534144[7]
		$\hat{\lambda}$	0.166973[2]	0.273881[4]	0.298537[7]	0.125512[1]	0.283187[5]	0.287671[6]	0.249034[3]
	MRE	$\hat{\theta}$	0.471986[2]	1.113486[6]	0.997155[5]	0.064014[1]	0.94383[4]	0.770542[3]	1.174786[7]
		$\hat{\beta}$	0.088661[2]	0.111063[3]	0.129633[6]	0.065338[1]	0.127639[5]	0.139889[7]	0.116933[4]
		$\hat{\alpha}$	0.312281[2]	0.636941[6]	0.610756[4]	0.070231[1]	0.594141[3]	0.632489[5]	0.68722[7]
		$\hat{\lambda}$	0.414974[2]	0.562522[4]	0.588668[6]	0.310312[1]	0.573544[5]	0.590721[7]	0.548036[3]
	$\sum$ *Ranks*		24[2]	56[4]	64[6]	12[1]	51[3]	66[7]	63[5]

Table 6. Simulation values of BIAS, MSE and MRE for ($\sigma = 0.75,\ \beta = 1.5,\ \alpha = 0.75,\ \lambda = 1.5$).

n	Est.	Est. Par.	MLE	ADE	CVME	MPSE	LSE	RTADE	WLSE
50	BIAS	$\hat{\sigma}$	0.533007[3]	0.525515[2]	0.723204[5]	0.768352[7]	0.669541[4]	0.735192[6]	0.467968[1]
		$\hat{\beta}$	0.696631[3]	0.678629[2]	0.919443[7]	0.314638[1]	0.820546[4]	0.845912[6]	0.83555[5]
		$\hat{\alpha}$	0.373187[1]	0.38736[2]	0.468738[7]	0.430852[5]	0.459504[6]	0.424095[4]	0.400041[3]
		$\hat{\lambda}$	1.235272[3]	1.233919[2]	1.310685[7]	0.852181[1]	1.258193[4]	1.30097[6]	1.258798[5]
	MSE	$\hat{\theta}$	0.538055[2]	0.588999[3]	1.283818[6]	1.342864[7]	1.038856[5]	0.968285[4]	0.41345[1]
		$\hat{\beta}$	0.925072[3]	0.894862[2]	1.938067[7]	0.240667[1]	1.571116[5]	1.852425[6]	1.243068[4]
		$\hat{\alpha}$	0.181722[1]	0.208587[2]	0.279792[7]	0.258366[5]	0.271894[6]	0.223819[3]	0.225754[4]
		$\hat{\lambda}$	1.915234[4]	1.876275[3]	2.075645[6]	1.170304[1]	1.92684[5]	2.166459[7]	1.8586[2]
	MRE	$\hat{\theta}$	0.710676[3]	0.700686[2]	0.964271[5]	1.02447[7]	0.892721[4]	0.980255[6]	0.623957[1]
		$\hat{\beta}$	0.464421[3]	0.45242[2]	0.612962[7]	0.209759[1]	0.547031[4]	0.563942[6]	0.557033[5]
		$\hat{\alpha}$	0.497583[1]	0.516479[2]	0.624984[7]	0.574469[5]	0.612672[6]	0.56546[4]	0.533389[3]
		$\hat{\lambda}$	0.823515[3]	0.822612[2]	0.87379[7]	0.568121[1]	0.838795[4]	0.867313[6]	0.839198[5]
	$\sum Ranks$		30[2]	26[1]	78[7]	42[4]	57[5]	64[6]	39[3]
100	BIAS	$\hat{\sigma}$	0.455966[3]	0.450245[2]	0.457172[4]	0.547988[6]	0.473082[5]	0.605352[7]	0.425103[1]
		$\hat{\beta}$	0.419928[2]	0.425707[3]	0.644488[7]	0.222528[1]	0.576533[6]	0.483531[5]	0.481736[4]
		$\hat{\alpha}$	0.367284[1]	0.384975[3]	0.430198[7]	0.380888[2]	0.409335[5]	0.405695[4]	0.412138[6]
		$\hat{\lambda}$	1.08248[2]	1.124127[3]	1.213522[7]	0.766751[1]	1.200679[6]	1.158919[5]	1.136341[4]
	MSE	$\hat{\theta}$	0.395709[2]	0.409277[3]	0.478773[4]	0.690754[7]	0.518087[5]	0.685493[6]	0.320726[1]
		$\hat{\beta}$	0.335538[2]	0.340951[3]	0.832996[7]	0.105629[1]	0.682518[6]	0.582234[5]	0.420308[4]
		$\hat{\alpha}$	0.194956[1]	0.222087[2]	0.26745[7]	0.24238[5]	0.240445[4]	0.229644[3]	0.267244[6]
		$\hat{\lambda}$	1.557265[2]	1.674066[4]	1.833067[5]	1.152476[1]	1.851618[6]	1.870229[7]	1.655101[3]
	MRE	$\hat{\theta}$	0.607955[3]	0.600327[2]	0.609563[4]	0.730651[6]	0.630776[5]	0.807137[7]	0.566804[1]
		$\hat{\beta}$	0.279952[2]	0.283805[3]	0.429658[7]	0.148352[1]	0.384355[6]	0.322354[5]	0.321157[4]
		$\hat{\alpha}$	0.489712[1]	0.5133[3]	0.573597[7]	0.507851[2]	0.545779[5]	0.540927[4]	0.549518[6]
		$\hat{\lambda}$	0.721653[2]	0.749418[3]	0.809015[7]	0.511167[1]	0.800453[6]	0.772612[5]	0.757561[4]
	$\sum Ranks$		23[1]	34[2.5]	73[7]	34[2.5]	65[6]	63[5]	44[4]
200	BIAS	$\hat{\sigma}$	0.402774[4]	0.402374[3]	0.377238[1]	0.417957[6]	0.380139[2]	0.491154[7]	0.41559[5]
		$\hat{\beta}$	0.245344[2]	0.275749[3]	0.406656[7]	0.149827[1]	0.368069[6]	0.321157[5]	0.304178[4]
		$\hat{\alpha}$	0.360952[2]	0.374004[3]	0.40199[7]	0.322812[1]	0.385975[5]	0.380121[4]	0.390848[6]
		$\hat{\lambda}$	0.946432[2]	1.022321[3]	1.095529[7]	0.58019[1]	1.09242[6]	1.029604[4]	1.060048[5]
	MSE	$\hat{\theta}$	0.302033[4]	0.309022[5]	0.27983[1]	0.437871[6]	0.282055[2]	0.456174[7]	0.301549[3]
		$\hat{\beta}$	0.111483[2]	0.138574[3]	0.319138[7]	0.042499[1]	0.251714[6]	0.23398[5]	0.169932[4]
		$\hat{\alpha}$	0.208275[2]	0.228173[4]	0.250129[7]	0.195911[1]	0.238967[5]	0.214321[3]	0.247718[6]
		$\hat{\lambda}$	1.344375[2]	1.523857[3]	1.637673[7]	0.793748[1]	1.626174[6]	1.601072[4]	1.606258[5]
	MRE	$\hat{\theta}$	0.537032[4]	0.536498[3]	0.502984[1]	0.557276[6]	0.506853[2]	0.654872[7]	0.55412[5]
		$\hat{\beta}$	0.163562[2]	0.183833[3]	0.271104[7]	0.099885[1]	0.24538[6]	0.214105[5]	0.202785[4]
		$\hat{\alpha}$	0.48127[2]	0.498672[3]	0.535987[7]	0.430416[1]	0.514634[5]	0.506829[4]	0.521131[6]
		$\hat{\lambda}$	0.630955[2]	0.681547[3]	0.730352[7]	0.386793[1]	0.72828[6]	0.686402[4]	0.706698[5]
	$\sum Ranks$		30[2]	39[3]	66[7]	27[1]	57[4]	59[6]	58[5]
300	BIAS	$\hat{\sigma}$	0.354316[3]	0.387339[6]	0.340595[2]	0.314435[1]	0.356816[4]	0.423753[7]	0.366491[5]
		$\hat{\beta}$	0.213491[2]	0.220535[3]	0.304576[7]	0.114722[1]	0.298278[6]	0.251673[5]	0.238859[4]
		$\hat{\alpha}$	0.347032[2]	0.371622[4]	0.387408[7]	0.300536[1]	0.376533[6]	0.351687[3]	0.37459[5]
		$\hat{\lambda}$	0.887239[2]	0.950702[3]	1.009539[6]	0.464702[1]	1.04187[7]	0.969555[4]	0.994709[5]
	MSE	$\hat{\theta}$	0.234554[4]	0.255777[5]	0.20785[1]	0.266827[6]	0.22302[2]	0.328572[7]	0.227776[3]
		$\hat{\beta}$	0.080081[2]	0.080411[3]	0.163303[6]	0.02384[1]	0.16766[7]	0.125845[5]	0.096789[4]
		$\hat{\alpha}$	0.202885[3]	0.225264[4]	0.239855[6]	0.195118[1]	0.23271[5]	0.196299[2]	0.250337[7]
		$\hat{\lambda}$	1.225221[2]	1.42731[3]	1.496599[6]	0.593533[1]	1.625164[7]	1.487977[5]	1.487614[4]
	MRE	$\hat{\theta}$	0.472421[3]	0.516452[6]	0.454126[2]	0.419247[1]	0.475754[4]	0.565004[7]	0.488654[5]
		$\hat{\beta}$	0.142327[2]	0.147024[3]	0.20305[7]	0.076481[1]	0.198852[6]	0.167782[5]	0.15924[4]
		$\hat{\alpha}$	0.462709[2]	0.495496[4]	0.516544[7]	0.400714[1]	0.502044[6]	0.468916[3]	0.499454[5]
		$\hat{\lambda}$	0.591493[2]	0.633801[3]	0.673026[6]	0.309801[1]	0.69458[7]	0.64637[4]	0.66314[5]
	$\sum Ranks$		29[2]	47[3]	63[6]	17[1]	67[7]	57[5]	56[4]
500	BIAS	$\hat{\sigma}$	0.319615[3]	0.337612[5]	0.318942[2]	0.226821[1]	0.324151[4]	0.352445[7]	0.341859[6]
		$\hat{\beta}$	0.152897[2]	0.161827[3]	0.232809[6]	0.084409[1]	0.239478[7]	0.199409[5]	0.183279[4]
		$\hat{\alpha}$	0.330652[2]	0.34606[4]	0.358393[5]	0.226347[1]	0.373944[7]	0.338023[3]	0.364009[6]
		$\hat{\lambda}$	0.725414[2]	0.781757[3]	0.941042[7]	0.307944[1]	0.93105[6]	0.892556[5]	0.82945[4]
	MSE	$\hat{\theta}$	0.171668[2]	0.197562[6]	0.173391[3]	0.160199[1]	0.173924[4]	0.224749[7]	0.18564[5]
		$\hat{\beta}$	0.037617[2]	0.043746[3]	0.093287[6]	0.014092[1]	0.098688[7]	0.072681[5]	0.0554[4]
		$\hat{\alpha}$	0.193338[2]	0.210329[4]	0.213739[5]	0.138737[1]	0.233195[6]	0.20139[3]	0.237107[7]
		$\hat{\lambda}$	0.923751[2]	1.044112[3]	1.35526[6]	0.32208[1]	1.362393[7]	1.323174[5]	1.154496[4]
	MRE	$\hat{\theta}$	0.426153[3]	0.45015[5]	0.425255[2]	0.302429[1]	0.432201[4]	0.469927[7]	0.455812[6]
		$\hat{\beta}$	0.101931[2]	0.107885[3]	0.155206[6]	0.056273[1]	0.159652[7]	0.132939[5]	0.122186[4]
		$\hat{\alpha}$	0.44087[2]	0.461413[4]	0.477857[5]	0.301796[1]	0.498592[7]	0.450697[3]	0.485345[6]
		$\hat{\lambda}$	0.48361[2]	0.521171[3]	0.627361[7]	0.205296[1]	0.6207[6]	0.595038[5]	0.552967[4]
	$\sum Ranks$		26[2]	46[3]	60[5]	12[1]	72[7]	60[5]	60[5]

Table 7. Simulation values of BIAS, MSE and MRE for ($\sigma = 1.5,\ \beta = 2.5,\ \alpha = 1.5,\ \lambda = 2.5$).

n	Est.	Est. Par.	MLE	ADE	CVME	MPSE	LSE	RTADE	WLSE
50	BIAS	$\hat{\sigma}$	3.071302$^{\{6\}}$	2.218067$^{\{3\}}$	3.407058$^{\{7\}}$	0.671389$^{\{1\}}$	2.611467$^{\{5\}}$	2.090763$^{\{2\}}$	2.304991$^{\{4\}}$
		$\hat{\beta}$	1.161666$^{\{6\}}$	0.855781$^{\{2\}}$	1.143568$^{\{5\}}$	0.476107$^{\{1\}}$	0.984626$^{\{4\}}$	1.279159$^{\{7\}}$	0.918149$^{\{3\}}$
		$\hat{\alpha}$	1.44144$^{\{7\}}$	1.217317$^{\{5\}}$	0.840631$^{\{3\}}$	0.35853$^{\{1\}}$	0.762359$^{\{2\}}$	1.190678$^{\{4\}}$	1.245395$^{\{6\}}$
		$\hat{\lambda}$	1.222605$^{\{5\}}$	1.208087$^{\{4\}}$	1.144089$^{\{2\}}$	0.853295$^{\{1\}}$	1.172626$^{\{3\}}$	1.257115$^{\{6\}}$	1.288674$^{\{7\}}$
	MSE	$\hat{\theta}$	34.936513$^{\{4\}}$	22.291755$^{\{2\}}$	200.181787$^{\{7\}}$	4.491111$^{\{1\}}$	143.181451$^{\{6\}}$	33.681688$^{\{3\}}$	35.74143$^{\{5\}}$
		$\hat{\beta}$	2.481203$^{\{5\}}$	1.364787$^{\{2\}}$	2.711632$^{\{6\}}$	0.495917$^{\{1\}}$	2.17304$^{\{4\}}$	3.112637$^{\{7\}}$	1.572973$^{\{3\}}$
		$\hat{\alpha}$	2.860903$^{\{7\}}$	2.563188$^{\{6\}}$	1.400073$^{\{3\}}$	0.335785$^{\{1\}}$	1.18094$^{\{2\}}$	2.271422$^{\{5\}}$	2.196576$^{\{4\}}$
		$\hat{\lambda}$	1.740274$^{\{5\}}$	1.704963$^{\{4\}}$	1.528254$^{\{2\}}$	1.206369$^{\{1\}}$	1.606718$^{\{3\}}$	1.790363$^{\{6\}}$	1.894384$^{\{7\}}$
	MRE	$\hat{\theta}$	2.047535$^{\{6\}}$	1.478711$^{\{3\}}$	2.271372$^{\{7\}}$	0.447592$^{\{1\}}$	1.740978$^{\{5\}}$	1.393842$^{\{2\}}$	1.536661$^{\{4\}}$
		$\hat{\beta}$	0.464667$^{\{6\}}$	0.342313$^{\{2\}}$	0.457427$^{\{5\}}$	0.190443$^{\{1\}}$	0.39385$^{\{4\}}$	0.511664$^{\{7\}}$	0.36726$^{\{3\}}$
		$\hat{\alpha}$	0.576576$^{\{7\}}$	0.486927$^{\{5\}}$	0.336253$^{\{3\}}$	0.143412$^{\{1\}}$	0.304944$^{\{2\}}$	0.476271$^{\{4\}}$	0.498158$^{\{6\}}$
		$\hat{\lambda}$	0.81507$^{\{5\}}$	0.805391$^{\{4\}}$	0.762726$^{\{2\}}$	0.568863$^{\{1\}}$	0.781751$^{\{3\}}$	0.838077$^{\{6\}}$	0.859116$^{\{7\}}$
	$\sum$ *Ranks*		70$^{\{7\}}$	43$^{\{3\}}$	47$^{\{4\}}$	18$^{\{1\}}$	38$^{\{2\}}$	60$^{\{5.5\}}$	60$^{\{5.5\}}$
100	BIAS	$\hat{\sigma}$	1.901263$^{\{7\}}$	1.856695$^{\{5\}}$	1.751922$^{\{4\}}$	0.374069$^{\{1\}}$	1.450007$^{\{3\}}$	1.362212$^{\{2\}}$	1.878976$^{\{6\}}$
		$\hat{\beta}$	0.611938$^{\{3\}}$	0.603637$^{\{2\}}$	0.727946$^{\{6\}}$	0.327749$^{\{1\}}$	0.648047$^{\{5\}}$	0.82756$^{\{7\}}$	0.637527$^{\{4\}}$
		$\hat{\alpha}$	1.248797$^{\{4\}}$	1.294143$^{\{6\}}$	0.739916$^{\{3\}}$	0.236211$^{\{1\}}$	0.701416$^{\{2\}}$	1.258424$^{\{5\}}$	1.382138$^{\{7\}}$
		$\hat{\lambda}$	1.105131$^{\{4\}}$	1.138658$^{\{5\}}$	1.076957$^{\{2\}}$	0.716659$^{\{1\}}$	1.09175$^{\{3\}}$	1.18777$^{\{7\}}$	1.148389$^{\{6\}}$
	MSE	$\hat{\theta}$	9.7488$^{\{4\}}$	8.653993$^{\{3\}}$	15.521894$^{\{7\}}$	0.299087$^{\{1\}}$	12.491631$^{\{6\}}$	4.683966$^{\{2\}}$	10.355477$^{\{5\}}$
		$\hat{\beta}$	0.684923$^{\{3\}}$	0.626635$^{\{2\}}$	1.037086$^{\{6\}}$	0.181031$^{\{1\}}$	0.794992$^{\{5\}}$	1.163807$^{\{7\}}$	0.726858$^{\{4\}}$
		$\hat{\alpha}$	2.40867$^{\{4\}}$	2.765037$^{\{7\}}$	1.246549$^{\{3\}}$	0.16602$^{\{1\}}$	1.139392$^{\{2\}}$	2.610954$^{\{5\}}$	2.76102$^{\{6\}}$
		$\hat{\lambda}$	1.59288$^{\{4\}}$	1.629489$^{\{5\}}$	1.409891$^{\{2\}}$	1.017869$^{\{1\}}$	1.444826$^{\{3\}}$	1.64664$^{\{6\}}$	1.647357$^{\{7\}}$
	MRE	$\hat{\theta}$	1.267509$^{\{7\}}$	1.237797$^{\{5\}}$	1.167948$^{\{4\}}$	0.249379$^{\{1\}}$	0.966671$^{\{3\}}$	0.908142$^{\{2\}}$	1.25265$^{\{6\}}$
		$\hat{\beta}$	0.244775$^{\{3\}}$	0.241455$^{\{2\}}$	0.291178$^{\{6\}}$	0.1311$^{\{1\}}$	0.259219$^{\{5\}}$	0.331024$^{\{7\}}$	0.255011$^{\{4\}}$
		$\hat{\alpha}$	0.499519$^{\{4\}}$	0.517657$^{\{6\}}$	0.295966$^{\{3\}}$	0.094484$^{\{1\}}$	0.280566$^{\{2\}}$	0.50337$^{\{5\}}$	0.552855$^{\{7\}}$
		$\hat{\lambda}$	0.736754$^{\{4\}}$	0.759105$^{\{5\}}$	0.717971$^{\{2\}}$	0.477772$^{\{1\}}$	0.727833$^{\{3\}}$	0.791847$^{\{7\}}$	0.765593$^{\{6\}}$
	$\sum$ *Ranks*		54$^{\{4\}}$	56$^{\{5\}}$	45$^{\{3\}}$	12$^{\{1\}}$	39$^{\{2\}}$	65$^{\{6.5\}}$	65$^{\{6.5\}}$
200	BIAS	$\hat{\sigma}$	1.489438$^{\{5\}}$	1.492649$^{\{6\}}$	1.251639$^{\{4\}}$	0.254702$^{\{1\}}$	1.124335$^{\{3\}}$	1.032508$^{\{2\}}$	1.631366$^{\{7\}}$
		$\hat{\beta}$	0.390916$^{\{2\}}$	0.417238$^{\{3\}}$	0.477833$^{\{6\}}$	0.239941$^{\{1\}}$	0.454654$^{\{5\}}$	0.529328$^{\{7\}}$	0.44636$^{\{4\}}$
		$\hat{\alpha}$	1.22106$^{\{4\}}$	1.243143$^{\{5\}}$	0.814897$^{\{3\}}$	0.165892$^{\{1\}}$	0.762155$^{\{2\}}$	1.261539$^{\{6\}}$	1.40163$^{\{7\}}$
		$\hat{\lambda}$	0.983139$^{\{2\}}$	1.046513$^{\{5\}}$	1.037743$^{\{4\}}$	0.559441$^{\{1\}}$	1.011137$^{\{3\}}$	1.07927$^{\{7\}}$	1.074943$^{\{6\}}$
	MSE	$\hat{\theta}$	4.861412$^{\{5\}}$	4.482581$^{\{4\}}$	5.358438$^{\{6\}}$	0.138289$^{\{1\}}$	3.857502$^{\{3\}}$	2.057535$^{\{2\}}$	5.46302$^{\{7\}}$
		$\hat{\beta}$	0.254524$^{\{2\}}$	0.277628$^{\{3\}}$	0.416379$^{\{6\}}$	0.098282$^{\{1\}}$	0.369749$^{\{5\}}$	0.460917$^{\{7\}}$	0.319359$^{\{4\}}$
		$\hat{\alpha}$	2.354586$^{\{4\}}$	2.456133$^{\{5\}}$	1.470366$^{\{3\}}$	0.090532$^{\{1\}}$	1.335057$^{\{2\}}$	2.586728$^{\{6\}}$	2.783177$^{\{7\}}$
		$\hat{\lambda}$	1.427093$^{\{4\}}$	1.53981$^{\{6\}}$	1.420709$^{\{3\}}$	0.7676$^{\{1\}}$	1.346116$^{\{2\}}$	1.502002$^{\{5\}}$	1.613239$^{\{7\}}$
	MRE	$\hat{\theta}$	0.992959$^{\{5\}}$	0.995099$^{\{6\}}$	0.834426$^{\{4\}}$	0.169801$^{\{1\}}$	0.749556$^{\{3\}}$	0.688339$^{\{2\}}$	1.087577$^{\{7\}}$
		$\hat{\beta}$	0.156367$^{\{2\}}$	0.166895$^{\{3\}}$	0.191133$^{\{6\}}$	0.095976$^{\{1\}}$	0.181861$^{\{5\}}$	0.211731$^{\{7\}}$	0.178544$^{\{4\}}$
		$\hat{\alpha}$	0.488424$^{\{4\}}$	0.497257$^{\{5\}}$	0.325959$^{\{3\}}$	0.066357$^{\{1\}}$	0.304862$^{\{2\}}$	0.504616$^{\{6\}}$	0.560652$^{\{7\}}$
		$\hat{\lambda}$	0.655426$^{\{2\}}$	0.697675$^{\{5\}}$	0.691829$^{\{4\}}$	0.37296$^{\{1\}}$	0.674091$^{\{3\}}$	0.719513$^{\{7\}}$	0.716629$^{\{6\}}$
	$\sum$ *Ranks*		41$^{\{3\}}$	56$^{\{5\}}$	52$^{\{4\}}$	12$^{\{1\}}$	38$^{\{2\}}$	64$^{\{6\}}$	73$^{\{7\}}$
300	BIAS	$\hat{\sigma}$	1.293435$^{\{5\}}$	1.405394$^{\{6\}}$	1.18707$^{\{4\}}$	0.18816$^{\{1\}}$	1.05779$^{\{3\}}$	1.013853$^{\{2\}}$	1.454799$^{\{7\}}$
		$\hat{\beta}$	0.301219$^{\{2\}}$	0.317693$^{\{3\}}$	0.398356$^{\{6\}}$	0.193541$^{\{1\}}$	0.376531$^{\{5\}}$	0.449381$^{\{7\}}$	0.345671$^{\{4\}}$
		$\hat{\alpha}$	1.191676$^{\{4\}}$	1.271217$^{\{5\}}$	0.90975$^{\{3\}}$	0.115396$^{\{1\}}$	0.826743$^{\{2\}}$	1.298864$^{\{6\}}$	1.383884$^{\{7\}}$
		$\hat{\lambda}$	0.883532$^{\{2\}}$	0.922988$^{\{3\}}$	0.986854$^{\{5\}}$	0.433424$^{\{1\}}$	0.986945$^{\{6\}}$	1.030789$^{\{7\}}$	0.942097$^{\{4\}}$
	MSE	$\hat{\theta}$	3.641669$^{\{5\}}$	3.869187$^{\{7\}}$	3.296032$^{\{4\}}$	0.082717$^{\{1\}}$	2.34463$^{\{3\}}$	1.912821$^{\{2\}}$	3.775563$^{\{6\}}$
		$\hat{\beta}$	0.143738$^{\{2\}}$	0.164582$^{\{3\}}$	0.263128$^{\{6\}}$	0.066753$^{\{1\}}$	0.216256$^{\{5\}}$	0.318415$^{\{7\}}$	0.184695$^{\{4\}}$
		$\hat{\alpha}$	2.285817$^{\{4\}}$	2.498052$^{\{5\}}$	1.718056$^{\{3\}}$	0.058219$^{\{1\}}$	1.473023$^{\{2\}}$	2.614885$^{\{6\}}$	2.718042$^{\{7\}}$
		$\hat{\lambda}$	1.234697$^{\{2\}}$	1.315659$^{\{3\}}$	1.325717$^{\{4\}}$	0.624418$^{\{1\}}$	1.339643$^{\{5\}}$	1.436265$^{\{7\}}$	1.349436$^{\{6\}}$
	MRE	$\hat{\theta}$	0.86229$^{\{5\}}$	0.936929$^{\{6\}}$	0.79138$^{\{4\}}$	0.12544$^{\{1\}}$	0.705193$^{\{3\}}$	0.675902$^{\{2\}}$	0.969866$^{\{7\}}$
		$\hat{\beta}$	0.120487$^{\{2\}}$	0.127077$^{\{3\}}$	0.159342$^{\{6\}}$	0.077417$^{\{1\}}$	0.150612$^{\{5\}}$	0.179752$^{\{7\}}$	0.138268$^{\{4\}}$
		$\hat{\alpha}$	0.476671$^{\{4\}}$	0.508487$^{\{5\}}$	0.3639$^{\{3\}}$	0.046158$^{\{1\}}$	0.330697$^{\{2\}}$	0.519546$^{\{6\}}$	0.553553$^{\{7\}}$
		$\hat{\lambda}$	0.589021$^{\{2\}}$	0.615326$^{\{3\}}$	0.657902$^{\{5\}}$	0.28895$^{\{1\}}$	0.657963$^{\{6\}}$	0.687193$^{\{7\}}$	0.628065$^{\{4\}}$
	$\sum$ *Ranks*		39$^{\{2\}}$	52$^{\{4\}}$	53$^{\{5\}}$	12$^{\{1\}}$	47$^{\{3\}}$	66$^{\{6\}}$	67$^{\{7\}}$
500	BIAS	$\hat{\sigma}$	1.07503$^{\{3\}}$	1.211186$^{\{6\}}$	1.107084$^{\{5\}}$	0.137255$^{\{1\}}$	1.081579$^{\{4\}}$	0.853268$^{\{2\}}$	1.319397$^{\{7\}}$
		$\hat{\beta}$	0.228821$^{\{2\}}$	0.256498$^{\{3\}}$	0.317303$^{\{6\}}$	0.137947$^{\{1\}}$	0.323822$^{\{7\}}$	0.314426$^{\{5\}}$	0.266151$^{\{4\}}$
		$\hat{\alpha}$	1.163324$^{\{4\}}$	1.3026$^{\{6\}}$	1.047082$^{\{3\}}$	0.06968$^{\{1\}}$	1.010481$^{\{2\}}$	1.188082$^{\{5\}}$	1.359189$^{\{7\}}$
		$\hat{\lambda}$	0.731742$^{\{2\}}$	0.814465$^{\{4\}}$	0.924263$^{\{6\}}$	0.297312$^{\{1\}}$	0.924078$^{\{5\}}$	0.937208$^{\{7\}}$	0.810166$^{\{3\}}$
	MSE	$\hat{\theta}$	2.473549$^{\{5\}}$	2.645847$^{\{6\}}$	2.210423$^{\{4\}}$	0.041337$^{\{1\}}$	2.195286$^{\{3\}}$	1.264111$^{\{2\}}$	3.143376$^{\{7\}}$
		$\hat{\beta}$	0.082146$^{\{2\}}$	0.09989$^{\{3\}}$	0.142998$^{\{5\}}$	0.036235$^{\{1\}}$	0.151097$^{\{7\}}$	0.146639$^{\{6\}}$	0.110179$^{\{4\}}$
		$\hat{\alpha}$	2.25719$^{\{4\}}$	2.625202$^{\{6\}}$	2.12372$^{\{3\}}$	0.014391$^{\{1\}}$	2.075987$^{\{2\}}$	2.308054$^{\{5\}}$	2.657415$^{\{7\}}$
		$\hat{\lambda}$	0.921386$^{\{2\}}$	1.101723$^{\{3\}}$	1.266424$^{\{5\}}$	0.352731$^{\{1\}}$	1.30509$^{\{6\}}$	1.343484$^{\{7\}}$	1.105024$^{\{4\}}$
	MRE	$\hat{\theta}$	0.716687$^{\{3\}}$	0.807457$^{\{6\}}$	0.738056$^{\{5\}}$	0.091503$^{\{1\}}$	0.721053$^{\{4\}}$	0.568845$^{\{2\}}$	0.879598$^{\{7\}}$
		$\hat{\beta}$	0.091528$^{\{2\}}$	0.102599$^{\{3\}}$	0.126921$^{\{6\}}$	0.055179$^{\{1\}}$	0.129529$^{\{7\}}$	0.12577$^{\{5\}}$	0.106461$^{\{4\}}$
		$\hat{\alpha}$	0.46533$^{\{4\}}$	0.52104$^{\{6\}}$	0.418833$^{\{3\}}$	0.027872$^{\{1\}}$	0.404192$^{\{2\}}$	0.475233$^{\{5\}}$	0.543676$^{\{7\}}$
		$\hat{\lambda}$	0.487828$^{\{2\}}$	0.542977$^{\{4\}}$	0.616175$^{\{6\}}$	0.198208$^{\{1\}}$	0.616052$^{\{5\}}$	0.624805$^{\{7\}}$	0.540111$^{\{3\}}$
	$\sum$ *Ranks*		35$^{\{2\}}$	56$^{\{4\}}$	57$^{\{5\}}$	12$^{\{1\}}$	54$^{\{3\}}$	58$^{\{6\}}$	64$^{\{7\}}$

Table 8. Simulation values of BIAS, MSE and MRE for ($\sigma = 0.5,\ \beta = 1.5,\ \alpha = 2.5,\ \lambda = 0.25$).

n	Est.	Est. Par.	MLE	ADE	CVME	MPSE	LSE	RTADE	WLSE
50	BIAS	$\hat{\sigma}$	2.893599$^{\{5\}}$	2.06015$^{\{3\}}$	5.432695$^{\{7\}}$	0.540312$^{\{1\}}$	2.900305$^{\{6\}}$	1.933509$^{\{2\}}$	2.470322$^{\{4\}}$
		$\hat{\beta}$	0.487959$^{\{5\}}$	0.407624$^{\{2\}}$	0.622881$^{\{7\}}$	0.257203$^{\{1\}}$	0.471101$^{\{4\}}$	0.566687$^{\{6\}}$	0.463078$^{\{3\}}$
		$\hat{\alpha}$	1.520295$^{\{7\}}$	1.387888$^{\{5\}}$	1.238646$^{\{3\}}$	0.505153$^{\{1\}}$	1.157777$^{\{2\}}$	1.348129$^{\{4\}}$	1.402493$^{\{6\}}$
		$\hat{\lambda}$	0.186321$^{\{1\}}$	0.212294$^{\{3\}}$	0.214501$^{\{4\}}$	0.207453$^{\{2\}}$	0.219211$^{\{6\}}$	0.224804$^{\{7\}}$	0.216843$^{\{5\}}$
	MSE	$\hat{\theta}$	31.399644$^{\{4\}}$	18.588935$^{\{2\}}$	456.281219$^{\{7\}}$	3.394397$^{\{1\}}$	199.087447$^{\{6\}}$	28.001065$^{\{3\}}$	39.865189$^{\{5\}}$
		$\hat{\beta}$	0.46565$^{\{3\}}$	0.334741$^{\{2\}}$	0.862602$^{\{7\}}$	0.149008$^{\{1\}}$	0.507983$^{\{5\}}$	0.685578$^{\{6\}}$	0.480383$^{\{4\}}$
		$\hat{\alpha}$	3.02738$^{\{7\}}$	2.789617$^{\{6\}}$	2.213728$^{\{3\}}$	0.600863$^{\{1\}}$	1.945237$^{\{2\}}$	2.638491$^{\{4\}}$	2.675085$^{\{5\}}$
		$\hat{\lambda}$	0.045601$^{\{1\}}$	0.052655$^{\{3\}}$	0.052244$^{\{2\}}$	0.053603$^{\{4.5\}}$	0.055113$^{\{6\}}$	0.057037$^{\{7\}}$	0.053603$^{\{4.5\}}$
	MRE	$\hat{\theta}$	5.787197$^{\{5\}}$	4.1203$^{\{3\}}$	10.86539$^{\{7\}}$	1.080624$^{\{1\}}$	5.800611$^{\{6\}}$	3.867018$^{\{2\}}$	4.940644$^{\{4\}}$
		$\hat{\beta}$	0.325306$^{\{5\}}$	0.271749$^{\{2\}}$	0.415254$^{\{7\}}$	0.171469$^{\{1\}}$	0.314067$^{\{4\}}$	0.377791$^{\{6\}}$	0.308719$^{\{3\}}$
		$\hat{\alpha}$	0.608118$^{\{7\}}$	0.555155$^{\{5\}}$	0.495459$^{\{3\}}$	0.202061$^{\{1\}}$	0.463111$^{\{2\}}$	0.539251$^{\{4\}}$	0.560997$^{\{6\}}$
		$\hat{\lambda}$	0.745285$^{\{1\}}$	0.849174$^{\{3\}}$	0.858004$^{\{4\}}$	0.829814$^{\{2\}}$	0.876843$^{\{6\}}$	0.899215$^{\{7\}}$	0.867372$^{\{5\}}$
	$\sum$ *Ranks*		53$^{\{4\}}$	39$^{\{2\}}$	56$^{\{5\}}$	20.5$^{\{1\}}$	52$^{\{3\}}$	59$^{\{7\}}$	56.5$^{\{6\}}$
100	BIAS	$\hat{\sigma}$	1.71046$^{\{4\}}$	1.543353$^{\{3\}}$	2.563169$^{\{7\}}$	0.324322$^{\{1\}}$	1.761025$^{\{5\}}$	1.348114$^{\{2\}}$	1.842607$^{\{6\}}$
		$\hat{\beta}$	0.271716$^{\{3\}}$	0.254277$^{\{2\}}$	0.365534$^{\{7\}}$	0.14494$^{\{1\}}$	0.303646$^{\{5\}}$	0.344037$^{\{6\}}$	0.276847$^{\{4\}}$
		$\hat{\alpha}$	1.378955$^{\{4\}}$	1.405776$^{\{6\}}$	1.171646$^{\{3\}}$	0.286991$^{\{1\}}$	1.122888$^{\{2\}}$	1.395265$^{\{5\}}$	1.433145$^{\{7\}}$
		$\hat{\lambda}$	0.165496$^{\{1\}}$	0.191167$^{\{3\}}$	0.199523$^{\{5\}}$	0.179456$^{\{2\}}$	0.204775$^{\{7\}}$	0.204683$^{\{6\}}$	0.19562$^{\{4\}}$
	MSE	$\hat{\theta}$	7.565284$^{\{3\}}$	8.06365$^{\{4\}}$	92.946325$^{\{7\}}$	0.270218$^{\{1\}}$	14.630448$^{\{6\}}$	5.312179$^{\{2\}}$	11.461603$^{\{5\}}$
		$\hat{\beta}$	0.137481$^{\{3\}}$	0.128691$^{\{2\}}$	0.291286$^{\{7\}}$	0.036228$^{\{1\}}$	0.191226$^{\{5\}}$	0.229521$^{\{6\}}$	0.155722$^{\{4\}}$
		$\hat{\alpha}$	2.642845$^{\{4\}}$	2.981665$^{\{6\}}$	2.146734$^{\{3\}}$	0.297547$^{\{1\}}$	1.933317$^{\{2\}}$	2.973921$^{\{5\}}$	3.020674$^{\{7\}}$
		$\hat{\lambda}$	0.038979$^{\{1\}}$	0.046183$^{\{3\}}$	0.049215$^{\{5\}}$	0.043477$^{\{2\}}$	0.051193$^{\{7\}}$	0.050262$^{\{6\}}$	0.04685$^{\{4\}}$
	MRE	$\hat{\theta}$	3.420919$^{\{4\}}$	3.086707$^{\{3\}}$	5.126338$^{\{7\}}$	0.648643$^{\{1\}}$	3.522051$^{\{5\}}$	2.696228$^{\{2\}}$	3.685214$^{\{6\}}$
		$\hat{\beta}$	0.181144$^{\{3\}}$	0.169518$^{\{2\}}$	0.243689$^{\{7\}}$	0.096627$^{\{1\}}$	0.20243$^{\{5\}}$	0.229358$^{\{6\}}$	0.184565$^{\{4\}}$
		$\hat{\alpha}$	0.551582$^{\{4\}}$	0.56231$^{\{6\}}$	0.468658$^{\{3\}}$	0.114797$^{\{1\}}$	0.449155$^{\{2\}}$	0.558106$^{\{5\}}$	0.573258$^{\{7\}}$
		$\hat{\lambda}$	0.661985$^{\{1\}}$	0.764667$^{\{3\}}$	0.798091$^{\{5\}}$	0.717824$^{\{2\}}$	0.819099$^{\{7\}}$	0.818733$^{\{6\}}$	0.78248$^{\{4\}}$
	$\sum$ *Ranks*		37$^{\{2\}}$	45$^{\{3\}}$	66$^{\{7\}}$	15$^{\{1\}}$	55$^{\{4\}}$	59$^{\{5.5\}}$	59$^{\{5.5\}}$
200	BIAS	$\hat{\sigma}$	1.317507$^{\{3\}}$	1.393451$^{\{4\}}$	1.807273$^{\{7\}}$	0.239625$^{\{1\}}$	1.417874$^{\{5\}}$	1.057839$^{\{2\}}$	1.572981$^{\{6\}}$
		$\hat{\beta}$	0.177342$^{\{2\}}$	0.186692$^{\{4\}}$	0.235386$^{\{7\}}$	0.102189$^{\{1\}}$	0.203973$^{\{5\}}$	0.207919$^{\{6\}}$	0.178855$^{\{3\}}$
		$\hat{\alpha}$	1.343016$^{\{4\}}$	1.506946$^{\{7\}}$	1.217309$^{\{3\}}$	0.180074$^{\{1\}}$	1.15298$^{\{2\}}$	1.397913$^{\{5\}}$	1.506771$^{\{6\}}$
		$\hat{\lambda}$	0.141567$^{\{1\}}$	0.175726$^{\{4\}}$	0.176276$^{\{5\}}$	0.155976$^{\{2\}}$	0.188878$^{\{7\}}$	0.186961$^{\{6\}}$	0.167679$^{\{3\}}$
	MSE	$\hat{\theta}$	4.217578$^{\{3\}}$	5.275548$^{\{4\}}$	11.273828$^{\{7\}}$	0.096368$^{\{1\}}$	7.968936$^{\{6\}}$	2.855553$^{\{2\}}$	7.32838$^{\{5\}}$
		$\hat{\beta}$	0.055162$^{\{2\}}$	0.062272$^{\{3\}}$	0.113856$^{\{7\}}$	0.017041$^{\{1\}}$	0.086531$^{\{6\}}$	0.082157$^{\{5\}}$	0.062758$^{\{4\}}$
		$\hat{\alpha}$	2.584102$^{\{4\}}$	3.404447$^{\{7\}}$	2.24374$^{\{3\}}$	0.196756$^{\{1\}}$	2.150459$^{\{2\}}$	3.080793$^{\{5\}}$	3.33868$^{\{6\}}$
		$\hat{\lambda}$	0.030984$^{\{1\}}$	0.041857$^{\{5\}}$	0.041581$^{\{4\}}$	0.036696$^{\{2\}}$	0.046431$^{\{7\}}$	0.04599$^{\{6\}}$	0.039497$^{\{3\}}$
	MRE	$\hat{\theta}$	2.635013$^{\{3\}}$	2.786902$^{\{4\}}$	3.614547$^{\{7\}}$	0.479251$^{\{1\}}$	2.835749$^{\{5\}}$	2.115678$^{\{2\}}$	3.145963$^{\{6\}}$
		$\hat{\beta}$	0.118228$^{\{2\}}$	0.124461$^{\{4\}}$	0.156924$^{\{7\}}$	0.068126$^{\{1\}}$	0.135982$^{\{5\}}$	0.138613$^{\{6\}}$	0.119237$^{\{3\}}$
		$\hat{\alpha}$	0.537206$^{\{4\}}$	0.602778$^{\{7\}}$	0.486923$^{\{3\}}$	0.07203$^{\{1\}}$	0.461192$^{\{2\}}$	0.559165$^{\{5\}}$	0.602709$^{\{6\}}$
		$\hat{\lambda}$	0.56627$^{\{1\}}$	0.702902$^{\{4\}}$	0.705102$^{\{5\}}$	0.623904$^{\{2\}}$	0.755514$^{\{7\}}$	0.747844$^{\{6\}}$	0.670717$^{\{3\}}$
	$\sum$ *Ranks*		31$^{\{2\}}$	58$^{\{5\}}$	60$^{\{6.5\}}$	15$^{\{1\}}$	60$^{\{6.5\}}$	57$^{\{4\}}$	55$^{\{3\}}$
300	BIAS	$\hat{\sigma}$	1.144979$^{\{3\}}$	1.146644$^{\{4\}}$	1.531675$^{\{7\}}$	0.193615$^{\{1\}}$	1.305878$^{\{5\}}$	0.901886$^{\{2\}}$	1.35253$^{\{6\}}$
		$\hat{\beta}$	0.136016$^{\{3\}}$	0.131749$^{\{2\}}$	0.185296$^{\{7\}}$	0.084294$^{\{1\}}$	0.164244$^{\{6\}}$	0.153234$^{\{5\}}$	0.139283$^{\{4\}}$
		$\hat{\alpha}$	1.351959$^{\{4\}}$	1.432239$^{\{5\}}$	1.222492$^{\{3\}}$	0.126176$^{\{1\}}$	1.127881$^{\{2\}}$	1.47839$^{\{6\}}$	1.569137$^{\{7\}}$
		$\hat{\lambda}$	0.135547$^{\{2\}}$	0.173872$^{\{6\}}$	0.169951$^{\{4\}}$	0.129027$^{\{1\}}$	0.177691$^{\{7\}}$	0.170295$^{\{5\}}$	0.15737$^{\{3\}}$
	MSE	$\hat{\theta}$	3.192105$^{\{3\}}$	3.487222$^{\{4\}}$	7.206876$^{\{7\}}$	0.062511$^{\{1\}}$	6.006768$^{\{6\}}$	1.745369$^{\{2\}}$	4.939705$^{\{5\}}$
		$\hat{\beta}$	0.032751$^{\{3\}}$	0.029968$^{\{2\}}$	0.066741$^{\{7\}}$	0.010959$^{\{1\}}$	0.053713$^{\{6\}}$	0.04456$^{\{5\}}$	0.03511$^{\{4\}}$
		$\hat{\alpha}$	2.684399$^{\{4\}}$	3.26821$^{\{5\}}$	2.300617$^{\{3\}}$	0.099863$^{\{1\}}$	2.089719$^{\{2\}}$	3.416702$^{\{6\}}$	3.431382$^{\{7\}}$
		$\hat{\lambda}$	0.028587$^{\{2\}}$	0.043915$^{\{7\}}$	0.04007$^{\{4\}}$	0.026484$^{\{1\}}$	0.042333$^{\{6\}}$	0.040364$^{\{5\}}$	0.036334$^{\{3\}}$
	MRE	$\hat{\theta}$	2.289959$^{\{3\}}$	2.293288$^{\{4\}}$	3.06335$^{\{7\}}$	0.38723$^{\{1\}}$	2.611756$^{\{5\}}$	1.803773$^{\{2\}}$	2.705059$^{\{6\}}$
		$\hat{\beta}$	0.090677$^{\{3\}}$	0.087833$^{\{2\}}$	0.123531$^{\{7\}}$	0.056196$^{\{1\}}$	0.109496$^{\{6\}}$	0.102156$^{\{5\}}$	0.092855$^{\{4\}}$
		$\hat{\alpha}$	0.540784$^{\{4\}}$	0.572895$^{\{5\}}$	0.488997$^{\{3\}}$	0.050471$^{\{1\}}$	0.451152$^{\{2\}}$	0.591356$^{\{6\}}$	0.627655$^{\{7\}}$
		$\hat{\lambda}$	0.542189$^{\{2\}}$	0.695487$^{\{6\}}$	0.679804$^{\{4\}}$	0.516109$^{\{1\}}$	0.710765$^{\{7\}}$	0.681181$^{\{5\}}$	0.629479$^{\{3\}}$
	$\sum$ *Ranks*		36$^{\{2\}}$	52$^{\{3\}}$	63$^{\{7\}}$	12$^{\{1\}}$	60$^{\{6\}}$	54$^{\{4\}}$	59$^{\{5\}}$
500	BIAS	$\hat{\sigma}$	0.77681$^{\{2\}}$	0.896938$^{\{4\}}$	1.248395$^{\{7\}}$	0.163887$^{\{1\}}$	1.091826$^{\{6\}}$	0.834705$^{\{3\}}$	1.07752$^{\{5\}}$
		$\hat{\beta}$	0.102285$^{\{3\}}$	0.100908$^{\{2\}}$	0.133641$^{\{7\}}$	0.069737$^{\{1\}}$	0.118822$^{\{5\}}$	0.127745$^{\{6\}}$	0.105786$^{\{4\}}$
		$\hat{\alpha}$	1.315699$^{\{4\}}$	1.505985$^{\{6\}}$	1.177099$^{\{2\}}$	0.074962$^{\{1\}}$	1.218619$^{\{3\}}$	1.48828$^{\{5\}}$	1.531011$^{\{7\}}$
		$\hat{\lambda}$	0.118643$^{\{2\}}$	0.150958$^{\{5\}}$	0.161525$^{\{7\}}$	0.11325$^{\{1\}}$	0.159578$^{\{6\}}$	0.144845$^{\{4\}}$	0.139587$^{\{3\}}$
	MSE	$\hat{\theta}$	1.414635$^{\{2\}}$	1.855396$^{\{4\}}$	4.459711$^{\{7\}}$	0.045603$^{\{1\}}$	3.281802$^{\{6\}}$	1.495434$^{\{3\}}$	3.14218$^{\{5\}}$
		$\hat{\beta}$	0.018327$^{\{3\}}$	0.017578$^{\{2\}}$	0.03204$^{\{7\}}$	0.007544$^{\{1\}}$	0.024952$^{\{5\}}$	0.028475$^{\{6\}}$	0.020264$^{\{4\}}$
		$\hat{\alpha}$	2.72946$^{\{4\}}$	3.524665$^{\{7\}}$	2.074643$^{\{2\}}$	0.043589$^{\{1\}}$	2.27618$^{\{3\}}$	3.349662$^{\{6\}}$	3.249412$^{\{5\}}$
		$\hat{\lambda}$	0.022884$^{\{2\}}$	0.033741$^{\{5\}}$	0.038651$^{\{7\}}$	0.022406$^{\{1\}}$	0.037913$^{\{6\}}$	0.03096$^{\{4\}}$	0.029755$^{\{3\}}$
	MRE	$\hat{\theta}$	1.553621$^{\{2\}}$	1.793875$^{\{4\}}$	2.496789$^{\{7\}}$	0.327773$^{\{1\}}$	2.183652$^{\{6\}}$	1.66941$^{\{3\}}$	2.15504$^{\{5\}}$
		$\hat{\beta}$	0.06819$^{\{3\}}$	0.067272$^{\{2\}}$	0.089094$^{\{7\}}$	0.046492$^{\{1\}}$	0.079215$^{\{5\}}$	0.085163$^{\{6\}}$	0.070524$^{\{4\}}$
		$\hat{\alpha}$	0.52628$^{\{4\}}$	0.602394$^{\{6\}}$	0.47084$^{\{2\}}$	0.029985$^{\{1\}}$	0.487448$^{\{3\}}$	0.595312$^{\{5\}}$	0.612404$^{\{7\}}$
		$\hat{\lambda}$	0.474573$^{\{2\}}$	0.603834$^{\{5\}}$	0.646098$^{\{7\}}$	0.452999$^{\{1\}}$	0.638314$^{\{6\}}$	0.579381$^{\{4\}}$	0.558348$^{\{3\}}$
	$\sum$ *Ranks*		33$^{\{2\}}$	52$^{\{3\}}$	69$^{\{7\}}$	12$^{\{1\}}$	60$^{\{6\}}$	55$^{\{4.5\}}$	55$^{\{4.5\}}$

Table 9. Partial and overall ranks of all the methods of estimation of proposed distribution by various values of model parameters.

Parameter	n	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSE
$\sigma = 0.25,\ \beta = 0.5,\ \alpha = 0.75,\ \lambda = 0.5$	50	3.0	2.0	5.0	1.0	7.0	4.0	6.0
	100	2.0	3.0	7.0	1.0	4.0	5.0	6.0
	200	2.0	3.0	5.0	1.0	7.0	4.0	6.0
	300	2.0	3.0	7.0	1.0	6.0	4.0	5.0
	500	2.0	4.0	6.0	1.0	7.0	3.0	5.0
$\sigma = 1.5,\ \beta = 0.25,\ \alpha = 1.5,\ \lambda = 0.75$	50	2.0	3.0	6.0	1.0	4.5	4.5	7.0
	100	2.0	4.0	5.0	1.0	3.0	6.0	7.0
	200	2.0	4.0	5.0	1.0	3.0	6.0	7.0
	300	2.0	3.0	5.0	1.0	4.0	6.0	7.0
	500	2.0	4.0	6.0	1.0	3.0	7.0	5.0
$\sigma = 0.75,\ \beta = 1.5,\ \alpha = 0.75,\ \lambda = 1.5$	50	2.0	1.0	7.0	4.0	5.0	6.0	3.0
	100	1.0	2.5	7.0	2.5	6.0	5.0	4.0
	200	2.0	3.0	7.0	1.0	4.0	6.0	5.0
	300	2.0	3.0	6.0	1.0	7.0	5.0	4.0
	500	2.0	3.0	5.0	1.0	7.0	5.0	5.0
$\sigma = 1.5,\ \beta = 2.5,\ \alpha = 1.5,\ \lambda = 2.5$	50	7.0	3.0	4.0	1.0	2.0	5.5	5.5
	100	4.0	5.0	3.0	1.0	2.0	6.5	6.5
	200	3.0	5.0	4.0	1.0	2.0	6.0	7.0
	300	2.0	4.0	5.0	1.0	3.0	6.0	7.0
	500	2.0	4.0	5.0	1.0	3.0	6.0	7.0
$\sigma = 0.5,\ \beta = 1.5,\ \alpha = 2.5,\ \lambda = 0.25$	50	4.0	2.0	5.0	1.0	3.0	7.0	6.0
	100	2.0	3.0	7.0	1.0	4.0	5.5	5.5
	200	2.0	5.0	6.5	1.0	6.5	4.0	3.0
	300	2.0	3.0	7.0	1.0	6.0	4.0	5.0
	500	2.0	3.0	7.0	1.0	6.0	4.5	4.5
$\sum$ Ranks		60.0	82.5	142.5	29.5	115.0	131.5	139.0
Overall Rank		2	3	7	1	4	5	6

7. Numerical simulation

In this section we will use all estimation methods presented in Section (4) with replacing out baseline model by Weibull distribution ($G(x,\xi) = 1 - e^{-\sigma x^{\beta}}$). Now, we will study the performance of the estimated parameters of the LocscW distribution by this estimation methods. Also, we do a comparison between all methods by using numerical values of average of bias (BIAS) $|Bias(\widehat{\mathbf{\Delta}})| = \frac{1}{M}\sum_{i=1}^{M}|\widehat{\mathbf{\Delta}} - \mathbf{\Delta}|$, mean squared errors (MSE), $MSE = \frac{1}{M}\sum_{i=1}^{M}(\widehat{\mathbf{\Delta}} - \mathbf{\Delta})^2$, and mean relative errors (MRE) $MRE = \frac{1}{M}\sum_{i=1}^{M}|\widehat{\mathbf{\Delta}} - \mathbf{\Delta}|/\mathbf{\Delta}$, $\mathbf{\Delta} = (\alpha, \lambda, \xi)$. The simulation results may be used to build and apply a guideline for choosing the best estimating approach for the specified model parameters. The R software (version 4.0.3) is used to produce $M = 10,000$ random samples from the proposed distribution for $n =$ 50, 100, 200, 300 and 500.

7.1. Concluding remarks on the simulation

The numerical results of simulations are reported in Tables 4–8 and the power of each value refers to its order in comparing all estimation methods with each other in the same line. Our estimators' partial and overall rankings are displayed in Table 9, in which we conclude that the best method for estimating proposed model parameters when having random samples from our model is MPSE, followed by MLE. Also, we found that as the sample increase, the absolute Bias and MSE and MRE diminishes

8. Applications

Researchers present two applications of the LocscW model in this section, one on hydrological data and the other on survival data. Using the approach of a limited-memory quasi-Newton code for bound-constrained optimization, we construct the log-likelihood function assessed at the MLEs ($\hat{\ell}$) (L-BFGS-B). We take into account many good statistics for model comparison, including the maximized log-likelihood ($\hat{\ell}$), Akaike information criterion (AIC), Corrected Akaike information criterion (CAIC), Bayesian information criterion (BIC), Hannan Quinn information criterion (HQIC), Anderson-Darling (A^*), Cramér–von Mises (W^*) and Kolmogorov-Smirnov (K-S) measures, where lower values of these statistics and higher p-values of K-S indicate good fits.

8.1. First application: Wheaton river data

The first data corresponds to the exceedances of flood peaks (in m3/s) of the Wheaton River near Carcross in Yukon Territory, Canada. They were analysed by Choulakian and Stephens (2001) and are listed below.

1.7, 2.2, 14.4, 1.1, 0.4, 20.6, 5.3, 0.7, 1.9, 13.0, 12.0, 9.3, 1.4, 18.7, 8.5, 25.5, 11.6, 14.1, 22.1, 1.1, 2.5, 14.4, 1.7, 37.6, 0.6, 2.2, 39.0, 0.3, 15.0, 11.0, 7.3, 22.9, 1.7, 0.1, 1.1, 0.6, 9.0, 1.7, 7.0, 20.1, 0.4, 2.8, 14.1, 9.9, 10.4, 10.7, 30.0, 3.6, 5.6, 30.8, 13.3, 4.2, 25.5, 3.4, 11.9, 21.5, 27.6, 36.4, 2.7, 64.0, 1.5, 2.5, 27.4, 1.0, 27.1, 20.2, 16.8, 5.3, 9.7, 27.5, 2.5, 27.0.

The summary statistics for these data are: $n = 60$, $\bar{x} = 2.19297$, $s = 1.920062$, skewness = 1.2614 and kurtosis = 2.23207.

The histogram, box plot, and kernel density plots of the aforementioned data are shown in Figure 10, which demonstrates that the distribution is right-skewed, while the TTT plot is first convex and subsequently concave, indicating a bathtub failure rate. As a result, the LocscW distribution may theoretically be used to represent the existing data.

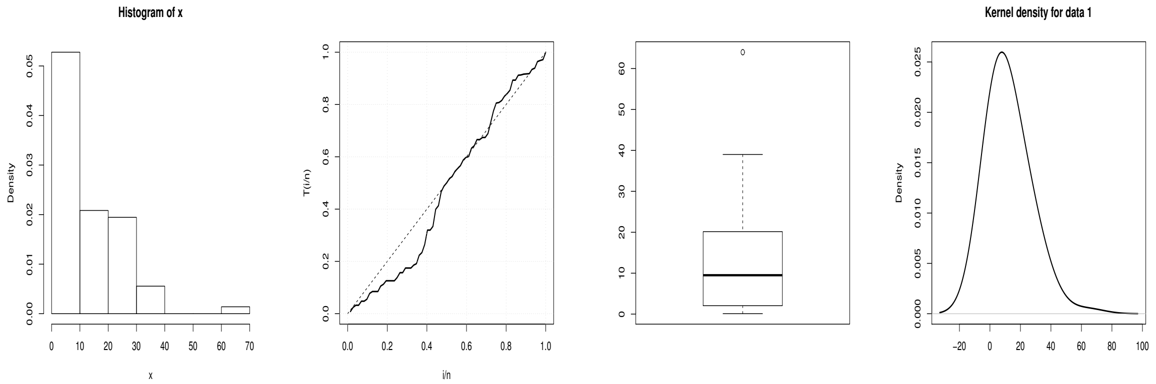


Figure 10. Histogram, TTT plot, box plot and Kernel density for Wheaton river data.

Table 10 provides the MLEs of the LocscW parameters with standard errors (in parentheses) along with the competitor Weibull models. The outputs attest that 4-parameter LocscW is the best fit because SEs are very small as compared to MLEs.

Table 10. MLEs and their standard errors (in parentheses) for Wheaton river data.

Distribution	a	b	c	α	β	λ
LocscW	0.06168	1.3654	-	0.0600	-	0.1505
	(0.0051)	(0.0066)	-	(0.0275)	-	(0.0234)
EKumW	0.2267	0.5664	0.1461	10.6840	8.6820	-
	(0.0906)	(0.0991)	(0.0407)	(0.2432)	(0.1311)	-
McW	0.2109	1.2298	-	0.2703	0.1560	2.6487
	(0.0043)	(0.0052)	-	(0.1034)	(0.0229)	(1.2659)
BurrW	0.2073	0.2261	-	60.3845	4.0139	-
	(5.9539)	(4.1151)	-	(140.5648)	(73.0488)	-
BW	0490	1.2944	-	0.5511	0.4996	-
	(0.1305)	(0.4677)	-	(0.2827)	(0.6805)	-
GoW	0.1638	0.7765	0.2871	-	0.7264	-
	(2.5000)	(0.1420)	(4.4033)	-	(11.0908)	-
EOW	0.2597	0.5971	-	2.1902	0.6213	-
	(0.3080)	(0.2603)	-	(2.0339)	(0.3781)	-
GW	0.0144	1.3645	-	-	0.5392	-
	(0.0227)	(0.3784)	-	-	(0.2152)	-
MOW	0.1261	0.8759	-	-	1.1534	-
	(0.1074)	(0.1686)	-	-	(0.9517)	-
OLLW	0.0353	1.3800	-	-	0.5914	-
	(0.0270)	(0.2800)	-	-	(1.4811)	-
LoW	0.3306	0.5792	-	-	2.0934	-
	(6.6968)	(10.6015)	-	-	(38.3131)	-
W	0.1096	0.9011	-	-	-	-
	(0.0301)	(0.0855)	-	-	-	-

Table 11 provides the values of AIC, CAIC, BIC, HQIC, A^*, W^*, K-S, and P-values for each model. We utilized all criteria of gof, and on the basis of these statistics outputs, the best fit model is LocscW (4-parameters only) than competitor Weibull models (having a greater number of parameters) and has the potential to fit right-skewed data with the bathtub failure rate.

Table 11. The statistics $\hat{\ell}$, AIC, CAIC, BIC, HQIC, A^*, W^*, K-S statistic and P-value for Wheaton river data.

Distribution	$\hat{\ell}$	AIC	CAIC	BIC	HQIC	A^*	W^*	K-S	P-value
LocscW	247.9148	503.8296	504.4266	510.9363	507.4550	0.2832	0.0458	0.0579	0.9694
EKumW	250.2966	510.5932	511.5023	521.9766	515.1250	0.5712	0.0961	0.0998	0.4696
McW	249.4471	508.8943	509.8034	520.2776	513.4260	0.4876	0.0802	0.0947	0.5383
BurrW	251.5910	511.1820	511.7790	520.2886	514.8073	0.8004	0.1408	0.1053	0.4018
BW	250.9795	509.9589	510.5560	519.0656	513.5843	0.6339	0.1031	0.1074	0.3770
GoW	250.7828	509.5656	510.1626	518.6722	513.1909	0.6201	0.1029	0.1037	0.4206
EOW	250.7625	509.5250	510.1220	518.6317	513.1504	0.6023	0.0981	0.1053	0.4014
GW	251.0818	508.1636	508.5165	514.9936	510.8826	0.6654	0.1103	0.1075	0.3764
MOW	251.4838	508.9675	509.3205	515.7975	511.6866	0.7762	0.1355	0.1071	0.3806
OLLW	249.9261	505.8522	506.2052	512.6822	508.5713	0.4738	0.0745	0.0962	0.5177
LoW	257.8391	521.6782	522.0311	528.5082	524.3972	1.6484	0.2961	0.1138	0.3090
W	251.4986	506.9973	507.1712	511.5506	508.8100	0.7855	0.1379	0.1052	0.4032

The plots of the estimated densities are shown in Figure 11 while the plots of the estimated densities are shown in Figure 12 for estimated distribution functions for Wheaton river data.

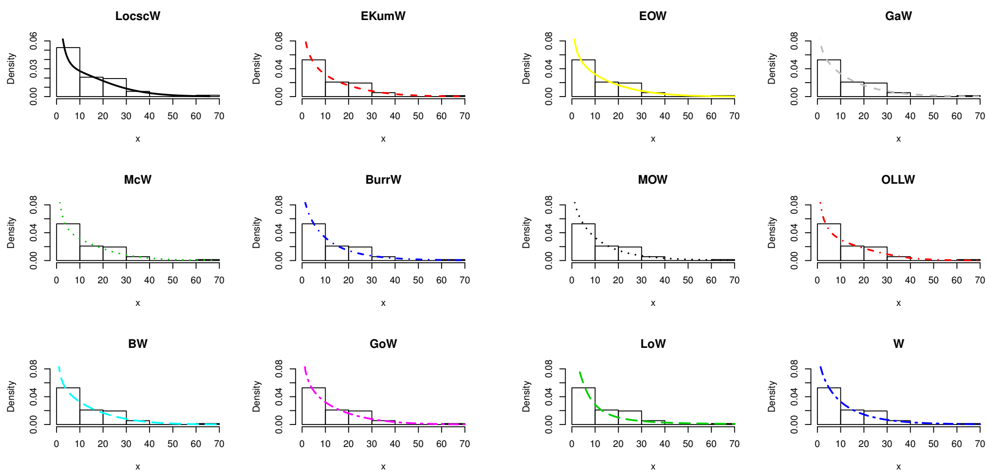


Figure 11. The density plots of LocscW and competitive models using Wheaton river data.

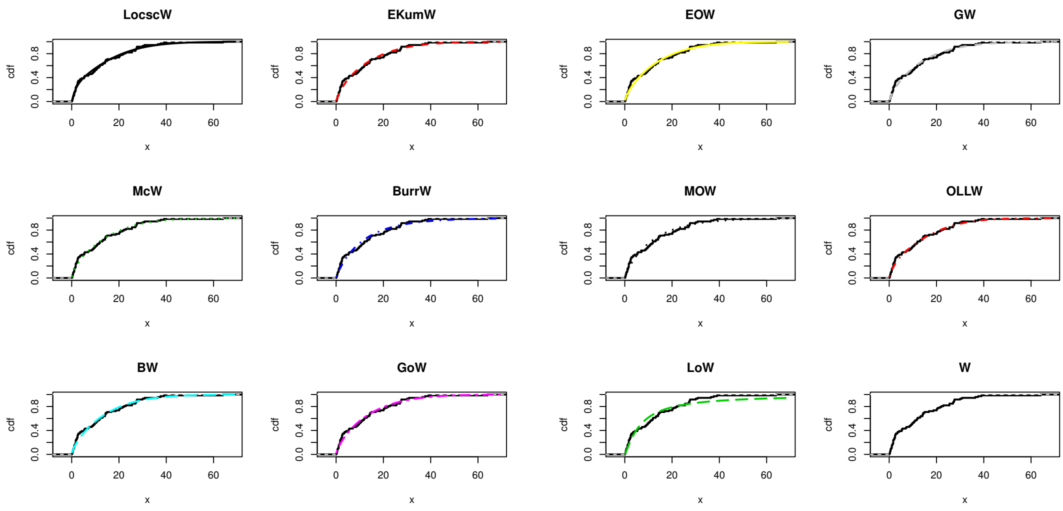
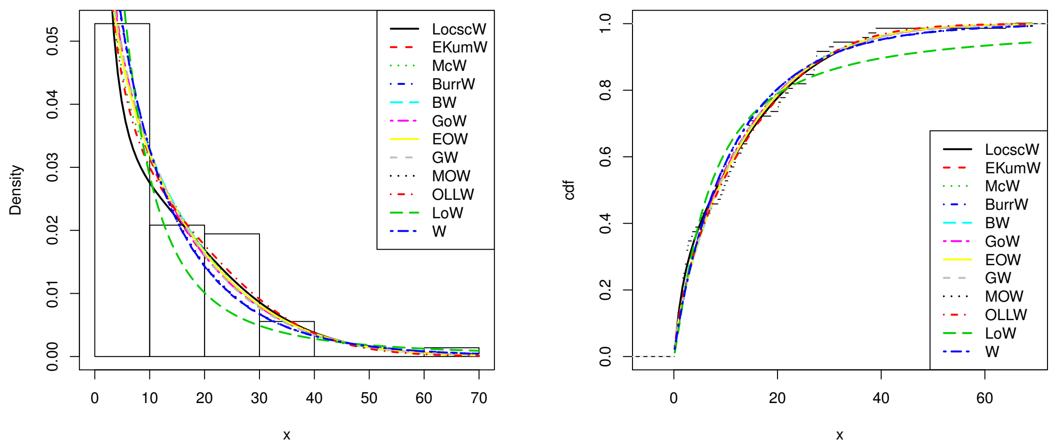


Figure 12. The cdf plots of LocscW and competitive models using Wheaton river data.

Figure 13 presents the plots of the estimated density in 13(a) while Figure 13(b) shows the plots the estimated cdf for LocscW model using Wheaton river data.



(a) (b)

Figure 13. (a) Estimated density (b) Estimated cdf for LocscW model and other competitive models for Wheaton river data.

Figure 14 presents the plot of the estimated density in 14(a) while Figure 14(b) shows the plot the estimated cdf and Figure 14(c) shows the P-P plot for LocscW model using Wheaton river data.

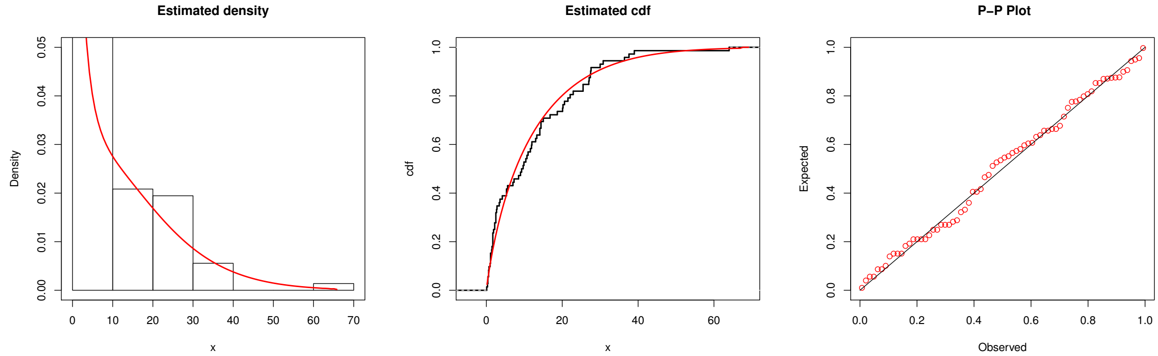


Figure 14. (a) Estimated density (b) Estimated cdf (c) P-P plot for the LocscW model for Wheaton river data.

8.2. Second application: Survival data

The next set of data reflects the survival times (in years) of a number of patients administered chemotherapy by Bekker et al. (2000). This data set's 47 values are as follows:

0.047, 0.115, 0.121, 0.132, 0.164, 0.197, 0.203, 0.260, 0.282, 0.296, 0.334, 0.395, 0.458, 0.466, 0.501, 0.507, 0.529, 0.534, 0.540, 0.641, 0.644, 0.696, 0.841, 0.863, 1.099, 1.219, 1.271, 1.326, 1.447, 1.485, 1.553, 1.581, 1.589, 2.178, 2.343, 2.416, 2.444, 2.825, 2.830, 3.578, 3.658, 3.743, 3.978, 4.003, 4.033.

The histogram, box plot, and kernel density of the above data are displayed in Figure 15 indicates that the distribution is right-skewed and unimodal while the TTT plot of the data is first convex and then concave (increasing-decreasing-increasing (confused type)), which suggests a model with heavy right tail is required, motivating the use of the LocscW model on these data.

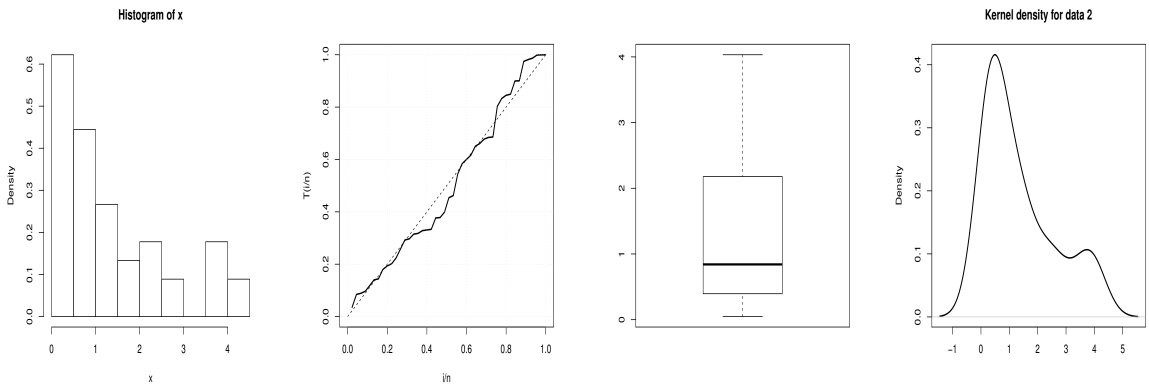


Figure 15. Histogram, TTT plot, box plot and Kernel density for survival data.

Table 12 provides the MLEs of the LocscW parameters with standard errors (in parentheses) along with the competitor Weibull models. The outputs attest that 4-parameter LocscW is the best fit because SEs are very small as compared to MLEs.

Table 12. MLEs and their standard errors (in parentheses) for survival data.

Distribution	a	b	c	α	β	λ
LocscW	0.2007 (0.1344)	2.4503 (0.4919)	- -	0.1914 (0.0673)	- -	0.0064 (0.0099)
EKumW	0.0044 (0.0050)	3.1433 (1.0831)	8.9653 (11.9282)	0.0596 (0.0.0681)	2.0517 (1.0049)	- -
McW	2.7531 (0.0440)	1.7413 (0.0383)	- -	0.0755 (0.0138)	0.8566 (0.0167)	10.9888 (0.1093)
BurrW	0.2652 (3.6580)	0.3311 (3.4383)	- -	51.2885 (188.1357)	3.2075 (33.2955)	- -
BW	9.3525 (0.0445)	0.8476 (0.0313)	- -	2.6799 (1.1839)	0.0987 (0.0158)	- -
GoW	0.2984 (13.3466)	0.9762 (0.1949)	0.4072 (18.2412)	- -	2.1798 (97.5099)	- -
EOW	1.3251 (1.5282)	0.3423 (0.2558)	- -	5.3644 (10.2286)	0.7698 (0.8812)	- -
GW	1.9216 (3.6782)	0.6738 (0.6229)	- -	- -	2.1270 (3.4430)	- -
MOW	0.4151 (0.3069)	1.2322 (0.2473)	- -	- -	0.4432 (0.4452)	- -
OLLW	0.6269 (0.2227)	1.4469 (0.5643)	- -	- -	0.6894 (0.3085)	- -
LoW	1.3361 (62.1457)	1.5872 (254.8173)	- -	- -	0.9498 (152.4854)	- -
W	0.7178 (0.1249)	1.0531 (0.1238)	- -	- -	- -	- -

Table 13 provides the values of AIC, CAIC, BIC, HQIC, A^*, W^*, K-S, and P-values for each model. We utilized all criteria of gof, and on the basis of these statistics outputs, the best fit model is LocscW (4-parameters only) than competitor Weibull models (having a greater number of parameters) and has the potential to fit right-skewed data.

Table 13. The statistics $\hat{\ell}$, AIC, CAIC, BIC, HQIC, A^*, W^*, K-S statistic and P-value for survival data.

Distribution	$\hat{\ell}$	AIC	CAIC	BIC	HQIC	A^*	W^*	K-S	P-value
LocscW	54.9274	117.8549	118.8549	125.0815	120.5489	0.2274	0.0284	0.0683	0.9752
EKumW	57.0848	124.1697	125.7081	133.2030	127.5372	0.4797	0.0709	0.1035	0.6815
McW	55.4733	120.9466	122.4851	129.9799	124.3142	0.2562	0.0301	0.0761	0.9390
BurrW	58.1607	124.3214	125.3214	131.5481	127.0154	0.5389	0.0805	0.1089	0.6207
BW	57.4021	122.8042	123.8042	130.0308	125.4982	0.3579	0.0491	0.0799	0.9141
GoW	57.9819	123.9639	124.9639	131.1905	126.6579	0.5951	0.0896	0.1139	0.5838
EOW	57.8992	123.7984	124.7984	131.0251	126.4924	0.4703	0.0691	0.1015	0.7049
GW	57.9877	121.9756	122.5609	127.3955	123.9961	0.4525	0.0660	0.0977	0.7465
MOW	57.8094	121.6188	122.2042	127.0388	123.6394	0.4599	0.0674	0.0961	0.7639
OLLW	57.8423	121.6847	122.2700	127.1047	123.7052	0.6352	0.0957	0.1234	0.4631
LoW	60.1856	126.3713	126.9566	131.7913	128.3918	0.5595	0.0808	0.0849	0.8746
W	58.1237	120.2474	120.5331	126.8608	121.5944	0.5436	0.0813	0.1094	0.6149

The plots of the estimated densities are shown in Figure 16, while the plots of the estimated CDFs for the LocscW model and its competing models utilizing survival data are shown in Figure 17.

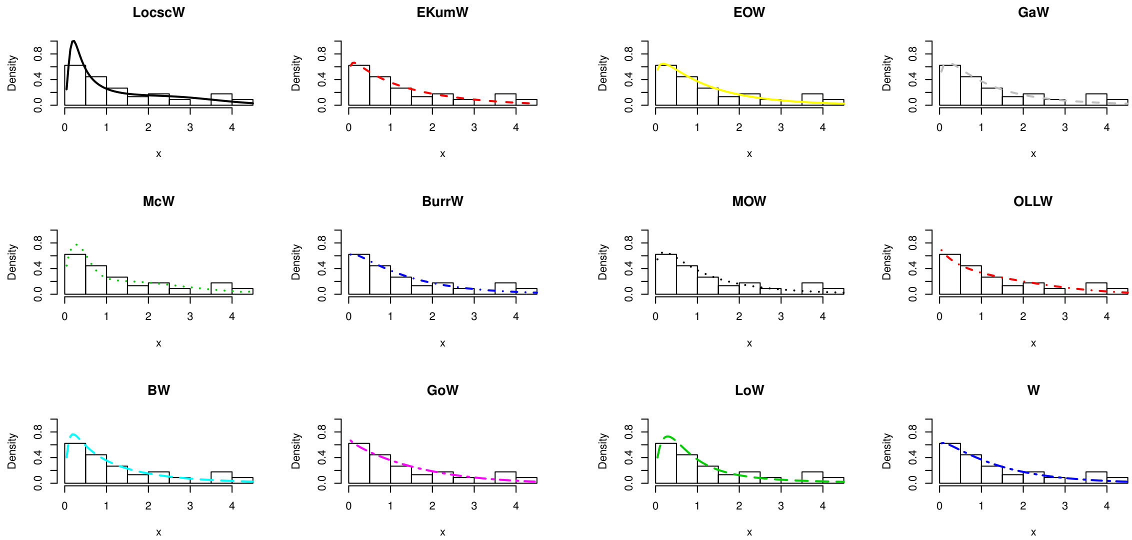


Figure 16. The density plots of LocscW and competitive models using survival data.

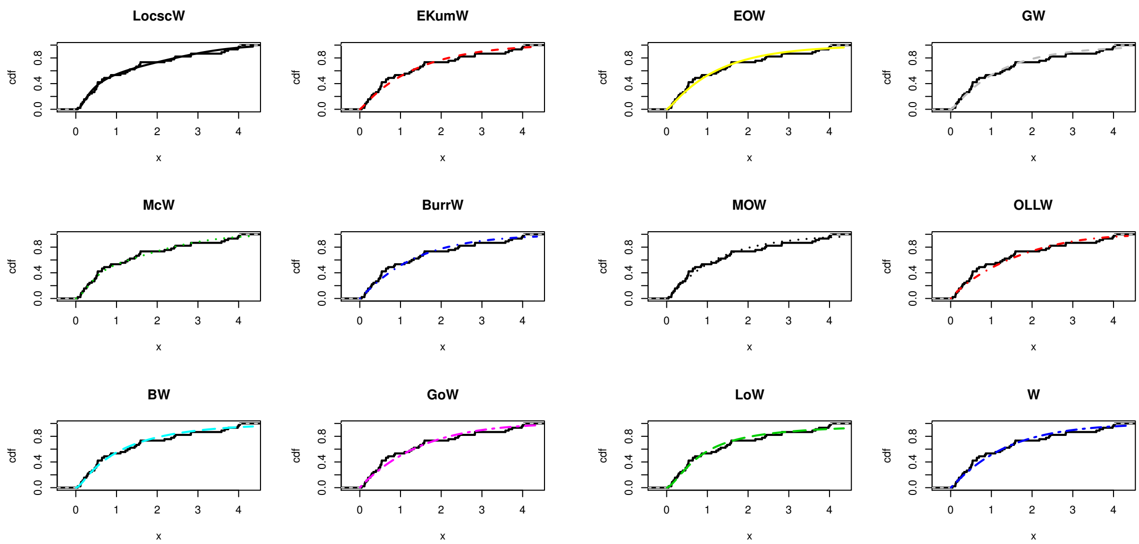
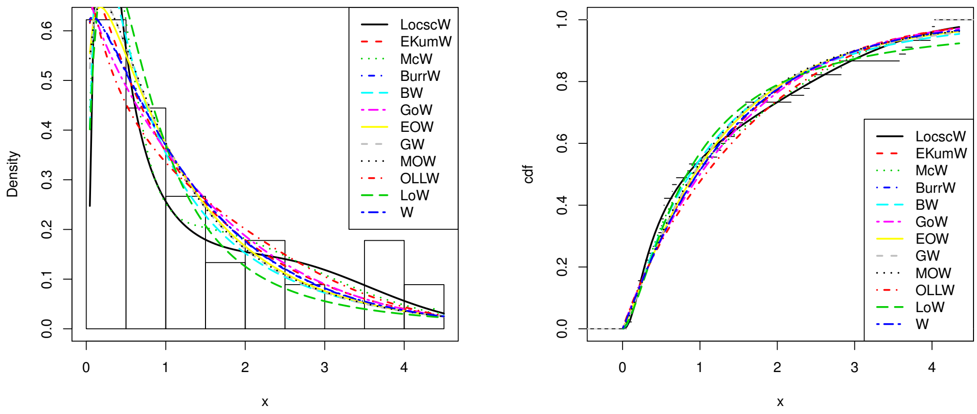


Figure 17. The distribution function plots of LocscW and competitive models using survival data.

Figure 18 presents the plots of the estimated density in 18(a) while Figure 18(b) shows the plots the estimated cdf for LocscW model for survival data.



(a) (b)

Figure 18. (a)Estimated density(b)Estimated cdf for the LocscW model and other competitive models for survival data.

Figure 19 presents the plot of the estimated density in 19(a) while Figure 19(b) shows the plot the estimated cdf and Figure 19(c) shows the P-P plot for LocscW model using Wheaton river data.

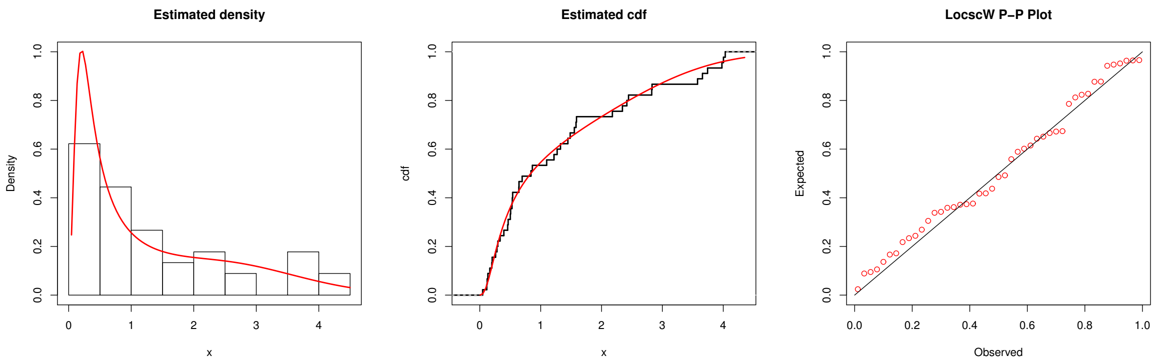


Figure 19. (a) Estimated density (b) Estimated cdf (c) P-P plot for the LocscW model for survival data.

9. Concluding remarks

In this paper, we presented a new lomax-G family of distributions using odd sine/cosecant function (Locsc-G) and obtained prominent mathematical properties such as reliability functions, linear representation for cdf and pdf in terms of exp-G distributions, ordinary and weighted moments, quantile and moment generating function, stress-strength reliability, stochastic ordering, and order statistics.Using well-known distributions, the graphical analysis is performed to observe the flexibility in the proposed family with almost all unimodal shapes of densities and hazard rate functions. Moreover, a new Lomax cosecant Weibull distribution (LocscW), a four-parameter model, is also discussed in detail. The model parameters are estimated by the method of maximum likelihood. We used almost all goodness-of-fit criteria to prove the usefulness of the proposed family and model (LocscW) by means of applications of two data sets. We forecast the wider utility of the new family and model in statistical fields, chiefly in hydrological studies, survival analysis, and reliability engineering.

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Conflict of interest

The authors declare there is no conflict of interest.

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