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Research article

On the crest factor and its relevance in detecting turbulent behaviour in solutions of partial differential equations

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Abstract: In this work we investigate the connection between two fundamental features of solutions of partial differential equations (PDEs), namely the crest factor and the length scale associated to each solution. We illustrate how the crest factor of solutions of some linear and non-linear PDEs, including the incompressible two-dimensional Navier-Stokes equations, has the capability for detecting turbulent and non-turbulent behaviour.

Keywords: partial differential equations; analysis of solutions; crest factor; length scales; turbulence

1. Introduction

In this work we study some properties of the so-called crest factor (CF) in solutions of partial differential equations (PDEs) and we aim to elucidate the connection between the CF and the length scale associated to each solution of PDEs. Furthermore we analyse how the CF can shed light on the nature of turbulence in non-linear PDEs flows.

First we define the CF as the ratio between the sup-norm and the L^2 norm of solutions, namely

$$\widetilde{C}_f := L^{\frac{d}{2}} \frac{\|u\|_\infty}{J_0^{\frac{1}{2}}},$$

where $J_0 := \int_\Omega u^2\,dx = \|u\|_2^2$, $\|\phi\|_\infty = \sup_{x\in\Omega} |\phi(x)|$ and L is the side length of the flat torus, that is $\Omega = [0, L]^d$, with d the spatial dimension. It is therefore by definition dimensionless and it contains important information on the "distortions" between the sup-norm (the amplitude) and the L^2 norm of the solution. It is in fact a standard measurement used in turbulence experiments in fluid dynamics [1]. It is also widely used in various fields of technology and engineering, where it has important applications. For instance in electrical engineering for describing the quality of an alternating current power waveform [2]; in vibration analysis for estimating the amount of impact wear in a bearing [3]; in music in order to generate smooth signals and harmonics [4]; in physiology where it is used to study human response and discomfort to vibrations [5], and to decipher the sound of snoring [6].

In the context of PDEs, the crest factor of a given solution "captures" the fluctuations and distortions between the "amplitude" measured by the sup-norm and the L^2 norm of that particular solution.

Hence effectively by measuring the crest factor of a solution one obtains some very important features shown by the solution itself. In particular if the crest factor is of order one then the dynamics is relatively "mild", in the sense that the solution does not have major excursions in space-time. However when the sup-norm of the solution becomes much larger with respect to its spatial average, it suggests that the solution does have significant fluctuations in space-time; these intermittent fluctuations away from the averages are one of the characteristic signature of hard turbulence.

In the light of this we will therefore say that a *given solution of a PDE shows turbulent behaviour if its corresponding CF shows turbulent behaviour, namely if it is time-dependent but not in a periodic or quasi-periodic fashion, and in addition if it is not of order one.*

This definition seems to capture the essential features of turbulent and non-turbulent solutions of PDEs. In the following we will endeavour to elucidate our definition of turbulent behaviour by analyzing some classical PDEs and their solutions in the various cases where turbulent features are either present or absent. But before doing that let us make the important remark that the crest factor can naturally be computed for the solutions of any PDE possessing the classical properties of existence, uniqueness and regularity of its solutions. The same can be said regarding the boundary conditions used for the solutions of the PDE under investigation, that is one can compute the crest factor using other types of boundary conditions such as (for example) Dirichlet boundary conditions.

The main tool for our analysis is computing various Sobolev norms of the solutions of appropriate PDEs; we will then compute the CF associated to these solutions. The aim is to show on the one end that when the CF is not time-dependent, then the corresponding solution does not have turbulent behaviour in its evolution; on the other end if the CF depends explicitly on time and in addition is not of order one, then the solution does possess turbulent behaviour in its dynamics.

2. Functional settings and notation

Let us first give a brief standard preliminary functional setting and notation [7–9]. Denote by $\Omega = [0, L]^d$ the $d-$dimensional torus; for any scalar function $\phi(x)$ with $x \in \Omega$ let

$$||\phi||_p = \left(\int_\Omega |\phi(x)|^p \, \mathrm{d}x\right)^{\frac{1}{p}}$$

be the norm associated with the Banach space of $\Omega-$periodic functions; we also define the L^∞ norm as

$$||\phi||_\infty = \sup_{x\in\Omega} |\phi(x)| \,.$$

For $p = 2$ we denote by $L^2(\Omega)$ the Hilbert space of $\Omega-$periodic functions ϕ with $||\phi||_2 < +\infty$. Given a multi-index $\vec{n} = (n_1, n_2, \ldots, n_d)$, with all the n_i non-negative integers, let $|\vec{n}| = n_1 + \ldots + n_d$ and

$$D^{\vec{n}} := \frac{\partial^{|\vec{n}|}}{\partial x_1^{n_1} \partial x_2^{n_2} \cdots \partial x_d^{n_d}},$$

and let define the subset of periodic functions on the $d-$dimensional torus Ω

$$\left\{\phi : \int_{\Omega} (D^{\vec{n}}\phi)^2 \mathrm{d}x < +\infty \text{ for all } \vec{n} \text{ such that } |\vec{n}| = n\right\}.$$

as the Sobolev space of $\Omega-$periodic functions with up to $n-$derivatives in $L^2(\Omega)$.

Furthermore, given a solution of a PDE, we define the seminorms

$$J_n := \sum_{\substack{n_1,\ldots,n_d \geq 0 \\ n_1+\ldots+n_d=n}} \frac{n!}{n_1! \cdot \cdot n_d!} \|D^{\vec{n}}u\|_2^2. \tag{2.1}$$

In (2.1), we naturally identify the functions having the same "mixed" partial derivatives, because it is well known that the solutions of regular PDEs are sufficiently smooth [8–10]; for example we identify the differential operators

$$\frac{\partial^{n_1+n_2+\cdots+n_d}}{\partial x_1^{n_1} \ldots \partial x_i^{n_i} \ldots \partial x_j^{n_j} \ldots \partial x_d^{n_d}} \equiv \frac{\partial^{n_1+n_2+\cdots+n_d}}{\partial x_1^{n_1} \ldots \partial x_j^{n_j} \ldots \partial x_i^{n_i} \ldots \partial x_d^{n_d}}, \tag{2.2}$$

and of course any other possible combination of the indices. Also from Parseval's identity we have that

$$J_n := \sum_{\substack{n_1,\ldots,n_d \geq 0 \\ n_1+\ldots+n_d=n}} \frac{n!}{n_1! \ldots n_d!} \|D^{\vec{n}}u\|_2^2 = L^d \left(\frac{2\pi}{L}\right)^{2n} \sum_{k\in\mathbb{Z}^d} |k|^{2n} |u_k|^2 . \tag{2.3}$$

In (2.3) the Fourier series expansion has been used,

$$\phi(x) = \sum_{k\in\mathbb{Z}^d} \phi_k \, e^{2\pi i k \cdot x/L},$$

and

$$|k|^2 = k \cdot k = k_1^2 + k_2^2 + \ldots + k_d^2.$$

By the same token the definition of Sobolev space can be extended to any *real* number s as

$$H^s := \left\{\phi = \sum_{k\in\mathbb{Z}^d} \phi_k \, e^{2\pi i k\cdot x/L} : \overline{\phi}_k = \phi_{-k} \text{ and } \sum_{k\in\mathbb{Z}^d} |k|^{2s} |\phi_k|^2 < +\infty\right\}, \tag{2.4}$$

and the corresponding norm is given by

$$\|\phi\|_{H^s}^2 := L^d \left(\frac{2\pi}{L}\right)^{2s} \sum_{k\in\mathbb{Z}^d} |k|^{2s} |\phi_k|^2 .$$

3. Connection between the CF and the length scale of solutions of PDEs

In this section we wish to analyse the connection between the CF and the length scale of solutions of PDEs. Let us take our formula for the CF, namely

$$\widetilde{C}_f := L^{\frac{d}{2}} \frac{\|u\|_\infty}{J_0^{\frac{1}{2}}}. \tag{3.1}$$

First of all, for the convenience of the reader, we report here the results contained in [11, 12], where we derived sharp estimates for the $\|u\|_\infty$ of typical solutions $u(x,t)$. Note that in general we cannot assume that the solutions of our equation have zero-mean. Hence we have to "carry along" the mean value of our solutions. Thus suppose that $\int_\Omega u(x)\,\mathrm{d}x \neq 0$ and write $u(x) = u^* + u'(x)$, where $u^* = \text{const} \neq 0$ and $\int_\Omega u'(x)\,\mathrm{d}x = 0$. Then using the inequality

$$|u^*| = L^{-d}\left|\int_\Omega u(x)\,\mathrm{d}x\right| \leq L^{-\frac{d}{2}} J_0^{\frac{1}{2}} \tag{3.2}$$

and defining $J_0' := \|u'\|_2^2$, we obtain [33]

$$\|u\|_\infty \leq |u^*| + \|u'\|_\infty \leq L^{-\frac{d}{2}} J_0^{\frac{1}{2}} + c(n)(J_0')^{\frac{2n-d}{4n}} J_n^{\frac{d}{4n}}\,, \tag{3.3}$$

with $n \geq 1$ and $c(n)$ a suitable constant, where we have used a Gagliardo-Nirenberg inequality to obtain the estimate on $\|u'\|_\infty$. By substituting $u = 1$ in (3.2) we see that the constant $L^{-\frac{d}{2}}$ is sharp. Therefore we obtain the following estimate

$$\frac{\|u\|_\infty}{J_0^{\frac{1}{2}}} \leq \frac{|u^*| + \|u'\|_\infty}{J_0^{\frac{1}{2}}} \leq L^{-\frac{d}{2}} + \frac{\|u'\|_\infty}{J_0^{\frac{1}{2}}}.$$

Hence by using (3.3) we obtain

$$\frac{\|u\|_\infty}{J_0^{\frac{1}{2}}} \leq L^{-\frac{d}{2}} + c(n)\left(\frac{J_n}{J_0}\right)^{\frac{d}{4n}} \left(\frac{J_0'}{J_0}\right)^{\frac{2n-d}{4n}}.$$

It is useful to concentrate on the "pure" distortion between the sup-norm and the L^2 norm for non-constant solutions (note that of course purely time dependent functions or constants have crest factor equal to 1). Bearing this in mind one obtains

$$\widetilde{C}_f = 1 + C_f, \qquad C_f := L^{\frac{d}{2}} \frac{\|u'\|_\infty}{J_0^{\frac{1}{2}}} \leq c(n) L^{\frac{d}{2}} \left(\frac{J_n}{J_0}\right)^{\frac{d}{4n}}, \tag{3.4}$$

where the last bound follows noting that $J_0' \leq J_0$. Since one has trivially $C_f = 0$ if $u(x,t)$ does not depend on x, in order to estimate the crest factor we may assume in the following that u' is not identically zero. Hence $J_n > 0$ for all $n \geq 0$.

We now wish to establish the close connection between the CF and the length scale of a given solution of a PDE. The definition of scale we use is that obtained in [13–15], namely

$$l^{-\frac{d}{2}} = c(n)\left(\frac{J_n}{J_0}\right)^{\frac{d}{4n}}. \tag{3.5}$$

Here with l we indicate the length scale and the $c(n)$ are the constants appearing in the Gagliardo-Nirenberg inequality. Thus by looking at (3.5) one can see the following

$$C_f := L^{\frac{d}{2}} \frac{\|u'\|_\infty}{J_0^{\frac{1}{2}}} \leq L^{\frac{d}{2}} c(n)\left(\frac{J_n}{J_0}\right)^{\frac{d}{4n}} = \left(\frac{L}{l}\right)^{\frac{d}{2}}. \tag{3.6}$$

The ideal result would be to have a time-pointwise estimate of C_f. However this is very difficult due to the nonlinearity of the equation. Alternatively one could try to estimate the time-asymptotic behaviour of C_f, but this also proves to be very hard to handle and it is essentially due to the lack of knowledge of a "decent" lower bound on the quantity J_0, namely an estimate of the form $J_0(t) \geq \alpha > 0$, where α is a "not too large" positive constant. The problem of estimating the lower bound appears in many contexts in the theory of nonlinear dissipative PDEs, such as for example in the theory of the Navier-Stokes equations where it is notoriously very hard to find a "proper" lower bound for the energy even on the torus [16]. Nevertheless one can generally compute the time-average of the crest-factor C_f. For instance, taking one of the most commonly used models as a representative PDE, namely the classical Fisher-Kolmogorov-Petrovski-Piskunov Equation (FKPPE) [17, 18]

$$u_t = \Delta u + \lambda u - u^3. \tag{3.7}$$

For simplicity we analyse the one-space dimension case (for more details see [19]):

$$u_t = u_{xx} + \lambda u - u^3, \tag{3.8}$$

with periodic boundary conditions on $\Omega = [0, L]$.

In one space dimension it is sufficient to have control on J_0 and J_1 in order to have control on the sup-norm of any solution of any PDE; this follows from the classical interpolation inequalities of Gagliardo-Nirenberg and Agmon [7]. Thus we start with the analysis of J_0.

Take the time-dependent quantity $J_0(t) = \int_\Omega u^2(x,t)\,\mathrm{d}x$ and differentiating it with respect to time one finds

$$\frac{1}{2}\dot{J}_0 = -J_1 + \lambda J_0 - \int_\Omega u^4\,\mathrm{d}x. \tag{3.9}$$

Note that, for non-trivial behaviour, we see that we must have a restriction on the values of the parameter λ; in fact, noting that $-\int_\Omega u^4\,\mathrm{d}x \leq -\frac{J_0^2}{L}$, it follows that (4.17a) becomes

$$\frac{1}{2}\dot{J}_0 \leq -J_1 + \lambda J_0 - \frac{J_0^2}{L}. \tag{3.10}$$

Hence one can see that if $\lambda \leq 0$ the zero solution becomes a global attractor. Also, from the linearisation of our PDE, one finds that the first eigenvalue of the Laplacian on periodic boundary conditions on a periodic domain $\Omega = [0, L]$ is $\left(\frac{2\pi}{L}\right)^2$. Therefore for non-trivial behaviour we take $\lambda > \left(\frac{2\pi}{L}\right)^2$. Also by taking the time average of (3.6) we see that for $d = 1$ we need the time average of $\frac{J_1}{J_0}$. So looking at (3.10), by dividing for J_0 and time averaging one obtains

$$\left\langle \frac{J_1}{J_0} \right\rangle \leq \lambda - \left\langle \frac{J_0}{L} \right\rangle. \tag{3.11}$$

A standard interpolation inequality gives

$$\|u\|_\infty \leq \left(\frac{J_1}{J_0}\right)^{\frac{1}{4}} J_0^{\frac{1}{2}} + L^{-\frac{1}{2}} J_0^{\frac{1}{2}}. \tag{3.12}$$

If we multiply through by $L^{\frac{1}{2}}J_0^{-\frac{1}{2}}$ and then we take the time average we obtain

$$\widetilde{C}_f = 1 + C_f = \left\langle L^{\frac{1}{2}}\frac{\|u\|_\infty}{J_0^{\frac{1}{2}}}\right\rangle \leq 1 + L^{\frac{1}{2}}\left\langle\left(\frac{J_1}{J_0}\right)^{\frac{1}{4}}\right\rangle \leq 1 + L^{\frac{1}{2}}\left\langle\frac{J_1}{J_0}\right\rangle^{\frac{1}{4}} \leq 1 + L^{\frac{1}{2}}\left[\lambda - \left\langle\frac{J_0}{L}\right\rangle\right]^{\frac{1}{4}}. \tag{3.13}$$

From now on we will focus on the term

$$C_f \;\leq\; L^{\frac{1}{2}}\left[\lambda - \left\langle\frac{J_0}{L}\right\rangle\right]^{\frac{1}{4}}, \tag{3.14}$$

that is we focus on the mean zero part of the solutions of our equations.

Formula (3.14) is quite expressive and reveals that the time average of the ratio between the "peak to the root mean square" scales like (3.14) as a function of the positive parameter λ. So for small λ it is small as it should be, and in general it increases following the estimate given above. This is in agreement with a qualitative analysis of the solutions of the FKPPE. Furthermore (3.14) reveals the intimate connection between the estimates for the crest factor, the dissipative length scale and the fractal dimension of the global attractor of any dissipative PDE. We believe this strong correlation should be further investigated with the aim to make clearer the relationship among these basic features of any solution of PDEs.

4. Analysis of solutions of PDEs and crest factor

In this section we will illustrate our definition of turbulent and non-turbulent solution of PDEs with some classical examples whose solutions provide the necessary properties to be used in the analysis of the CF. The problem of turbulence is well known to everyone working in the field of analysis of the NSE and more generally to people studying nonlinear PDEs [20, 21]. Nevertheless, to the best of our knowledge, all the results on the crest factor reported in this paper (and the related ones [19, 22, 23]) are new.

Linear PDEs

1. We start with the steady states solutions of PDEs such as the Laplace equation. Here the CF is constant and so the solutions of this class of equations cannot have turbulent behaviour. This is of course quite expected.
2. Our second example is the set of linear parabolic separable PDEs. The prototype of this class of PDEs is the heat equation on the torus. Here the solutions $u(x,t)$, in one spatial dimension (by using standard notation for x and t), can be cast into the form $u(x,t) = \sum_n X_n(x)T_n(t)$, with the $X_n(x)$ depending upon the initial datum and the $T_n(t)$ standard decaying exponentials. So one can see that the CF of these solutions is damped away in time and it will tend to a constant limit. Thus the solutions of linear parabolic PDEs such as the heat equation on the torus cannot have turbulent behaviour.
3. We now consider the class of linear hyperbolic separable PDEs. The prototype of this set of equations is the wave equation on the torus. Here the fundamental solution is made up of an infinite superposition of trigonometric oscillating modes both in space and time. So the

corresponding CF for these solutions will follow a time-periodic evolution and so according to our definition of turbulent solutions one can see that the solutions of the wave equation cannot have turbulent behaviour. Similar results are obtained for any other linear hyperbolic separable PDEs on the torus.

4. If one considers other type of boundary conditions or infinite domains then by using similar arguments one can see that the solutions of linear PDEs do not show turbulent behaviour; think for example of the d'Alembert solution for the wave equation in one or more spatial dimensions.

Nonlinear PDEs

1. For nonlinear PDEs let us start with the class of equations having the so-called soliton solutions [24,25]. The prototype for this class of nonlinear PDEs is the Korteweg-de Vrie (KdV) equation, namely
$$u_t - 6uu_x + u_{xxx} = 0.$$
On the line a classical solution of the KdV equation is given by
$$u(x,t) = -\frac{c}{2}\mathrm{sech}^2\Big[\frac{\sqrt{c}}{2}(x - ct - a)\Big],$$
where c and a are constants. Here one can see that the CF on a moving frame is again constant and so a solitonic solutions cannot have turbulent behaviour. This is obvious as this class of nonlinear PDEs belongs to the so-called class of integrable infinite-dimensional PDEs and so there is absence of any chaotic behaviour (let alone turbulent behaviour) in their solutions.
2. Another class of widely studied nonlinear PDEs is the so-called class of reaction-diffusion equations. We have already analysed the FKPPE in section 3 as a model representative to illustrate the computation of the crest factor and in one space dimension we found the estimate
$$C_f \;\leq\; L^{\frac{1}{2}}\left[\lambda - \Big\langle\frac{J_0}{L}\Big\rangle\right]^{\frac{1}{4}}. \tag{4.1}$$
Another classical model which we use to understand how the crest factor gives useful information on the nature of solutions of the set of reaction-diffusion PDEs is the Bi-Laplacian FKPPE, namely
$$u_t = -u_{xxxx} + \lambda u - u^3, \tag{4.2}$$
in one space dimension and with periodic boundary conditions on $\Omega = [0, L]$.
The presence of the Bi-Laplacian instead of the Laplacian changes the contribution of the first term in all our differential inequalities. We begin with J_0, that is
$$\frac{1}{2}\dot{J}_0 = -J_2 + \lambda J_0 - \int_\Omega u^4. \tag{4.3}$$
Observing that $-J_2 \leq -\frac{J_1^2}{J_0}$ and that $-\int_\Omega u^4 \leq -\frac{J_0^2}{L}$, and dividing by J_0 and taking the time average one has
$$\Big\langle\Big(\frac{J_1}{J_0}\Big)^2\Big\rangle \leq \lambda - \Big\langle\frac{J_0}{L}\Big\rangle.$$

Therefore the crest factor is bounded above by (use that in (3.6) one has $c(1) = 1$ (see [27, 28]))

$$C_f \le \left\langle L^{\frac{1}{2}} \left(\frac{J_1}{J_0}\right)^{\frac{1}{4}} \right\rangle \le L^{\frac{1}{2}} \left\langle \left(\frac{J_1}{J_0}\right)^2 \right\rangle^{\frac{1}{8}} \le L^{\frac{1}{2}} \left(\lambda - \left\langle \frac{J_0}{L} \right\rangle\right)^{\frac{1}{8}}. \tag{4.4}$$

The estimates above shed light on some of the features of the dynamics of the solutions of our two models of reaction diffusion equations. In fact for the FKPPE in one space dimension, the time average of the ratio between the "peak to the root mean square" (i.e. the crest factor) scales like (4.1) as a function of the positive parameter λ.
So the dependence is somewhat mild and does not grow very fast such as an exponential in λ. Note that the quantity $\lambda - \left\langle \frac{J_0}{L} \right\rangle$ is always positive. This in turn suggests that our model PDE cannot have hard turbulent behaviour unless one takes a relatively large parameter λ.
For solutions of the Bi-Laplacian FKPPE the crest factor scales like (4.4). This estimate shows again that the solutions of the Bi-Laplacian FKPPE cannot possess hard turbulent behaviour unless λ is very large. So from these two typical reaction diffusion PDEs one can conclude that the solutions of this set of dissipative PDEs have in general soft turbulent behaviour provided the bifurcation parameter λ is not too large. For more details about these two model PDEs see [19].

The incompressible two-dimensional Navier-Stokes equations on the torus with large Reynolds number

We now consider one of the most important systems of classical dynamics, namely the incompressible two spatial dimensions Navier-Stokes equations (NSE) on the periodic domain $\Omega = [0, 2\pi]^2$,

$$u_t + (u \cdot \nabla)u = \nu\Delta u - \nabla\mathcal{P} + f, \;\; \nabla \cdot u = 0, \;\; \nabla \cdot f = 0, \;\; u(0) = u_0. \tag{4.5}$$

We call $u = (u_1, u_2)$ the velocity vector, $\mathcal{P}$ the pressure, f the external force applied to the fluid and ν the kinematic viscosity. We assume that our force is periodic in space and independent of time. In addition we suppose that it belongs to the Sobolev space $L^2(\Omega)$ together with its n spatial derivatives. We note however that our analysis still holds for force functions which are bounded above and below both in space and time. Such a class includes the bounded (above and below) time periodic and quasi-periodic functions. Rigorous results [8–10, 26] for the Navier-Stokes flow on the two-dimensional torus show that for any periodic and divergence free initial condition $u_0 \in J_1$ and any force $f \in L^2(\Omega)$ there is a unique solution which depends continuously on the initial condition u_0. We take the spatial average of both the velocity field and the force field to be zero, namely we suppose that (all the integrals are evaluated on the domain $\Omega = [0, 2\pi]^2$)

$$\int u(x,t)dx = 0, \quad \text{and} \quad \int f(x)dx = 0.$$

Below we are going to use the Brezis-Gallouet inequality with explicit constants, and so we need to have an as accurate as possible estimate of the $J_1(t)$ norm which appears in the denominator.

Recall that the Brezis-Gallouet inequality states that for any J_2 mean-zero function $\phi(x, y)$ on the two dimensional torus we have (see formula (48) in [11])

$$||\phi||_\infty^2 \le \frac{||\nabla\phi||_2^2}{4\pi} \left[\eta + \ln\delta\right], \tag{4.6}$$

where $1 \ll \delta := \dfrac{\|\Delta \phi\|_2^2}{\|\nabla \phi\|_2^2}$, $\epsilon := \dfrac{1}{\ln \delta} \ll 1$, and $\eta := 1.24 + \gamma + O(\epsilon) \simeq 1.82 + O(\epsilon)$, with $\gamma \simeq 0.58$ being the Euler-Mascheroni constant.

In the vector case $u = (u_1, u_2)$, as for example in our case of the two-dimensional Navier-Stokes equations, the explicit constants are essentially the same (see ([12]) for a detailed analysis). Note that here we use the vector version of the J_n, that is

$$J_n := \sum_{i=1}^{2} \sum_{\substack{n_1,\dots,n_d \geq 0 \\ n_1+\dots+n_d=n}} \frac{n!}{n_1!\dots n_d!} \|D^{\vec{n}} u_i\|_2^2. \tag{4.7}$$

Using the J_n above, in [12] it is shown that

$$\|u\|_\infty^2 \leq \frac{J_1}{4\pi}\left[\hat{\eta} + \frac{1}{2}\ln\left(\frac{J_2}{J_1}\right)\right], \tag{4.8}$$

where $\hat{\eta} = \eta - \dfrac{1}{4}\ln(4c_1c_2)$, and c_1, c_2 are two positive constants such that $c_1 + c_2 = 1$.

We can cast (4.8) into the form

$$\|u\|_\infty \leq \sqrt{\frac{J_1}{4\pi}}\left[\hat{\eta} + \frac{1}{2}\ln J_2 - \frac{1}{2}\ln J_1\right]^{\frac{1}{2}}. \tag{4.9}$$

Analysing the term $-J_1 \ln J_1$ one can see that it has a maximum at the value $\dfrac{1}{e}$ and hence

$$\|u\|_\infty \leq \sqrt{\frac{1}{4\pi}}\left[J_1(\hat{\eta} + \frac{1}{2}\ln J_2) + \frac{1}{2e}\right]^{\frac{1}{2}}. \tag{4.10}$$

In order to compute the crest factor we need to divide by some seminorm which in principle can become very small, causing unwanted complications. Therefore a way to control this division is to include the forcing term into the evolution of the J_n by adding the set of seminorms

$$\Phi_n := \sum_{i=1}^{2} \sum_{\substack{n_1,\dots,n_d \geq 0 \\ n_1+\dots+n_d=n}} \frac{n!}{n_1!\dots n_d!} \|D^{\vec{n}} f_i\|_2^2, \tag{4.11}$$

where f_i are the components of the periodic time independent forcing function. We wish to add the Φ_n to the J_n for each n, and so in order to make the involved quantities dimensionally equivalent we multiply each Φ_n by the quantity $\tau^2 = L^4\nu^{-2}$ with $L = 2\pi$ in our case (for more details see [13, 14]). Hence we obtain

$$F_n = J_n + \tau^2\Phi_n.$$

Note that $J_n \leq F_n$ and $\tau^2\Phi_n \leq F_n$.

We can now compute the crest factor for the solutions of the two-dimensional NSE on the torus. Using the F_n we can rewrite (4.10) as follows:

$$\|u\|_\infty \leq \sqrt{\frac{F_1}{4\pi}}\left[\hat{\eta} + \frac{1}{2}\ln J_2 + \frac{1}{2e}\frac{1}{F_1}\right]^{\frac{1}{2}} \leq \sqrt{\frac{F_1}{4\pi}}\left[\hat{\eta} + \frac{1}{2}\ln J_2 + \frac{1}{2e}\frac{1}{\tau^2\Phi_1}\right]^{\frac{1}{2}}. \tag{4.12}$$

We then multiply by $\dfrac{L}{F_0^{\frac{1}{2}}}$ with $L = 2\pi$ and we time average obtaining

$$C_f = \left\langle L\frac{\|u\|_\infty}{F_0^{\frac{1}{2}}}\right\rangle \leq \frac{L}{\sqrt{4\pi}}\left\langle\left[\frac{F_1}{F_0}\left(\hat{\eta} + \frac{1}{2}\ln J_2 + \frac{1}{2e}\frac{1}{\tau^2\Phi_1}\right)\right]^{\frac{1}{2}}\right\rangle. \tag{4.13}$$

Splitting the time average

$$\left\langle\left[\frac{F_1}{F_0}\left(\hat{\eta} + \frac{1}{2}\ln J_2 + \frac{1}{2e}\frac{1}{\tau^2\Phi_1}\right)\right]^{\frac{1}{2}}\right\rangle \leq \left\langle\frac{F_1}{F_0}\right\rangle^{\frac{1}{2}}\left\langle\hat{\eta} + \frac{1}{2}\ln J_2 + \frac{1}{2e}\frac{1}{\tau^2\Phi_1}\right\rangle^{\frac{1}{2}}$$

we obtain

$$C_f = \left\langle L\frac{\|u\|_\infty}{F_0^{\frac{1}{2}}}\right\rangle \leq \frac{L}{\sqrt{4\pi}}\left\langle\frac{F_1}{F_0}\right\rangle^{\frac{1}{2}}\left[\hat{\eta} + \frac{1}{2}\ln\left\langle J_2\right\rangle + \frac{1}{2e}\frac{1}{\tau^2\Phi_1}\right]^{\frac{1}{2}}, \tag{4.14}$$

where in the logarithm term we have used the Jensen inequality and $L = 2\pi$.

The formula above expresses the non-dimensional crest factor for solutions of the two dimensional NSE. So we need to estimate the quantities

$$\langle\frac{F_1}{F_0}\rangle, \quad \langle J_2\rangle$$

and substitute them into (4.14). To this aim we use the following result with the details contained in [13, 14]:

$$\frac{1}{2}\dot{F}_n \leq -\nu F_{n+1} + \left[c_n\|Du\|_\infty + \nu\lambda_n^{-2}\right]F_n. \tag{4.15}$$

were c_n are the constants appearing in the embedding theorem of Gagliardo and Nirenberg and $\lambda_n^{-2} := \left[\dfrac{\Phi_{n+1}}{\Phi_n} + \dfrac{1}{L^2}\right]$. Considering the above differential inequality in the two dimensional case for $n = 0, 1$ one obtains

$$\frac{1}{2}\dot{F}_0 \leq -\nu F_1 + \nu\lambda_0^{-2}F_0, \tag{4.16a}$$

$$\frac{1}{2}\dot{F}_1 \leq -\nu F_2 + \nu\lambda_1^{-2}F_1. \tag{4.16b}$$

We also need the differential inequalities for J_0 and J_1 which are

$$\frac{1}{2}\dot{J}_0 \leq -\nu J_1 + J_0^{\frac{1}{2}}\Phi_0^{\frac{1}{2}}, \tag{4.17a}$$

$$\frac{1}{2}\dot{J}_1 \leq -\nu J_2 + J_1^{\frac{1}{2}}\Phi_1^{\frac{1}{2}}. \tag{4.17b}$$

In order to obtain $\langle\dfrac{F_1}{F_0}\rangle$ we divide (4.16a) by F_0, we then time average to obtain

$$\left\langle\frac{F_1}{F_0}\right\rangle \leq \lambda_0^{-2}. \tag{4.18}$$

To compute $\langle J_2\rangle$ we take the time average of (4.17b)

$$\langle J_2\rangle \le \frac{\langle J_1^{\frac{1}{2}}\Phi_1^{\frac{1}{2}}\rangle}{\nu} \le \frac{\langle J_1\rangle^{\frac{1}{2}}\Phi_1^{\frac{1}{2}}}{\nu}. \tag{4.19}$$

Similarly from (4.17a) one has

$$\langle J_1\rangle \le \frac{\langle J_0^{\frac{1}{2}}\Phi_0^{\frac{1}{2}}\rangle}{\nu} \le \frac{\langle J_0\rangle^{\frac{1}{2}}\Phi_0^{\frac{1}{2}}}{\nu}. \tag{4.20}$$

Here we need the result that for any function $f(t)$ with a bounded time average and initial condition inside the global attractor one has $\langle f\rangle \le \overline{f}$ where $\overline{f} := \lim_{t\to\infty} f$.

The time-asymptotic behaviour of $J_0(t)$ can be obtained from (4.17a) as follows: first we take a Poincaré inequality on the torus with side length $L = 2\pi$ giving $J_0 \le J_1$ and use it in (4.17a), then solve the closed differential inequality for J_0 obtaining

$$\overline{J}_0 := \limsup_{t\to\infty} J_0(t) \le \frac{\Phi_0}{\nu^2}, \tag{4.21}$$

where the limit superior is taken over all the initial data inside the attractor. Inserting the above inequality in (4.20) we finally get

$$\langle J_1\rangle \le \frac{\Phi_0}{\nu^2}. \tag{4.22}$$

Substituting into (4.19) one has

$$\langle J_2\rangle \le \frac{\Phi_0\Phi_1}{\nu^2}. \tag{4.23}$$

Thus by inserting the estimates (4.18) and (4.23) in the estimate (4.14), we can now state the following result:

Theorem: *For large Reynolds numbers, the time-averaged crest factor for the two-dimensional incompressible NSE on the torus obeys the estimate*

$$C_f = \langle L\frac{||u||_\infty}{F_0^{\frac{1}{2}}}\rangle \le \frac{1}{\sqrt{4\pi}}\frac{L}{\lambda_0}\left[\hat{\eta} + \frac{1}{2}\ln\left(\frac{\Phi_0\Phi_1}{\nu^2}\right) + \frac{1}{2e}\frac{1}{\tau^2\Phi_1}\right]^{\frac{1}{2}}, \tag{4.24}$$

where

$$\lambda_0^{-2} := \left[\frac{\Phi_1}{\Phi_0} + \frac{1}{L^2}\right], \quad \tau := L^2\nu^{-1}, \quad L = 2\pi.$$

Looking at the theorem above one can see that, for very small ν, the crest factor for the solutions of the two-dimensional NSE on the torus has the term $\ln\left(\frac{\Phi_0\Phi_1}{\nu^2}\right)$ and so goes like $\frac{1}{\nu^2}$ logarithmically; this is effectively saying that the crest factor of the solutions "follows" the scale of the forcing function through the term $\frac{L}{\lambda_0}$ modulus a logarithmic correction. In other words one can say that the solutions of the the two-dimensional NSE on the torus do not generally show strong turbulent behaviour unless the term (involving the forcing) $\left[\frac{\Phi_1}{\Phi_0}\right]$ is very large, or the viscosity parameter is extremely small, and so very close to the inviscid limit which coincide (on the torus) with the two-dimensional Euler equations [29].

5. Conclusions

Given the universally acknowledged consensus that understanding the nature of turbulence is an extremely complicated issue, we can certainly say that any little light shed on this problem ought to be very welcome. That said, in this work we have stated a criterion for discerning if a solution of a particular PDE shows turbulent or non-turbulent behaviour. Our criterion (and it may help reporting it here again) is:

A solution of a PDE shows turbulent behaviour if its corresponding crest factor (CF) shows turbulent behaviour, namely if it is time-dependent but not in a periodic or quasi-periodic fashion, and in addition if is not of order one.

We believe this definition captures the intrinsic character of turbulence, namely its erratic and intermittent space-time evolution. We believe that one way of measuring the intermittent behaviour contained in a given solution of a PDE is analysing its corresponding CF. Hence if its CF is time-independent or time periodic or quasi-periodic then the given solution does not show turbulent behaviour. According to this, we have shown that the solutions of linear PDEs and the solutions of the so-called completely integrable non-linear PDEs do not have any turbulent behaviour in their evolution. On the other hand, in the case where the CF is time-dependent but not in a periodic or quasi-periodic way, we distinguish between two fundamentally different regimes:

1. The CF is of order one. We call this regime *soft* or *mild turbulent regime*. This appears to be quite natural because we are in a situation where the sup-norm and the L^2 norm of the solution are close to one another. Therefore no strong intermittent excursions are experienced by the solution.
2. The CF is not of order one. We call this regime *strong* or *hard turbulent regime*. Again this is also quite natural because here the solution does have relevant and intermittent excursion away from its mean square space average. These strong fluctuations are one of the fundamental hallmark of turbulence. Thus, it appears that the CF is able to extract this feature in any turbulent solution in non-linear PDEs.

In the light of the two regimes above we have found that the solutions of the so-called non-linear reaction-diffusion PDEs can show both regimes depending upon the value of the "bifurcation" parameter λ. More precisely for λ relatively small we are in a soft regime. As the parameter λ increases we tend to go to a potentially hard turbulent regime.

The same can be said for the incompressible two-dimensional Navier-Stokes equations on the torus. Here the solutions normally have mild turbulent behaviour unless one chooses forcing functions which make the term $\left[\frac{\Phi_1}{\Phi_0}\right]$ large, or take the kinematic viscosity extremely small, thereby going towards the two-dimensional forced Euler equations [29].

It would be also very interesting to compute the crest factor for the three-dimensional NSE on the torus. Here we are still plagued by the fact that we do not yet know about existence, uniqueness and regularity of the solution for all time for large Reynolds numbers. This open problem is one of the seven Clay Millennium Prize Problems. To its solution there is attached the prize of one million USA dollars. However one can compute a "conditional" CF for the three-dimensional NSE as a function of the vorticity field. It turns out that in this case the time averaged CF would depends in an essential way upon the sup-norm of the vorticity field.

We conclude with the remark that it would be very desirable to generate very efficient programs to compute the crest factor for typical nonlinear PDEs. It is well known that, due to the non-linearity of some PDEs, it is generally difficult to achieve accurate, let alone sharp, analysis of their solutions. Hence constructing efficient numerical and software programs for computing the crest factor would be very welcome. Some interesting results addressing this topic are in the papers [30, 31].

Conflict of interest

The authors declare there is no conflict of interest.

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