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Research article

Optimal harvesting for a periodic n-dimensional food chain model with size structure in a polluted environment

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Abstract: This study examines an optimal harvesting problem for a periodic n-dimensional food chain model that is dependent on size structure in a polluted environment. This is closely related to the protection of biodiversity, as well as the development and utilization of renewable resources. The model contains state variables representing the density of the ith population, the concentration of toxicants in the ith population, and the concentration of toxicants in the environment. The well-posedness of the hybrid system is proved by using the fixed point theorem. The necessary optimality conditions are derived by using the tangent-normal cone technique in nonlinear functional analysis. The existence and uniqueness of the optimal control pair are verified via the Ekeland variational principle. The finite difference scheme and the chasing method are used to approximate the nonnegative T-periodic solution of the state system corresponding to a given initial datum. Some numerical tests are given to illustrate that the numerical solution has good periodicity. The objective functional here represents the total profit obtained from harvesting n species.

Keywords: food chain model; size structure; optimal harvesting; pollution; finite difference method

1. Introduction

In today's world of industrial pollution, toxicants are pervading the air, ecological problems have become increasingly prominent, and environmental pollution has become a major problem. SARS, Ebola virus, AIV, H1N1 influenza, and COVID-19 are threatening the ecological balance as well as the survival of human beings and other creatures. It is necessary to study the effects of toxicants on the ecosystem. Hallam et al. proposed using a dynamic methodology to examine ecotoxicology in [1–3]. They established a model of the interaction between toxicants and population, and provided sufficient conditions for the persistence and extinction of a population stressed by a toxicant. Researchers have been studying ecotoxicology since the 1980s, and a large amount of literature has been devoted to problems in the area. Luo et al. [4–6] studied a new age-dependent model of toxicant population in an environment with a small capacity for toxicants. The threshold between persistence in the mean and extinction was obtained for each species in a polluted environment [7, 8]. The results of thresholding in [9] were then extended to a stochastic Lotka–Volterra cooperative model for n-species.

The effects of environmental pollution on biological population, the dynamical behavioral analysis of ecosystem models, and the control problem have attracted the attention of many scholars. A large number of ecological studies have shown that differences in individual size structures have a more important effect on population development than those in the age structure. This kind of model has achieved remarkable results through theory, numerical calculations, and experimental methods. Among them, the maximum principle and bang-bang structure of the optimal control were established in [10] for a size-structured forestry model with the benefit of carbon sequestration. In [11–13], several authors investigated size-structured population models with separable mortality rates. He and Liu [14] studied an optimal birth control problem for a size-structured population model that takes fertility as the control variable. Later, they studied a nonlinear egg-juvenile-adult model in which eggs and juveniles were structured by age while adults were structured by size [15]. In [16], Liu et al. proposed a unified size-structured PDE model for the growth of metastatic tumors and provided several numerical examples. In [17], Mansal et al. studied the dynamics of the predator and the prey in a fishery model by using the fractional-order derivative. In [18], Thiao et al. discussed the fractional optimal economic control of a continuous game theory model described by the fractional-order derivative. Optimal harvesting problems for a size-structured population model were analyzed in [19, 20]. However, most of these studies have focused on a single species, and few have examined interactions among species, especially predator-prey models of two species. In addition, due to the influence of seasonal changes and other factors, the living environment of populations often undergoes periodic changes. Research on optimal harvesting problems dependent on the model of individual size in a periodic environment has been reported in [21, 22]. In this study, we establish mathematical population models to consider the size factor to render them reasonable.

Few studies to date have examined optimal control problems of size-dependent population models and periodic effects in a polluted environment. In order to bridge this gap, we discuss optimal harvesting for a periodic, n-dimensional food chain model dependent on size structure in a polluted environment, and detail a simulation and an example based on the population density.

The remainder of this paper is organized as follows: Section 2 describes a population model with size structure in a polluted environment. Its well-posedness is proved in Section 3 and its optimality conditions are established in Section 4. In Section 5, we discuss the existence and uniqueness of the optimal control pair. Section 6 is devoted to approximating the T-periodic solution by a numerical algorithm. A discussion of the results and the conclusions of this study are provided in Section 7.

2. The basic model

In [1–3], Hallam et al. proposed the following dynamic population model with toxicant effects:

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = x[r_0 - r_1 C_0 - f x], \\ \frac{\mathrm{d}C_0}{\mathrm{d}t} = kC_E - gC_0 - mC_0, \\ \frac{\mathrm{d}C_E}{\mathrm{d}t} = -k_1 C_E x + g_1 C_0 x - hC_E + u, \end{cases} \tag{2.1}$$

where $x = x(t)$ is the population biomass at time t; $C_0 = C_0(t)$ is the concentration of toxicants in the organism at time t; $C_E = C_E(t)$ is the concentration of toxicants in the environment of the population at time t. The exogenous rate of input of toxicants into the environment was represented by u. They investigated the persistence and extinction of a population in a polluted environment.

Luo et al. [23] studied optimal birth control for the following age-dependent n-dimensional food chain model:

$$\begin{cases} \frac{\partial p_1}{\partial t} + \frac{\partial p_1}{\partial a} = -\mu_1(a,t)p_1 - \lambda_1(a,t)P_2(t)p_1, \\ \frac{\partial p_i}{\partial t} + \frac{\partial p_i}{\partial a} = -\mu_i(a,t)p_i + \lambda_{2i-2}(a,t)P_{i-1}(t)p_i - \lambda_{2i-1}(a,t)P_{i+1}(t)p_i, \\ \frac{\partial p_n}{\partial t} + \frac{\partial p_n}{\partial a} = -\mu_n(a,t)p_n + \lambda_{2n-2}(a,t)P_{n-1}(t)p_n, \\ p_i(0,t) = \beta_i(t)\int_{a_1}^{a_2} m_i(a,t)p_i(a,t)\mathrm{d}a, \\ p_i(a,0) = p_{i0}(a), \\ P_i(t) = \int_0^{a_+} p_i(a,t)\mathrm{d}a, \quad (a,t) \in Q, \end{cases} \tag{2.2}$$

where $Q = (0, a_+) \times (0, +\infty)$, $[a_1, a_2]$ is the fertility interval. $p_i(a,t)$ represents the density of the ith population of age a at time t, and a_+ is the life expectancy of individuals. The control variable $\beta_i(t)$ is the average fertility of the ith population. The vitality rate $\mu_i(a,t)$ denotes the average mortality of the ith population. $\lambda_k(a,t)$ is the interaction coefficient, and $m_i(a,t)$ is the ratio of females in the ith population. Maximum principles for the control problems with free terminal states, infinite horizons and target sets have been derived.

By combining (2.1) and (2.2), we propose the following periodic, n-dimensional food chain model with size structure in a polluted environment:

$$\begin{cases} \frac{\partial p_1}{\partial t} + \frac{\partial (V_1(x,t)p_1)}{\partial x} = f_1(x,t) - \mu_1(x, c_{10}(t))p_1 - \lambda_1(x,t)P_2(t)p_1 - u_1(x,t)p_1, \\ \frac{\partial p_i}{\partial t} + \frac{\partial (V_i(x,t)p_i)}{\partial x} = f_i(x,t) - \mu_i(x, c_{i0}(t))p_i + \lambda_{2i-2}(x,t)P_{i-1}(t)p_i \\ \qquad\qquad - \lambda_{2i-1}(x,t)P_{i+1}(t)p_i - u_i(x,t)p_i, \quad i = 2, 3, \ldots n-1, \\ \frac{\partial p_n}{\partial t} + \frac{\partial (V_n(x,t)p_n)}{\partial x} = f_n(x,t) - \mu_n(x, c_{n0}(t))p_n + \lambda_{2n-2}(x,t)P_{n-1}(t)p_n - u_n(x,t)p_n, \\ \frac{\mathrm{d}c_{i0}}{\mathrm{d}t} = k_1 c_e(t) - g_1 c_{i0}(t) - m c_{i0}(t), \\ \frac{\mathrm{d}c_e}{\mathrm{d}t} = -k_2 c_e(t)\sum_{i=1}^{n} P_i(t) + g_2 \sum_{i=1}^{n} c_{i0}(t)P_i(t) - h_1 c_e(t) + v(t), \\ V_i(0,t)p_i(0,t) = \int_0^l \beta_i(x, c_{i0}(t))p_i(x,t)\mathrm{d}x, \\ p_i(x,t) = p_i(x, t+T), \\ P_i(t) = \int_0^l p_i(x,t)\mathrm{d}x, \quad (x,t) \in Q, \end{cases} \tag{2.3}$$

where $Q = (0,l)\times R_+, l \in R_+$ is the maximal size of an individual in the population, $T \in R_+$ is the period of evolution of the population. The meanings of the other parameters are as follows: $p_i(x,t)$: the density of the ith population of size x at time t; $c_{i0}(t)$: the concentration of toxicants in the ith population; $c_e(t)$: the concentration of toxicants in the environment; $V_i(x,t)$: the growth rate of the ith population; $\mu_i(x, c_{i0}(t)), \beta_i(x, c_{i0}(t))$: the mortality and fertility rates of the ith population, respectively; $\lambda_k(x,t)$: the interaction coefficient ($k = 1, 2, \ldots, 2n-2$); $v(t)$: the input rate of exogenous toxicants; $P_i(t)$: total number of individuals in the ith population; $f_i(x,t)$: the immigration rate of the ith population; $u_i(x,t)$: function of the harvesting efforts of the ith population. k_1, g_1, m, h_1, k_2, and g_2 are nonnegative constants. For the sake of convenience, if there is no special description, $i = 1, 2, \ldots, n$. The model represents the population dynamics, and couples toxicants with the population. Moreover, it is more realistic for the fertility and mortality rates to depend on the size and the concentration of toxicants in an organism.

This paper considers the following objective functional:

$$\max\{J(u,v) : u = (u_1(x,t), u_2(x,t), \dots, u_n(x,t), v = v(t), (u,v) \in \Omega)\}, \tag{2.4}$$

where

$$J(u,v) = \sum_{i=1}^{n} \int_0^T \int_0^l w_i(x,t) u_i(x,t) p_i(x,t) \mathrm{d}x\mathrm{d}t - \frac{1}{2} \sum_{i=1}^{n} \int_0^T \int_0^l c_i u_i^2(x,t) \mathrm{d}x\mathrm{d}t - \frac{1}{2} \int_0^T c_{n+1} v^2(t) \mathrm{d}t,$$

represents the revenue obtained from harvesting less the costs of harvesting and curbing environmental pollution. The weight function $w_i(x,t)$ is the selling price factor of an individual belonging to the ith population. The positive constants c_i and c_{n+1} are the cost factors of the ith harvested population and the curbing of environmental pollution, respectively. Therefore, the objective function represents the total profit from the harvested populations during period T. To utilize the resources of the species over a long time, we must develop them rationally and manage them scientifically. We should not only consider the maximization of current economic benefits, but should also consider the ecological balance to ensure the maximization of long-term economic benefits. Our optimal control pair (u^*, v^*) in Ω satisfies $J(u^*, v^*) = \max_{(u,v)\in\Omega} J(u,v)$. The admissible control set Ω is as follows:

$$\Omega = \{(u,v) \in [L_T^\infty(Q)]^n \times L_T^\infty(R_+) : 0 \le u_i(x,t) \le N_i, \text{a.e. } (x,t) \in Q,\ 0 \le v_0 \le v(t) \le v_1, \text{a.e. } t \in R_+\},$$

where

$$L_T^\infty(Q) = \{\eta \in L^\infty(Q) : \eta(x,t) = \eta(x,t+T), \text{a.e. } (x,t) \in Q\},$$

$$L_T^\infty(R_+) = \{\eta \in L^\infty(R_+) : \eta(t) = \eta(t+T), \text{a.e. } t \in\in R_+\}.$$

This paper makes the following assumptions:

(A_1) $V_i : [0,l) \times R_+ \to R_+$ are bounded continuous functions, $V_i(x,t) > 0$ and $V_i(x,t) = V_i(x,t+T)$ for $(x,t) \in Q$, $\lim_{x\uparrow l} V_i(x,t) = 0$, and $V_i(0,t) = 1$ for $t \in R_+$. There are Lipschitz constants L_{V_i} such that

$$|V_i(x_1,t) - V_i(x_2,t)| \le L_{V_i}|x_1 - x_2| \text{ for } x_1, x_2 \in [0,l],\ t \in R_+.$$

(A_2) $0 \le \beta_i(x, c_{i0}(t)) = \beta_i(x, c_{i0}(t+T)) \le \overline{\beta}_i, \overline{\beta}_i$ are constants.

(A_3) $$\begin{cases} \mu_i(x, c_{i0}(t)) = \mu_0(x) + \overline{\mu}_i(x, c_{i0}(t)) \text{ a.e. } (x,t) \in Q, \text{ where } \mu_0 \in L^1_{loc}([0,l)), \\ \mu_0(x) \ge 0 \text{ a.e. } x \in [0,l),\ \int_0^l \mu_0(x)\mathrm{d}x = +\infty, \overline{\mu}_i \in L^\infty(Q), \\ \overline{\mu}_i(x, c_{i0}(t)) \ge 0 \text{ and } \overline{\mu}_i(x, c_{i0}(t)) = \overline{\mu}_i(x, c_{i0}(t+T)) \text{ a.e. } (x,t) \in Q. \end{cases}$$

(A_4) $f_i \in L^\infty(Q), 0 \le f_i(x,t) = f_i(x,t+T), 0 \le w_i(x,t) \le w_i(x,t+T) \le \overline{w}_i, 0 \le \lambda_i(x,t) \le \overline{\lambda}_i, P_i(t) \le M_0, \overline{w}_i, \overline{\lambda}_i$ and M_0 are constants.

(A_5) $\mid \beta_i(x, c_{i0}^1(t)) - \beta_i(x, c_{i0}^2(t)) \mid \le L_\beta \mid c_{i0}^1(t) - c_{i0}^2(t) \mid, \mid \mu_i(x, c_{i0}^1(t)) - \mu_i(x, c_{i0}^2(t)) \mid \le L_\mu \mid c_{i0}^1(t) - c_{i0}^2(t) \mid .$

(A_6) $g_1 \le k_1 \le g_1 + m, v_1 \le h_1$. (see [24])

3. Well-posedness of the state system

This section discusses the existence and uniqueness of the solution to the state system (2.3). For convenience of research, the following definitions are introduced:

Definition 3.1 For $i = 1, 2, \ldots, n$, the unique solution $x = \varphi_i(t; t_0, x_{i0})$ of the initial value problem $x'(t) = V_i(x, t), x(t_0) = x_{i0}$ is said to be a characteristic curve of (2.3) through (t_0, x_{i0}). In particular, let $z_i(t) := \varphi_i(t; 0, 0)$ be the characteristic curve through (0,0) in the $x - t$ plane.

Definition 3.2 Suppose $g(x)$ is an essentially bounded measurable function on E; let

$$\|g\|_\infty = \inf_{E_0 \subset E} \left(\sup_{E - E_0} |g(x)| \right),$$

where the infimum is taken for all zero sets E_0 in E that make $f(x)$ a bounded function on $E - E_0$. This is also denoted by

$$\operatorname{Ess}\sup_{x \in E} |g(x)|.$$

For any point $(x, t) \in [0, l) \times [0, T]$ such that $x \le z_i(t)$, define the initial time $\tau := \tau(t, x)$; then, $\varphi_i(t; \tau, 0) = x \Leftrightarrow \varphi_i(\tau; t, x) = 0$. The solution of (2.3) is

$$p_i(x,t) = p_i(0, t - z_i^{-1}(x))\Pi_i(x; x, t) + \int_0^x \frac{f_i(r, \varphi_i^{-1}(r; t, x))}{V_i(r, \varphi_i^{-1}(r; t, x))} \frac{\Pi_i(x; x, t)}{\Pi_i(r; x, t)} \mathrm{d}r, \tag{3.1}$$

where

$$\begin{aligned}\Pi_1(s; x, t) = \exp\Bigg\{ -\int_0^s & \frac{\mu_1(r, c_{10}(\varphi_1^{-1}(r; t, x))) + \lambda_1(r, \varphi_1^{-1}(r; t, x))P_2(\varphi_1^{-1}(r; t, x))}{V_1(r, \varphi_1^{-1}(r; t, x))} \\ & + \frac{u_1(r, \varphi_1^{-1}(r; t, x)) + V_{1x}(r, \varphi_1^{-1}(r; t, x))}{V_1(r, \varphi_1^{-1}(r; t, x))} \mathrm{d}r \Bigg\},\end{aligned}$$

$$\begin{aligned}\Pi_i(s; x, t) = \exp\Bigg\{ -\int_0^s & \frac{\mu_i(r, c_{i0}(\varphi_i^{-1}(r; t, x))) - \lambda_{2i-2}(r, \varphi_i^{-1}(r; t, x))P_{i-1}(\varphi_i^{-1}(r; t, x))}{V_i(r, \varphi_i^{-1}(r; t, x))} \\ & + \frac{\lambda_{2i-1}(r, \varphi_i^{-1}(r; t, x))P_{i+1}(\varphi_i^{-1}(r; t, x)) + u_i(r, \varphi_i^{-1}(r; t, x)) + V_{ix}(r, \varphi_i^{-1}(r; t, x))}{V_i(r, \varphi_i^{-1}(r; t, x))} \mathrm{d}r \Bigg\} \\ & i = 2, 3, \ldots, n-1,\end{aligned}$$

$$\begin{aligned}\Pi_n(s; x, t) = \exp\Bigg\{ -\int_0^s & \frac{\mu_n(r, c_{n0}(\varphi_n^{-1}(r; t, x))) - \lambda_{2n-2}(r, \varphi_n^{-1}(r; t, x))P_{n-1}(\varphi_n^{-1}(r; t, x))}{V_n(r, \varphi_n^{-1}(r; t, x))} \\ & + \frac{u_n(r, \varphi_n^{-1}(r; t, x)) + V_{nx}(r, \varphi_n^{-1}(r; t, x))}{V_n(r, \varphi_n^{-1}(r; t, x))} \mathrm{d}r \Bigg\}.\end{aligned}$$

$$c_{i0}(t) = c_{i0}(0)\exp\{-(g_1 + m)t\} + k_1 \int_0^t c_e(\sigma) \exp\{(\sigma - t)(g_1 + m)\} \mathrm{d}\sigma. \tag{3.2}$$

$$\begin{aligned} c_e(t) =& c_e(0)\exp\left\{-\int_0^t\left[k_2\sum_{i=1}^n P_i(\tau)+h_1\right]\mathrm{d}\tau\right\}+\int_0^t\left[g_2\sum_{i=1}^n c_{i0}(\sigma)P_i(\sigma)+v(\sigma)\right] \\ &\cdot\exp\left\{\int_t^\sigma\left[k_2\sum_{i=1}^n P_i(\tau)+h_1\right]\mathrm{d}\tau\right\}\mathrm{d}\sigma. \end{aligned} \tag{3.3}$$

Theorem 3.1 Assume that $(A_1)-(A_6)$ hold; then, the hybrid system (2.3) has a nonnegative and unique solution $(p_1(x,t), p_2(x,t), \ldots, p_n(x,t), c_{10}(t), c_{20}(t), \ldots, c_{n0}(t), c_e(t))$, such that
(i) $(p_i(x,t), c_{i0}(t), c_e(t)) \in L^\infty(Q)\times L^\infty(0,T)\times L^\infty(0,T)$.
(ii) $0\le c_{i0}(t)\le 1, 0\le c_e(t)\le 1, \forall t\in(0,T),\ \ 0\le p_i(x,t), \int_0^l p_i(x,t)\mathrm{d}x\le M, \forall (x,t)\in Q$.

Proof. Without loss of generality, we assume that $u_i(x,t)\equiv 0$. $p(x,t) = (p_1(x,t), p_2(x,t), \ldots, p_n(x,t))$, $c_0(t) = (c_{10}(t), c_{20}(t), \ldots, c_{n0}(t))$, $X = [L_T^\infty(R_+, L^1(0,l))]^n\times[L^\infty(R_+)]^n\times L^\infty(R_+)$. Then, the state space is defined as follows:

$$Y=\left\{(p,c_0,c_e)\in X \mid p_i(x,t)\ge 0 \text{ a.e. in } Q, \int_0^l p_i(x,t)\mathrm{d}x\le M,\ 0\le c_{i0}(t)\le 1,\ 0\le c_e(t)\le 1.\right\},$$

where

$$\begin{aligned} M := \max&\Big\{\overline{\beta}_1 T\|f_1(\cdot,\cdot)\|_{L^1(Q)}\exp\{\overline{\beta}_1 T\}+\|f_1(\cdot,\cdot)\|_{L^1(Q)}, \overline{\beta}_i T\exp\{\overline{\lambda}_{2i-2}M_0T\}\|f_i(\cdot,\cdot)\|_{L^1(Q)} \\ &\cdot\exp\{\overline{\beta}_i T\exp\{\overline{\lambda}_{2i-2}M_0T\}\}+\exp\{\overline{\lambda}_{2i-2}M_0T\}\|f_i(\cdot,\cdot)\|_{L^1(Q)}, i=2,3,\ldots,n\Big\}. \end{aligned}$$

Define a mapping

$$G: Y\to X, G(p,c_0,c_e) = (G_1(p,c_0,c_e), G_2(p,c_0,c_e),\ldots,G_{2n+1}(p,c_0,c_e)).$$

where

$$G_i(p,c_0,c_e)(x,t) = p_i(0, t-z_i^{-1}(x))\Pi_i(x;x,t)+\int_0^x \frac{f_i(r,\varphi_i^{-1}(r;t,x))}{V_i(r,\varphi_i^{-1}(r;t,x))}\frac{\Pi_i(x;x,t)}{\Pi_i(r;x,t)}\mathrm{d}r, i=1,2,\ldots,n.$$

$$\begin{aligned} G_j(p,c_0,c_e)(t) =& c_{j0}(0)\exp\{-(g_1+m)t\}+k_1\int_0^t c_e(\sigma)\exp\{(\sigma-t)(g_1+m)\}\mathrm{d}\sigma, \\ & j=n+1,n+2,\ldots,2n. \end{aligned}$$

$$\begin{aligned} G_{2n+1}(p,c_0,c_e)(t) =& c_e(0)\exp\left\{-\int_0^t\left[k_2\sum_{i=1}^n P_i(\tau)+h_1\right]\mathrm{d}\tau\right\}+\int_0^t\left[g_2\sum_{i=1}^n c_{i0}(\sigma)P_i(\sigma)+v(\sigma)\right] \\ &\cdot\exp\left\{\int_t^\sigma\left[k_2\sum_{i=1}^n P_i(\tau)+h_1\right]\mathrm{d}\tau\right\}\mathrm{d}\sigma. \end{aligned}$$

By assumption (A_1), we have $V_i(0,t)=1$. Let $b_i(t)=p_i(0,t)$, then, noting $\varphi_i^{-1}=t-z_i^{-1}(x)$, we have

$$\begin{aligned}
b_1(t) =&V_1(0,t)p_1(0,t)\\
=&\int_0^l \beta_1(x,c_{10}(t))p_1(x,t)\mathrm{d}x\\
=&\int_0^l \beta_1(x,c_{10}(t))b_1(\varphi_1^{-1}(0;t,x))\mathrm{d}x+\int_0^l \beta_1(x,c_{10}(t))\int_0^x \frac{f_1(r,\varphi_1^{-1}(r;t,x))}{V_1(r,\varphi_1^{-1}(r;t,x))}\mathrm{d}r\mathrm{d}x\\
=&\int_0^l \beta_1(x,c_{10}(t))b_1(t-z_1^{-1}(x))\mathrm{d}x+\int_0^l \beta_1(x,c_{10}(t))\int_{\varphi_1^{-1}(0;t,x)}^t f_1(\varphi_1(\sigma;t,x),\sigma)\mathrm{d}\sigma\mathrm{d}x\\
\le&\overline{\beta}_1\int_0^l b_1(t-z_1^{-1}(x))\mathrm{d}x+\overline{\beta}_1\int_0^l\int_0^t f_1(\varphi_1(\sigma;t,x),\sigma)\mathrm{d}\sigma\mathrm{d}x\\
\le&\overline{\beta}_1\int_0^t b_1(\sigma)\mathrm{d}\sigma+\overline{\beta}_1\|f_1(\cdot,\cdot)\|_{L^1(Q)}.
\end{aligned}$$

It follows from Bellman's lemma that

$$b_1(t)\le\overline{\beta}_1\|f_1(\cdot,\cdot)\|_{L^1(Q)}\exp\{\overline{\beta}_1T\}.$$

Then we can see that

$$\begin{aligned}
&\int_0^l |\,G_1(p,c_0,c_e)\,|\,(x,t)\mathrm{d}x\\
=&\int_0^l p_1(0,\varphi_1^{-1}(0;t,x))\Pi_1(x;x,t)\mathrm{d}x+\int_0^l\int_0^x \frac{f_1(r,\varphi_1^{-1}(r;t,x))}{V_1(r,\varphi_1^{-1}(r;t,x))}\frac{\Pi_1(x;x,t)}{\Pi_1(r;x,t)}\mathrm{d}r\mathrm{d}x\\
\le&\int_0^l p_1(0,\varphi_1^{-1}(0;t,x))\mathrm{d}x+\int_0^l\int_0^x \frac{f_1(r,\varphi_1^{-1}(r;t,x))}{V_1(r,\varphi_1^{-1}(r;t,x))}\mathrm{d}r\mathrm{d}x\\
=&\int_0^l b_1(t-z_1^{-1})\mathrm{d}x+\int_0^l\int_{\varphi_1^{-1}(0;t,x)}^t f_1(\varphi_1(\sigma;t,x),\sigma)\mathrm{d}\sigma\mathrm{d}x\\
\le&\int_0^t b_1(\sigma)\mathrm{d}\sigma+\|f_1(\cdot,\cdot)\|_{L^1(Q)}\\
\le&\bar{\beta}_1T\|f_1(\cdot,\cdot)\|_{L^1(Q)}\exp\{\bar{\beta}_1T\}+\|f_1(\cdot,\cdot)\|_{L^1(Q)}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\int_0^l |\,G_i(p,c_0,c_e)\,|\,(x,t)\mathrm{d}x\\
\le&\overline{\beta}_iT\exp\{\overline{\lambda}_{2i-2}M_0T\}\|f_i(\cdot,\cdot)\|_{L^1(Q)}\exp\{\overline{\beta}_iT\exp\{\overline{\lambda}_{2i-2}M_0T\}\}+\exp\{\overline{\lambda}_{2i-2}M_0T\}\|f_i(\cdot,\cdot)\|_{L^1(Q)},\\
&i=2,3,\dots,n.
\end{aligned}$$

It follows that G is a mapping from Y to Y. We now discuss the compressibility of the mapping G.

$$G_1(p,c_0,c_e)(x,t) = b_1(\varphi_1^{-1}(0;t,x))E_1(\varphi_1^{-1}(0;t,x);x,t,P_2(t)) \\ + \int_{\varphi_1^{-1}(0;t,x)}^{l} f_1(\varphi_1(\sigma;t,x),\sigma)E_1(\sigma;x,t,P_2(t))\mathrm{d}\sigma,$$

where

$$E_1(r;x,t,P_2(t)) = \exp\left\{-\int_r^t \mu_1(\varphi_1(\sigma;t,x),c_{10}(\sigma)) + \lambda_1(\varphi_1(\sigma;t,x),\sigma)P_2(t) + V_{1x}(\varphi_1(\sigma;t,x),\sigma)\mathrm{d}\sigma\right\}.$$

Then, we have

$$\begin{aligned}
&\int_0^l |\, G_1(p^1,c_0^1,c_e^1) - G_1(p^2,c_0^2,c_e^2)\,|\,\mathrm{d}x \\
=&\int_0^l |\, b_1^1(\varphi_1^{-1}(0;t,x))E_1^1(\varphi_1^{-1}(0;t,x);x,t,P_2^1(t)) - b_1^2(\varphi_1^{-1}(0;t,x))E_1^2(\varphi_1^{-1}(0;t,x);x,t,P_2^2(t))\,|\,\mathrm{d}x \\
&+\int_0^l \int_{\varphi_1^{-1}(0;t,x)}^t |\, f_1(\varphi_1(\sigma;t,x),\sigma)E_1^1(\sigma;x,t,P_2^1(t)) - f_1(\varphi_1(\sigma;t,x),\sigma)E_1^2(\sigma;x,t,P_2^2(t))\,|\,\mathrm{d}\sigma\mathrm{d}x \\
\le&\int_0^l |\, b_1^1(\varphi_1^{-1}(0;t,x)) - b_1^2(\varphi_1^{-1}(0;t,x))\,|\,\mathrm{d}x \\
&+\int_0^l b_1^1(\varphi_1^{-1}(0;t,x))\int_{\varphi_1^{-1}(0;t,x)}^t |\,\mu_1(\varphi_1(\sigma;t,x),c_{10}^1(\sigma)) - \mu_1(\varphi_1(\sigma;t,x),c_{10}^2(\sigma))\,|\,\mathrm{d}\sigma\mathrm{d}x \\
&+\int_0^l b_1^1(\varphi_1^{-1}(0;t,x))\int_{\varphi_1^{-1}(0;t,x)}^t \overline{\lambda}_1\,|\,P_2^1(t) - P_2^2(t)\,|\,\mathrm{d}\sigma\mathrm{d}x \\
&+\int_0^l \int_{\varphi_1^{-1}(0;t,x)}^t f_1(\varphi_1(\sigma;t,x),\sigma)\int_\sigma^t |\,\mu_1(\varphi_1(\sigma;t,x),c_{10}^1(\sigma)) - \mu_1(\varphi_1(\sigma;t,x),c_{10}^2(\sigma))\,|\,\mathrm{d}r\mathrm{d}\sigma\mathrm{d}x \\
&+\int_0^l \int_{\varphi_1^{-1}(0;t,x)}^t f_1(\varphi_1(\sigma;t,x),\sigma)\int_\sigma^t |\,P_2^1(t) - P_2^2(t)\,|\,\mathrm{d}r\mathrm{d}\sigma\mathrm{d}x \\
\le&\int_0^t |\, b_1^1(\sigma) - b_1^2(\sigma)\,|\,\mathrm{d}\sigma + ML_\mu\int_0^t |\, c_{10}^1(\sigma) - c_{10}^2(\sigma)\,|\,\mathrm{d}\sigma \\
&+\overline{\lambda}_1 MT\int_0^l |\, p_2^1(x,\sigma) - p_2^2(x,\sigma)\,|\,\mathrm{d}x + lL_\mu\|f_1(\cdot,\cdot)\|_{L^1(Q)}\int_0^t |\, c_{10}^1(\sigma) - c_{10}^2(\sigma)\,|\,\mathrm{d}\sigma \\
&+lT\|f_1(\cdot,\cdot)\|_{L^1(Q)}\int_0^l |\, p_2^1(x,\sigma) - p_2^2(x,\sigma)\,|\,\mathrm{d}x \\
\le&\overline{\beta}_1\int_0^t\int_0^l |\, p_1^1(x,\sigma) - p_1^2(x,\sigma)\,|\,\mathrm{d}x\mathrm{d}\sigma \\
&+\left[ML_\mu + lL_\mu\|f_1(\cdot,\cdot)\|_{L^1(Q)} + ML_\beta\right]\int_0^t |\, c_{10}^1(\sigma) - c_{10}^2(\sigma)\,|\,\mathrm{d}\sigma \\
&+\left[\overline{\lambda}_1 MT + lT\|f_1(\cdot,\cdot)\|_{L^1(Q)}\right]\int_0^l |\, p_2^1(x,\sigma) - p_2^2(x,\sigma)\,|\,\mathrm{d}x.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\int_0^l \mid G_i(p^1, c_0^1, c_e^1) - G_i(p^2, c_0^2, c_e^2) \mid \mathrm{d}x \\
\le &\exp\left\{\overline{\lambda}_{2i-2} M_0 T\right\} \left\{\overline{\beta}_i \int_0^t \int_0^l \mid p_i^1(x,\sigma) - p_i^2(x,\sigma) \mid \mathrm{d}x\mathrm{d}\sigma \right.\\
&+ \left[ML_\mu + lL_\mu \|f(\cdot,\cdot)\|_{L^1(Q)} + ML_\beta\right] \int_0^t \mid c_{i0}^1(\sigma) - c_{i0}^2(\sigma) \mid \mathrm{d}\sigma \\
&\left. + \left[\overline{\lambda}_{2i-1} MT + lT\|f(\cdot,\cdot)\|_{L^1(Q)}\right] \int_0^l \mid p_{i+1}^1(x,\sigma) - p_{i+1}^2(x,\sigma) \mid \mathrm{d}x \right\}, i = 2, 3, \ldots, n-1.
\end{aligned}$$

$$\begin{aligned}
&\int_0^l \mid G_n(p^1, c_0^1, c_e^1) - G_n(p^2, c_0^2, c_e^2) \mid \mathrm{d}x \\
\le &\exp\left\{\overline{\lambda}_{2n-2} M_0 T\right\} \left\{\overline{\beta}_n \int_0^t \int_0^l \mid p_n^1(x,\sigma) - p_n^2(x,\sigma) \mid \mathrm{d}x\mathrm{d}\sigma \right.\\
&\left. + \left[ML_\mu + lL_\mu \|f(\cdot,\cdot)\|_{L^1(Q)} + ML_\beta\right] \int_0^t \mid c_{n0}^1(\sigma) - c_{n0}^2(\sigma) \mid \mathrm{d}\sigma \right\}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\int_0^l \mid G_i(p^1, c_0^1, c_e^1) - G_i(p^2, c_0^2, c_e^2) \mid \mathrm{d}x, \\
\le &M_1 \left(\sum_{i=1}^n \int_0^t \int_0^l \mid p_i^1(x,\sigma) - p_i^2(x,\sigma) \mid \mathrm{d}x\mathrm{d}\sigma + \int_0^t \mid c_{i0}^1(\sigma) - c_{i0}^2(\sigma) \mid \mathrm{d}\sigma \right),
\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}
M_1 = \max &\left\{\overline{\beta}_1, \left[1 + \overline{\lambda}_1 MT + lT\|f(\cdot,\cdot)\|_{L^1(Q)}\right] \overline{\beta}_2 \exp\left\{\overline{\lambda}_2 M_0 T\right\}, \ldots, \right.\\
&\left. \left[1 + \overline{\lambda}_{2n-3} MT + lT\|f(\cdot,\cdot)\|_{L^1(Q)}\right] \overline{\beta}_n \exp\left\{\overline{\lambda}_{2n-2} M_0 T\right\}, \left[ML_\mu + lL_\mu \|f(\cdot,\cdot)\|_{L^1(Q)} + ML_\beta\right] \right\}.
\end{aligned}$$

By (3.2) and (3.3), we obtain

$$\begin{aligned}
&\mid G_j(p^1, c_0^1, c_e^1) - G_j(p^2, c_0^2, c_e^2) \mid (t) \quad (j = n+1, n+2, \ldots, 2n) \\
\le &M_2 \int_0^t \mid c_e^1(\sigma) - c_e^2(\sigma) \mid \mathrm{d}\sigma,
\end{aligned} \tag{3.5}$$

where $M_2 = k_1$.

$$\begin{aligned}
&\mid G_{2n+1}(p^1, c_0^1, c_e^1) - G_{2n+1}(p^2, c_0^2, c_e^2) \mid (t) \\
\le &M_3 \left(\sum_{i=1}^n \int_0^t \int_0^l \mid p_i^1(x,\sigma) - p_i^2(x,\sigma) \mid \mathrm{d}x\mathrm{d}\sigma + \int_0^t \mid c_{i0}^1(\sigma) - c_{i0}^2(\sigma) \mid \mathrm{d}\sigma \right),
\end{aligned} \tag{3.6}$$

where $M_3 = \max\{k_2 + g_2 + k_2 h_1 T + k_2 g_2 MT, g_2 M\}$.

We now show that the mapping G has a unique fixed point. Due to the periodicity of elements in the set Y, we consider the case $[0,T]$ only. We define an equivalent norm in $L^\infty(0,T)$ as follows:

$$\|(p,c_0,c_e)\|_* = \operatorname{Ess}\sup_{t\in[0,T]} e^{-\lambda t}\left\{\sum_{i=1}^{n}\int_0^l \mid p_i(x,t)\mid \mathrm{d}x + \sum_{i=1}^{n}\mid c_{i0}(t)\mid + \mid c_e(t)\mid\right\},$$

where $\lambda > 0$ is large enough. Then, we have

$$\begin{aligned}
&\|G(p^1,c_0^1,c_e^1) - G(p^2,c_0^2,c_e^2)\|_* \\
\leq & M_4 \operatorname{Ess}\sup_{t\in[0,T]} e^{-\lambda t}\int_0^t\left\{\sum_{i=1}^{n}\int_0^l \mid p_i^1(x,\sigma) - p_i^2(x,\sigma)\mid \mathrm{d}x\right. \\
&\left. + \sum_{i=1}^{n}\mid c_{i0}^1(\sigma) - c_{i0}^2(\sigma)\mid + \mid c_e^1(\sigma) - c_e^2(\sigma)\mid\right\}\mathrm{d}\sigma \\
\leq & M_4 \operatorname{Ess}\sup_{t\in[0,T]} e^{-\lambda t}\int_0^t e^{\lambda\sigma}\left\{e^{-\lambda\sigma}\left[\sum_{i=1}^{n}\int_0^l \mid p_i^1(x,\sigma) - p_i^2(x,\sigma)\mid \mathrm{d}x\right.\right. \\
&\left.\left. + \sum_{i=1}^{n}\mid c_{i0}^1(\sigma) - c_{i0}^2(\sigma)\mid + \mid c_e^1(\sigma) - c_e^2(\sigma)\mid\right]\right\}\mathrm{d}\sigma \\
\leq & M_4\|(p^1 - p^2, c_0^1 - c_0^2, c_e^1 - c_e^2)\|_* \operatorname{Ess}\sup_{t\in[0,T]} e^{-\lambda t}\int_0^t e^{\lambda\sigma}\mathrm{d}\sigma \\
\leq & \frac{M_4}{\lambda}\|(p^1 - p^2, c_0^1 - c_0^2, c_e^1 - c_e^2)\|_*,
\end{aligned}$$

where $M_4 = \max\{M_1, M_2, M_3\}$. Thus, choosing $\lambda > M_4$ yields G as a strict contraction on the space of $(Y, \|\cdot\|_*)$. By the Banach fixed point theory, G has a unique fixed point, which is the solution of the system (2.3).

Theorem 3.2 If T is small enough, then there are constants $K_j(T)$ with $\lim_{T\to 0} K_j(T) > 0$, $j = 1, 2$, such that

$$\begin{aligned}
&\sum_{i=1}^{n}\|p_i^1 - p_i^2\|_{L^\infty(0,T;L^1(0,l))} + \sum_{i=1}^{n}\|c_{i0}^1 - c_{i0}^2\|_{L^\infty(0,T)} + \|c_e^1 - c_e^2\|_{L^\infty(0,T)} \\
\leq & K_1(T)T\left(\sum_{i=1}^{n}\|u_i^1 - u_i^2\|_{L^\infty(0,T;L^1(0,l))} + \|v^1 - v^2\|_{L^\infty(0,T)}\right).
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
&\sum_{i=1}^{n}\|p_i^1 - p_i^2\|_{L^1(Q)} + \sum_{i=1}^{n}\|c_{i0}^1 - c_{i0}^2\|_{L^1(0,T)} + \|c_e^1 - c_e^2\|_{L^1(0,T)} \\
\leq & K_2(T)T\left(\sum_{i=1}^{n}\|u_i^1 - u_i^2\|_{L^1(Q)} + \|v^1 - v^2\|_{L^1(0,T)}\right).
\end{aligned} \tag{3.8}$$

Proof. Let (p^j, c_0^j, c_e^j) be the solution of (2.3) corresponding to (u^j, v^j), $j = 1, 2$, it follows from (3.1) that

$$
\begin{aligned}
&\int_0^l \mid p_1^1(x,t) - p_1^2(x,t) \mid \mathrm{d}x \\
&= \int_0^l \mid b_1^1(\varphi_1^{-1}(0;t,x))\Pi_1^1(x;x,t) - b_1^2(\varphi_1^{-1}(0;t,x))\Pi_1^2(x;x,t) \mid \mathrm{d}x \\
&\quad + \int_0^l \int_0^x \left| \frac{f_1(r,\varphi_1^{-1}(r;t,x))}{V_1(r,\varphi_1^{-1}(r;t,x))} \frac{\Pi_1^1(x;x,t)}{\Pi_1^1(r;x,t)} - \frac{f_1(r,\varphi_1^{-1}(r;t,x))}{V_1(r,\varphi_1^{-1}(r;t,x))} \frac{\Pi_1^2(x;x,t)}{\Pi_1^2(r;x,t)} \right| \mathrm{d}r\mathrm{d}x \\
&\leq \int_0^l \mid b_1^1(\varphi_1^{-1}(0;t,x)) - b_1^1(\varphi_1^{-1}(0;t,x)) \mid \mathrm{d}x \\
&\quad + \int_0^l b_1^1(\varphi_1^{-1}(0;t,x)) \left[\int_{\varphi_1^{-1}(0;t,x)}^t \mid \mu_1(\varphi_1(r;t,x), c_{10}^1(r)) - \mu_1(\varphi_1(r;t,x), c_{10}^2(r)) \mid \mathrm{d}r \right] \mathrm{d}x \\
&\quad + \int_0^l b_1^1(\varphi_1^{-1}(0;t,x)) \left[\int_{\varphi_1^{-1}(0;t,x)}^t \mid \lambda_1(\varphi_1(r;t,x), r)(P_2^1(t) - P_2^2(t)) \mid \mathrm{d}r \right] \mathrm{d}x \\
&\quad + \int_0^l b_1^1(\varphi_1^{-1}(0;t,x)) \left[\int_{\varphi_1^{-1}(0;t,x)}^t \mid u_1^1(\varphi_1(r;t,x), r) - u_1^2(\varphi_1(r;t,x), r) \mid \mathrm{d}r \right] \mathrm{d}x \\
&\quad + \int_0^l \int_{\varphi_1^{-1}(0;t,x)}^t f_1(\varphi_1(\sigma;t,x),\sigma) \left[\int_\sigma^t \mid \mu_1(\varphi_1(\sigma;t,x), c_{10}^1(\sigma)) - \mu_1(\varphi_1(\sigma;t,x), c_{10}^2(\sigma)) \mid \mathrm{d}r \right] \mathrm{d}\sigma\mathrm{d}x \\
&\quad + \int_0^l \int_{\varphi_1^{-1}(0;t,x)}^t f_1(\varphi_1(\sigma;t,x),\sigma) \left[\int_\sigma^t \mid \lambda_1(\varphi_1(\sigma;t,x), \sigma)(P_2^1(t) - P_2^2(t)) \mid \mathrm{d}r \right] \mathrm{d}\sigma\mathrm{d}x \\
&\quad + \int_0^l \int_{\varphi_1^{-1}(0;t,x)}^t f_1(\varphi_1(\sigma;t,x),\sigma) \left[\int_\sigma^t \mid u_1^1(\varphi_1(\sigma;t,x), \sigma) - u_1^2(\varphi_1(\sigma;t,x), \sigma) \mid \mathrm{d}r \right] \mathrm{d}\sigma\mathrm{d}x \\
&\leq \int_0^t \mid b_1^1(\sigma) - b_1^2(\sigma) \mid \mathrm{d}x + \overline{\beta}_1 M L_\mu \int_0^t \mid c_{10}^1(\sigma) - c_{10}^2(\sigma) \mid \mathrm{d}\sigma \\
&\quad + \overline{\beta}_1 \overline{\lambda}_1 M T \int_0^l \mid p_2^1(x,\sigma) - p_2^1(x,\sigma) \mid \mathrm{d}x \\
&\quad + \overline{\beta}_1 M \int_0^t \int_0^l \mid u_1^1(\varphi_1(\sigma;t,x),\sigma) - u_1^2(\varphi_1(\sigma;t,x),\sigma) \mid \mathrm{d}x\mathrm{d}\sigma \\
&\quad + l L_\mu \|f_1(\cdot,\cdot)\|_{L^1(Q)} \int_0^t \mid c_{10}^1(\sigma) - c_{10}^2(\sigma) \mid \mathrm{d}\sigma + \overline{\lambda}_1 l T \|f_1(\cdot,\cdot)\|_{L^1(Q)} \int_0^l \mid p_2^1(x,\sigma) - p_2^2(x,\sigma) \mid \mathrm{d}x \\
&\quad + \|f_1(\cdot,\cdot)\|_{L^1(Q)} \int_0^t \int_0^l \mid u_1^1(\varphi_1(\sigma;t,x),\sigma) - u_1^2(\varphi_1(\sigma;t,x),\sigma) \mid \mathrm{d}x\mathrm{d}\sigma \\
&\leq \overline{\beta}_1 \int_0^t \int_0^l \mid p_1^1(x,\sigma) - p_1^2(x,\sigma) \mid \mathrm{d}x\mathrm{d}\sigma + \overline{\lambda}_1 [MT + lT\|f_1(\cdot,\cdot)\|_{L^1(Q)}] \int_0^l \mid p_2^1(x,\sigma) - p_2^2(x,\sigma) \mid \mathrm{d}x \\
&\quad + \left[\overline{\beta}_1 M L_\mu + l L_\mu \|f_1(\cdot,\cdot)\|_{L^1(Q)} + M L_\beta \right] \int_0^t \mid c_{10}^1(\sigma) - c_{10}^2(\sigma) \mid \mathrm{d}\sigma \\
&\quad + \left[\overline{\beta}_1 M + \|f_1(\cdot,\cdot)\|_{L^1(Q)} \right] \int_0^t \int_0^l \mid u_1^1(\varphi_1(\sigma;t,x),\sigma) - u_1^2(\varphi_1(\sigma;t,x),\sigma) \mid \mathrm{d}x\mathrm{d}\sigma.
\end{aligned} \tag{3.9}
$$

Similarly, we can see that

$$
\begin{aligned}
&\int_0^l \mid p_i^1(x,t) - p_i^2(x,t) \mid \mathrm{d}x \\
\leq& \overline{\beta}_i \exp\left\{\overline{\lambda}_{2i-2} M_0 T\right\} \int_0^t \int_0^l \mid p_i^1(x,\sigma) - p_i^2(x,\sigma) \mid \mathrm{d}x\mathrm{d}\sigma \\
&+ \exp\left\{\overline{\lambda}_{2i-2} M_0 T\right\} \left[\overline{\beta}_i M L_\mu + l L_\mu \|f_i(\cdot,\cdot)\|_{L^1(Q)} + M L_\beta\right] \int_0^t \mid c_{i0}^1(\sigma) - c_{i0}^2(\sigma) \mid \mathrm{d}\sigma \\
&+ \overline{\lambda}_i \exp\left\{\overline{\lambda}_{2i-2} M_0 T\right\} [MT + lT \|f_i(\cdot,\cdot)\|_{L^1(Q)}] \int_0^l \mid p_{i+1}^1(x,\sigma) - p_{i+1}^2(x,\sigma) \mid \mathrm{d}x \\
&+ \exp\left\{\overline{\lambda}_{2i-2} M_0 T\right\} \left[\overline{\beta}_i M + \|f_i(\cdot,\cdot)\|_{L^1(Q)}\right] \int_0^t \int_0^l \mid u_i^1(\varphi_i(\sigma;t,x),\sigma) - u_i^2(\varphi_i(\sigma;t,x),\sigma) \mid \mathrm{d}x\mathrm{d}\sigma, \\
& i = 2,3,\ldots,n-1.
\end{aligned} \tag{3.10}
$$

$$
\begin{aligned}
&\int_0^l \mid p_n^1(x,t) - p_n^2(x,t) \mid \mathrm{d}x \\
\leq& \overline{\beta}_n \exp\left\{\overline{\lambda}_{2n-2} M_0 T\right\} \int_0^t \int_0^l \mid p_n^1(x,\sigma) - p_n^2(x,\sigma) \mid \mathrm{d}x\mathrm{d}\sigma \\
&+ \exp\left\{\overline{\lambda}_{2n-2} M_0 T\right\} \left[\overline{\beta}_n M L_\mu + l L_\mu \|f_n(\cdot,\cdot)\|_{L^1(Q)} + M L_\beta\right] \int_0^t \mid c_{n0}^1(\sigma) - c_{n0}^2(\sigma) \mid \mathrm{d}\sigma \\
&+ \exp\left\{\overline{\lambda}_{2n-2} M_0 T\right\} \left[\overline{\beta}_n M + \|f_n(\cdot,\cdot)\|_{L^1(Q)}\right] \int_0^t \int_0^l \mid u_n^1(\varphi_n(\sigma;t,x),\sigma) - u_n^2(\varphi_n(\sigma;t,x),\sigma) \mid \mathrm{d}x\mathrm{d}\sigma.
\end{aligned} \tag{3.11}
$$

From (3.2) and (3.3), we obtain

$$
\mid c_{i0}^1(t) - c_{i0}^2(t) \mid \leq k_1 \int_0^t \mid c_e^1(s) - c_e^2(s) \mid \mathrm{d}s. \tag{3.12}
$$

$$
\begin{aligned}
\mid c_e^1(t) - c_e^2(t) \mid \leq& (k_2 + g_2 + k_2 h_1 T + k_2 g_2 M T) \sum_{i=1}^n \int_0^l \int_0^t \mid p_i^1(x,s) - p_i^2(x,s) \mid \mathrm{d}s\mathrm{d}x \\
&+ g_2 M \sum_{i=1}^n \int_0^t \mid c_{i0}^1(s) - c_{i0}^2(s) \mid \mathrm{d}s + \int_0^t \mid v^1(s) - v^2(s) \mid \mathrm{d}s.
\end{aligned} \tag{3.13}
$$

Finally, from (3.9)–(3.13), we can deduce (3.7), where $K_1(T)$ is a constant depending on the bounds of parameters in (2.3). In addition, combined with (3.9)–(3.13), there exists $K_2(T)$, which enables us to obtain (3.8) of the standard norm in L^1 space.

4. Optimality conditions

In this section, we employ tangent-normal cone techniques in nonlinear functional analysis to deduce the necessary conditions for the optimal control pair.

Theorem 4.1 If (u^*, v^*) is an optimal control pair and (p^*, c_0^*, c_e^*) is the corresponding optimal state, then

$$u_i^*(x,t) = \mathcal{F}_i\left(\frac{[w_i(x,t) - \xi_i(x,t)]p_i^*(x,t)}{c_i}\right), \tag{4.1}$$

$$v^*(t) = \mathcal{F}_{n+1}\left(\frac{\xi_{2n+1}(t)}{c_{n+1}}\right), \tag{4.2}$$

in which $\mathcal{F}_j$ are given by

$$\mathcal{F}_j(\eta) = \begin{cases} 0, & \eta < 0, \\ \eta, & 0 \le \eta \le N_j, \quad j = 1,2,\ldots,n+1, \\ N_j, & \eta > N_j, \end{cases} \tag{4.3}$$

and $(\xi_1, \xi_2, \ldots, \xi_{2n+1})$ is the solution of the following adjoint system corresponding to (u^*, v^*):

$$\begin{cases} \frac{\partial \xi_1}{\partial t} + V_1\frac{\partial \xi_1}{\partial x} = [\mu_1(x, c_{10}^*(t)) + \lambda_1 P_2^*(t) + u_1^*]\xi_1 + [k_2c_e^*(t) - g_2c_{10}^*(t)]\xi_{2n+1} - \xi_1(0,t)\beta_1(x, c_{10}^*(t)) \\ \qquad - \int_0^l (\lambda_2 p_2^* \xi_2)(x,t)\mathrm{d}x + w_1u_1^*, \\ \frac{\partial \xi_i}{\partial t} + V_i\frac{\partial \xi_i}{\partial x} = [\mu_i(x, c_{i0}^*(t)) - \lambda_{2i-2}P_{i-1}^*(t) + \lambda_{2i-1}P_{i+1}^*(t) + u_i^*]\xi_i + [k_2c_e^*(t) - g_2c_{i0}^*(t)]\xi_{2n+1} \\ \qquad - \xi_i(0,t)\beta_i(x, c_{i0}^*(t)) + \int_0^l (\lambda_{2i-3}p_{i-1}^*\xi_{i-1} - \lambda_{2i}p_{i+1}^*\xi_{i+1})(x,t)\mathrm{d}x + w_iu_i^*, \\ \qquad i = 2,3,\ldots n-1, \\ \frac{\partial \xi_n}{\partial t} + V_n\frac{\partial \xi_n}{\partial x} = [\mu_n(x, c_{n0}(t)) - \lambda_{2n-2}P_{n-1}^*(t) + u_n^*]\xi_n + [k_2c_e^*(t) - g_2c_{n0}^*(t)]\xi_{2n+1} \\ \qquad - \xi_n(0,t)\beta_n(x, c_{n0}^*(t)) + \int_0^l (\lambda_{2n-3}p_{n-1}^*\xi_{n-1})(x,t)\mathrm{d}x + w_nu_n^*, \\ \frac{\mathrm{d}\xi_{n+i}}{\mathrm{d}t} = \int_0^l \frac{\partial \mu_i(x, c_{i0}^*(t))}{\partial c_{i0}} p_i^*\xi_i \mathrm{d}t + (g_1 + m)\xi_{n+i} - g_2P_i^*(t)\xi_{2n+1} - \xi_i(0,t)\int_0^l \frac{\partial \beta_i(x, c_{i0}^*(t))}{\partial c_{i0}} p_i^* \mathrm{d}x, \\ \frac{\mathrm{d}\xi_{2n+1}}{\mathrm{d}t} = -k_1\sum_{i=1}^n \xi_{i+n} + \left[k_2\sum_{i=1}^n P_i^*(t) + h_1\right]\xi_{2n+1}, \\ \xi_i(l,t) = 0, \xi_i(x,t) = \xi_i(x,t+T), \\ \xi_j(T) = 0, \quad j = n+1, n+2, \ldots, 2n+1. \end{cases} \tag{4.4}$$

Proof. The existence of a unique, bounded solution to the system (4.4) can be treated in the same manner as that for (2.3). For any given $(v_1, v_2) \in \mathcal{T}_\Omega(u^*, v^*)$ (the tangent cone of Ω at (u^*, v^*)), $v_1 = (v_{11}, v_{21}, \ldots, v_{n1})$, such that $(u^* + \varepsilon v_1, v^* + \varepsilon v_2) \in \Omega$, where $\varepsilon > 0$ is small enough. Then, from $J(u^* + \varepsilon v_1, v^* + \varepsilon v_2) \le J(u^*, v^*)$, we derive

$$\begin{aligned} &\sum_{i=1}^n \int_0^T \int_0^l w_i(u_i^* + \varepsilon v_{i1})p_i^\varepsilon \mathrm{d}x\mathrm{d}t - \frac{1}{2}\sum_{i=1}^n \int_0^T \int_0^l c_i(u_i^* + \varepsilon v_{i1})^2 \mathrm{d}x\mathrm{d}t - \frac{1}{2}\int_0^T c_{n+1}(v^* + \varepsilon v_2)^2 \mathrm{d}t \\ \le &\sum_{i=1}^n \int_0^T \int_0^l w_iu_i^*p_i^* \mathrm{d}x\mathrm{d}t - \frac{1}{2}\sum_{i=1}^n \int_0^T \int_0^l c_iu_i^{*^2} \mathrm{d}x\mathrm{d}t - \frac{1}{2}\int_0^T c_{n+1}v^{*^2} \mathrm{d}t, \end{aligned}$$

that is

$$\sum_{i=1}^n \int_0^T \int_0^l w_iu_i^* \frac{p_i^\varepsilon - p_i^*}{\varepsilon} \mathrm{d}x\mathrm{d}t + \sum_{i=1}^n \int_0^T \int_0^l w_iv_{i1}p_i^\varepsilon \mathrm{d}x\mathrm{d}t$$

$$\leq \sum_{i=1}^{n}\int_0^T\int_0^l c_i u_i^* v_{i1}\mathrm{d}x\mathrm{d}t + \int_0^T c_{n+1}v^* v_2\mathrm{d}t + \sum_{i=1}^{n}\int_0^T\int_0^l c_i\varepsilon v_{i1}^2\mathrm{d}x\mathrm{d}t + \int_0^T c_{n+1}\varepsilon v_2^2\mathrm{d}t,$$

as $\varepsilon \to 0^+$,

$$\sum_{i=1}^{n}\int_0^T\int_0^l w_i(u_i^* z_i + v_{i1}p_i^*)\mathrm{d}x\mathrm{d}t - \sum_{i=1}^{n}\int_0^T\int_0^l c_i u_i^* v_{i1}\mathrm{d}x\mathrm{d}t - \int_0^T c_{n+1}v^* v_2\mathrm{d}t \leq 0, \tag{4.5}$$

where

$$\frac{1}{\varepsilon}(p_i^\varepsilon - p_i^*) \to z_i, \frac{1}{\varepsilon}(c_{i0}^\varepsilon - c_{i0}^*) \to z_{i+n}, \frac{1}{\varepsilon}(c_e^\varepsilon - c_e^*) \to z_{2n+1}, \quad \text{as } \varepsilon \to 0.$$

By Theorem 3.2 we see that $z_1, z_2, \ldots, z_{2n+1}$ makes sense ([25], p.18). $(p^\varepsilon, c_0^\varepsilon, c_e^\varepsilon)$ is the state corresponding to $(u^* + \varepsilon v_1, v^* + \varepsilon v_2)$. It follows from (2.3) that $(z_1, z_2, \ldots, z_{2n+1})$ satisfies

$$\begin{cases} \frac{\partial z_1}{\partial t} + V_1\frac{\partial z_1}{\partial x} = -[\mu_1(x, c_{10}^*(t)) + \lambda_1 P_2^*(t) + V_{1x} + u_1^*]z_1 - \lambda_1 Z_2(t)p_1^* - \frac{\partial\mu_1(x,c_{10}^*(t))}{\partial c_{10}}p_1^* z_{n+1} - v_{11}p_1^*, \\ \frac{\partial z_i}{\partial t} + V_i\frac{\partial z_i}{\partial x} = -[\mu_i(x, c_{i0}(t)) - \lambda_{2i-2}P_{i-1}^*(t) + \lambda_{2i-1}P_{i+1}^*(t) + V_{ix} + u_i^*]z_i + \lambda_{2i-2}Z_{i-1}(t)p_i^* \\ \qquad - \lambda_{2i-1}(x,t)Z_{i+1}(t)p_i^* - \frac{\partial\mu_i(x,c_{i0}^*(t))}{\partial c_{i0}}p_i^* z_{n+i} - v_{i1}p_i^*, \quad i = 2, 3, \ldots n-1, \\ \frac{\partial z_n}{\partial t} + V_n\frac{\partial z_n}{\partial x} = -[\mu_n(x, c_{n0}(t)) - \lambda_{2n-2}P_{n-1}^*(t) + V_{nx} + u_n^*]z_n + \lambda_{2n-2}Z_{n-1}(t)p_n^* \\ \qquad - \frac{\partial\mu_n(x,c_{n0}^*(t))}{\partial c_{n0}}p_n^* z_{2n} - v_n p_n^*, \\ \frac{\mathrm{d}z_{n+i}}{\mathrm{d}t} = k_1 z_{2n+1} - g_1 z_{n+i} - m z_{n+i}, \\ \frac{\mathrm{d}z_{2n+1}}{\mathrm{d}t} = -k_2 c_e^*(t)\sum_{i=1}^{n} Z_i(t) + g_2\sum_{i=1}^{n}(c_{i0}^* Z_i(t) + z_{n+i}P_i^*(t)) - \left[k_2\sum_{i=1}^{n} P_i^*(t) + h_1\right]z_{2n+1} + v_2, \\ V_i(0,t)z_i(0,t) = \int_0^l \beta_i(x, c_{i0}^*(t))z_i\mathrm{d}x + \int_0^l \frac{\partial\beta_i(a,c_{i0}^*(t))}{\partial c_{i0}}p_i^* z_{n+i}\mathrm{d}x, \\ z_i(x,t) = z_i(x, t+T), \ z_{n+i}(0) = z_{2n+1}(0) = 0, \\ P_i^*(t) = \int_0^l p_i^*(x,t)\mathrm{d}x, \ Z_i(t) = \int_0^l z_i(x,t)\mathrm{d}x. \end{cases} \tag{4.6}$$

We multiply the first five equations in (4.6) by $\xi_i (i = 1, 2, \ldots, 2n+1)$ respectively, and integrate on Q and $(0, T)$. By using (4.4), we have

$$\sum_{i=1}^{n}\int_0^T\int_0^l w_i u_i^* z_i\mathrm{d}x\mathrm{d}t = -\sum_{i=1}^{n}\int_0^T\int_0^l v_{i1}\xi_i p_i^*\mathrm{d}x\mathrm{d}t + \int_0^T v_2\xi_{2n+1}\mathrm{d}t. \tag{4.7}$$

Substituting (4.7) and (4.5) gives

$$\sum_{i=1}^{n}\int_0^T\int_0^l [(w_i - \xi_i)p_i^* - c_i u_i^*]v_{i1}\mathrm{d}x\mathrm{d}t + \int_0^T(-c_{n+1}v^* + \xi_{2n+1})v_2\mathrm{d}t \leq 0,$$

for any $(v_1, v_2) \in \mathcal{T}_\Omega(u^*, v^*)$. Consequently, the structure of normal cone tells us that $[(w_i - \xi_i)p_i^* - c_i u_i^*, -c_{n+1}v^* + \xi_{2n+1}] \in \mathcal{N}_\Omega(u^*, v^*)$, (the normal cone of Ω at (u^*, v^*)), which gives the desired result.

Theorem 4.2 If T is small enough, then there is a constant K_3 such that

$$\begin{aligned} &\sum_{i=1}^{n}\|\xi_i^1 - \xi_i^2\|_{L^\infty(Q)} + \sum_{q=n+1}^{2n}\|\xi_q^1 - \xi_q^2\|_{L^\infty(0,T)} + \|\xi_{2n+1}^1 - \xi_{2n+1}^2\|_{L^\infty(0,T)} \\ \leq &K_3(T)T\left(\sum_{i=1}^{n}\|u_i^1 - u_i^2\|_{L^\infty(Q)} + \|v^1 - v^2\|_{L^\infty(0,T)}\right). \end{aligned} \tag{4.8}$$

The proof process of Theorem 4.2 is similar to that of Theorem 3.2, and is omitted here.

5. Existence of optimal control pair

In order to show that there exists a unique optimal control pair by means of the Ekeland variational principle, we embed the functional $\widetilde{J}(u,v)$ into $[L^1(Q)]^n \times L^1(0,T)$. We define

$$\widetilde{J}(u,v) = \begin{cases} J(u,v), & (u,v) \in \Omega, \\ -\infty, & \text{otherwise.} \end{cases}$$

Lemma 5.1([6], Lemma 4.1) $\widetilde{J}(u,v)$ is upper semi-continuous in $[L^1(Q)]^n \times L^1(0,T)$.

Theorem 5.1 If T is sufficiently small, there exists one and only one optimal control pair (u^*, v^*), which is in the feedback, and is determined by (4.1)–(4.4) and (2.3), where C_1 and C_2 are the supremum of $|p_i|$ and $|\xi_j|$, $i = 1,2,\ldots,n$, $j = 1,2,\ldots,2n+1$, respectively.

Proof. Define the mapping $\mathcal{L} : \Omega \to \Omega$ as follows:

$$\begin{aligned}\mathcal{L}(u,v) &= \mathcal{F}\left(\frac{(w_1-\xi_1)p_1}{c_1}, \frac{(w_2-\xi_2)p_2}{c_2}, \ldots, \frac{(w_n-\xi_n)p_n}{c_n}, \frac{\xi_{2n+1}}{c_{n+1}}\right) \\ &= \left(\mathcal{F}_1\left(\frac{(w_1-\xi_1)p_1}{c_1}\right), \mathcal{F}_2\left(\frac{(w_2-\xi_1)p_2}{c_2}\right), \ldots, \mathcal{F}_n\left(\frac{(w_n-\xi_n)p_n}{c_n}\right), \mathcal{F}_{n+1}\left(\frac{\xi_{2n+1}}{c_{n+1}}\right)\right),\end{aligned}$$

where (p, c_0, c_e) and $(\xi_1, \xi_2, \ldots, \xi_{2n+1})$ are the state and adjoint state, respectively, corresponding to the control (u,v). We show that $\mathcal{L}$ admits a unique fixed point, which maximizes the functional $\widetilde{J}$.

From Lemma 5.1 and the Ekeland variational principle ([26], p.180), for any given $\varepsilon > 0$, there exists $(u^\varepsilon, v^\varepsilon) \in \Omega \subset [L^1(Q)]^n \times L^1(0,T)$ such that

$$\widetilde{J}(u^\varepsilon, v^\varepsilon) \geq \sup_{(u,v)\in\Omega} \widetilde{J}(u,v) - \varepsilon, \tag{5.1}$$

$$\widetilde{J}(u^\varepsilon, v^\varepsilon) \geq \sup_{(u,v)\in\Omega} \left\{\widetilde{J}(u,v) - \sqrt{\varepsilon}\left(\sum_{i=1}^n \|u_i^\varepsilon - u_i\|_{L^1(Q)} + \|v^\varepsilon - v\|_{L^1(0,T)}\right)\right\}, \tag{5.2}$$

Thus, the perturbed functional

$$\widetilde{J}_\varepsilon(u,v) = \widetilde{J}(u,v) - \sqrt{\varepsilon}\left(\sum_{i=1}^n \|u_i^\varepsilon - u_i\|_{L^1(Q)} + \|v^\varepsilon - v\|_{L^1(0,T)}\right),$$

attains its supremum at $(u^\varepsilon, v^\varepsilon)$. Then, we argue as in Theorem 4.1:

$$\begin{aligned}(u^\varepsilon, v^\varepsilon) = \mathcal{L}(u^\varepsilon, v^\varepsilon) = &\left(\mathcal{F}_1\left(\frac{(w_1-\xi_1^\varepsilon)p_1^\varepsilon + \sqrt{\varepsilon}\theta_1^\varepsilon}{c_1}\right), \mathcal{F}_2\left(\frac{(w_2-\xi_2^\varepsilon)p_2^\varepsilon + \sqrt{\varepsilon}\theta_2^\varepsilon}{c_2}\right), \ldots,\right. \\ &\left.\mathcal{F}_n\left(\frac{(w_n-\xi_n^\varepsilon)p_n^\varepsilon + \sqrt{\varepsilon}\theta_n^\varepsilon}{c_n}\right), \mathcal{F}_{n+1}\left(\frac{\xi_{2n+1}^\varepsilon + \sqrt{\varepsilon}\theta_{n+1}^\varepsilon}{c_{n+1}}\right)\right),\end{aligned} \tag{5.3}$$

where $(p^\varepsilon, c_0^\varepsilon, c_e^\varepsilon)$ and $(\xi_1^\varepsilon, \xi_2^\varepsilon, \ldots, \xi_{2n+1}^\varepsilon)$ are the state and adjoint state, respcetively, corresponding to the control $(u^\varepsilon, v^\varepsilon)$, $\theta_1^\varepsilon, \theta_2^\varepsilon, \ldots, \theta_n^\varepsilon \in L^\infty(Q)$, and $\theta_{n+1}^\varepsilon \in L^\infty(0,T)$ with $|\ \theta_i^\varepsilon\ | \leq 1, i = 1,2,\ldots,n+1$.

First, we show that $\mathcal{L}$ has only one fixed point. Let (p^j, c_0^j, c_e^j) and $(\xi_1^j, \xi_2^j, \ldots, \xi_{2n+1}^j)$ be the state and adjoint state corresponding to the control (u^j, v^j), $j = 1, 2$. By (3.7) and (4.8), we have

$$
\begin{aligned}
&\|\mathcal{L}(u^1, v^1) - \mathcal{L}(u^2, v^2)\|_\infty \\
=&\sum_{i=1}^{n} \left\| \mathcal{F}_i\left(\frac{(w_i - \xi_i^1)p_i^1}{c_i}\right) - \mathcal{F}_i\left(\frac{(w_i - \xi_i^2)p_i^2}{c_i}\right) \right\|_{L^\infty(Q)} + \left\| \mathcal{F}_{n+1}\left(\frac{\xi_{2n+1}^1}{c_{n+1}}\right) - \mathcal{F}_{n+1}\left(\frac{\xi_{2n+1}^1}{c_{n+1}}\right) \right\|_{L^\infty(0,T)} \\
\le&\sum_{i=1}^{n} \left\| \frac{(w_i - \xi_i^1)p_i^1}{c_i} - \frac{(w_i - \xi_i^1)p_i^1}{c_i} \right\|_{L^\infty(Q)} + \left\| \frac{\xi_{2n+1}^1}{c_{n+1}} - \frac{\xi_{2n+1}^2}{c_{n+1}} \right\|_{L^\infty(0,T)} \\
\le&\sum_{i=1}^{n} \left\| \frac{w_i(p_i^1 - p_i^2)}{c_i} + \frac{|\, \xi_i^1 \,|\,(p_i^1 - p_i^2)}{c_i} + \frac{|\, p_i^2 \,|\,(\xi_i^1 - \xi_i^2)}{c_i} \right\|_{L^\infty(Q)} + \left\| \frac{\xi_{2n+1}^1 - \xi_{2n+1}^2}{c_{n+1}} \right\|_{L^\infty(0,T)} \\
\le& T\left(\sum_{i=1}^{n} \frac{1}{c_i}(\overline{w}_i K_1 + C_2 K_1 + C_1 K_3) + K_3\right) \cdot \left(\sum_{i=1}^{n} \|u_i^1 - u_i^2\|_{L^\infty(Q)} + \|v^1 - v^2\|_{L^\infty(0,T)}\right).
\end{aligned}
$$

Clearly, $\mathcal{L}$ is a contraction if T is sufficiently small. Hence, $\mathcal{L}$ has a unique fixed point (u^*, v^*).

Next, we prove $(u^\varepsilon, v^\varepsilon) \to (u^*, v^*)$ as $\varepsilon \to 0^+$. The relations (4.1), (4.2), and (5.3) lead to

$$
\begin{aligned}
&\|\mathcal{L}(u^\varepsilon, v^\varepsilon) - (u^\varepsilon, v^\varepsilon\|_\infty \\
=&\left\| \left(\mathcal{F}_1\left(\frac{(w_1 - \xi_1^\varepsilon)p_1^\varepsilon}{c_1}\right), \mathcal{F}_2\left(\frac{(w_2 - \xi_2^\varepsilon)p_2^\varepsilon}{c_2}\right), \ldots, \mathcal{F}_n\left(\frac{(w_n - \xi_n^\varepsilon)p_n^\varepsilon}{c_n}\right), \mathcal{F}_{n+1}\left(\frac{\xi_{2n+1}^\varepsilon}{c_{n+1}}\right) \right) \right. \\
&- \left(\mathcal{F}_1\left(\frac{(w_1 - \xi_1^\varepsilon)p_1^\varepsilon}{c_1} + \frac{\sqrt{\varepsilon}\theta_1^\varepsilon}{c_1}\right), \mathcal{F}_2\left(\frac{(w_2 - \xi_2^\varepsilon)p_2^\varepsilon}{c_2} + \frac{\sqrt{\varepsilon}\theta_2^\varepsilon}{c_2}\right), \ldots, \right. \\
&\left.\left. \mathcal{F}_n\left(\frac{(w_n - \xi_n^\varepsilon)p_n^\varepsilon}{c_n} + \frac{\sqrt{\varepsilon}\theta_n^\varepsilon}{c_n}\right),\ \mathcal{F}_{n+1}\left(\frac{\xi_{2n+1}^\varepsilon}{c_{n+1}} + \frac{\sqrt{\varepsilon}\theta_{n+1}^\varepsilon}{c_{n+1}}\right) \right) \right\|_\infty \\
\le&\sum_{i=1}^{n} \left\| \left(\mathcal{F}_i\left(\frac{(w_i - \xi_i^\varepsilon)p_i^\varepsilon}{c_i}\right) - \mathcal{F}_i\left(\frac{(w_i - \xi_i^\varepsilon)p_i^\varepsilon}{c_i} + \frac{\sqrt{\varepsilon}\theta_i^\varepsilon}{c_i}\right) \right) \right\|_{L^\infty(Q)} \\
&+ \left\| \mathcal{F}_{n+1}\left(\frac{\xi_{2n+1}^\varepsilon}{c_{n+1}}\right) - \mathcal{F}_{n+1}\left(\frac{\xi_{2n+1}^\varepsilon}{c_{n+1}} + \frac{\sqrt{\varepsilon}\theta_{n+1}^\varepsilon}{c_{n+1}}\right) \right\|_{L^\infty(0,T)} \\
=&\sum_{i=1}^{n} \left\| \frac{(w_i - \xi_i^\varepsilon)p_i^\varepsilon}{c_i} - \frac{(w_i - \xi_i^\varepsilon)p_i^\varepsilon}{c_i} - \frac{\sqrt{\varepsilon}\theta_i^\varepsilon}{c_i} \right\|_{L^\infty(Q)} + \left\| \frac{\xi_{2n+1}^\varepsilon}{c_{n+1}} - \frac{\xi_{2n+1}^\varepsilon}{c_{n+1}} - \frac{\sqrt{\varepsilon}\theta_{n+1}^\varepsilon}{c_{n+1}} \right\|_{L^\infty(0,T)} \\
\le& \sqrt{\varepsilon} \sum_{i=1}^{n} \frac{\|\theta_i^\varepsilon(x,t)\|_{L^\infty(Q)}}{c_i} + \sqrt{\varepsilon} \frac{\|\theta_{n+1}^\varepsilon(t)\|_{L^\infty(0,T)}}{c_{n+1}} \\
\le& \sqrt{\varepsilon} \sum_{i=1}^{n+1} \frac{1}{c_i},
\end{aligned}
$$

it is easy to derive that

$$
\begin{aligned}
&\|(u^*, v^*) - (v^\varepsilon, v^\varepsilon)\|_\infty \\
\le&\|\mathcal{L}((u^*, v^*)) - \mathcal{L}(u^\varepsilon, v^\varepsilon)\|_\infty + \|\mathcal{L}(u^\varepsilon, v^\varepsilon) - (u^\varepsilon, v^\varepsilon)\|_\infty \\
\le&T\left(\sum_{i=1}^{n}\frac{1}{c_i}(\overline{w}_iK_1 + C_2K_1 + C_1K_3) + K_3\right)\cdot\left(\sum_{i=1}^{n}\|u_i^* - u_i^\varepsilon\|_{L^\infty(Q)} + \|v^* - v^\varepsilon\|_{L^\infty(0,T)}\right) + \sqrt{\varepsilon}\sum_{i=1}^{n+1}\frac{1}{c_i}.
\end{aligned}
$$

So, if T is small enough, the following result holds:

$$
\sum_{i=1}^{n}\|u_i^* - u_i^\varepsilon\|_{L^\infty(Q)} + \|v^* - v^\varepsilon\|_{L^\infty(0,T)} \le \frac{\sqrt{\varepsilon}\sum\limits_{i=1}^{n+1}\frac{1}{c_i}}{1 - T\left(\sum\limits_{i=1}^{n}\frac{1}{c_i}(\overline{w}_iK_1 + C_2K_1 + C_1K_3) + K_3\right)}.
$$

which gives the desired result.

Finally, passing to the limit $\varepsilon \to 0^+$ in the inequality of (5.2) and using Lemma 5.1 yield $\widetilde{J}(u^*, v^*) \ge \limsup_{(u,v)\in\Omega}\widetilde{J}(u, v)$, which finishes the proof.

6. Numerical approximation

6.1. Difference scheme

Our goal is to obtain a numerical approximation for the nonnegative T-periodic solution of the system (2.3). We discuss the problem of evolution of a single species in a polluted environment. Suppose the computational domain $\tilde{Q} = [0, l] \times [0, \tilde{T}]$ is divided into an $J \times N$ mesh with the spatial step size $h = \frac{l}{J} = 0.01$ in the x direction and time step size $\tau = \frac{\tilde{T}}{N} = 0.03$. The grid points (x_j, t_n) are defined by

$$
\begin{aligned}
x_j &= jh, \quad j = 0, 1, 2, \ldots, J; \\
t_n &= n\tau, \quad n = 0, 1, 2, \ldots, N,
\end{aligned}
$$

where J and N are two integers. The notation p_j^n denotes the solution $p(jh, n\tau)$ of the difference equation.

Based on the above analysis, the finite difference scheme of system (2.3) can be written as follows:

$$
\frac{p_j^n - p_j^{n-1}}{\tau} + V\frac{p_j^n - p_{j-1}^n}{h} + V_xp_j^n + \mu p_j^n - f_j^n = 0, \tag{6.1}
$$

where $j = 1, 2, \ldots, J; n = 1, 2, \ldots, N$. It follows from (6.1) that

$$
-\lambda Vp_{j-1}^n + [1 + \lambda V + \tau(V_x + \mu)]p_j^n = p_j^{n-1} + \tau f_j^n, \tag{6.2}
$$

where $\lambda = \frac{\tau}{h}$.

The boundary and initial conditions can be discretized as

$$
\begin{cases}
p_j^0 = p_{0j}, \\
p_0^n = \sum\limits_{j=1}^{J}\beta_jp_jh.
\end{cases} \tag{6.3}
$$

From (6.2) and (6.3), we have

$$\mathbf{A}\mathbf{P}^n = \mathbf{P}^{n-1} + \tau\mathbf{F}, \tag{6.4}$$

where

$$\mathbf{A} = \begin{bmatrix} 1+\lambda V+\tau(V_x+\mu)-\lambda V\beta h & -\lambda V\beta h & \dots & -\lambda V\beta h & -\lambda V\beta h \\ -\lambda V & 1+\lambda V+\tau(V_x+\mu) & & & \\ & \ddots & \ddots & \ddots & \\ & & & -\lambda V & 1+\lambda V+\tau(V_x+\mu) \\ & & & & -\lambda V \end{bmatrix},$$

$$\mathbf{P}^n = (p_1^n, p_2^n, \dots, p_J^n)^{\mathrm{T}}, \ \ \mathbf{F} = (f_1^n, f_2^n, \dots, f_J^n)^{\mathrm{T}}.$$

Note that $\mathbf{A}$ is an upper triangular matrix, so the nonlinear algebraic equation (6.4) have solutions.

6.2. Stability and convergence

We first prove the stability of the implicit difference scheme (6.1) by using the Fourier method. We suppose that $f \equiv 0$. The domain of the function defined on the grid point is extended according to the usual method, that is, when $x \in (x_j - \frac{h}{2}, x_j + \frac{h}{2})$, $p^n(x) = p_j^n$; then, we have

$$-\lambda V p^n(x-h) + [1+\lambda V+\tau(V_x+\mu)]p^n(x) = p^{n-1}(x), x \in R. \tag{6.5}$$

Taking the Fourier transform of both sides of (6.5), we obtain

$$-\lambda V \hat{U}^n(k)e^{-\mathrm{i}kh} + [1+\lambda V+\tau(V_x+\mu)]\hat{U}^n(k) = \hat{U}^{n-1}(k). \tag{6.6}$$

By (6.6), we get

$$\hat{U}^n(k) = \frac{\hat{U}^{n-1}(k)}{-\lambda V e^{-\mathrm{i}kh} + 1 + \lambda V + \tau(V_x+\mu)}.$$

Thus, the growth factor is as follows:

$$\begin{aligned} G(\tau,k) &= \frac{1}{-\lambda V e^{-\mathrm{i}kh} + 1 + \lambda V + \tau(V_x+\mu)} \\ &= \frac{1}{1+\lambda V(1-\cos kh)+\tau(V_x+\mu)+\lambda V \mathrm{i}\sin kh}. \end{aligned}$$

Then, we have

$$\begin{aligned} |G(\tau,k)|^2 &= \frac{1}{[1+\lambda V(1-\cos kh)+\tau(V_x+\mu)]^2+\lambda^2V^2\sin^2 kh} \\ &= \frac{1}{[1+2\lambda V\sin^2\frac{kh}{2}+\tau(V_x+\mu)]^2+4\lambda^2V^2\sin^2\frac{kh}{2}(1-\sin^2\frac{kh}{2})}. \end{aligned}$$

If $V_x + \mu > 0$, there is $|G(\tau,k)| \le 1$, that is, the von Neumann condition is satisfied, the difference scheme (6.1) is stable under condition $V_x + \mu > 0$.

Next, we analyze the convergence of the difference scheme (6.1). Note that the difference scheme is compatible, we thus use the Lax equivalence theorem. The difference scheme must be convergent and have first-order accuracy.

6.3. Numerical test

We chose the following parameters:

$$\begin{cases} \beta(x, c_0(t)) = 100x^2(1-x)(1+\sin(\pi x))\left|\sin\frac{2\pi c_0(t)}{T}\right|, \\ \mu(x, c_0(t)) = e^{-4x}(1-x)^{-1.4}(2+\cos\frac{2\pi c_0(t)}{T}), \\ V(t,x) = 1-x,\ f(x,t) = 2+(1+x)\sin(\frac{2\pi t}{T}), \\ p_0(x) = e^x,\ u(x,t) = 0,\ x = 1,\ T = \frac{1}{3},\ \tilde{T} = 3. \end{cases}$$

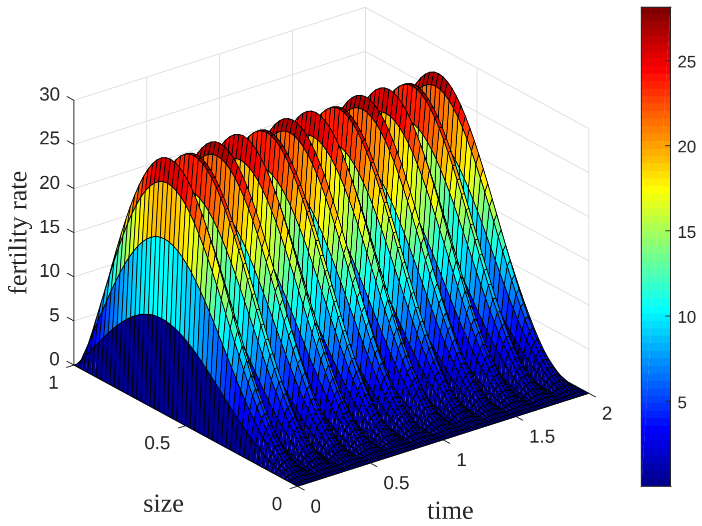

Figure 1. Fertility rate of the population.

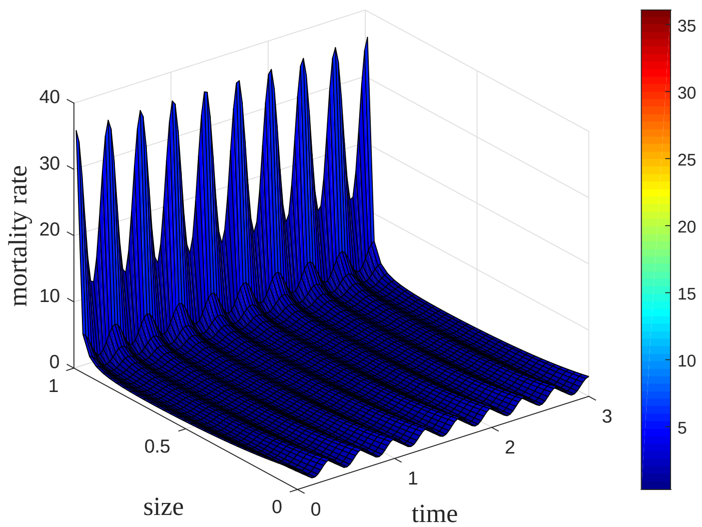

Figure 2. Mortality rate of the population.

In this paper, we used the backward difference scheme and chasing method, and (6.4) was solved through programming. The fertility rate, mortality rate, and immigration rate were T-periodic and were all greater than zero, which is consistent with the assumptions. We considered $T = \frac{1}{3}$. Their graphs are given in Figures 1–3, respectively. The fertility rate was the highest when the size was half and the mortality rate was the highest when the size was the maximum, which conformed to the empirical situation. Therefore, the selection of parameters β, μ, and f was reasonable.

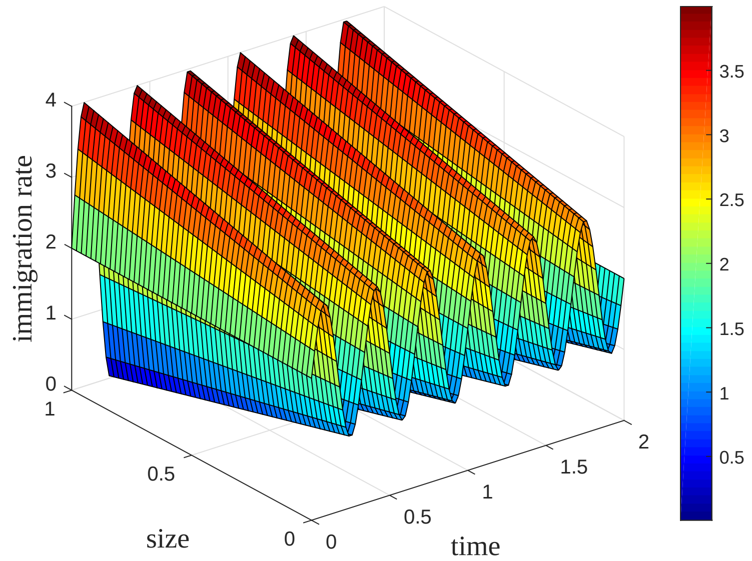

Figure 3. Immigration rate of the population.

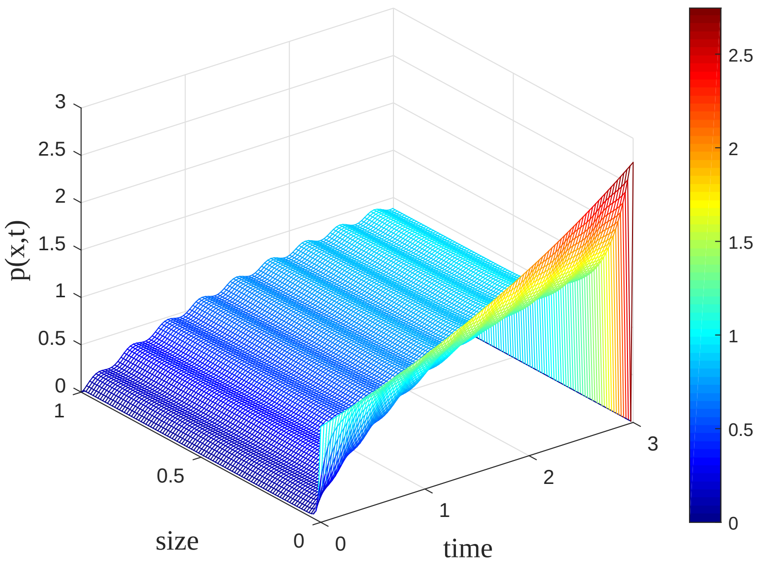

Figure 4. Numerical solution of the system.

The graph of the numerical solution p is given in Figure 4. Over time, solution p showed T-periodic changes. We take the numerical solution of system (2.3), corresponding to an arbitrary positive initial datum p_0, on some interval $[kT, (k+1)T]$, where k is large enough. We can then get the periodic solution of system (2.3) by extending the numerical solution p. Over time, the density of the population increased first and then decreased. With respect to size, the density of the population gradually decreased, and finally tended to be stable. The maximum value was reached at $x = 0$. On such an interval with a sufficiently large k, the numerical solution was already stable. During computation we found that any positive initial datum p_0 was appropriate for use.

7. Discussion and conclusions

In this paper, we considered the problem of optimal harvesting for a periodic n-dimensional food chain model dependent on the size structure, that combined toxicants with the population. In the foregoing, we have discussed the existence and uniqueness of a nonnegative solution of the state system. The necessary optimality conditions were provided, and the existence of the optimal policy was investigated. Some numerical results were finally given. The results implied that the solution of (2.3) always maintains the pattern of increasing periodically, and any positive initial datum p_0 is appropriate. From Theorem 4.1, the optimal strategy had a bang-bang structure and provided threshold conditions for the optimal control problems (2.3) and (2.4). The bang-bang structure of solutions is much more common in optimal population management.

We now comment on the differences between the methods and results of this paper as well as closely related work. The existence of the optimal strategy was proved by compactness and the extreme sequence in [13]. In this paper, the corresponding problem was treated by using Ekeland variational principle. The authors [10] established the maximum principle and bang-bang structure for optimal control but paid no attention to existence and uniqueness results. The existence of the optimal harvesting rate in the harvesting problem was only given in [11]. Furthermore, if $V_i(x,t) = 1$ for $Q = (0,l) \times R_+, i = 1, 2 \ldots, n$, the state system degenerates into an age-structured model, and our results cover the corresponding results [4–6].

Note that the individual price factor $w_i(x,t)$ plays an important role in the structure of the optimal controller (4.1). However, as we do not have a clear biological meaning for the solutions $\xi_i (i = 1, 2, \ldots, 2n+1)$ of the adjoint system (4.4), it is difficult to give a precise explanation of the threshold conditions (4.1) and (4.2). In specific applications, the optimal population density and optimal policy are calculated by combining the state system and the adjoint system. In this way, we enable the objective functional (2.4) to maximize the total profit during period T. This is a challenging problem, and future work in the area should address it. In addition, inspired by [17, 18], we intend to consider the problem of fractional optimal control problem of the population model in future research.

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Conflict of interest

None of the authors has a conflict of interest in the publication of this paper.

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