Research article

Stationary patterns in bistable reaction-diffusion cellular automata

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Abstract: In this paper, we study stationary patterns of bistable reaction-diffusion cellular automata, i.e., models with discrete time, space and state. We show the rich variability based on the interplay of the capacity and viability and the specific form of reaction functions. While stationary k-periodic patterns occur naturally in many situations in large (exponential) numbers, there exist extreme situations for which there are no heterogeneous patterns. Moreover, nonmonotone dependence of the number of stationary patterns on the diffusion parameter is shown to be natural in the fully discrete setting.

Keywords: patterns; reaction-diffusion; cellular automata; discrete dynamics; difference equations

1. Introduction

In this paper, we study the rich structure of stationary solutions for a class of bistable reaction-diffusion cellular automata

$$u(t + 1) = F(u(t)), \quad t \in \mathbb{N}_0, \ u(t) \in S^{\mathbb{Z}},$$

with the state space $S \subseteq \mathbb{N}_0$ and the mapping $F : S^{\mathbb{Z}} \to S^{\mathbb{Z}}$. Such a model can be seen as a nonlinear discrete dynamical system with discrete space, time and state set.

Classical reaction-diffusion equations

$$u_t(x, t) = d u_{xx}(x, t) + \lambda f(u(x, t)), \quad x \in \mathbb{R}, t \in [0, \infty), d, \lambda > 0 \tag{1.1}$$

represent an important class of nonlinear partial differential equations which describes the evolution of numerous natural quantities (population densities, chemical concentrations, temperatures, etc.). A large number of phenomena share common features which combine a local dynamics (via the reaction function f) and a spatial dynamics (via the diffusion), see, e.g., [1].

It has been shown that the difference among the behaviour of continuous, semi-discrete and discrete reaction-diffusion dynamical systems is very significant especially with respect to the existence of

stationary patterns and properties of travelling waves. Spatially discrete models exhibit the so-called topological chaos [2] which, roughly speaking, implies the existence of a large number of stationary patterns in regimes with small diffusion. Their existence consequently prevents travelling fronts to propagate, a phenomenon which is commonly referred to as pinning.

In order to properly motivate our model, we sum up first the existing literature and key properties for bistable reaction-diffusion models from this perspective.

Bistable partial differential equations. When we consider f to be a bistable function in (1.1) we get a bistable reaction-diffusion equation (PDE). In a special case of the cubic nonlinearity $f(u) = u(1-u)(u-a)$ with $a \in (0, 1)$ we get the famous Nagumo equation. These equations serve as prototypes for important dynamical concepts like travelling waves, [3]. From our point of view, stationary patterns are rather rare and exist only in the special case of standing waves when

$$\int_0^1 f(u)\mathrm{d}u = 0, \tag{1.2}$$

i.e., $a = \frac{1}{2}$ for the cubic nonlinearity.

Bistable lattice differential equations. There is, however, a stark contrast once the corresponding model with discrete space is considered. The bistable lattice (or semidiscrete) differential equation (LDE)

$$\dot{u}_i(t) = d[u_{j-1}(t) - 2u_i(t) + u_{j+1}(t)] + \lambda f(u_i(t)), \quad i \in \mathbb{Z}, \ t \in [0, \infty), d, \lambda > 0, \tag{1.3}$$

can be seen as a spatial discretization of the bistable PDE but arises naturally in the modelling of numerous phenomena in discrete media. Mathematically, LDE (1.3) yields a huge number of stationary solutions if $d > 0$ is sufficiently small, [4, 5]. Their existence ensures, among other things, that the travelling waves do not travel once the diffusion parameter is small enough for all bistable f, even if (1.2) is violated. This phenomenon is known as pinning, [6]. Such a behaviour is typical and has been shown for discrete-space dynamical systems, [7, 8]. Among other spatially heterogeneous stationary patterns, k-periodic patterns play special role. They can be counted, there exist 3^k stationary patterns with period k, put into equivalence classes based on numerous symmetries and these classes can be partially ordered, [9, 10].

Bistable coupled map lattices. Going one step further, time discretization of LDE (1.3) provides the reaction diffusion partial difference equation or a coupled map lattice (CML)

$$u_{i,j+1} = u_{i,j} + d[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}] + \lambda f(u_{i,j}), \quad i \in \mathbb{Z}, \ j \in \mathbb{N}_0, \ d, \lambda > 0. \tag{1.4}$$

While the properties of the travelling waves become even more difficult to analyse, it is quite natural, that the rich structure of stationary solutions is independent of the fact whether discrete or continuous time is considered, see [2].

Table 1. Spatial, temporal and state structures in dynamical systems mentioned in Section 1.

	space X	time $\mathbb{T}$	state $\mathcal{S}$
partial differential eqns. (PDEs)	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$
lattice differential eqns. (LDEs)	$\mathbb{Z}$	$\mathbb{R}$	$\mathbb{R}$
partial difference eqns. (CMLs)	$\mathbb{Z}$	$\mathbb{N}_0$	$\mathbb{R}$
cellular automata	$\mathbb{Z}$	$\mathbb{N}_0$	$\mathbb{N}_0$ (finite)

Bistable cellular automata. Cellular automata are discrete dynamical systems with discrete time, space and even state sets, see Table 1. Spatial points x (typically $x \in \mathbb{Z}$) are called cells and are updated simultaneously in discrete time (typically $t \in \mathbb{N}_0$) and cells can typically attain values only from a finite state set $\mathcal{S}$. The bistable CAs could thus be seen as variants of (1.4) with a finite alphabet. Models based on cellular automata are now omnipresent in most research fields, [11]. At the same time, cellular automata represent a considerable challenge for mathematical analysis, complete characterization is usually restricted to special classes, [12]. This is nicely illustrated on the case of reaction-diffusion cellular automata. Quite naturally, most attention is devoted to the study of Turing-like patterns and the computational simplicity and elegance enable wide range of possible phenomena, [13]. There are particular results of a bistable reaction-diffusion cellular automaton with a three letter alphabet $\mathcal{S} = \{-1, 0, 1\}$ and visually impressive system of stationary solutions, the so-called mosaics, [14].

In this paper, we devote our attention to simpler models, bistable reaction-diffusion cellular automata. At the same time, we provide their analytical characterizations and compare them to the above mentioned standard models in the form of partial differential equations (PDEs), lattice differential equation (LDEs) or coupled map lattices (CMLs).

Paper structure. This paper is structured as follows. In Section 2, we construct in detail our class of reaction-diffusion automata. We define the diffusion automata, outline a class of bistable reaction functions and combine these to construct the bistable reaction-diffusion automata with interval invariance. In Section 3, we study stationary 2-periodic patterns, provide a necessary and sufficient condition for their existence and compute the maximum possible number of stationary 2-periodic patterns. In Section 4, we extend these ideas and prove a sufficient condition for the existence of stationary k-periodic patterns. Finally, Section 5 provides a lower bound for the number of stationary k-periodic patterns and we compare this estimate to the number of periodic patterns of bistable LDEs or CMLs. We conclude with a short summary in Section 6.

2. Preliminaries

In this section, we define bistable reaction-diffusion cellular automata in several steps. First, we focus on discrete-state diffusion, next we introduce discrete-state bistable reactions and then we construct reaction-diffusion systems which preserve interval invariance.

Discrete-state diffusion. In order to define a symmetric discrete-state diffusion with diffusion parameter $\delta \in \mathbb{N}$, we consider an auxiliary function $h_\delta : (\mathbb{N}_0)^2 \to \mathbb{N}_0$ defined by

$$h_\delta(m, n) := \begin{cases} \delta, & m - n \geq 2\delta, \\ -\delta, & m - n \leq -2\delta, \\ \delta - 1 & m - n \in \{2(\delta - 1), 2\delta - 1\}, \\ -(\delta - 1) & n - m \in \{2(\delta - 1), 2\delta - 1\}, \\ \vdots \\ 1, & m - n \in \{2, 3\}, \\ -1, & n - m \leq \{2, 3\}, \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

Consequently, we define the discrete-state diffusion $d_\delta : (\mathbb{N}_0)^3 \to \mathbb{N}_0$ by

$$d_\delta(m, n, p) = n + \underbrace{h_\delta(m, n)}_{\text{left-side dynamics}} + \underbrace{h_\delta(p, n)}_{\text{right-side dynamics}}. \tag{2.2}$$

A basic diffusion automaton is then defined

$$D(u) = \begin{pmatrix} \vdots \\ d_\delta(u_{x-2}, u_{x-1}, u_x) \\ d_\delta(u_{x-1}, u_x, u_{x+1}) \\ d_\delta(u_x, u_{x+1}, u_{x+2}) \\ \vdots \end{pmatrix}, \quad u \in \mathbb{N}_0^{\mathbb{Z}}, \tag{2.3}$$

Naturally, this is not a unique approach. While the naïve definition (2.2) is clearly symmetric and behaves similarly to the standard continuous diffusion, it leads to some counterintuitive situations. To illustrate, note that the definition (2.1) ensures that the neighbouring states with lower or equal values do not change if they differ by one. As the simplest example, let us mention

$$u = (\dots, 0, 0, 1, 0, 0, \dots).$$

While this seems perfectly natural, such a definition yields other, unnatural, stationary states, e.g., the following stairwise configuration

$$u = (\dots, 0, 0, 1, 2, 3, 4, \dots).$$

The exact approach to discrete-state diffusion is not unified and may depend on a specific application. In this paper, we use the minimalist approach via (2.1)–(2.2).

Discrete-state bistable reaction function. In order to define a discrete-state nonspatial bistable local dynamics $v(t + 1) = f(v(t))$, $v(t) \in \mathbb{N}_0$, we consider a capacity $K \in \mathbb{N}$, and a viability constant $a \in \mathbb{N}$ with $0 < a < K$.
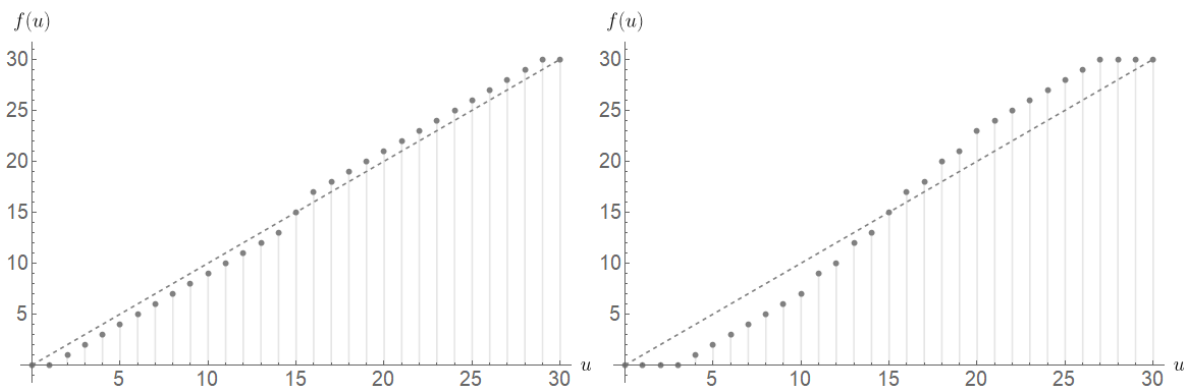
Figure 1. Two examples for bistable reaction functions with $a = 15$ and $K = 30$

Definition 2.1. A function $f : \mathbb{N}_0 \to \mathbb{N}_0$ is said to be a bistable reaction function if it is nondecreasing and there exist $a, K, 0 < a < K$ such that $f(0) = 0, f(a) = a, f(K) = K$, and

$$f(u) \in \begin{cases} [0, u)_{\mathbb{Z}} & \text{if } u \in (0, a)_{\mathbb{Z}}, \\ (u, K]_{\mathbb{Z}} & \text{if } u \in (a, K)_{\mathbb{Z}}, \\ [K, u)_{\mathbb{Z}} & \text{if } u \in (K, \infty)_{\mathbb{Z}}. \end{cases}$$

Two examples of bistable reaction functions are depicted in Figure 1.

Reaction-diffusion cellular automata. Putting together the spatial and local dynamics (diffusion and reaction), we construct bistable reaction-diffusion automata in the following way. Consider a time scale $\mathbb{T} = \mathbb{N}_0$, one-dimensional discrete lattice $X = \mathbb{Z}$, a state set $S \subseteq \mathbb{N}_0$ and an update rule $F : S^X \to S^X$ given by

$$F(u) = \begin{pmatrix} \vdots \\ d_\delta(f(u_{x-2}), f(u_{x-1}), f(u_x)) \\ d_\delta(f(u_{x-1}), f(u_x), f(u_{x+1})) \\ d_\delta(f(u_x), f(u_{x+1}), f(u_{x+2})) \\ \vdots \end{pmatrix}, \quad u \in S^{\mathbb{Z}}, \tag{2.4}$$

with $d_\delta : S^3 \to S$ given by (2.2) and $f : S \to S$ being a bistable map (see Def. 2.1). We then call the quadruple $C = (\mathbb{T}, X, S, F)$ the bistable reaction-diffusion cellular automaton (abbreviated by RDCA) and study the induced discrete dynamical system

$$u(t + 1) = F(u(t)), \quad t \in \mathbb{T}, \ u \in S^{\mathbb{Z}}. \tag{2.5}$$

We construct the reaction-diffusion automaton via the composition of the reaction and diffusion (as opposed to summing them up as in continuous-state models, e.g., (1.1)). Our motivation stems from the fact that such an approach ensures the interval invariance.

Lemma 2.2 (Interval invariance)**.** *Consider a bistable RDCA (2.5) $u, v \in S^{\mathbb{Z}}$ such that $F(u) = v$. If $u \in ([0, K]_{\mathbb{Z}})^{\mathbb{Z}}$, then $v \in ([0, K]_{\mathbb{Z}})^{\mathbb{Z}}$.*

Proof. Let $u \in ([0, K]_\mathbb{Z})^\mathbb{Z}$ and assume by contradiction that $v \notin ([0, K]_\mathbb{Z})^\mathbb{Z}$. Then there exists $x \in \mathbb{Z}$ such that $v_x \notin [0, K]_\mathbb{Z}$. Without loss of generality, let us assume that $v_x > K$. This implies that

$$d_\delta(f(u_{x-1}), f(u_x), f(u_{x+1})) > K.$$

We show that this is not possible. Indeed, we use (2.2) to get

$$d_\delta(f(u_{x-1}), f(u_x), f(u_{x+1})) = f(u_x) + h_\delta(f(u_{x-1}), f(u_x)) + h_\delta(f(u_{x+1}), f(u_x)).$$

If $u_{x-1} < u_x$ or $u_{x+1} < u_x$, we have $h_\delta(f(u_{x-1}), f(u_x)) \le 0$, or $h_\delta(f(u_{x+1}), f(u_x)) \le 0$, which simplifies the situation. In the worst case scenario, we have

$$
\begin{aligned}
d_\delta(f(u_{x-1}), f(u_x), f(u_{x+1})) &= f(u_x) + h_\delta(f(u_{x-1}), f(u_x)) + h_\delta(f(u_{x+1}), f(u_x)) \\
&\le f(u_x) + \frac{f(u_{x-1}) - f(u_x)}{2} + \frac{f(u_{x+1}) - f(u_x)}{2} \\
&= f(u_x) + \frac{f(u_{x-1}) + f(u_{x+1}) - 2f(u_x)}{2} \\
&= \frac{f(u_{x-1}) + f(u_{x+1})}{2} \\
&\le K,
\end{aligned}
$$

a contradiction. Other configurations lead to similar arguments. $\square$

Remark 2.3. Note that the seemingly more natural additive approach motivated by continuous state models like (1.1) leads to additive reaction-diffusion automata of the form

$$F_{\text{add}}(u) = \begin{pmatrix} \dots \\ d_\delta(u_{x-2}, u_{x-1}, u_x) - u_{x-1} + f(u_{x-1}) \\ d_\delta(u_{x-1}, u_x, u_{x+1}) - u_x + f(u_x) \\ d_\delta(u_x, u_{x+1}, u_{x+2}) - u_{x+1} + f(u_{x+1}) \\ \dots \end{pmatrix}, \quad u \in S^\mathbb{Z}. \tag{2.6}$$

We can easily show that such automata are quite naturally not interval invariant.

To illustrate, let us consider any bistable reaction function and a state $u \in ([0, K]_\mathbb{Z})^\mathbb{Z}$ with $u_0 = K - 2 > a$ and $u_{\pm 1} = K$. Then $F_{\text{add}}(u) \notin ([0, K]_\mathbb{Z})^\mathbb{Z}$ since Def. 2.1 implies that

$$f(K - 2) > K - 2$$

and consequently,

$$
\begin{aligned}
[F_{\text{add}}(u)]_0 &= d_\delta(u_{-1}, u_0, u_1) - u_0 + f(u_0) \\
&= d_\delta(K, K - 2, K) - (K - 2) + f(K - 2) \\
&= (K - 2) + 2h_\delta(K, K - 2) - (K - 2) + f(K - 2) \\
&= 2 + f(K - 2) \\
&\ge 2 + K - 1 = K + 1.
\end{aligned}
$$

We refer to [15] for interesting insights about composite and additive reaction-diffusion systems.

3. Stationary 2-periodic patterns

Consider a bistable RDCA (2.5). We call a pattern $u^* \in S^{\mathbb{Z}}$ stationary if $F(u) = u$. In other words, for all $x \in \mathbb{Z}$ the equality

$$d_\delta(f(u^*_{x-1}), f(u^*_x), f(u^*_{x+1})) = u^*_x \tag{3.1}$$

is satisfied.

We say that a stationary pattern $u^* \in S^{\mathbb{Z}}$ is homogeneous if u^* is constant. Otherwise, u^* is said to be heterogeneous.

Moreover, a stationary pattern $u^* \in S^{\mathbb{Z}}$ is k-periodic (with $k \in \mathbb{N}$), if $u^*_{x+k} = u^*_x$ holds for all $x \in \mathbb{Z}$.

Lemma 3.1 (A priori estimate). *Consider a bistable RDCA (2.5) with a, K such that $1 < a < K - 1$ and a heterogeneous stationary pattern $u^* \in S^{\mathbb{Z}}$. Then there exist $x_1, x_2 \in \mathbb{Z}$, such that $u^*_{x_1} < a$ and $u^*_{x_2} > a$.*

Proof. Without loss of generality, let us assume by contradiction that $u^*_x \geq a$ for all $x \in \mathbb{Z}$. Then there exists $x_0 \in \mathbb{Z}$ such that

$$u^*_{x_0} = \min_{x \in \mathbb{Z}} u^*_x.$$

Let us assume first that $u^*_{x_0} > a$. The definition of h_δ in (2.1) then implies that

$$h_\delta(f(u^*_{x_0-1}), f(u^*_{x_0})) \geq 0, \text{ and } h_\delta(f(u^*_{x_0+1}), f(u^*_{x_0})) \geq 0.$$

Applying (2.2) we obtain

$$\begin{aligned}
d(f(u^*_{x_0-1}), f(u^*_{x_0}), f(u^*_{x_0+1})) &= f(u^*_{x_0}) + h_\delta(f(u^*_{x_0-1}), f(u^*_{x_0})) + h_\delta(f(u^*_{x_0+1}), f(u^*_{x_0})) \\
&\geq f(u^*_{x_0}) + 0 + 0 > u^*_{x_0},
\end{aligned}$$

a contradiction.

If $u^*_{x_0} = a$, the fact that the pattern is heterogeneous implies that we can choose x_0 such that $u^*_{x_0-1} > a$. Consequently, $f(u^*_{x_0-1}) \geq a + 2$ and we have

$$\begin{aligned}
d(f(u^*_{x_0-1}), f(u^*_{x_0}), f(u^*_{x_0+1})) &= f(a) + h_\delta(f(u^*_{x_0-1}), f(u^*_{x_0})) + h_\delta(f(u^*_{x_0+1}), f(u^*_{x_0})) \\
&\geq f(a) + 1 + 0 = a + 1 > a = u^*_{x_0},
\end{aligned}$$

a contradiction. $\square$

Remark 3.2. Note that the seemingly counterintuitive assumption $1 < a < K - 1$ is essential in this setting with finite alphabet $\mathcal{S}$. Indeed, let us assume that $a = 1$, $\delta \in \mathbb{N}$. Then the bistable RDCA (2.5) has periodic stationary configurations

$$(\ldots, 0, 1, 0, 1, 0, \ldots), \ (\ldots, 0, 1, 1, 0, 1, 1, 0, \ldots), \ \ldots$$

Similarly, if $a = K - 1$, $\delta \in \mathbb{N}$, then we can construct periodic stationary configurations

$$(\ldots, K, K - 1, K, K - 1, K, \ldots), \ (\ldots, K, K - 1, K - 1, K, K - 1, K - 1, K, \ldots), \ \ldots$$

Our key auxilliary statement is the characterization of stationary 2-periodic solutions.

Lemma 3.3 (Necessary and sufficient condition for the existence of stationary 2-periodic solutions)**.**
Consider a bistable RDCA (2.5) *with $\delta \in \mathbb{N}$ and $1 < a < K - 1$. There exists a 2-periodic stationary
pattern $u^* = (\dots, p, q, p, q, p, q, \dots)$ if and only if $p \in (a, K)_{\mathbb{Z}}$, $q \in (0, a)_{\mathbb{Z}}$ satisfy*

$$h_\delta(f(p), f(q)) = \frac{f(p) - p}{2} = \frac{q - f(q)}{2} = \delta. \tag{3.2}$$

Proof. Let us assume first that $u^* = (\dots, p, q, p, q, p, q, \dots)$ is a 2-periodic stationary pattern. Based on
Lemma 3.1 we have $p > a$, $q < a$. Then $F(u^*) = u^*$ with Eq. (2.2) imply

$$q = d_\delta(f(p), f(q), f(p)) = f(q) + 2h_\delta(f(p), f(q)),$$

which implies $\frac{q - f(q)}{2} = h_\delta(f(p), f(q))$. Similarly

$$p = d_\delta(f(q), f(p), f(q)) = f(p) + 2h_\delta(f(q), f(p))$$

yields $h_\delta(f(p), f(q)) = -h_\delta(f(q), f(p)) = \frac{f(p) - p}{2}$.

Let us now prove the last equality in Eq. (3.2), i.e., $h_\delta(f(p), f(q)) = \delta$. Since $h_\delta \leq \delta$, see (2.1), we
assume by contradiction that $h_\delta(f(p), f(q)) = \hat{\delta} < \delta$. This implies that

$$h_\delta(f(p), f(q)) = \frac{f(p) - p}{2} = \frac{q - f(q)}{2} = \hat{\delta} \tag{3.3}$$

and consequently

$$f(p) = p + 2\hat{\delta}, \text{ and } q = f(q) + 2\hat{\delta}.$$

Hence, we can estimate

$$f(p) - f(q) = p + 2\hat{\delta} - q + 2\hat{\delta} = p - q + 4\hat{\delta} \geq 2(2\hat{\delta} + 1).$$

Applying the definition of h_δ in (2.1) yields

$$h_\delta(f(p), f(q)) > \hat{\delta},$$

a contradiction with (3.3).

The reverse implication of the statement is straightforward since (3.2) implies

$$d_\delta(f(p), f(q), f(p)) = f(q) + 2h_\delta(f(p), f(q)) = f(q) + 2\frac{q - f(q)}{2} = q,$$

and similarly

$$d_\delta(f(q), f(p), f(q)) = f(p) + 2h_\delta(f(q), f(p)) = f(p) - 2\frac{f(p) - p}{2} = p.$$

$\square$

The condition (3.2) immediately implies that there could be parameter regimes in which there are
no stationary 2-periodic solutions.

Example 3.4. Let a, K be such that $0 < a < K$ and let us consider the bistable reaction function defined by

$$f(u) = \begin{cases} u + 1 \text{ if } u \in (a, K)_{\mathbb{Z}}, \\ u - 1 \text{ if } u \in (0, a)_{\mathbb{Z}}. \end{cases} \tag{3.4}$$

Then clearly

$$f(u) - u = \begin{cases} 1 & \text{if } u \in (a, K)_{\mathbb{Z}} \\ -1 & \text{if } u \in (0, a)_{\mathbb{Z}}, \end{cases}$$

and the condition (3.2) is always violated. Consequently, the bistable RDCA (2.5) with f given by (3.4) has no 2-periodic stationary patterns. See the left panel of Figure 1 for an example of a function f given by (3.4).

Let $k, a, K \in \mathbb{N}$ such that $0 < a < K$. We denote by

$$N_{a,K}(k, \delta) \in \mathbb{N}_0$$

the largest possible number of stationary heterogeneous k-periodic patterns in the set of all bistable RDCAs (2.5) with $a, K, \delta \in \mathbb{N}$.

Theorem 3.5 (Maximum number of heterogeneous stationary 2-periodic patterns)**.** *Let $\delta, a, K \in \mathbb{N}$ with $a < K$ and consider a bistable RDCA (2.5). Then*

$$N_{a,K}(2, \delta) = \begin{cases} 2(a - 2\delta)(K - a - 2\delta) & \text{if } 2\delta \leq a \leq K - 2\delta \\ 0 & \text{if } a < 2\delta \text{ or } a > K - 2\delta. \end{cases} \tag{3.5}$$

Proof. Let us consider 2-periodic stationary patterns $u^* = (\ldots, p, q, p, q, p, q, \ldots)$ with $p \in (a, K)_{\mathbb{Z}}$, $q \in (0, a)_{\mathbb{Z}}$. The necessary and sufficient condition (3.2) from Lemma 3.2 implies that

$$f(p) = p + 2\delta, \tag{3.6}$$
$$f(q) = q - 2\delta. \tag{3.7}$$

If $2\delta \leq a \leq K - 2\delta$ holds then the equality (3.7) may be satisfied for all $q \in [2\delta, a - 1]_{\mathbb{Z}}$. i.e., in $a - 2\delta$ cases. Similarly, the equality (3.6) may hold for any $p \in [a + 1, K - 2\delta]$, i.e., in $K - a - 2\delta$ cases. Realizing that we can always shift the pattern so that p is either at an odd or even position finishes the first part of (3.5).

In order to prove the latter part of (3.5), we realize that if $a < 2\delta$ then (3.7) cannot be satisfied. Similarly, if $a > K - 2\delta$ then (3.6) is always violated. □

While the maximum number $N_{a,K}(2, \delta)$ decreases quadratically in δ, the following example shows that the number of stationary patterns can be, quite naturally, non-monotone in bistable RDCAs (2.5). In LDEs (1.3) this can also happen but only for small parameter regions, see [16], and the phenomenon is closely connected to intricate bifurcations of stationary patterns.

Example 3.6. Consider a bistable RDCA (2.5) with $a = 15$, $K = 30$, $p \in (a, K)_{\mathbb{Z}}$, $q \in (0, a)_{\mathbb{Z}}$ and the following bistable reaction f (illustrated in Figure 2):

m	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
$f(m)$	0	0	1	1	1	1	1	1	1	1	1	1	2	3	4	15	26	27	28	29	29	29	29	29	29	29	29	29	30	30

We can see that Eq. (3.2) is satisfied
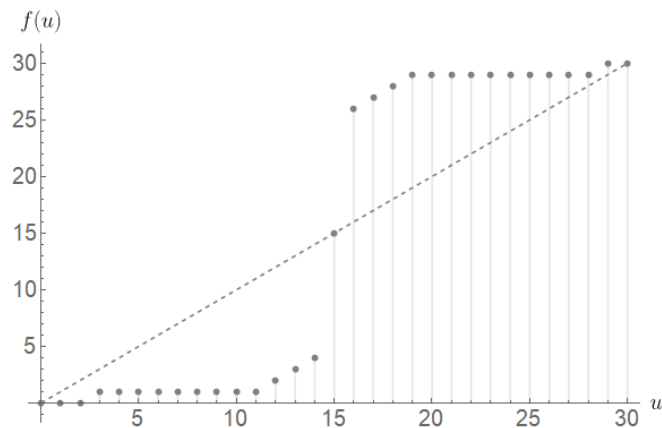
Figure 2. Bistable reaction function from Example 3.6

1. for $\delta = 1$ for two pairs $\{p, q\} \in \{\{27, 2\}, \{27, 3\}\}$,
2. for $\delta = 2$ for one pair $\{p, q\} \in \{\{25, 5\}\}$,
3. for $\delta = 3$ for one pair $\{p, q\} \in \{\{23, 7\}\}$,
4. for $\delta = 4$ for one pair $\{p, q\} \in \{\{21, 9\}\}$ and
5. for $\delta = 5$ for 16 pairs, as both $f(q) = q - 10$ and $f(p) = p + 10$ hold for 4 values each which results in 16 different pairs,
6. for $\delta \geq 6$ for no pair, as both $f(q) = q - 2\delta$ and $f(p) = p + 2\delta$ are not satisfied for any p, q.

In conclusion, we get to the following results for $N_{a,K}(2, \delta)$ (using Eq. (3.5)) and for the actual number of stationary 2-periodic patterns for our specific function f (denoted by $n_f(2, \delta)$ in Table 2). Note again that any pattern can be shifted so that p is either in odd x's or even ones.

Table 2. Maximum number $N_{a,K}(2, \delta)$ of 2-periodic stationary patterns of bistable RDCA (2.5) with $\delta = 1$ and the actual number of 2-periodic stationary patterns for the function f from Ex. 3.6, see Figure 2.

δ	1	2	3	4	5	6	7	8	...
$N_{a,K}(2, \delta)$	338	242	162	98	50	18	2	0	...
$n_f(2, \delta)$	4	2	2	2	32	0	0	0	...

4. Stationary k-periodic patterns

In this section, we extend the ideas from the previous sections to patterns with arbitrary period $k \in \mathbb{N}$.

Theorem 4.1 (Sufficient condition for the existence of k-periodic stationary patterns). *Consider a bistable RDCA (2.5). Let $\delta \in \mathbb{N}$, $p \in (a, K)_{\mathbb{Z}}$ and $q \in (0, a)_{\mathbb{Z}}$ be such that*

$$h_\delta(f(p), f(q)) = h_\delta(f(p), a) = h_\delta(a, f(q)) = \frac{f(p) - p}{2} = \frac{q - f(q)}{2}, \tag{4.1}$$

then there exist k-periodic stationary patterns for all $k \in \mathbb{N}$.

Proof. The existence of 2-periodic patterns follows directly from Lemma 3.3.

Let us next show that (4.1) ensures the existence of a 3-periodic solution. We construct a solution in the form

$$u^* = (\ldots, p, a, q, p, a, q, p, a, q, \ldots).$$

Its stationarity follows by direct computations

$$d_\delta(f(p), f(a), f(q)) = a + h_\delta(f(p), a) + h_\delta(f(q), a) = a,$$

$$d_\delta(f(a), f(q), f(p)) = f(q) + h_\delta(a, f(q)) + h_\delta(f(p), f(q)) = f(q) + 2\frac{q - f(q)}{2} = q,$$

$$d_\delta(f(q), f(p), f(a)) = f(p) + h_\delta(f(q), f(p)) + h_\delta(a, f(p)) = f(p) - 2\frac{f(p) - p}{2} = p.$$

Let $k \in \mathbb{N}$ satisfy $k > 3$. Then there exist $m_2, m_3 \in \mathbb{N}_0$ such that $k = 2m_2 + 3m_3$. We define a vector v as a combination of m_2 blocks $b_2 = (p, q)$ and m_3 blocks $b_3 = (p, a, q)$ (in arbitrary order), e.g.,

$$v = (\underbrace{p, q, \ldots, p, q}_{m_2 \text{ times}}, \underbrace{p, a, q, \ldots, p, a, q}_{m_3 \text{ times}}).$$

Consequently, the sequence $u^* = (\ldots, v, v, v, \ldots)$, which arises as an infinite repetition of vectors v, is k-periodic and satisfies $F(u^*) = u^*$. □

Remark 4.2. The condition (4.1) is no longer necessary because there could be (for example) 3-stationary stationary solutions of the form

$$u^* = (\ldots, p_1, p_2, q_1, p_1, p_2, q_1, p_1, p_2, q_1, \ldots)$$

or

$$u^* = (\ldots, q_1, q_2, p_1, q_1, q_2, p_1, q_1, q_2, p_1, \ldots)$$

for some $q_1, q_2 \in (0, a)_\mathbb{Z}$ and $p_1, p_2 \in (a, K)_\mathbb{Z}$, and these solutions can exist even if the condition (4.1) is violated.

To illustrate, let us consider a reaction function such that there exists $p \in (a, K)_\mathbb{Z}$ such that $f(p) = p + 2$. Then the following 3-periodic configuration is stationary if $\delta = 1$

$$u^* = (\ldots, p, 1, 1, p, 1, 1, \ldots),$$

since $f(1) = 0$ and $h_1(p + 2, 0) = 1$.

Another counterexample is the 4-periodic pattern

$$u^* = (\ldots, K - 1, K - 1, 1, 1, K - 1, K - 1, 1, 1, \ldots), \tag{4.2}$$

which is stationary for any f and $\delta = 1$. In other words, there always exists a 4-periodic stationary pattern of (2.5) if $\delta = 1$. Comparing this fact with Ex. 3.4, we get that the number of k-stationary solutions of (2.5) need not be monotone in k. Indeed, if $\delta = 1$ and f in (3.4) is considered, there are no 2-periodic patterns but we have, e.g., the 4-periodic pattern (4.2).

Remark 4.3. The ideas from the proof of Thm. 4.1 can be modified to get a slightly more general result.

The assumption (4.1) suffices for the existence of primitive k-periodic stationary configurations, i.e., periodic configurations which are k-periodic but are not $\bar{k}$-periodic for any $\bar{k} < k$. Now observe that any $k \in \mathbb{N}$, $k \geq 5$ can be then obtained as $k = 2m_2 + 3m_3 + 4m_4$ for some $m_2, m_3, m_4 \in \mathbb{N}_0$. Consequently, we construct a primitive k-periodic stationary solution using m_2 blocks $b_2 = (p, q)$, m_3 blocks $b_3 = (p, a, q)$ and m_4 blocks $b_4 = (p, a, q, a)$

5. Lower bound on number of stationary k-periodic patterns for small diffusion

In this section, we consider $\delta = 1$ and study the lower bound of the number of stationary k-periodic patterns. Let us denote the number of stationary k-periodic patterns for $\delta = 1$ by $P_1(k)$. In contrast to Thm. 3.5, our goal is to estimate $P_1(k)$ from below, i.e., find a sequence $P_1^*(k)$ such that $P_1(k) \geq P_1^*(k)$.

Motivated by Thm. 4.1, we define sets

$$\mathcal{L} := \{i \in [2, a-1]_{\mathbb{Z}} : f(i) = i - 2\}, \quad \mathcal{U} := \{i \in [a+1, K-2]_{\mathbb{Z}} : f(i) = i + 2\}, \tag{5.1}$$

and their respective cardinalities

$$L := |\mathcal{L}|, \quad U := |\mathcal{U}|. \tag{5.2}$$

Straightforwardly, we get $0 \leq L \leq a - 2$ and $0 \leq U \leq K - a - 2$.

Theorem 5.1 (Lower bound for number of heterogeneous k-periodic stationary patterns). *Let $k \in \mathbb{N}$, $k > 1$, $\delta = 1$ and L, U be defined by Eq. (5.2). Then there exist at least*

$$P_1^*(k) = \begin{cases} \sum\limits_{i=0}^{\lfloor \frac{k}{4} \rfloor} 2\binom{k-2i}{2i} L^{k/2-i} U^{k/2-i} & \text{if } k \text{ is even,} \\ \sum\limits_{i=0}^{\lfloor \frac{k-1}{4} \rfloor} 2\binom{k-2i-1}{2i+1} L^{(k-1)/2-i} U^{(k-1)/2-i} & \text{if } k \text{ is odd,} \end{cases} \tag{5.3}$$

k-periodic stationary heterogeneous patterns of a bistable RDCA (2.5).

Proof. We consider heterogeneous patterns of the form

$$u^* = (\ldots, q_{-1}, [a], p_{-1}, [a], q_0, [a], p_0, [a], q_1, [a], p_1, \ldots), \quad q_i \in (1, a)_{\mathbb{Z}}, \ p_i \in (a, K-1)_{\mathbb{Z}}, i \in \mathbb{Z},$$

where $[a]$ indicates that the entry a between q_i and p_i can be both present as well as avoided.

Let us first focus on k even and patterns consisting of a periodically repeating block B_k^l with $l \in \mathbb{N}$ being the number of a's in B_k^l and $|B_k^l| = k$, i.e.,

$$u^* = (\ldots, B_k^l, B_k^l, B_k^l, \ldots).$$

In the simplest case, there are no a's in $B_k^l = B_k^0$, i.e., $l = 0$ and

$$B_k^0 = \{p_1, q_1, p_2, q_2, \ldots, p_{k/2}, q_{k/2}\},$$

with $p_j \in [2, a-1]_{\mathbb{Z}}$, $q_j \in [a+1, K-2]_{\mathbb{Z}}$ for $j \in \{1, 2, \ldots, k/2\}$.

For each j, there are L ways to choose p_j and U ways to choose q_j. Moreover, a block can start either with p_j or q_j. Therefore, we have

$$n_0 := 2L^{k/2}U^{k/2}$$

such blocks.

Furthermore, let us consider blocks $B_k^l = B_k^2$ with exactly 2 a's, i.e. $l = 2$ and

$$B_k^2 = \{[a], p_1, [a], q_1, [a], p_2, [a], q_2, [a], \ldots, p_{k/2-2}, [a], q_{k/2-2}\},$$

where $[a]$ symbolizes that no or one a can occur at this position in B_k^2.

The block B_k^2 consists of $k/2 - 1$ p_j's, $k/2 - 1$ q_j's and 2 a's which can be placed at $k - 2$ gaps. Therefore, there are at least

$$n_2 := 2\binom{k-2}{2}L^{k/2-1}U^{k/2-1}$$

such blocks.

Similarly, if we consider $B_k^l = B_k^{2m}$ with exactly $2m$ a's, identical reasoning yields the existence of at least

$$n_{2m} := 2\binom{k-2m}{2m}L^{k/2-m}U^{k/2-m}$$

such blocks. Naturally, these blocks exist only for $2m \leq k/2$, i.e. $m \leq k/4$.

Therefore, the total number of stationary k-periodic patterns for k even is bounded from below by

$$P_1(k_{\text{even}}) \geq P_1^*(k_{\text{even}}) = \sum_{i=0}^{\lfloor \frac{k}{4} \rfloor} n_{2i} = \sum_{i=0}^{\lfloor \frac{k}{4} \rfloor} 2\binom{k-2i}{2i}L^{k/2-i}U^{k/2-i}. \tag{5.4}$$

We can proceed similarly if $k > 2$ is odd. We construct blocks B_k^1 with exactly one a

$$B_k^1 = \{[a], p_1, [a], q_1, [a], p_2, [a], q_2, [a], \ldots, p_{(k-1)/2}, [a], q_{(k-1)/2}\}.$$

Repeating the same arguments we get the existence of

$$n_1 := 2(k-1)L^{(k-1)/2}U^{(k-1)/2}$$

blocks B_k^1.

In general, we consider $l = 2m + 1$ and obtain the existence of

$$n_{2m+1} := 2\binom{k-2m-1}{2m+1}L^{(k-1)/2-m}U^{(k-1)/2-m}$$

blocks B_k^{2m+1}.

Again, such a construction only makes sense if $2m \leq (k-1)/2$, i.e., $m \leq (k-1)/4$. Consequently,

$$P_1(k_{\text{odd}}) \geq P_1^*(k_{\text{odd}}) = \sum_{i=0}^{\lfloor \frac{k-1}{4} \rfloor} n_{2i+1} = \sum_{i=0}^{\lfloor \frac{k}{4} \rfloor} 2\binom{k-2i-1}{2i+1}L^{(k-1)/2-i}U^{(k-1)/2-i}. \tag{5.5}$$

Combining Eqns. (5.4) and (5.5) we get (5.3) which finishes the proof. □

The following remark illustrates the fact that (5.3) is indeed a mere lower bound of the number of k-periodic stationary patterns.

Remark 5.2. The following patterns are not included in the estimates from the proof of Thm. 5.1:

$$u^* = (\dots, K-1, K-1, q, K-1, K-1, q, \dots),$$
$$u^* = (\dots, 1, 1, p, 1, 1, p, \dots),$$
$$u^* = (\dots, 1, 1, K-1, K-1, 1, 1, K-1, K-1, \dots),$$
$$u^* = (\dots, 1, 1, a, K-1, K-1, 1, 1, a, K-1, K-1, \dots),$$
$$u^* = (\dots, 1, 1, a, K-1, K-1, a, 1, 1, a, K-1, K-1, \dots),$$
$$\vdots$$

where p and q are such that $f(p) = p+2$ and $f(q) = q-2$. Note that these patterns are not considered in Thm. 5.1 since 1 and $K-1$ are not in the sets $\mathcal{L}, \mathcal{U}$ defined by (5.1).

Remark 5.3. If $L = U$, the expression (5.3) can be rewritten as

$$P_1^*(k) = \begin{cases} \sum_{i=0}^{\lfloor \frac{k}{4} \rfloor} 2\binom{k-2i}{2i} L^{k-2i} & \text{if } k \text{ is even,} \\ \sum_{i=0}^{\lfloor \frac{k-1}{4} \rfloor} 2\binom{k-2i-1}{2i+1} L^{k-2i-1} & \text{if } k \text{ is odd.} \end{cases} \tag{5.6}$$

The following example evaluates the rather opaque lower bound (5.3) and compares it to the number of stationary periodic patterns for lattice differential and difference equations.

Example 5.4. Let us consider a bistable RDCA (2.5) with $\delta = 1$. Table 3 enumerates the lower bound $P_1^*(k)$ from (5.3) and the numbers are visualised in Figure 3.

Table 3. Lower estimate $P_1^*(k)$ (5.3) on the number of k-periodic stationary patterns of bistable RDCA (2.5) with $\delta = 1$. The last row lists the values 3^k which correspond to the number of stationary k-periodic patterns of the bistable lattice differential equation (1.3). See Figure 3 for visualization.

k		2	3	4	5	6	7	8	9	10
$P_1^*(k)$	$L = U = 1$	2	4	4	8	14	20	34	56	88
	$L = U = 2$	8	16	40	128	320	896	2464	6656	18304
	$L = 2, U = 3$	12	24	84	288	864	2880	9144	29376	94608
	$L = U = 3$	18	36	180	648	2430	9396	35154	134136	507384
	$\vdots$	$\vdots$								
3^k		9	27	81	243	729	2187	6561	19683	59049

Apparently, the expression for the lower bound (5.3) ensures the exponential growth in k. Moreover, we estimate from Table 3 that $P_1^*(k)$ grows asymptotically faster than 3^k, a value implying the number of k-periodic stationary patterns of the bistable LDE (1.3), see [4, 9].
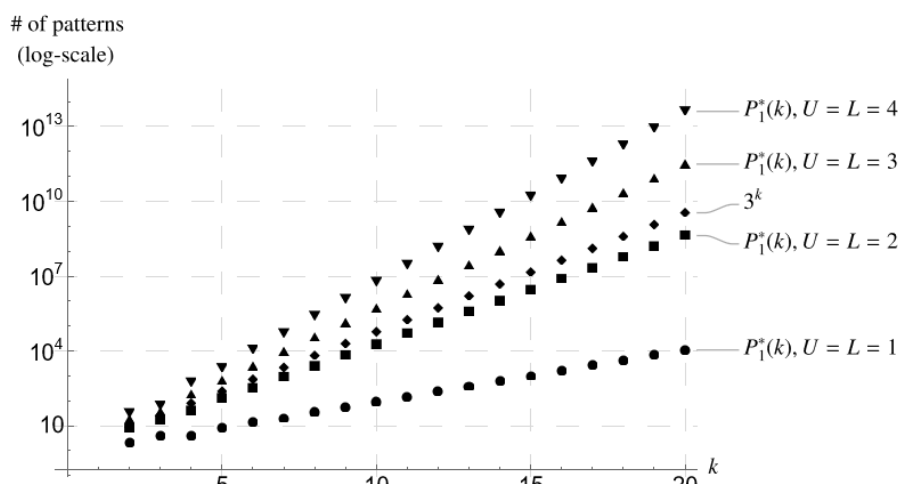
Figure 3. Lower bound on the number $P_1^*(k)$ of stationary k-periodic patterns of the bistable RDCA (2.5) given by (5.3) compared to 3^k stationary k-periodic patterns of LDE (1.3)

6. Final remarks

In this paper, we have analyzed stationary patterns of bistable reaction-diffusion automata. We have revealed some interesting properties. Let us highlight few of them. The number of k-periodic stationary patterns need not be monotone in the diffusion strength δ (see Ex. 3.6) and stationary 2-periodic solutions need not exist. Similarly, if we fix the diffusion parameter δ the number of k-periodic stationary patterns need not be increasing in k (see Rem. 4.2). On the other hand, Thm. 5.1 shows that the number of k-periodic stationary solutions can be higher than 3^k which represents the number of k-periodic stationary solutions of LDE (1.3) and CML (1.4).

More importantly, our analysis shows that even if the number of stationary solutions can be either much higher or lower, their nature is different. Whereas the stationary solutions of LDE (1.3) and CML (1.4) can be represented via infinite double sequences of words with 0, 1-alphabet, the stationary solutions of the bistable RDCA (2.5) are built via blocks. See the proof of Thm. 5.1 for more details.

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Conflict of interest

The authors declare there is no conflict of interest.

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