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*Research article*

# Resource dependent scheduling with truncated learning effects

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**Abstract:** In this article, we investigate the single-machine scheduling problem with truncated learning effect and resource allocation, where the actual processing time of a job is a general function of its additional resources and position in a sequence. The goal is to determine the optimal resource allocation and optimal sequence such that a weighted sum of scheduling cost and resource consumption cost is minimized. We show that the problem can be solved in $O(n^3)$ time by using an assignment formulation, where $n$ is the number of jobs.

**Keywords:** scheduling; resource allocation; learning effect; single-machine

## 1. Introduction

Planning and scheduling are important problems in industrial engineering, logistics and supply chain management (see Wu et al. [1], Zhang et al. [2], Polverini et al. [3], Yang et al. [4]). In the traditional scheduling models, the processing time of jobs is a fixed constant, but in many real productions, the processing time of jobs often decreases with the repetition of certain jobs, this phenomenon is called learning effects (see the survey Azzouz et al. [5]). Recently, Pei et al. [6] examined the single-machine and parallel-machine serial batching scheduling with a learning effect and time-dependent setup time. The objective functions is to minimize maximum earliness and total number of tardy jobs, respectively. For the single-machine scheduling, they proved that the maximum earliness and number of tardy jobs minimizations can be solved in polynomial time. For the parallel-machine scheduling, they proposed some algorithms to solve the problems. Qian et al. [7] addressed single-machine release times scheduling with a learning effect. For the weighted sum of makespan, total completion time and maximum tardiness, they proposed several heuristic algorithms and a branch-and-bound algorithm to solve the problem. Wang et al. [8] investigated single-machine release dates scheduling with a position-weighted learning effect. For the total completion time minimization, they proposed the heuristic and branch-and-bound algorithms. Wang et al. [9] considered flow shop scheduling with truncated learning effects. For the makespan and total weighted completion time minimizations, they proposed some heuristics and a branch-and-bound algorithm. Liang et al. [10] studied flow shop scheduling with sum-of-logarithm-processing-times-based learning effects. For the total (weighted) completion time, the makespan, and the sum of the quadratic job completion times minimizations, they proposed several heuristic algorithms to solve the problems and the worst-case bounds of heuristic algorithms were analyzed. Sun et al. [11] addressed single-machine scheduling with general position weighted learning effects. For the total weighted completion time minimization, they proposed two heuristics and their worst-case bounds were analysed. They also proposed some complex heuristics and a branch-and-bound algorithm.

Another, increasing attention has been paid to scheduling problems with controllable processing times (resource allocation) (see Shabtay and Steiner [12], Wei et al. [13], Wang and Wang [14], Mashayekhy et al. [15], Liang et al. [16] and Liu and Xiong [17]) and learning effects (see Lu et al. [18]). Recently, Zhu et al. [19] studied single-machine scheduling with learning effects and resource allocation. Under past-sequence-dependent setup times and general resource allocation, they showed that a regular objective function minimization can be solved in polynomial time. Pei et al. [20] investigated scheduling with learning effects and resource allocation on a single-machine. For the makespan minimization under the serial-batching production, they proposed a hybrid GSA-TS algorithm. Liu and Jiang [21] addressed single machine scheduling problems with learning effects and resource allocation. For the common and slack due-date assignments with position-dependent weights, they gave some results. Geng et al. [22] and Liu and Jiang [23] considered flow shop scheduling with learning effects and resource allocation. Under due date assignments, they proved that some two machine no-wait flow shop problems can be solved in polynomial time. Wang et al. [24] considered single machine scheduling with learning effects and resource allocation. Under a convex resource allocation function, they provided a bicriteria analysis for the total weighted flow time cost and the total resource consumption cost. Lv and Wang [25] considered flow shop scheduling problem with learning effect and resource allocation. Under different due-window assignment and two machine no-wait setting, they provided a bicriteria analysis for the scheduling cost and the resource consumption cost. They demonstrated that four versions about these both costs remain polynomially solvable. In this article, we study scheduling problems in a single-machine environment with truncated learning effects and resource allocation. Under the job processing times function is a general resource consumption function, we provide a unified approach for a large scheduling objective functions. We show that all these problems can be solved in polynomial time.

The remaining of this paper is as follows: Section 2 gives the description of the problem. In Section 3, we give some positional weights results. In Section 4, the optimal properties and the optimal solution algorithms are proposed. A special case is given in Section 5. Section 6 concludes the paper.

## 2. Problem description

A set $J = \{J_1, J_2, ..., J_{\breve{n}}\}$ of $\breve{n}$ independent jobs are processed on a single machine, and all the jobs are available at time 0 and not allowed to be preempted. Suppose that job $J_h$ is scheduled in $r$th position, the actual processing time is

$$P_h^A(u_h) = P_h(u_h)\max\{r^{\beta_h}, \delta\}, \tag{1}$$

where $\beta_h \leq 0$ is the learning rate of job $J_h$, $0 < \delta \leq 1$ is a truncation parameter, function $P_h(u_h)$ satisfies $P'_h(u_h) \leq 0$, $P''_h(u_h) \geq 0$, $P'_h(u_h) = \frac{dP_h(u_h)}{du_h}$ is the first derivative of $u_h$, $P''_h(u_h) = \frac{d^2P_h(u_h)}{d(u_h)^2}$ is the second derivative of $u_h$, $u_h$ is the amount of resource allocated to job $J_h$ such that $u_h^{min} \leq u_h \leq u_h^{max}$, $u_h^{min}$ and $u_h^{max}$ are the lower and upper bound of the resource allocation $u_h$. Note that the linear resource allocation $P_h^A(u_h) = a_h - b_h u_h$ and convex resource allocation $P_h^A = a_h + \left(\frac{w_h}{u_h}\right)^\theta$ are special cases of Eq (1), where $a_h$ is the basic processing time of job $J_h$, $b_h$ is the compression rate of job $J_h$, $\theta > 0$ is a constant, and $w_h$ is the workload of job $J_h$.

Let $[h]$ be the job that is placed in $h$th position, $P_{[h]}^A$ denote the actual processing time of job $J_{[h]}$, the scheduling cost of this article can be expressed as $\sum_{h=1}^{\breve{n}} \eta_h P_{[h]}^A$, and the resource consumption cost is $\sum_{h=1}^{\breve{n}} g_{[h]} u_{[h]}$, where $\eta_h$ is the positional weight of $h$th position and $g_h$ is the cost allocated to job $J_h$. The goal is to find the optimal sequence $\pi$ of all jobs, and optimal resource allocation to minimize

$$F(\pi, u_{[h]}) = \widehat{\alpha} \sum_{h=1}^{\breve{n}} \eta_h P_{[h]}^A + \widehat{\beta} \sum_{h=1}^{\breve{n}} g_{[h]} u_{[h]}, \tag{2}$$

where $\widehat{\alpha} \geq 0$ and $\widehat{\beta} \geq 0$ are given constants. By using extensions of the notation, the problem can be denoted:

$$1 \left| P_h^A(u_h) = P_h(u_h) \max\{r^{\beta_h}, \delta\} \right| \widehat{\alpha} \sum_{h=1}^{\breve{n}} \eta_h P_{[h]}^A + \widehat{\beta} \sum_{h=1}^{\breve{n}} g_{[h]} u_{[h]}. \tag{3}$$

## 3. Positional weights results

### 3.1. Case 1

Note that the scheduling cost $\sum_{h=1}^{\breve{n}} \eta_h P_{[h]}^A$ can be applied to many scheduling cost, such as, for the makespan $C_{\max} = \sum_{h=1}^{\breve{n}} P_{[h]}^A$, i.e., $\eta_h = 1$; for the total completion time $\sum_{h=1}^{\breve{n}} C_h = \sum_{h=1}^{\breve{n}} \sum_{j=1}^{h} P_{[j]}^A = \sum_{h=1}^{\breve{n}} (\breve{n} - h + 1) P_{[h]}^A$, i.e., $\eta_h = \breve{n} - h + 1$; for the total absolute differences in completion times (see Kanet [26]) $\sum_{h=1}^{\breve{n}} \sum_{j=1}^{h} |C_h - C_j| = \sum_{h=1}^{\breve{n}} (h-1)(\breve{n} - h + 1) P_{[h]}^A$, i.e., $\eta_h = (h-1)(\breve{n} - h + 1)$; for the total absolute differences in waiting times (see Bagchi [27]) $\sum_{h=1}^{\breve{n}} \sum_{j=1}^{h} |W_h - W_j| = \sum_{h=1}^{\breve{n}} h(\breve{n} - h) P_{[h]}^A$, i.e., $\eta_h = h(\breve{n} - h)$, where $W_h$ is the waiting time of job $J_h$.

Under due date assignment, let $d_h$ be the due date of job $J_h$, $E_h = \max\{0, d_h - C_h\}$ and $T_h = \max\{0, C_h - d_h\}$ be the earliness and tardiness of job $J_h$. For the common (CON) due date assignment (see Panwalkar et al. [28]), $\sum_{h=1}^{n} (\phi E_h + \varphi T_h + \chi d) = \sum_{h=1}^{\breve{n}} \eta_h P_{[h]}^A$, where $\phi$, $\varphi$, and $\chi$ are given constants, $d_h = d$ is the common due date ($d$ is a decision variable) and $\eta_h = \min\{\breve{n}\chi + (h-1)\phi, (\breve{n}+1-h)\varphi\}$. For the slack due date assignment (see Adamopoulos and Pappis [29]), $\sum_{h=1}^{n} (\phi E_h + \varphi T_h + \chi q) = \sum_{h=1}^{\breve{n}} \eta_h P_{[h]}^A$, where $d_h = P_h^A + q$, $q$ is the common flow allowance ($q$ is a decision variable) and $\eta_h = \min\{\breve{n}\chi + h\phi, (\breve{n} - h)\varphi\}$. For the different due date assignment (see Seidmann et al. [30]), $\sum_{h=1}^{n} (\phi E_h + \varphi T_h + \chi d_h) = \sum_{h=1}^{\breve{n}} \eta_h P_{[h]}^A$, where $d_h$ is a decision variable and $\eta_h = \min\{\chi(\breve{n} + 1 - h), \varphi(\breve{n} + 1 - h)\}$.

Under due window assignment, let $[d'_h, d''_h]$ be the due window of job $J_h$, $d'_h$ ($d''_h$) is the starting (finishing) time of the due window of job $J_h$, and $D_h = d''_h - d'_h$ is the size of due window $[d'_h, d''_h]$, $E_h = \max\{0, d'_h - C_h\}$ and $T_h = \max\{0, C_h - d''_h\}$) are the earliness and tardiness of job $J_h$. For the common due window assignment (see Liman et al. [31]), $d'_h = a'$ ($d''_h = d''$, such $D = d''_h - d'$ ), $\sum_{h=1}^{n} (\phi E_h + \varphi T_h + \chi d' + \psi D) = \sum_{h=1}^{\breve{n}} \eta_h P_{[h]}^A$, where $\psi$ is a given constant, and $\eta_h = \min\{\breve{n}\chi + (h -$

$1)\phi, \breve{n}\psi, (\breve{n}+1-h)\varphi\}$. For the slack due window assignment (see Wu et al. [32]), $\sum_{h=1}^{n}(\phi E_h + \varphi T_h + \chi d' + \psi D) = \sum_{h=1}^{\breve{n}} \eta_h P_{[h]}^A$, where $d_i' = p_i + q'$, $d_i'' = p_i + q''$, $D = q'' - q'$, $\eta_h = \min\{\breve{n}\chi + h\phi, \breve{n}\psi, (\breve{n}-h)\varphi\}$; For the different due window assignment (see Wang et al. [33]), $d_h'$ and $d_h''$ are decision variables, $\sum_{h=1}^{n}(\phi E_h + \varphi T_h + \chi d_h' + \psi D_h) = \sum_{h=1}^{\breve{n}} \eta_h P_{[h]}^A$, where $\eta_h = \min\{\chi(\breve{n}+1-h), (\breve{n}+1-h)\varphi\}$.

### 3.2. *Case 2*

For the scheduling problems with past-sequence-dependent setup times (see Koulamas and Kyparisis [34], Liu et al. [35] and Wang et al. [36]), i.e., the setup time of job $J_h$ is $s_{[h]} = \epsilon \sum_{j=1}^{h-1} P_{[j]}^A$, where $\epsilon \geq 0$ is a constant, we have the following results:

For the makespan $C_{\max} = \sum_{h=1}^{\breve{n}}(s_{[h]} + P_{[h]}^A) = \sum_{h=1}^{\breve{n}}[1 + (\breve{n}-h)\varepsilon]P_{[h]}^A$, i.e., $\eta_h = 1 + (\breve{n}-h)\varepsilon$; for the total completion time $\sum_{h=1}^{\breve{n}} C_h = \sum_{h=1}^{\breve{n}} \sum_{j=1}^{h}(s_{[h]} + P_{[h]}^A) = \sum_{h=1}^{\breve{n}}(\breve{n}-h+1)\left[1 + \frac{\varepsilon(n-h)}{2}\right]P_{[h]}^A$, i.e., $\eta_h = (\breve{n}-h+1)\left[1 + \frac{\varepsilon(n-h)}{2}\right]$; for the total absolute differences in completion times $\sum_{h=1}^{\breve{n}} \sum_{j=1}^{h} |C_h - C_j| = \sum_{h=1}^{\breve{n}}(h-1)(\breve{n}-h+1)(s_{[h]} + P_{[h]}^A) = \sum_{h=1}^{\breve{n}}\left[(h-1)(\breve{n}-h+1) + \epsilon\sum_{j=h+1}^{\breve{n}}(j-1)(\breve{n}-j+1)\right]P_{[h]}^A$, i.e., $\eta_h = (h-1)(\breve{n}-h+1) + \epsilon\sum_{j=h+1}^{\breve{n}}(j-1)(\breve{n}-j+1)$; for the total absolute differences in waiting times $\sum_{h=1}^{\breve{n}} \sum_{j=1}^{h} |W_h - W_j| = \sum_{h=1}^{\breve{n}} h(\breve{n}-h)(s_{[h]} + P_{[h]}^A) = \sum_{h=1}^{\breve{n}}\left[h(\breve{n}-h) + \epsilon\sum_{j=h+1}^{\breve{n}} j(\breve{n}-j)\right]P_{[h]}^A$, i.e., $\eta_h = h(\breve{n}-h) + \epsilon\sum_{j=h+1}^{\breve{n}} j(\breve{n}-j)$.

For the common due date assignment, $\sum_{h=1}^{n}(\phi E_h + \varphi T_h + \chi d) = \sum_{h=1}^{\breve{n}} \varpi_h(s_{[h]} + P_{[h]}^A) = \sum_{h=1}^{\breve{n}}\left[\varpi_h + \epsilon\sum_{j=h+1}^{\breve{n}} \varpi_j\right]P_{[h]}^A$, where $\varpi_h = \min\{\breve{n}\chi + (h-1)\phi, (\breve{n}+1-h)\varphi\}$ and $\eta_h = \varpi_h + \epsilon\sum_{j=h+1}^{\breve{n}} \varpi_j$. For the slack due date assignment, $\sum_{h=1}^{n}(\phi E_h + \varphi T_h + \chi d_h) = \sum_{h=1}^{\breve{n}} \varpi_h(s_{[h]} + P_{[h]}^A) = \sum_{h=1}^{\breve{n}}\left[\varpi_h + \epsilon\sum_{j=h+1}^{\breve{n}} \varpi_j\right]P_{[h]}^A$, where $\varpi_h = \min\{\breve{n}\chi + h\phi, (\breve{n}-h)\varphi\}$ and $\eta_h = \varpi_h + \epsilon\sum_{j=h+1}^{\breve{n}} \varpi_j$. For the different due date assignment, $\sum_{h=1}^{n}(\phi E_h + \varphi T_h + \chi d_h) = \sum_{h=1}^{\breve{n}} \varpi_h(s_{[h]} + P_{[h]}^A) = \sum_{h=1}^{\breve{n}}\left[\varpi_h + \epsilon\sum_{j=h+1}^{\breve{n}} \varpi_j\right]P_{[h]}^A$, where $\varpi_h = \min\{\chi(\breve{n}+1-h), \varphi(\breve{n}+1-h)\}$ and $\eta_h = \varpi_h + \epsilon\sum_{j=h+1}^{\breve{n}} \varpi_j$.

Under the common due window assignment, $\sum_{h=1}^{n}(\phi E_h + \varphi T_h + \chi d' + \psi D) = \sum_{h=1}^{\breve{n}} \varpi_h(s_{[h]} + P_{[h]}^A) = \sum_{h=1}^{\breve{n}}\left[\varpi_h + \epsilon\sum_{j=h+1}^{\breve{n}} \varpi_j\right]P_{[h]}^A$, where $\varpi_h = \min\{\breve{n}\chi + (h-1)\phi, \breve{n}\psi, (\breve{n}+1-h)\varphi\}$. For the slack due window assignment, $\sum_{h=1}^{n}(\phi E_h + \varphi T_h + \chi d' + \psi D) = \sum_{h=1}^{\breve{n}} \eta_h(s_{[h]} + P_{[h]}^A) = \sum_{h=1}^{\breve{n}}\left[\varpi_h + \epsilon\sum_{j=h+1}^{\breve{n}} \varpi_j\right]P_{[h]}^A$, where $\varpi_h = \min\{\breve{n}\chi + h\phi, \breve{n}\psi, (\breve{n}-h)\varphi\}$; For the different due window assignment, $d_h'$ and $d_h''$ are decision variables, $\sum_{h=1}^{n}(\phi E_h + \varphi T_h + \chi d_h' + \psi D_h) = \sum_{h=1}^{\breve{n}} \varpi_h(s_{[h]} + P_{[h]}^A) = \sum_{h=1}^{\breve{n}}\left[\varpi_h + \epsilon\sum_{j=h+1}^{\breve{n}} \varpi_j\right]P_{[h]}^A$, where $\varpi_h = \min\{\chi(\breve{n}+1-h), (\breve{n}+1-h)\varphi\}$.

## 4. Main results

**Lemma 1.** *For a given sequence $\pi = [J_{[1]}, J_{[2]}, \ldots, J_{[\breve{n}]}]$, the optimal resource allocation of the problem $1\left|P_h^A(u_h) = P_h(u_h)\max\{r^{\beta_h}, \delta\}\right|\widehat{\alpha}\sum_{h=1}^{\breve{n}} \eta_h P_{[h]}^A + \widehat{\beta}\sum_{h=1}^{\breve{n}} g_{[h]}u_{[h]}$ is:*

$$u_{[h]}^* = \begin{cases} u_{[h]}^{min}, & if \ \ P_{[h]}'(u_{[h]}^{min}) \geq \frac{-\widehat{\beta}g_{[h]}}{\widehat{\alpha}\eta_h}, \\ \ddot{u}_{[h]}, & if \ \ P_{[h]}'(u_{[h]}^{min}) < \frac{-\widehat{\beta}g_{[h]}}{\widehat{\alpha}\eta_h} < P_{[h]}'(u_{[h]}^{max}), \\ u_{[h]}^{max}, & if \ \ P_{[h]}'(u_{[h]}^{max}) \leq \frac{-\widehat{\beta}g_{[h]}}{\widehat{\alpha}\eta_h}. \end{cases} \tag{4}$$

*Proof.* Taking the derivative of Eq (2) with respect to $u_{[h]}$ and setting the derivative value as 0, it follows that

$$\frac{\partial F(\pi, u_{[h]})}{\partial u_{[h]}} = \frac{d(\widehat{\alpha}\sum_{h=1}^{\breve{n}} \eta_h P^A_{[h]} + \widehat{\beta}\sum_{h=1}^{\breve{n}} g_{[h]}u_{[h]})}{du_{[h]}} = \widehat{\alpha}\eta_h P'_{[h]}(u_{[h]}) + \widehat{\beta} g_{[h]} = 0, \tag{5}$$

then we obtain:

$$P'_{[h]}(u_{[h]}) = \frac{-\widehat{\beta} g_{[h]}}{\widehat{\alpha}\eta_h}. \tag{6}$$

Let the solution of Eq (6) is $\ddot{u}_{[h]}$, if $P'_{[h]}(u^{min}_{[h]}) \geq 0$, it follows that $\frac{\partial F(\pi,u_{[h]})}{\partial u_{[h]}} \geq \frac{-\widehat{\beta} g_{[h]}}{\widehat{\alpha}\eta_h}$, which indicates $F(\pi, u_{[h]})$ is a non-decreasing function of $u_{[h]}$, the optimal solution of minimizing $F(\pi, u_{[h]})$ is $u^*_{[h]} = u^{min}_{[h]}$.

If $P'_{[h]}(u^{max}_{[h]}) \leq \frac{-\widehat{\beta} g_{[h]}}{\widehat{\alpha}\eta_h}$, it follows that $\frac{\partial F(\pi,u_{[h]})}{\partial u_{[h]}} \leq 0$, which indicates $F(\pi, u_{[h]})$ is a non-increasing function of $u_{[h]}$, the optimal solution of minimizing $F(\pi, u_{[h]})$ is $u^*_{[h]} = u^{max}_{[h]}$.

If $P'_{[h]}(u^{min}_{[h]}) < \frac{-\widehat{\beta} g_{[h]}}{\widehat{\alpha}\eta_h} < P'_{[h]}(u^{max}_{[h]})$, based on the properties of $P'_{[h]}(u_{[h]})$, the optimal solution of minimizing $F(\pi, u_{[h]})$ is $u^*_{[h]} = \ddot{u}_{[h]}$.

Let $z_{hr} = 1$ if job $J_h$ is placed in position $r$, and $z_{hr} = 0$; otherwise. Then, the optimal job sequence of the problem $1\left|P^A_h(u_h) = P_h(u_h)\max\{r^{\beta_h}, \delta\}\right|\widehat{\alpha}\sum_{h=1}^{\breve{n}} \eta_h P^A_{[h]} + \widehat{\beta}\sum_{h=1}^{\breve{n}} g_{[h]}u_{[h]}$ can be formulated as the following assignment problem (denoted by $\widehat{AP}$):

$$\text{Min} \sum_{h=1}^{\breve{n}}\sum_{r=1}^{\breve{n}} \widehat{\Omega}_{hr} z_{hr} \tag{7}$$

$$s.t. \begin{cases} \sum_{h=1}^{n} z_{hr} = 1, & r = 1, 2, ..., \breve{n}, \\ \sum_{r=1}^{n} z_{hr} = 1, & h = 1, 2, ..., \breve{n}, \\ z_{hr} = 0 \ \text{ or } \ 1, \end{cases} \tag{8}$$

where

$$\widehat{\Omega}_{hr} = \begin{cases} \widehat{\alpha}\eta_r P_h(u^{min}_h)\max\{r^{\beta_h}, \delta\} + \widehat{\beta} g_h u^{min}_h, & if \ \ P'_h(u^{min}_h) \geq \frac{-\widehat{\beta} g_h}{\widehat{\alpha}\eta_r}, \\ \widehat{\alpha}\eta_r P_h(\ddot{u}_h)\max\{r^{\beta_h}, \delta\} + \widehat{\beta} g_h \ddot{u}_h, & if \ \ P'_h(u^{min}_h) < \frac{-\widehat{\beta} g_h}{\widehat{\alpha}\eta_r} < P'_h(u^{max}_h), \\ \widehat{\alpha}\eta_r P_h(u^{max}_h)\max\{r^{\beta_h}, \delta\} + \widehat{\beta} g_h u^{max}_h, & if \ \ P'_h(u^{max}_h) \leq \frac{-\widehat{\beta} g_h}{\widehat{\alpha}\eta_r}. \end{cases} \tag{9}$$

From Lemma 1 and the above analysis, for solving $1\left|P^A_h(u_h) = P_h(u_h)\max\{r^{\beta_h}, \delta\}\right|\widehat{\alpha}\sum_{h=1}^{\breve{n}} \eta_h P^A_{[h]} + \widehat{\beta}\sum_{h=1}^{\breve{n}} g_{[h]}u_{[h]}$, the following algorithm can be summarized.

**Algorithm 1**

*Step 1.* Calculate $\widehat{\Omega}_{hr}$ (see Eq (9), $h, r = 1, 2, ..., \breve{n}$), solve $\widehat{AP}$ Eq (7)–(8) to determine an optimal sequence.

*Step 2.* Calculate optimal resource allocation $u^*_{[h]}$ by using Lemma 1 (see Eq (4)).

**Theorem 1.** *The problem* $1\left|P^A_h(u_h) = P_h(u_h)\max\{r^{\beta_h}, \delta\}\right|\widehat{\alpha}\sum_{h=1}^{\breve{n}} \eta_h P^A_{[h]} + \widehat{\beta}\sum_{h=1}^{\breve{n}} g_{[h]}u_{[h]}$ *can be solved by Algorithm 1 in* $O(\breve{n}^3)$ *time.*

*Proof.* In Step 1, solving $\widehat{AP}$ needs $O(\breve{n}^3)$ time; Step 2 requires $O(\breve{n})$ time, hence, the total time of Algorithm 1 is $O(\breve{n}^3)$.

**Example 1**. Assume a 8-job instance, where $P_h(u_h) = a_h + \left(\frac{w_h}{u_h}\right)^{\theta}$, $\widehat{\alpha} = \widehat{\beta} = 1, \delta = 0.65, \theta = 2$, the positional weights are $\eta_1 = 26, \eta_2 = 5, \eta_3 = 27, \eta_4 = 9, \eta_5 = 4, \eta_6 = 25, \eta_7 = 8, \eta_8 = 3$, and the other parameters are given in Table 1.

**Table 1**. Data of Example 1.

| $J_h$ | $J_1$ | $J_2$ | $J_3$ | $J_4$ | $J_5$ | $J_6$ | $J_7$ | $J_8$ |
|---|---|---|---|---|---|---|---|---|
| $a_h$ | 4 | 9 | 7 | 6 | 11 | 5 | 10 | 8 |
| $w_h$ | 2 | 4 | 12 | 8 | 14 | 7 | 9 | 5 |
| $g_h$ | 3 | 5 | 10 | 9 | 9 | 11 | 6 | 7 |
| $\beta_h$ | -0.25 | -0.32 | -0.28 | -0.3 | -0.35 | -0.27 | -0.33 | -0.29 |
| $u_h^{min}$ | 1 | 2 | 1 | 3 | 3 | 1 | 2 | 1 |
| $u_h^{max}$ | 4 | 5 | 5 | 6 | 8 | 4 | 5 | 3 |

**Solution**: By Algorithm 1, $P'_h(u_h) = \frac{dP_h(u_h)}{du_h} = -\theta\,(w_h)^{\theta}\,(u_h)^{-(\theta+1)}$, $\widehat{\Omega}_{hr}$ are given in Table 2, optimal sequence is $\pi^* = J_6 \to J_3 \to J_4 \to J_2 \to J_7 \to J_1 \to J_8 \to J_5$, optimal resource allocations (see Table 3) are $u_6^* = 3.2105, u_3^* = 5, u_4^* = 4.5789, u_2^* = 3.8620, u_7^* = 2.2894, u_1^* = 4, u_8^* = 2.2525, u_5^* = 3$ and $\widehat{\alpha}\sum_{h=1}^{\breve{n}} \eta_h P_{[h]}^A + \widehat{\beta}\sum_{h=1}^{\breve{n}} g_{[h]}u_{[h]} = 963.5719$.

**Table 2.** Values $\widehat{\Omega}_{hr}$ of Example 1.

| $J_h \backslash r$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $J_1$ | 122.5000 | 26.9227 | 99.1911 | 37.1688 | 19.5118 | 23.0500 | 19.8360 | 20.0723 |
| $J_2$ | 275.6400 | 58.2802 | 208.1311 | 78.2355 | 42.9253 | 181.6500 | 71.4005 | 35.2899 |
| $J_3$ | 381.7600 | 102.5451 | 303.2914 | 127.8962 | 82.6715 | 257.3500 | 116.3520 | 72.2260 |
| $J_4$ | 278.0840 | 80.2477 | 217.0011 | 101.9026 | 61.0889 | 189.8816 | 94.3878 | 52.5667 |
| $J_5$ | 597.3798 | 155.5846 | 429.4433 | 216.0268 | 112.2222 | 391.9662 | 197.4444 | 90.9167 |
| $J_6$ | 288.9171 | 100.2855 | 229.1521 | 115.4353 | 74.9721 | 195.4044 | 104.0816 | 66.0552 |
| $J_7$ | 400.9958 | 107.5475 | 295.1536 | 129.1500 | 79.9169 | 261.8131 | 119.9299 | 68.4855 |
| $J_8$ | 301.2222 | 73.7615 | 232.6059 | 91.9896 | 53.6511 | 196.1389 | 82.9895 | 45.4476 |

**Table 3.** Values $u_h$ scheduled at $r$ position of Example 1.

| $J_h \backslash r$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $J_1$ | 4 | 2.3713 | 4 | 2.8845 | 2.2013 | 4 | 2.6527 | 2.7734 |
| $J_2$ | 5 | 3.1748 | 5 | 3.8620 | 2.9472 | 5 | 3.7133 | 2.6777 |
| $J_3$ | 5 | 5 | 5 | 5 | 4.8658 | 5 | 5 | 4.4208 |
| $J_4$ | 4.5216 | 3 | 4.5789 | 3.1748 | 3 | 4.4629 | 3.0526 | 3 |
| $J_5$ | 4.3248 | 3 | 4.3795 | 3.0366 | 3 | 4.2686 | 3 | 3 |
| $J_6$ | 3.2105 | 1.8531 | 3.2511 | 2.2542 | 1.7203 | 3.1688 | 2.1674 | 1.5630 |
| $J_7$ | 4.2727 | 2.4662 | 4.3267 | 3 | 2.2894 | 4.2172 | 2.8845 | 2.0801 |
| $J_8$ | 3 | 1.9259 | 3 | 2.3427 | 1.7878 | 3 | 2.2525 | 1.6243 |

## 5. A special case

**Lemma 2.** *(Hardy et al. [37]). "The sum of products $\sum_{h=1}^{\breve{n}} X_h Y_h$ is minimized if sequence $X_1, X_2, \ldots, X_n$ is ordered non-decreasingly and sequence $Y_1, Y_2, \ldots, Y_n$ is ordered non-increasingly or vice versa."*

For the convex resource allocation function $P_h^A(u_h) = \left(\frac{w_h}{u_h}\right)^\theta \max\{r^\beta, \delta\}$ $(u_h > 0)$, i.e., $a_h = 0, \beta_h = \beta$, it follows that

**Lemma 3.** *For a given sequence $\pi = [J_{[1]}, J_{[2]}, \ldots, J_{[\breve{n}]}]$, the optimal resource allocation of the problem $1\left|P_h^A(u_h) = \left(\frac{w_h}{u_h}\right)^\theta \max\{r^\beta, \delta\}\right|\widehat{\alpha}\sum_{h=1}^{\breve{n}} \eta_h P_{[h]}^A + \widehat{\beta}\sum_{h=1}^{\breve{n}} g_{[h]} u_{[h]}$ is:*

$$u_{[h]}^* = \left(\frac{\theta\widehat{\alpha}\eta_h \max\{h^\beta, \delta\}\left(w_{[h]}\right)^\theta}{\widehat{\beta} g_{[h]}}\right)^{\frac{1}{1+\theta}}. \tag{10}$$

*Proof.* Taking the derivative of $F(\pi, u_{[h]}) = \widehat{\alpha}\sum_{h=1}^{\breve{n}} \eta_h \max\{h^\beta, \delta\}\left(\frac{w_{[h]}}{u_{[h]}}\right)^\theta + \widehat{\beta}\sum_{h=1}^{\breve{n}} g_{[h]} u_{[h]}$ with respect to $u_{[h]}$ and setting the derivative value as 0, it follows that

$$\frac{\partial F(\pi, u_{[h]})}{\partial u_{[h]}} = -\theta\widehat{\alpha}\eta_h \max\{h^\beta, \delta\}\left(w_{[h]}\right)^\theta \left(u_{[h]}\right)^{-(1+\theta)} + \widehat{\beta} g_{[h]} = 0, \tag{11}$$

then we obtain: $u_{[h]} = \left(\frac{\theta\widehat{\alpha}\eta_h \max\{h^\beta, \delta\}\left(w_{[h]}\right)^\theta}{\widehat{\beta} g_{[h]}}\right)^{\frac{1}{1+\theta}}$.

By substituting Eq (10) into $F(\pi, u_{[h]}) = \widehat{\alpha}\sum_{h=1}^{\breve{n}} \eta_h \max\{h^\beta, \delta\}\left(\frac{w_{[h]}}{u_{[h]}}\right)^\theta + \widehat{\beta}\sum_{h=1}^{\breve{n}} g_{[h]} u_{[h]}$, we have:

$$\begin{aligned} F(\pi, u_{[h]}) &= \widehat{\alpha}\sum_{h=1}^{\breve{n}} \eta_h \max\{h^\beta, \delta\}\left(\frac{w_{[h]}}{u_{[h]}}\right)^\theta + \widehat{\beta}\sum_{h=1}^{\breve{n}} g_{[h]} u_{[h]} \\ &= \widehat{\alpha}^{\frac{1}{1+\theta}}\widehat{\beta}^{\frac{\theta}{1+\theta}}(\theta^{\frac{1}{1+\theta}} + \theta^{\frac{-\theta}{1+\theta}})\sum_{h=1}^{\breve{n}}(\eta_h \max\{h^\beta, \delta\})^{\frac{1}{1+\theta}}\left(g_{[h]} w_{[h]}\right)^{\frac{\theta}{1+\theta}}. \end{aligned} \tag{12}$$

Equation (12) can be minimized by Lemma 2, i.e., $X_h = (\eta_h \max\{h^\beta, \delta\})^{\frac{1}{1+\theta}}$, $Y_h = \left(g_{[h]} w_{[h]}\right)^{\frac{\theta}{1+\theta}}$, hence, we introduce the following algorithm to solve the problem $1\left|P_h^A(u_h) = \left(\frac{w_h}{u_h}\right)^\theta \max\{h^\beta, \delta\}\right|\widehat{\alpha}\sum_{h=1}^{\breve{n}} \eta_h P_{[h]}^A + \widehat{\beta}\sum_{h=1}^{\breve{n}} g_{[h]} u_{[h]}$.

**Algorithm 2**
*Step 1.* By using Lemma 2 (let $X_h = (\eta_h \max\{h^\beta, \delta\})^{\frac{1}{1+\theta}}$ and $Y_h = (g_h w_h)^{\frac{\theta}{1+\theta}}$) to determine an optimal job sequence.
*Step 2.* Calculate optimal resource allocation by Lemma 3 (see Eq (10)).

**Theorem 2.** *The problem $1\left|P_h^A(u_h) = \left(\frac{w_h}{u_h}\right)^\theta \max\{r^\beta, \delta\}\right|\widehat{\alpha}\sum_{h=1}^{\breve{n}} \eta_h P_{[h]}^A + \widehat{\beta}\sum_{h=1}^{\breve{n}} g_{[h]} u_{[h]}$ can be solved by Algorithm 2 in $O(n \log n)$ time.*

**Example 2**. Assume a 8-job instance, where $P_h^A(u_h) = \left(\frac{w_h}{u_h}\right)^\theta \max\{r^\beta, \delta\}$, $\widehat{\alpha} = \widehat{\beta} = 1, \beta = -0.3, \delta = 0.65, \theta = 2$, the positional weights are $\eta_1 = 26, \eta_2 = 5, \eta_3 = 27, \eta_4 = 9, \eta_5 = 4, \eta_6 = 25, \eta_7 = 8, \eta_8 = 3$, and the other parameters are given in Table 4.

**Table 4.** Data of Example 2.

| $J_h$ | $J_1$ | $J_2$ | $J_3$ | $J_4$ | $J_5$ | $J_6$ | $J_7$ | $J_8$ |
|---|---|---|---|---|---|---|---|---|
| $w_h$ | 2 | 4 | 12 | 8 | 14 | 7 | 9 | 5 |
| $g_h$ | 3 | 5 | 10 | 9 | 9 | 11 | 6 | 7 |

**Solution**: By Algorithm 2, $X_1 = (26\max\{1^{-0.3}, 0.65\})^{\frac{1}{1+2}} = 2.9625$, $X_2 = (5\max\{2^{-0.3}, 0.65\})^{\frac{1}{1+2}} = 1.5955$, $X_3 = (27\max\{3^{-0.3}, 0.65\})^{\frac{1}{1+2}} = 2.6879$, $X_4 = (9\max\{4^{-0.3}, 0.65\})^{\frac{1}{1+2}} = 1.8108$, $X_5 = (4\max\{5^{-0.3}, 0.65\})^{\frac{1}{1+2}} = 1.3751$, $X_6 = (25\max\{6^{-0.3}, 0.65\})^{\frac{1}{1+2}} = 2.5329$, $X_7 = (8\max\{7^{-0.3}, 0.65\})^{\frac{1}{1+2}} = 1.7325$, $X_8 = (3\max\{8^{-0.3}, 0.65\})^{\frac{1}{1+2}} = 1.2493$, $Y_1 = (2\times 3)^{\frac{2}{1+2}} = 3.3019$, $Y_2 = (4\times 5)^{\frac{2}{1+2}} = 7.3681$, $Y_3 = (12\times 10)^{\frac{2}{1+2}} = 24.3288$, $Y_4 = (8\times 9)^{\frac{2}{1+2}} = 17.3070$, $Y_5 = (14\times 9)^{\frac{2}{1+2}} = 25.1332$, $Y_6 = (7\times 11)^{\frac{2}{1+2}} = 18.0992$, $Y_7 = (9\times 6)^{\frac{2}{1+2}} = 14.2886$, $Y_8 = (5\times 7)^{\frac{2}{1+2}} = 10.6999$, optimal sequence is $\pi^* = J_1 \to J_6 \to J_2 \to J_7 \to J_3 \to J_8 \to J_4 \to J_5$, optimal resource allocations are
$u_1^* = \left(\frac{2\times 26\times\max\{1^{-0.3},0.65\}\times(2)^2}{3}\right)^{\frac{1}{1+2}} = 4.1082$, $u_6^* = \left(\frac{2\times 7\times\max\{2^{-0.3},0.65\}\times(7)^2}{11}\right)^{\frac{1}{1+2}} = 3.7000$,
$u_2^* = \left(\frac{2\times 27\times\max\{3^{-0.3},0.65\}\times(4)^2}{5}\right)^{\frac{1}{1+2}} = 4.9904$, $u_7^* = \left(\frac{2\times 9\times\max\{4^{-0.3},0.65\}\times(9)^2}{6}\right)^{\frac{1}{1+2}} = 5.4325$,
$u_3^* = \left(\frac{2\times 4\times\max\{5^{-0.3},0.65\}\times(12)^2}{10}\right)^{\frac{1}{1+2}} = 4.2149$, $u_8^* = \left(\frac{2\times 25\times\max\{6^{-0.3},0.65\}\times(5)^2}{7}\right)^{\frac{1}{1+2}} = 4.8780$,
$u_4^* = \left(\frac{2\times 8\times\max\{7^{-0.3},0.65\}\times(8)^2}{9}\right)^{\frac{1}{1+2}} = 4.1975$, $u_5^* = \left(\frac{2\times 3\times\max\{8^{-0.3},0.65\}\times(14)^2}{9}\right)^{\frac{1}{1+2}} = 4.3957$ and
$\widehat{\alpha}\sum_{h=1}^{\breve{n}} \eta_h P_{[h]}^A + \widehat{\beta}\sum_{h=1}^{\breve{n}} g_{[h]}u_{[h]} = (2^{\frac{1}{1+2}} + 2^{\frac{-2}{1+2}})\sum_{h=1}^{\breve{n}} X_h Y_{[h]} = (2^{\frac{1}{1+2}} + 2^{\frac{-2}{1+2}}) \times (2.9625\times 3.3019 + 1.5955\times 18.0992 + 2.6879\times 7.3681 + 1.8108\times 14.2886 + 1.3751\times 24.3288 + 2.5329\times 10.6999 + 1.7325\times 17.3070 + 1.2493\times 25.1332) = 389.8396$.

## 6. Conclusions

This article discussed the single-machine scheduling problems with truncated learning effect and resource allocation. Under a general resource consumption function, some optimal properties are given and it is showed that the problem can be solved in $O(n^3)$ time. Further research might investigate multi-machine (e.g., flow shop setting) scheduling with learning effect and resource allocation (most of the multi-machine problems are NP-hard, and it's hard to find a quick and exact solution), consider serial-batching scheduling problems with learning effects and resource allocation (most of the serial-batching problems are NP-hard, and it's hard to find a quick and exact solution), study medical (hospital) scheduling problem (Wu et al. [38]) or open job scheduling problem (see Zhuang et al. [39]).

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## Conflict of interest

The authors declare that they have no conflicts of interest.

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