



Research article

Deterministic-Stochastic modeling: A new direction in modeling real world problems with crossover effect

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Abstract: Many real world problems depict processes following crossover behaviours. Modelling processes following crossover behaviors have been a great challenge to mankind. Indeed real world problems following crossover from Markovian to randomness processes have been observed in many scenarios, for example in epidemiology with spread of infectious diseases and even some chaos. Deterministic and stochastic methods have been developed independently to develop the future state of the system and randomness respectively. Very recently, Atangana and Seda introduced a new concept called piecewise differentiation and integration, this approach helps to capture processes with crossover effects. In this paper, an example of piecewise modelling is presented with illustration to chaos problems. Some important analysis including a piecewise existence and uniqueness and piecewise numerical scheme are presented. Numerical simulations are performed for different cases.

Keywords: Stochastic-deterministic models; piecewise existence and uniqueness; piecewise numerical scheme; new classes of chaotic attractors

1. Introduction

Deterministic models have been initiated to replicate processes in nature. They have been applied with great success in the last decades. However, when dealing with processes following randomness, indeed these deterministic models cannot be in use as they have failed to model random behaviors. However they are very useful to depict processes exhibiting power-law, fading memory to power law, passage from fading memory to power-law. They are also useful to depict memory processes. Stochastic differential equations have been introduced to deal with processes exhibiting randomness. They

have been used in many real world problems with great success. Noting that, these models are obtained using the concept of differentiation and integral. In particular, the concept of differentiation with the generalized Mittag-Leffler function was developed in [1], while Caputo and Fabrizio suggested a differential operator with exponential decay function [2]. Atangana generalized the concept of rate of change in his paper [3]. Atangana and Seda have suggested piecewise differential and integral operators in their work [4]. An analysis of a new chaos model was presented in [5]; Atangana and Gomez presented some important properties of fractional differential and integral operators with no singular kernel in [6]. Chaos in a turbulent swirling flow was investigated in [7], another important analysis of chaos with stochastic properties was presented in [8]. A comparative study between chaos and stochastic dynamics was presented in [9], Daniel et al, presented a simple method to detect chaos in nature [10], another important analysis of stochastic chaos was performed in [11]. However there exist in nature processes that exhibit passage from power-law to fading memory, or even from memoriless to power-law, indeed such processes cannot be replicated using either stochastic models or deterministic models. Very recently Atangana and Seda introduced new concepts called piecewise differentiation and integration, where a derivative or an integral is defined as a piecewise within a given interval [4]. Such a concept is indeed a new mathematical weapon to model complex real world problems exhibiting complex cross-over behaviors. The new type of modeling will be able to handle many complex problems. For example, we observe a very strange way of spread exhibiting by Covid-19, where in some cases, the spread shows a sign of deterministic within a given period of time then later it shows sign of stochastic randomness. A clear indication of complex cross-over. In this paper, we aim at modeling behaviors of real world exhibiting cross-over from deterministic to stochastic or vice versa.

2. Modeling cross-over from deterministic to stochastic and vice versa

In this section, we discuss different possible scenarios that could possibly occur in nature [4].

Case 1. Assume that a real world problem exhibiting cross-over from memoriless process to randomness with no steady state. A general Cauchy problem associated to this is given as

$$\left\{ \begin{array}{l} \frac{dy(t)}{dt} = f(t, y(t)), 0 \leq t \leq t_1 \\ y(0) = y_0 \\ dy(t) = f(t, y(t)) dt + \sigma y(t) dB(t), t_1 \leq t \leq T \\ y(t_1) = y_1 \end{array} \right. \quad (2.1)$$

where σ and $B(t)$, are density of randomness and environmental noise respectively.

Case 2. Assuming that a real world problem exhibits cross-over from power-law to stochastic processes, then the Cauchy problem associated to this problem can be defined as

$$\left\{ \begin{array}{l} {}^C_0 D_t^\alpha y(t) = f(t, y(t)), 0 < t \leq t_1 \\ y(0) = y_0 \\ dy(t) = f(t, y(t)) dt + \sigma y(t) dB(t), t_1 \leq t \leq T \\ y(t_1) = y_1 \end{array} \right. \quad (2.2)$$

where $0 < \alpha \leq 1$.

Case 3. Assuming that a real world problem exhibits cross-over from fading memory to stochastic processes, then the general Cauchy problem associated to this real world problem can be defined as

$$\begin{cases} {}_0^{CF}D_t^\alpha y(t) = f(t, y(t)), 0 < t \leq t_1 \\ y(0) = y_0 \\ dy(t) = f(t, y(t)) dt + \sigma y(t) dB(t), t_1 \leq t \leq T \\ y(t_1) = y_1 \end{cases} \quad (2.3)$$

where $0 < \alpha \leq 1$.

Case 4. Assuming that a complex real world problem is exhibiting cross-over from Mittag-Leffler process to stochastic processes, then the general Cauchy problem associated to such problem is defined as

$$\begin{cases} {}_0^{ABC}D_t^\alpha y(t) = f(t, y(t)), 0 < t \leq t_1 \\ y(0) = y_0 \\ dy(t) = f(t, y(t)) dt + \sigma y(t) dB(t), t_1 \leq t \leq T \\ y(t_1) = y_1 \end{cases} \quad (2.4)$$

where $0 < \alpha \leq 1$.

In the stochastic part, the classical derivative can be replaced by global derivative to capture more complex behaviors. To accommodate readers that are not use to fractional calculus, we present the following definitions

$${}_0^CD_t^\alpha y(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t y'(\tau) (t-\tau)^{-\alpha} d\tau \quad (2.5)$$

where $0 < \alpha \leq 1$.

$${}_0^{CF}D_t^\alpha y(t) = \frac{M(\alpha)}{1-\alpha} \int_0^t y'(\tau) \exp\left[-\frac{\alpha}{1-\alpha}(t-\tau)\right] d\tau \quad (2.6)$$

where $0 < \alpha \leq 1$, $M(0) = M(1)$ and $M(\alpha)$ is defined as normalized function. The space within which such is defined is well presented in [12]

$${}_0^{ABC}D_t^\alpha y(t) = \frac{AB(\alpha)}{1-\alpha} \int_0^t y'(\tau) E_\alpha\left[-\frac{\alpha}{1-\alpha}(t-\tau)^\alpha\right] d\tau \quad (2.7)$$

where $0 < \alpha \leq 1$, $AB(0) = AB(1)$ and $AB(\alpha) = 1 - \alpha + \frac{\alpha}{\Gamma(\alpha)}$.

Finally a global derivative of a function f with respect to a function g where g is positive continuously increasing function is defined as

$$D_g f(t) = \lim_{t \rightarrow t_1} \frac{f(t) - f(t_1)}{g(t) - g(t_1)}. \quad (2.8)$$

If f and g are differentiable with $g'(t) \neq 0$, then

$$D_g f(t) = \frac{f'(t)}{g'(t)}. \quad (2.9)$$

Case 5. Assuming that a real world problem exhibits random walk processes from $[0, t_1]$, then later the process follows changes within region in $[t_1, T]$. A mathematical model associate to this will be

$$\begin{cases} dy(t) = f(t, y(t)) dt + \sigma y(t) dB(t), 0 < t \leq t_1 \\ y(0) = y_0 \\ D_g y(t) = f(t, y(t)), t_1 \leq t \leq T \\ y(t_1) = y_1. \end{cases} \quad (2.10)$$

where D_g can be fractional also.

3. Analysis of deterministic-stochastic models

In this section, we present a discussion regarding numerical solution of each defined deterministic-stochastic model. For each suggested deterministic-stochastic model, the analysis of existence and uniqueness of solutions can be performed piecewisely according to each interval. A proper space of the function can be chosen and a suitable methodology for proving existence and uniqueness of such a model can be performed within each interval. This will not be the object of discussion in our paper, however some examples will be considered later and the existence and uniqueness of their solutions will be presented accordingly. In this section, our focus will be to provide a numerical solution to each model using the Newton polynomial interpolation formula [12].

3.1. Numerical scheme for Case 1

We consider the following problem to present numerical scheme

$$\begin{cases} \frac{dy(t)}{dt} = f(t, y(t)), 0 \leq t \leq t_1 \\ y(0) = y_0 \\ dy(t) = f(t, y(t))dt + \sigma y(t)dB(t), t_1 \leq t \leq T \\ y(t_1) = y_1 \end{cases} \quad (3.1)$$

The above is converted to

$$y(t) = \begin{cases} y(0) + \int_0^t f(\tau, y(\tau))d\tau \\ y(t_1) + \int_{t_1}^t f(\tau, y(\tau))d\tau + \int_{t_1}^t \sigma y(\tau)dB(\tau) \end{cases} \quad (3.2)$$

or

$$y(t_{n+1}) = \begin{cases} y(0) + \int_0^t f(\tau, y(\tau))d\tau \\ y(t_1) + \int_{t_1}^t f(\tau, y(\tau))d\tau + \int_{t_1}^t \sigma y(\tau)B'(\tau)d\tau \end{cases} \quad (3.3)$$

At $t = t_{n+1}$, we write

$$y(t_{n+1}) = \begin{cases} y(0) + \int_0^t f(\tau, y(\tau))d\tau \\ y(t_1) + \int_{t_1}^{t_{n+1}} f(\tau, y(\tau))d\tau + \int_{t_1}^{t_{n+1}} \sigma y(\tau)B'(\tau)d\tau \end{cases} \quad (3.4)$$

and

$$y(t_{n+1}) = \begin{cases} y(0) + \sum_{k=0}^i \int_{t_k}^{t_{k+1}} f(\tau, y(\tau))d\tau \\ y(t_1) + \sum_{k=i+1}^n \int_{t_k}^{t_{k+1}} f(\tau, y(\tau))d\tau + \sum_{k=i+1}^n \int_{t_k}^{t_{k+1}} \sigma y(\tau)B'(\tau)d\tau \end{cases} \quad (3.5)$$

We approximate $f(t, y)$ within $[t_k, t_{k+1}]$ when $0 < t \leq t_1$ using the Newton interpolation polynomial

$$\begin{aligned} f(t, y) = P_k(t) &\simeq f(t_{k-1}, y(t_{k-1})) + \frac{f(t_{k-1}, y(t_{k-1})) - f(t_{k-2}, y(t_{k-2}))}{\Delta t} (t - t_{k-2}) \\ &+ \frac{f(t_k, y(t_k)) - 2f(t_{k-1}, y(t_{k-1})) + f(t_{k-2}, y(t_{k-2}))}{2(\Delta t)^2} (t - t_{k-1})(t - t_{k-2}). \end{aligned} \quad (3.6)$$

Replacing $f(t, y)$ by $P_k(t)$ within $[t_k, t_{k+1}]$, we have following scheme

$$y(t_{n+1}) = \left[\begin{array}{l} y(0) + \sum_{k=2}^i \left\{ \frac{5}{12} f(t_{k-2}, y^{k-2}) \Delta t - \frac{4}{3} f(t_{k-1}, y^{k-1}) \Delta t \right. \\ \left. + \frac{23}{12} f(t_k, y^k) \Delta t \right\} \\ y(t_1) + \sum_{k=i+3}^n \left\{ \frac{5}{12} f(t_{k-2}, y^{k-2}) \Delta t - \frac{4}{3} f(t_{k-1}, y^{k-1}) \Delta t \right. \\ \left. + \frac{23}{12} f(t_k, y^k) \Delta t \right\} \\ + \sum_{k=i+3}^n \left\{ \frac{5}{12} (B(t_{k-1}) - B(t_{k-2})) \sigma y^{k-2} \right. \\ \left. - \frac{4}{3} (B(t_k) - B(t_{k-1})) \sigma y^{k-1} \right. \\ \left. + \frac{23}{12} (B(t_{k+1}) - B(t_k)) \sigma y^k \right\} \end{array} \right] . \quad (3.7)$$

3.2. Numerical scheme for Case 2

We deal with the following problem where first part is with Caputo fractional derivative and second part is stochastic. Such model is given by

$$\left\{ \begin{array}{l} {}^C_0 D_t^\alpha y(t) = f(t, y(t)), 0 < t \leq t_1 \\ y(0) = y_0 \\ dy(t) = f(t, y(t)) dt + \sigma y(t) dB(t), t_1 \leq t \leq T \\ y(t_1) = y_1. \end{array} \right. \quad (3.8)$$

The above is converted to

$$y(t) = \left\{ \begin{array}{l} \frac{1}{\Gamma(\alpha)} \int_0^t f(\tau, y) (t_1 - \tau)^{\alpha-1} d\tau \\ y(t_1) + \int_{t_1}^t f(\tau, y(\tau)) d\tau + \int_{t_1}^t \sigma y(\tau) B'(\tau) d\tau \end{array} \right. \quad (3.9)$$

At $t = t_{n+1}$, we write

$$y(t_{n+1}) = \left\{ \begin{array}{l} \frac{1}{\Gamma(\alpha)} \int_0^{t_{n+1}} f(\tau, y) (t_1 - \tau)^{\alpha-1} d\tau \\ y(t_1) + \int_{t_1}^{t_{n+1}} f(\tau, y(\tau)) d\tau + \int_{t_1}^{t_{n+1}} \sigma y(\tau) B'(\tau) d\tau \end{array} \right. \quad (3.10)$$

and

$$y(t_{n+1}) = \left\{ \begin{array}{l} \frac{1}{\Gamma(\alpha)} \sum_{k=0}^i \int_{t_k}^{t_{k+1}} f(\tau, y) (t_{n+1} - \tau)^{\alpha-1} d\tau \\ y(t_1) + \sum_{k=i+1}^n \int_{t_k}^{t_{k+1}} f(\tau, y(\tau)) d\tau + \sum_{k=i+1}^n \int_{t_k}^{t_{k+1}} \sigma y(\tau) B'(\tau) d\tau \end{array} \right. \quad (3.11)$$

Replacing $f(t, y)$ by its Newton polynomial, we can approximate $f(t, y)$ within $[t_k, t_{k+1}]$ as follows

$$y(t_{n+1}) = \left[\begin{array}{l} \left\{ \begin{array}{l} \frac{(\Delta t)^{\alpha-1}}{\Gamma(\alpha+1)} \sum_{k=2}^i f(t_{k-2}, y^{k-2}) \Pi \\ + \frac{(\Delta t)^{\alpha-1}}{\Gamma(\alpha+2)} \sum_{k=2}^i \left[f(t_{k-1}, y^{k-1}) - f(t_{k-2}, y^{k-2}) \right] \Sigma \\ + \frac{\alpha(\Delta t)^{\alpha-1}}{2\Gamma(\alpha+3)} \sum_{k=2}^i \left[f(t_k, y^k) - 2f(t_{k-1}, y^{k-1}) \right. \\ \left. + f(t_{k-2}, y^{k-2}) \right] \Delta \end{array} \right\} \\ y(t_1) + \sum_{k=i+3}^n \left\{ \frac{5}{12} f(t_{k-2}, y^{k-2}) \Delta t - \frac{4}{3} f(t_{k-1}, y^{k-1}) \Delta t \right. \\ \left. + \frac{23}{12} f(t_k, y^k) \Delta t \right\} \\ + \sum_{k=i+3}^n \left\{ \frac{5}{12} (B(t_{k-1}) - B(t_{k-2})) \sigma y^{k-2} \right. \\ \left. - \frac{4}{3} (B(t_k) - B(t_{k-1})) \sigma y^{k-1} \right. \\ \left. + \frac{23}{12} (B(t_{k+1}) - B(t_k)) \sigma y^k \right\} \end{array} \right] . \quad (3.12)$$

3.3. Numerical scheme for Case 3

In this subsection, we present numerical approximation for stochastic-deterministic Cauchy problem given by

$$\begin{cases} {}_0^{CF}D_t^\alpha y(t) = f(t, y(t)), 0 < t \leq t_1 \\ y(0) = y_0 \\ dy(t) = f(t, y(t))dt + \sigma y(t)dB(t), t_1 \leq t \leq T \\ y(t_1) = y_1 \end{cases} \quad (3.13)$$

where the first part is Caputo-Fabrizio fractional derivative and second part is stochastic. Such problem can be integrated as

$$y(t) = \begin{cases} \frac{1-\alpha}{M(\alpha)}f(t, y) + \frac{\alpha}{M(\alpha)} \int_0^t f(\tau, y) d\tau \\ y(t_1) + \int_{t_1}^t f(\tau, y(\tau)) d\tau + \int_{t_1}^t \sigma y(\tau) dB(\tau) \end{cases} \quad (3.14)$$

or

$$y(t) = \begin{cases} \frac{1-\alpha}{M(\alpha)}f(t, y) + \frac{\alpha}{M(\alpha)} \int_0^t f(\tau, y) d\tau \\ y(t_1) + \int_{t_1}^t f(\tau, y(\tau)) d\tau + \int_{t_1}^t \sigma y(\tau) B'(\tau) d\tau \end{cases} \quad (3.15)$$

At $t = t_{n+1}$, we write

$$y(t_{n+1}) = \begin{cases} \frac{1-\alpha}{M(\alpha)}f(t, y) + \frac{\alpha}{M(\alpha)} \int_0^{t_{n+1}} f(\tau, y) d\tau \\ y(t_1) + \int_{t_1}^{t_{n+1}} f(\tau, y(\tau)) d\tau + \int_{t_1}^{t_{n+1}} \sigma y(\tau) B'(\tau) d\tau \end{cases} \quad (3.16)$$

and

$$y(t_{n+1}) = \begin{cases} \frac{1-\alpha}{M(\alpha)}f(t_n, y^n) + \frac{\alpha}{M(\alpha)} \sum_{k=0}^i \int_{t_k}^{t_{k+1}} f(\tau, y) d\tau \\ y(t_1) + \sum_{k=i+1}^n \int_{t_k}^{t_{k+1}} f(\tau, y(\tau)) d\tau + \sum_{k=i+1}^n \int_{t_k}^{t_{k+1}} \sigma y(\tau) B'(\tau) d\tau \end{cases} \quad (3.17)$$

We can approximate $f(t, y)$ within $[t_k, t_{k+1}]$ as follows

$$y(t_{n+1}) = \begin{cases} \frac{1-\alpha}{M(\alpha)}f(t_n, y^n) + \frac{\alpha}{M(\alpha)} \sum_{k=2}^i \left\{ \frac{5}{12}f(t_{k-2}, y^{k-2})\Delta t - \frac{4}{3}f(t_{k-1}, y^{k-1})\Delta t + \frac{23}{12}f(t_k, y^k)\Delta t \right\} \\ \left[y(t_1) + \sum_{k=i+3}^n \left\{ \frac{5}{12}f(t_{k-2}, y^{k-2})\Delta t - \frac{4}{3}f(t_{k-1}, y^{k-1})\Delta t + \frac{23}{12}f(t_k, y^k)\Delta t \right\} \right. \\ \left. + \sum_{k=i+3}^n \left\{ \frac{5}{12}(B(t_{k-1}) - B(t_{k-2}))\sigma y^{k-2} - \frac{4}{3}(B(t_k) - B(t_{k-1}))\sigma y^{k-1} + \frac{23}{12}(B(t_{k+1}) - B(t_k))\sigma y^k \right\} \right] \end{cases} \quad (3.18)$$

3.4. Numerical scheme for Case 4

To obtain numerical scheme for following problem with piecewise derivative, we consider the following problem

$$\begin{cases} {}_0^{ABC}D_t^\alpha y(t) = f(t, y(t)), 0 < t \leq t_1 \\ y(0) = y_0 \\ dy(t) = f(t, y(t))dt + \sigma y(t)dB(t), t_1 \leq t \leq T \\ y(t_1) = y_1 \end{cases} \quad (3.19)$$

The above problem can be converted as

$$y(t) = \begin{cases} \frac{1-\alpha}{AB(\alpha)} f(t, y) + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t f(\tau, y) (t_1 - \tau)^{\alpha-1} d\tau \\ y(t_1) + \int_{t_1}^t f(\tau, y(\tau)) d\tau + \int_{t_1}^t \sigma y(\tau) B'(\tau) d\tau \end{cases} \quad (3.20)$$

At $t = t_{n+1}$, we write

$$y(t_{n+1}) = \begin{cases} \frac{1-\alpha}{AB(\alpha)} f(t, y) + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^{t_{n+1}} f(\tau, y) (t_1 - \tau)^{\alpha-1} d\tau \\ y(t_1) + \int_{t_1}^{t_{n+1}} f(\tau, y(\tau)) d\tau + \int_{t_1}^{t_{n+1}} \sigma y(\tau) B'(\tau) d\tau \end{cases} \quad (3.21)$$

and

$$y(t_{n+1}) = \begin{cases} \frac{1-\alpha}{AB(\alpha)} f(t_n, y^n) + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \sum_{k=0}^i \int_{t_k}^{t_{k+1}} f(\tau, y) (t_{n+1} - \tau)^{\alpha-1} d\tau \\ y(t_1) + \sum_{k=i+1}^n \int_{t_k}^{t_{k+1}} f(\tau, y(\tau)) d\tau + \sum_{k=i+1}^n \int_{t_k}^{t_{k+1}} \sigma y(\tau) B'(\tau) d\tau \end{cases} \quad (3.22)$$

Replacing $f(t, y)$ by its Newton polynomial, we can get the following scheme

$$y(t_{n+1}) = \begin{cases} \left\{ \begin{aligned} &\frac{1-\alpha}{AB(\alpha)} f(t_n, y^n) + \frac{\alpha}{AB(\alpha)\Gamma(\alpha+1)} \sum_{k=2}^i f(t_{k-2}, y^{k-2}) \Pi \\ &+ \frac{\alpha}{AB(\alpha)\Gamma(\alpha+2)} \sum_{k=2}^i \left[f(t_{k-1}, y^{k-1}) - f(t_{k-2}, y^{k-2}) \right] \Sigma \\ &+ \frac{\alpha}{AB(\alpha)\Gamma(\alpha+3)} \sum_{k=2}^i \left[\begin{aligned} &f(t_k, y^k) - 2f(t_{k-1}, y^{k-1}) \\ &+ f(t_{k-2}, y^{k-2}) \end{aligned} \right] \Delta \end{aligned} \right\} \\ \left[\begin{aligned} &y(t_1) + \sum_{k=i+3}^n \left\{ \begin{aligned} &\frac{5}{12} f(t_{k-2}, y^{k-2}) \Delta t - \frac{4}{3} f(t_{k-1}, y^{k-1}) \Delta t \\ &+ \frac{23}{12} f(t_k, y^k) \Delta t \end{aligned} \right\} \\ &+ \sum_{k=i+3}^n \left\{ \begin{aligned} &\frac{5}{12} (B(t_{k-1}) - B(t_{k-2})) \sigma y^{k-2} \\ &-\frac{4}{3} (B(t_k) - B(t_{k-1})) \sigma y^{k-1} \\ &+ \frac{23}{12} (B(t_{k+1}) - B(t_k)) \sigma y^k \end{aligned} \right\} \end{aligned} \right] \end{cases} \quad (3.23)$$

3.5. Numerical scheme for Case 5

In this paper, we consider the following problem with piecewise derivative

$$\begin{cases} dy(t) = f(t, y(t)) dt + \sigma y(t) dB(t), 0 < t \leq t_1 \\ y(0) = y_0 \\ D_g y(t) = f(t, y(t)), t_1 \leq t \leq T \\ y(t_1) = y_1 \end{cases} \quad (3.24)$$

which is transformed into

$$y(t) = \begin{cases} y(0) + \int_0^t f(\tau, y(\tau)) d\tau + \int_0^t \sigma y(\tau) dB(\tau) \\ y(t_1) + \int_{t_1}^t f(\tau, y(\tau)) dg(\tau) \end{cases} \quad (3.25)$$

or

$$y(t) = \begin{cases} y(0) + \int_0^t f(\tau, y(\tau)) d\tau + \int_0^t \sigma y(\tau) B'(\tau) d\tau \\ y(t_1) + \int_{t_1}^t f(\tau, y(\tau)) g'(\tau) d\tau \end{cases} \quad (3.26)$$

At $t = t_{n+1}$, we write

$$y(t_{n+1}) = \begin{cases} y(0) + \int_0^t f(\tau, y(\tau)) d\tau + \int_0^t \sigma y(\tau) B'(\tau) d\tau \\ y(t_1) + \int_{t_1}^{t_{n+1}} f(\tau, y) g'(\tau) d\tau \end{cases} \quad (3.27)$$

and

$$y(t_{n+1}) = \begin{cases} y(0) + \sum_{k=0}^i \int_{t_k}^{t_{k+1}} f(\tau, y(\tau)) d\tau + \sum_{k=0}^i \int_{t_k}^{t_{k+1}} \sigma y(\tau) B'(\tau) d\tau \\ y(t_1) + \sum_{k=i+1}^n \int_{t_k}^{t_{k+1}} f(\tau, y(\tau)) g'(\tau) d\tau \end{cases}. \quad (3.28)$$

The function $f(t, y)$ can be approximated as follows

$$y(t_{n+1}) = \begin{cases} \left[\begin{aligned} &y(0) + \sum_{k=2}^i \left\{ \frac{5}{12} f(t_{k-2}, y^{k-2}) \Delta t - \frac{4}{3} f(t_{k-1}, y^{k-1}) \Delta t + \frac{23}{12} f(t_k, y^k) \Delta t \right\} \\ &+ \sum_{k=2}^i \left\{ \frac{5}{12} (B(t_{k-1}) - B(t_{k-2})) \sigma y^{k-2} - \frac{4}{3} (B(t_k) - B(t_{k-1})) \sigma y^{k-1} \right. \\ &\quad \left. + \frac{23}{12} (B(t_{k+1}) - B(t_k)) \sigma y^k \right\} \end{aligned} \right] \\ y(t_1) + \sum_{k=i+3}^n \left\{ \begin{aligned} &\frac{5}{12} f(t_{k-2}, y^{k-2}) (g(t_{k-1}) - g(t_{k-2})) \\ &- \frac{4}{3} f(t_{k-1}, y^{k-1}) (g(t_k) - g(t_{k-1})) \\ &+ \frac{23}{12} f(t_k, y^k) (g(t_{k+1}) - g(t_k)) \end{aligned} \right\} \end{cases}. \quad (3.29)$$

4. Analysis with illustrative example with Hindmarsh-Rose model of neuronal activity

Case 1, we assume that the process follow classical mechanical behavior from $[0, 12]$, then later follows the randomness from $[12, 25]$. Thus the model can be formulated as

$$\begin{cases} \frac{dx}{dt} = y + \phi(x) - z + I \\ \frac{dy}{dt} = \psi(x) - y \\ \frac{dz}{dt} = r[s(x - x_R) - z] \\ x(0) = x_0, y(0) = y_0, z(0) = z_0 \end{cases} \quad \text{if } 0 \leq t \leq 12 \quad (4.1)$$

$$\begin{cases} dx = (y + \phi(x) - z + I) dt + \sigma_1 x dB_1(t) \\ dy = (\psi(x) - y) dt + \sigma_2 y dB_2(t) \\ dz = (r[s(x - x_R) - z]) dt + \sigma_3 z dB_3(t) \\ x(12) = x_{12}, y(12) = y_{12}, z(12) = z_{12} \end{cases} \quad \text{if } 12 \leq t \leq 25. \quad (4.2)$$

Here $\phi(x) = -ax^3 + bx^2$, $\psi(x) = c - dx^2$.

We define the norm $\|\varphi\|_\infty = \sup_{t \in D_\varphi} |\varphi(t)|$, we consider a Banach space. We present here the existence and uniqueness of the solution piecewisely. However to achieve this, we verify the linear-growth and Lipschitz condition properties. We also assume that $\forall t \in [0, 12]$, there exists 3 positive constant M_1, M_2 and $M_3 < \infty$ such that $\|x\|_\infty < M_1, \|y\|_\infty < M_2$ and $\|z\|_\infty < M_3$.

$$\begin{cases} \dot{x} = f_1(x, y, z, t) \\ \dot{y} = f_2(x, y, z, t) \\ \dot{z} = f_3(x, y, z, t) \end{cases} \quad \text{if } 0 \leq t \leq 12. \quad (4.3)$$

$\forall i = 1, 2, 3$, we first verify that

$$|f_i(x, t)|^2 < k_i(|x_i|^2 + 1) \quad (4.4)$$

and

$$|f_i(x^1, t) - f_i(x^2, t)|^2 < \bar{k}_i |x^1 - x^2|^2. \quad (4.5)$$

For proof, we consider the function $f_1(x, y, z, t)$

$$|f_1(x, y, z, t)|^2 = |y + bx^2 - ax^3 - z + I|^2 \quad (4.6)$$

$$\begin{aligned} &\leq 4|y|^2 + 4|bx^2 - ax^3|^2 + 4|z|^2 + 4I^2 \\ &\leq 4\left(\sup_{t \in [0,12]} |y|^2 + |bx - ax^2|^2 |x|^2 + \sup_{t \in [0,12]} |z|^2 + I^2\right) \\ &\leq 4\left(\|y^2\|_\infty + \|bx - ax^2\|_\infty |x|^2 + \|z^2\|_\infty + I^2\right) \\ &\leq 4\left(\|y^2\|_\infty + \|z^2\|_\infty + I^2\right) \left(1 + \frac{\|bx - ax^2\|_\infty}{\|y^2\|_\infty + \|z^2\|_\infty + I^2} |x|^2\right) \end{aligned} \quad (4.7)$$

under the condition that $\frac{\|bx - ax^2\|_\infty}{\|y^2\|_\infty + \|z^2\|_\infty + I^2} < 1$, then

$$|f_1(x, y, z, t)|^2 \leq k_1(1 + |x|^2).$$

Using same routine,

$$|f_2(x, y, z, t)|^2 = |c - dx^2 - y|^2 \quad (4.8)$$

$$\begin{aligned} &\leq 3c^2 + 3d^2|x^2|^2 + 3|y|^2 \\ &\leq 3\left(c^2 + d^2 \sup_{t \in [0,12]} |x^4| + |y|^2\right) \\ &\leq 3\left(c^2 + d^2 \|x^4\|_\infty + |y|^2\right) \\ &\leq 3\left(c^2 + d^2 \|x^4\|_\infty\right) \left(1 + \frac{1}{c^2 + d^2 \|x^4\|_\infty} |y|^2\right) \end{aligned} \quad (4.9)$$

under the condition that $\frac{1}{c^2 + d^2 \|x^4\|_\infty} < 1$, then

$$|f_2(x, y, z, t)|^2 \leq k_2(1 + |y|^2).$$

For the function F_3 ,

$$\begin{aligned} |f_3(x, y, z, t)|^2 &= |r[s(x - x_R) - z]|^2 \\ &\leq 2r^2 s^2 |x - x_R|^2 + 2r^2 |z|^2 \\ &\leq 2\left(r^2 s^2 \sup_{t \in [0,12]} |x - x_R|^2 + |z|^2\right) \\ &\leq 2r^2 (s^2 \|x - x_R\|_\infty^2 + |z|^2) \\ &\leq 2r^2 (s^2 \|x - x_R\|_\infty^2) \left(1 + \frac{1}{s^2 \|x - x_R\|_\infty^2} |z|^2\right) \end{aligned} \quad (4.10)$$

under the condition that $\frac{1}{s^2 \|(x - x_R)^2\|_\infty} < 1$, then

$$|f_3(x, y, z, t)|^2 \leq k_3(1 + |z|^2).$$

Therefore the condition of linear growth is verified if

$$\max \left\{ \frac{\|bx - ax^2\|_\infty}{\|y^2\|_\infty + \|z^2\|_\infty + I^2}, \frac{1}{c^2 + d^2 \|x^4\|_\infty}, \frac{1}{s^2 \|x - x_R\|_\infty^2} \right\} < 1. \quad (4.11)$$

4.1. Numerical scheme for stochastic-deterministic model of Hindmarsh-Rose model

We consider the Hindmarsh-Rose model in piecewise where the first part is classical and the second part is stochastic such that the stochastic-deterministic model of Hindmarsh-Rose model[13] is given as

$$\begin{cases} \frac{dx}{dt} = y - ax^3 + bx^2 - z + I \\ \frac{dy}{dt} = c - dx^2 - y \\ \frac{dz}{dt} = rs(x - x_R) - rz \\ x(0) = x_0, y(0) = y_0, z(0) = z_0 \end{cases} \quad \text{if } 0 \leq t \leq t_1 \quad (4.12)$$

$$\begin{cases} dx = (y - ax^3 + bx^2 - z + I)dt + \sigma_1 x dB_1(t) \\ dy = (c - dx^2 - y)dt + \sigma_2 y dB_2(t) \\ dz = (rs(x - x_R) - rz)dt + \sigma_3 z dB_3(t) \\ x(t_1) = x_1, y(t_1) = y_1, z(t_1) = z_1 \end{cases} \quad \text{if } t_1 \leq t \leq T. \quad (4.13)$$

Using the numerical scheme presented in Case 1, the numerical solution of the stochastic-deterministic Hindmarsh-Rose model is given as follows:

$$x(t_{n+1}) = \begin{cases} x^0 + \sum_{k=2}^i \left\{ \begin{aligned} &\frac{5}{12} (y^{k-2} - ax^{3k-6} + bx^{2k-4} - z^{k-2} + I) \Delta t \\ &-\frac{4}{3} (y^{k-1} - ax^{3k-3} + bx^{2k-2} - z^{k-1} + I) \Delta t \\ &+\frac{23}{12} (y^k - ax^{3k} + bx^{2k} - z^k + I) \Delta t \end{aligned} \right\}, & \text{if } 0 < t \leq t_1 \\ \left[\begin{aligned} &x^1 + \sum_{k=i+3}^n \left\{ \begin{aligned} &\frac{5}{12} (y^{k-2} - ax^{3k-6} + bx^{2k-4} - z^{k-2} + I) \Delta t \\ &-\frac{4}{3} (y^{k-1} - ax^{3k-3} + bx^{2k-2} - z^{k-1} + I) \Delta t \\ &+\frac{23}{12} (y^k - ax^{3k} + bx^{2k} - z^k + I) \Delta t \end{aligned} \right\} \\ &+ \sum_{k=i+3}^n \left\{ \begin{aligned} &\frac{5}{12} (B_1(t_{k-1}) - B_1(t_{k-2})) \sigma_1 x^{k-2} \\ &-\frac{4}{3} (B_1(t_k) - B_1(t_{k-1})) \sigma_1 x^{k-1} \\ &+\frac{23}{12} (B_1(t_{k+1}) - B_1(t_k)) \sigma_1 x^k \end{aligned} \right\} \end{aligned} \right], & \text{if } t_1 \leq t \leq T \end{cases} \quad (4.14)$$

$$y(t_{n+1}) = \begin{cases} y^0 + \sum_{k=2}^i \left\{ \begin{aligned} &\frac{5}{12} (c - dx^{2k-4} - y^{k-2}) \Delta t \\ &-\frac{4}{3} (c - dx^{2k-2} - y^{k-1}) \Delta t \\ &+\frac{23}{12} (c - dx^{2k} - y^k) \Delta t \end{aligned} \right\}, & \text{if } 0 < t \leq t_1 \\ \left[\begin{aligned} &y^1 + \sum_{k=i+3}^n \left\{ \begin{aligned} &\frac{5}{12} (c - dx^{2k-4} - y^{k-2}) \Delta t \\ &-\frac{4}{3} (c - dx^{2k-2} - y^{k-1}) \Delta t \\ &+\frac{23}{12} (c - dx^{2k} - y^k) \Delta t \end{aligned} \right\} \\ &+ \sum_{k=i+3}^n \left\{ \begin{aligned} &\frac{5}{12} (B_2(t_{k-1}) - B_2(t_{k-2})) \sigma_2 y^{k-2} \\ &-\frac{4}{3} (B_2(t_k) - B_2(t_{k-1})) \sigma_2 y^{k-1} \\ &+\frac{23}{12} (B_2(t_{k+1}) - B_2(t_k)) \sigma_2 y^k \end{aligned} \right\} \end{aligned} \right], & \text{if } t_1 \leq t \leq T \end{cases} \quad (4.15)$$

$$z(t_{n+1}) = \begin{cases} z^0 + \sum_{k=2}^i \left\{ \begin{array}{l} \frac{5}{12} (rs(x^{k-2} - x_R) - rz^{k-2}) \Delta t \\ -\frac{4}{3} (rs(x^{k-1} - x_R) - rz^{k-1}) \Delta t \\ +\frac{23}{12} (rs(x^k - x_R) - rz^k) \Delta t \end{array} \right\}, & \text{if } 0 \leq t \leq t_1 \\ \left[\begin{array}{l} z^1 + \sum_{k=i+3}^n \left\{ \begin{array}{l} \frac{5}{12} (rs(x^{k-2} - x_R) - rz^{k-2}) \Delta t \\ -\frac{4}{3} (rs(x^{k-1} - x_R) - rz^{k-1}) \Delta t \\ +\frac{23}{12} (rs(x^k - x_R) - rz^k) \Delta t \end{array} \right\} \\ + \sum_{k=i+3}^n \left\{ \begin{array}{l} \frac{5}{12} (B_3(t_{k-1}) - B_3(t_{k-2})) \sigma_3 z^{k-2} \\ -\frac{4}{3} (B_3(t_k) - B_3(t_{k-1})) \sigma_3 z^{k-1} \\ +\frac{23}{12} (B_3(t_{k+1}) - B_3(t_k)) \sigma_3 z^k \end{array} \right\} \end{array} \right], & \text{if } t_1 \leq t \leq T \end{cases}. \quad (4.16)$$

4.2. Numerical simulation for stochastic-deterministic model of Hindmarsh-Rose model

We provide the numerical simulations for the Hindmarsh-Rose model[13] with piecewise derivative

$$\begin{cases} \frac{dx}{dt} = y - ax^3 + bx^2 - z + I \\ \frac{dy}{dt} = c - dx^2 - y \\ \frac{dz}{dt} = rs(x - x_R) - rz \\ x(0) = x_0, y(0) = y_0, z(0) = z_0 \end{cases} \quad \text{if } 0 \leq t \leq t_1 \quad (4.17)$$

$$\begin{cases} dx = (y - ax^3 + bx^2 - z + I) dt + \sigma_1 x dB_1(t) \\ dy = (c - dx^2 - y) dt + \sigma_2 y dB_2(t) \\ dz = (rs(x - x_R) - rz) dt + \sigma_3 z dB_3(t) \\ x(t_1) = x_1, y(t_1) = y_1, z(t_1) = z_1 \end{cases} \quad \text{if } t_1 \leq t \leq T \quad (4.18)$$

where the first part is classical and the second part is stochastic. Here initial data is taken as

$$x(0) = 0.1, y(0) = 0.1, z(0) = 0.1. \quad (4.19)$$

In Figure 1, the numerical simulations are depicted with the parameters $a = 1, b = 3, c = 1, d = 5, s = 4, x_R = -8/5, r = 10^{-3}, I = 6$.

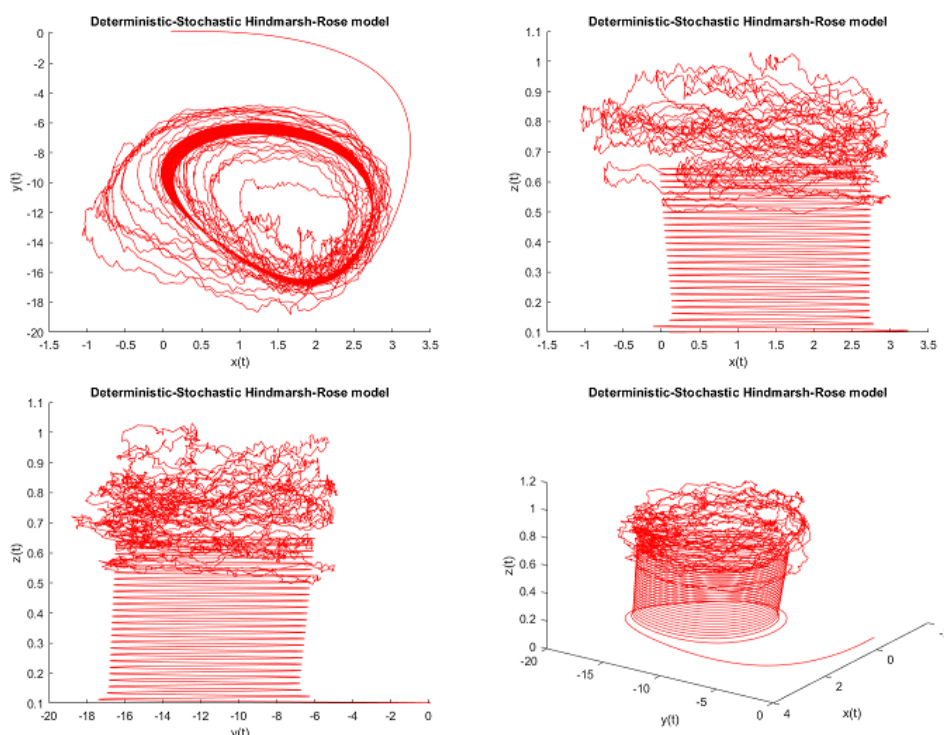


Figure 1. Numerical visualization of Hindmarsh-Rose model for $\alpha = 1, \sigma_1 = 0.01, \sigma_2 = 0.015, \sigma_3 = 0.012$.

4.3. Numerical scheme for stochastic-deterministic model of Hindmarsh-Rose model

We present the numerical solution for Hindmarsh-Rose model[13] in piecewise where the first part is with Caputo fractional derivative and the second part is stochastic. Such model is represented by

$$\begin{cases} {}^C_0 D_t^\alpha x = y - ax^3 + bx^2 - z + I \\ {}^C_0 D_t^\alpha y = c - dx^2 - y \\ {}^C_0 D_t^\alpha z = rs(x - x_R) - rz \end{cases} \quad \text{if } 0 < t \leq t_1 \quad (4.20)$$

$$\begin{cases} dx = (y - ax^3 + bx^2 - z + I)dt + \sigma_1 x dB_1(t) \\ dy = (c - dx^2 - y)dt + \sigma_2 y dB_2(t) \\ dz = (rs(x - x_R) - rz)dt + \sigma_3 z dB_3(t) \end{cases} \quad \text{if } t_1 \leq t \leq T.$$

The numerical solution of such model is given by

$$x(t_{n+1}) = \begin{cases} \left[\begin{aligned} & \left(\frac{(\Delta t)^{\alpha-1}}{\Gamma(\alpha+1)} \sum_{k=2}^i (y^{k-2} - ax^{3k-6} + bx^{2k-4} - z^{k-2} + I) \Pi \right. \\ & + \frac{(\Delta t)^{\alpha-1}}{\Gamma(\alpha+2)} \sum_{k=2}^i \left[\begin{aligned} & (y^{k-1} - ax^{3k-3} + bx^{2k-2} - z^{k-1} + I) \\ & - (y^{k-2} - ax^{3k-6} + bx^{2k-4} - z^{k-2} + I) \end{aligned} \right] \Sigma \\ & + \frac{\alpha(\Delta t)^{\alpha-1}}{2\Gamma(\alpha+3)} \sum_{k=2}^i \left[\begin{aligned} & (y^{k-2} - ax^{3k-6} + bx^{2k-4} - z^{k-2} + I) \\ & - 2(y^{k-1} - ax^{3k-3} + bx^{2k-2} - z^{k-1} + I) \\ & + (y^k - ax^{3k} + bx^{2k} - z^k + I) \end{aligned} \right] \Delta \end{aligned} \right] \right\}, \text{ if } 0 < t \leq t_1 \\ \left[\begin{aligned} & x^1 + \sum_{k=i+3}^n \left\{ \begin{aligned} & \frac{5}{12} (y^{k-2} - ax^{3k-6} + bx^{2k-4} - z^{k-2} + I) \Delta t \\ & - \frac{4}{3} (y^{k-1} - ax^{3k-3} + bx^{2k-2} - z^{k-1} + I) \Delta t \\ & + \frac{23}{12} (y^k - ax^{3k} + bx^{2k} - z^k + I) \Delta t \end{aligned} \right\} \\ & + \sum_{k=i+3}^n \left\{ \begin{aligned} & \frac{5}{12} (B_1(t_{k-1}) - B_1(t_{k-2})) \sigma_1 x^{k-2} \\ & - \frac{4}{3} (B_1(t_k) - B_1(t_{k-1})) \sigma_1 x^{k-1} \\ & + \frac{23}{12} (B_1(t_{k+1}) - B_1(t_k)) \sigma_1 x^k \end{aligned} \right\} \end{aligned} \right] \right\}, \text{ if } t_1 \leq t \leq T \end{cases} \quad (4.21)$$

$$y(t_{n+1}) = \begin{cases} \left[\begin{aligned} & \left(\frac{(\Delta t)^{\alpha-1}}{\Gamma(\alpha+1)} \sum_{k=2}^i (c - dx^{2k-4} - y^{k-2}) \Pi \right. \\ & + \frac{(\Delta t)^{\alpha-1}}{\Gamma(\alpha+2)} \sum_{k=2}^i \left[\begin{aligned} & (c - dx^{2k-2} - y^{k-1}) \\ & - (c - dx^{2k-4} - y^{k-2}) \end{aligned} \right] \Sigma \\ & + \frac{\alpha(\Delta t)^{\alpha-1}}{2\Gamma(\alpha+3)} \sum_{k=2}^i \left[\begin{aligned} & (c - dx^{2k-4} - y^{k-2}) \\ & - 2(c - dx^{2k-2} - y^{k-1}) \\ & + (c - dx^{2k} - y^k) \end{aligned} \right] \Delta \end{aligned} \right] \right\}, \text{ if } 0 < t \leq t_1 \\ \left[\begin{aligned} & x^1 + \sum_{k=i+3}^n \left\{ \begin{aligned} & \frac{5}{12} (c - dx^{2k-4} - y^{k-2}) \Delta t \\ & - \frac{4}{3} (c - dx^{2k-2} - y^{k-1}) \Delta t \\ & + \frac{23}{12} (c - dx^{2k} - y^k) \Delta t \end{aligned} \right\} \\ & + \sum_{k=i+3}^n \left\{ \begin{aligned} & \frac{5}{12} (B_2(t_{k-1}) - B_2(t_{k-2})) \sigma_2 y^{k-2} \\ & - \frac{4}{3} (B_2(t_k) - B_2(t_{k-1})) \sigma_2 y^{k-1} \\ & + \frac{23}{12} (B_2(t_{k+1}) - B_2(t_k)) \sigma_2 y^k \end{aligned} \right\} \end{aligned} \right] \right\}, \text{ if } t_1 \leq t \leq T \end{cases} \quad (4.22)$$

$$z(t_{n+1}) = \begin{cases} \left[\begin{aligned} & \left(\frac{(\Delta t)^{\alpha-1}}{\Gamma(\alpha+1)} \sum_{k=2}^i (rs(x^{k-2} - x_R) - rz^{k-2}) \Pi \right. \\ & + \frac{(\Delta t)^{\alpha-1}}{\Gamma(\alpha+2)} \sum_{k=2}^i \left[\begin{aligned} & (rs(x^{k-2} - x_R) - rz^{k-2}) \\ & - (rs(x^{k-1} - x_R) - rz^{k-1}) \end{aligned} \right] \Sigma \\ & + \frac{\alpha(\Delta t)^{\alpha-1}}{2\Gamma(\alpha+3)} \sum_{k=2}^i \left[\begin{aligned} & (rs(x^{k-2} - x_R) - rz^{k-2}) \\ & - 2(rs(x^{k-1} - x_R) - rz^{k-1}) \\ & + (rs(x^k - x_R) - rz^k) \end{aligned} \right] \Delta \end{aligned} \right] \right\}, \text{ if } 0 < t \leq t_1 \\ \left[\begin{aligned} & x^1 + \sum_{k=i+3}^n \left\{ \begin{aligned} & \frac{5}{12} (rs(x^{k-2} - x_R) - rz^{k-2}) \Delta t \\ & - \frac{4}{3} (rs(x^{k-1} - x_R) - rz^{k-1}) \Delta t \\ & + \frac{23}{12} (rs(x^k - x_R) - rz^k) \Delta t \end{aligned} \right\} \\ & + \sum_{k=i+3}^n \left\{ \begin{aligned} & \frac{5}{12} (B_3(t_{k-1}) - B_3(t_{k-2})) \sigma_3 z^{k-2} \\ & - \frac{4}{3} (B_3(t_k) - B_3(t_{k-1})) \sigma_3 z^{k-1} \\ & + \frac{23}{12} (B_3(t_{k+1}) - B_3(t_k)) \sigma_3 z^k \end{aligned} \right\} \end{aligned} \right] \right\}, \text{ if } t_1 \leq t \leq T \end{cases} \quad (4.23)$$

4.4. Numerical simulation for stochastic-deterministic model of Hindmarsh-Rose model

We present the numerical simulations for the Hindmarsh-Rose model[13] in piecewise

$$\begin{cases} {}^C_0D_t^\alpha x = y - ax^3 + bx^2 - z + I \\ {}^C_0D_t^\alpha y = c - dx^2 - y \\ {}^C_0D_t^\alpha z = rs(x - x_R) - rz \end{cases} \quad \text{if } 0 < t \leq t_1 \quad (4.24)$$

$$\begin{cases} dx = (y - ax^3 + bx^2 - z + I)dt + \sigma_1 x dB_1(t) \\ dy = (c - dx^2 - y)dt + \sigma_2 y dB_2(t) \\ dz = (rs(x - x_R) - rz)dt + \sigma_3 z dB_3(t) \end{cases} \quad \text{if } t_1 \leq t \leq T.$$

where initial data is considered as

$$x(0) = 0.1, y(0) = 0.1, z(0) = 0.1. \quad (4.25)$$

In Figure 2, the numerical simulations are provided with the parameters $a = 1, b = 3, c = 1, d = 5, s = 4, x_R = -8/5, r = 10^{-3}, I = 6$.

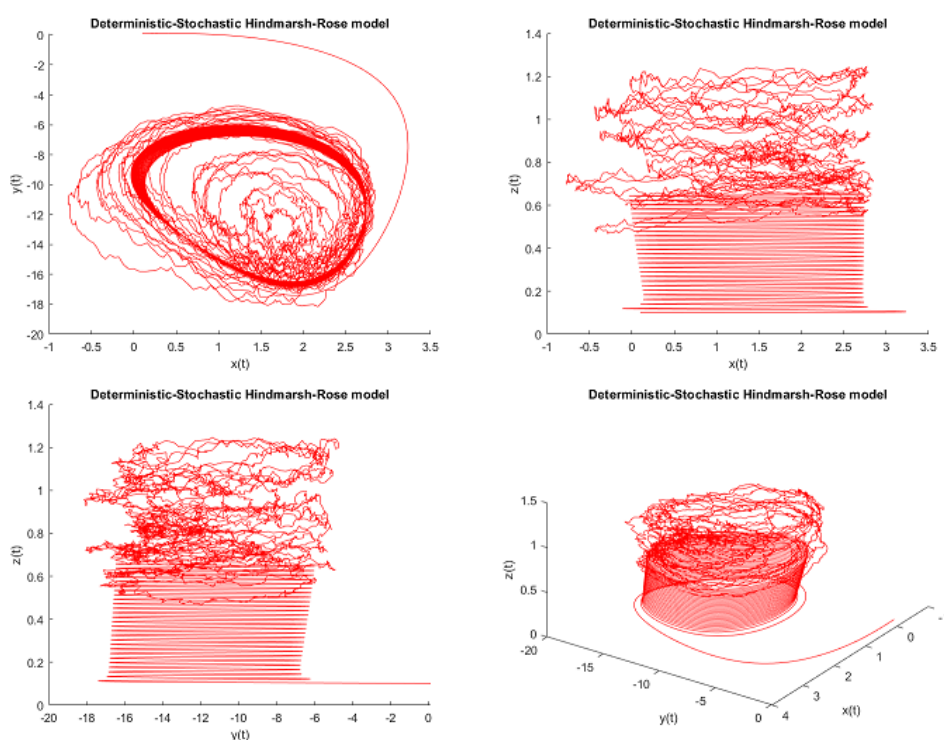


Figure 2. Numerical visualization of Hindmarsh-Rose model for $\alpha = 1, \sigma_1 = 0.01, \sigma_2 = 0.015, \sigma_3 = 0.012$.

4.5. Numerical scheme for stochastic-deterministic model of Hindmarsh-Rose model

We consider the Hindmarsh-Rose model[13] where the first part is Caputo-Fabrizio derivative and the second part is stochastic as follows

$$\begin{cases} {}_0^{CF}D_t^\alpha x = y - ax^3 + bx^2 - z + I \\ {}_0^{CF}D_t^\alpha y = c - dx^2 - y \\ {}_0^{CF}D_t^\alpha z = rs(x - x_R) - rz \end{cases} \quad \text{if } 0 < t \leq t_1 \quad (4.26)$$

$$\begin{cases} dx = (y - ax^3 + bx^2 - z + I)dt + \sigma_1 x dB_1(t) \\ dy = (c - dx^2 - y)dt + \sigma_2 y dB_2(t) \\ dz = (rs(x - x_R) - rz)dt + \sigma_3 z dB_3(t) \end{cases} \quad \text{if } t_1 \leq t \leq T. \quad (4.27)$$

Using the numerical scheme presented in Case 3, the numerical scheme for stochastic-deterministic Hindmarsh-rose model is given as follows:

$$x(t_{n+1}) = \begin{cases} + \frac{\alpha}{M(\alpha)} \sum_{k=2}^i \left\{ \begin{aligned} & \frac{5}{12} (y^{k-2} - ax^{3k-6} + bx^{2k-4} - z^{k-2} + I) \Delta t \\ & - \frac{4}{3} (y^{k-1} - ax^{3k-3} + bx^{2k-2} - z^{k-1} + I) \Delta t \\ & + \frac{23}{12} (y^k - ax^{3k} + bx^{2k} - z^k + I) \Delta t \end{aligned} \right\}, \text{ if } 0 < t \leq t_1 \\ \left[\begin{aligned} & x^1 + \sum_{k=i+3}^n \left\{ \begin{aligned} & \frac{5}{12} (y^{k-2} - ax^{3k-6} + bx^{2k-4} - z^{k-2} + I) \Delta t \\ & - \frac{4}{3} (y^{k-1} - ax^{3k-3} + bx^{2k-2} - z^{k-1} + I) \Delta t \\ & + \frac{23}{12} (y^k - ax^{3k} + bx^{2k} - z^k + I) \Delta t \end{aligned} \right\} \\ & + \sum_{k=i+3}^n \left\{ \begin{aligned} & \frac{5}{12} (B_1(t_{k-1}) - B_1(t_{k-2})) \sigma_1 x^{k-2} \\ & - \frac{4}{3} (B_1(t_k) - B_1(t_{k-1})) \sigma_1 x^{k-1} \\ & + \frac{23}{12} (B_1(t_{k+1}) - B_1(t_k)) \sigma_1 x^k \end{aligned} \right\} \end{aligned} \right], \text{ if } t_1 \leq t \leq T \end{cases} \quad (4.28)$$

$$y(t_{n+1}) = \begin{cases} + \frac{\alpha}{M(\alpha)} \sum_{k=2}^i \left\{ \begin{aligned} & \frac{5}{12} (c - dx^{2k-4} - y^{k-2}) \Delta t \\ & - \frac{4}{3} (c - dx^{2k-2} - y^{k-1}) \Delta t \\ & + \frac{23}{12} (c - dx^{2k} - y^k) \Delta t \end{aligned} \right\}, \text{ if } 0 < t \leq t_1 \\ \left[\begin{aligned} & y^1 + \sum_{k=i+3}^n \left\{ \begin{aligned} & \frac{5}{12} (c - dx^{2k-4} - y^{k-2}) \Delta t \\ & - \frac{4}{3} (c - dx^{2k-2} - y^{k-1}) \Delta t \\ & + \frac{23}{12} (c - dx^{2k} - y^k) \Delta t \end{aligned} \right\} \\ & + \sum_{k=i+3}^n \left\{ \begin{aligned} & \frac{5}{12} (B_2(t_{k-1}) - B_2(t_{k-2})) \sigma_2 y^{k-2} \\ & - \frac{4}{3} (B_2(t_k) - B_2(t_{k-1})) \sigma_2 y^{k-1} \\ & + \frac{23}{12} (B_2(t_{k+1}) - B_2(t_k)) \sigma_2 y^k \end{aligned} \right\} \end{aligned} \right], \text{ if } t_1 \leq t \leq T \end{cases} \quad (4.29)$$

$$z(t_{n+1}) = \begin{cases} \frac{1-\alpha}{M(\alpha)} (rs(x^n - x_R) - rz^n) \\ + \frac{\alpha}{M(\alpha)} \sum_{k=2}^i \left\{ \begin{aligned} &\frac{5}{12} (rs(x^{k-2} - x_R) - rz^{k-2}) \Delta t \\ &-\frac{4}{3} (rs(x^{k-1} - x_R) - rz^{k-1}) \Delta t \\ &+\frac{23}{12} (rs(x^k - x_R) - rz^k) \Delta t \end{aligned} \right\}, & \text{if } 0 < t \leq t_1 \\ \left[\begin{aligned} &z^1 + \sum_{k=i+3}^n \left\{ \begin{aligned} &\frac{5}{12} (rs(x^{k-2} - x_R) - rz^{k-2}) \Delta t \\ &-\frac{4}{3} (rs(x^{k-1} - x_R) - rz^{k-1}) \Delta t \\ &+\frac{23}{12} (rs(x^k - x_R) - rz^k) \Delta t \end{aligned} \right\} \\ &+ \sum_{k=i+3}^n \left\{ \begin{aligned} &\frac{5}{12} (B_3(t_{k-1}) - B_3(t_{k-2})) \sigma_3 z^{k-2} \\ &-\frac{4}{3} (B_3(t_k) - B_3(t_{k-1})) \sigma_3 z^{k-1} \\ &+\frac{23}{12} (B_3(t_{k+1}) - B_3(t_k)) \sigma_3 z^k \end{aligned} \right\} \end{aligned} \right], & \text{if } t_1 \leq t \leq T \end{cases}. \quad (4.30)$$

4.6. Numerical simulation for stochastic-deterministic model of Hindmarsh-Rose model

In this subsection, we consider stochastic-deterministic the Hindmarsh-Rose model[13]

$$\begin{cases} {}_0^C D_t^\alpha x = y - ax^3 + bx^2 - z + I \\ {}_0^C D_t^\alpha y = c - dx^2 - y \\ {}_0^C D_t^\alpha z = rs(x - x_R) - rz \end{cases} \quad \text{if } 0 < t \leq t_1 \quad (4.31)$$

$$\begin{cases} dx = (y - ax^3 + bx^2 - z + I) dt + \sigma_1 x dB_1(t) \\ dy = (c - dx^2 - y) dt + \sigma_2 y dB_2(t) \\ dz = (rs(x - x_R) - rz) dt + \sigma_3 z dB_3(t) \end{cases} \quad \text{if } t_1 \leq t \leq T$$

where initial data is considered as

$$x(0) = 0.1, y(0) = 0.1, z(0) = 0.1. \quad (4.32)$$

In Figure 3, we perform the numerical simulations with the parameters $a = 1, b = 3, c = 1, d = 5, s = 4, x_R = -8/5, r = 10^{-3}, I = 6$.

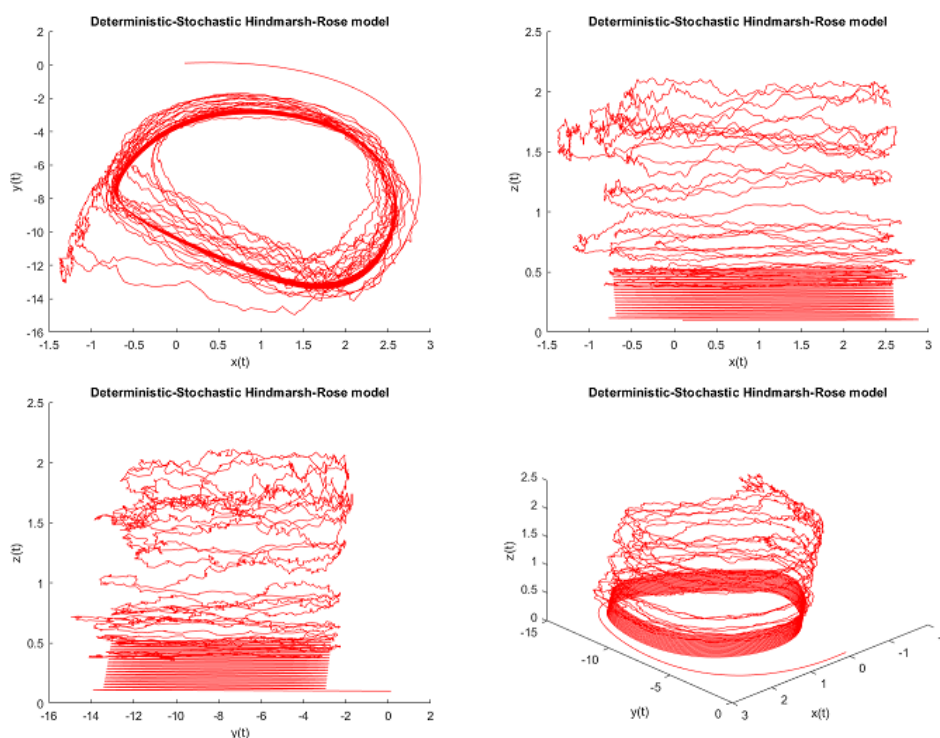


Figure 3. Numerical visualization of Hindmarsh-Rose model for $\alpha = 1, \sigma_1 = 0.01, \sigma_2 = 0.015, \sigma_3 = 0.012$.

4.7. Numerical scheme for stochastic-deterministic model of Hindmarsh-Rose model

In this section, we present the numerical scheme for Hindmarsh-Rose model[13] with piecewise derivative

$$\begin{cases} {}_0^{ABC}D_t^\alpha x = y - ax^3 + bx^2 - z + I \\ {}_0^{ABC}D_t^\alpha y = c - dx^2 - y \\ {}_0^{ABC}D_t^\alpha z = rs(x - x_R) - rz \end{cases} \quad \text{if } 0 < t \leq t_1 \quad (4.33)$$

$$\begin{cases} dx = (y - ax^3 + bx^2 - z + I)dt + \sigma_1 x dB_1(t) \\ dy = (c - dx^2 - y)dt + \sigma_2 y dB_2(t) \\ dz = (rs(x - x_R) - rz)dt + \sigma_3 z dB_3(t) \end{cases} \quad \text{if } t_1 \leq t \leq T.$$

Stochastic-deterministic Hindmarsh-Rose model can be solved by the following scheme

$$x(t_{n+1}) = \begin{cases} \left\{ \begin{aligned} & \frac{1-\alpha}{AB(\alpha)} (y^n - ax^{3n} + bx^{2n} - z^n + I) \\ & + \frac{\alpha}{AB(\alpha)} \frac{(\Delta t)^{\alpha-1}}{\Gamma(\alpha+1)} \sum_{k=2}^i (y^{k-2} - ax^{3k-6} + bx^{2k-4} - z^{k-2} + I) \Pi \\ & + \frac{\alpha}{AB(\alpha)} \frac{(\Delta t)^{\alpha-1}}{\Gamma(\alpha+2)} \sum_{k=2}^i \left[\begin{aligned} & (y^{k-1} - ax^{3k-3} + bx^{2k-2} - z^{k-1} + I) \\ & - (y^{k-2} - ax^{3k-6} + bx^{2k-4} - z^{k-2} + I) \end{aligned} \right] \Sigma \\ & + \frac{\alpha}{AB(\alpha)} \frac{(\Delta t)^{\alpha-1}}{2\Gamma(\alpha+3)} \sum_{k=2}^i \left[\begin{aligned} & (y^{k-2} - ax^{3k-6} + bx^{2k-4} - z^{k-2} + I) \\ & - 2(y^{k-1} - ax^{3k-3} + bx^{2k-2} - z^{k-1} + I) \\ & + (y^k - ax^{3k} + bx^{2k} - z^k + I) \end{aligned} \right] \Delta \end{aligned} \right\}, \text{ if } 0 < t \leq t_1 \\ \left[\begin{aligned} & x^1 + \sum_{k=i+3}^n \left\{ \begin{aligned} & \frac{5}{12} (y^{k-2} - ax^{3k-6} + bx^{2k-4} - z^{k-2} + I) \Delta t \\ & - \frac{4}{3} (y^{k-1} - ax^{3k-3} + bx^{2k-2} - z^{k-1} + I) \Delta t \\ & + \frac{23}{12} (y^k - ax^{3k} + bx^{2k} - z^k + I) \Delta t \end{aligned} \right\} \\ & + \sum_{k=i+3}^n \left\{ \begin{aligned} & \frac{5}{12} (B_1(t_{k-1}) - B_1(t_{k-2})) \sigma_1 x^{k-2} \\ & - \frac{4}{3} (B_1(t_k) - B_1(t_{k-1})) \sigma_1 x^{k-1} \\ & + \frac{23}{12} (B_1(t_{k+1}) - B_1(t_k)) \sigma_1 x^k \end{aligned} \right\} \end{aligned} \right], \text{ if } t_1 \leq t \leq T \end{cases}, \quad (4.34)$$

$$y(t_{n+1}) = \begin{cases} \left\{ \begin{aligned} & \frac{1-\alpha}{AB(\alpha)} (c - dx^{2n} - y^n) + \frac{\alpha}{AB(\alpha)} \frac{(\Delta t)^{\alpha-1}}{\Gamma(\alpha+1)} \sum_{k=2}^i (c - dx^{2k-4} - y^{k-2}) \Pi \\ & + \frac{\alpha}{AB(\alpha)} \frac{(\Delta t)^{\alpha-1}}{\Gamma(\alpha+2)} \sum_{k=2}^i \left[\begin{aligned} & (c - dx^{2k-2} - y^{k-1}) \\ & - (c - dx^{2k-4} - y^{k-2}) \end{aligned} \right] \Sigma \\ & + \frac{\alpha}{AB(\alpha)} \frac{(\Delta t)^{\alpha-1}}{2\Gamma(\alpha+3)} \sum_{k=2}^i \left[\begin{aligned} & (c - dx^{2k-4} - y^{k-2}) \\ & - 2(c - dx^{2k-2} - y^{k-1}) \\ & + (c - dx^{2k} - y^k) \end{aligned} \right] \Delta \end{aligned} \right\}, \text{ if } 0 < t \leq t_1 \\ \left[\begin{aligned} & x^1 + \sum_{k=i+3}^n \left\{ \begin{aligned} & \frac{5}{12} (c - dx^{2k-4} - y^{k-2}) \Delta t \\ & - \frac{4}{3} (c - dx^{2k-2} - y^{k-1}) \Delta t \\ & + \frac{23}{12} (c - dx^{2k} - y^k) \Delta t \end{aligned} \right\} \\ & + \sum_{k=i+3}^n \left\{ \begin{aligned} & \frac{5}{12} (B_2(t_{k-1}) - B_2(t_{k-2})) \sigma_2 y^{k-2} \\ & - \frac{4}{3} (B_2(t_k) - B_2(t_{k-1})) \sigma_2 y^{k-1} \\ & + \frac{23}{12} (B_2(t_{k+1}) - B_2(t_k)) \sigma_2 y^k \end{aligned} \right\} \end{aligned} \right], \text{ if } t_1 \leq t \leq T \end{cases}, \quad (4.35)$$

$$z(t_{n+1}) = \begin{cases} \left(\begin{aligned} & \frac{1-\alpha}{AB(\alpha)} (rs(x^n - x_R) - rz^n) \\ & + \frac{\alpha}{AB(\alpha)} \frac{(\Delta t)^{\alpha-1}}{\Gamma(\alpha+1)} \sum_{k=2}^i \left(rs(x^{k-2} - x_R) - rz^{k-2} \right) \Pi \\ & + \frac{\alpha}{AB(\alpha)} \frac{(\Delta t)^{\alpha-1}}{\Gamma(\alpha+2)} \sum_{k=2}^i \left[\begin{aligned} & \left(rs(x^{k-2} - x_R) - rz^{k-2} \right) \\ & - \left(rs(x^{k-1} - x_R) - rz^{k-1} \right) \end{aligned} \right] \Sigma \\ & + \frac{\alpha}{AB(\alpha)} \frac{(\Delta t)^{\alpha-1}}{2\Gamma(\alpha+3)} \sum_{k=2}^i \left[\begin{aligned} & \left(rs(x^{k-2} - x_R) - rz^{k-2} \right) \\ & - 2 \left(rs(x^{k-1} - x_R) - rz^{k-1} \right) \\ & + \left(rs(x^k - x_R) - rz^k \right) \end{aligned} \right] \Delta \end{aligned} \right) \Bigg\}, \text{ if } 0 < t \leq t_1 \\ \left[\begin{aligned} & x^1 + \sum_{k=i+3}^n \left\{ \begin{aligned} & \frac{5}{12} (rs(x^{k-2} - x_R) - rz^{k-2}) \Delta t \\ & - \frac{4}{3} (rs(x^{k-1} - x_R) - rz^{k-1}) \Delta t \\ & + \frac{23}{12} (rs(x^k - x_R) - rz^k) \Delta t \end{aligned} \right\} \\ & + \sum_{k=i+3}^n \left\{ \begin{aligned} & \frac{5}{12} (B_3(t_{k-1}) - B_3(t_{k-2})) \sigma_3 z^{k-2} \\ & - \frac{4}{3} (B_3(t_k) - B_3(t_{k-1})) \sigma_3 z^{k-1} \\ & + \frac{23}{12} (B_3(t_{k+1}) - B_3(t_k)) \sigma_3 z^k \end{aligned} \right\} \end{aligned} \right] \Bigg\}, \text{ if } t_1 \leq t \leq T \end{cases}. \quad (4.36)$$

4.8. Numerical simulation for stochastic-deterministic model of Hindmarsh-Rose model

We provide the numerical simulations for the Hindmarsh-Rose model[13] in piecewise

$$\begin{cases} \begin{aligned} & {}_0^{ABC}D_t^\alpha x = y - ax^3 + bx^2 - z + I \\ & {}_0^{ABC}D_t^\alpha y = c - dx^2 - y \\ & {}_0^{ABC}D_t^\alpha z = rs(x - x_R) - rz \end{aligned} & \text{if } 0 < t \leq t_1 \\ \begin{cases} dx = (y - ax^3 + bx^2 - z + I) dt + \sigma_1 x dB_1(t) \\ dy = (c - dx^2 - y) dt + \sigma_2 y dB_2(t) \\ dz = (rs(x - x_R) - rz) dt + \sigma_3 z dB_3(t) \end{cases} & \text{if } t_1 \leq t \leq T \end{cases} \quad (4.37)$$

where the first part is with Mittag-Leffler kernel and the second part is stochastic. Initial data is considered as

$$x(0) = 0.1, y(0) = 0.1, z(0) = 0.1. \quad (4.38)$$

In Figure 4, the numerical simulations are depicted with the parameters $a = 1, b = 3, c = 1, d = 5, s = 4, x_R = -8/5, r = 10^{-3}, I = 6$.

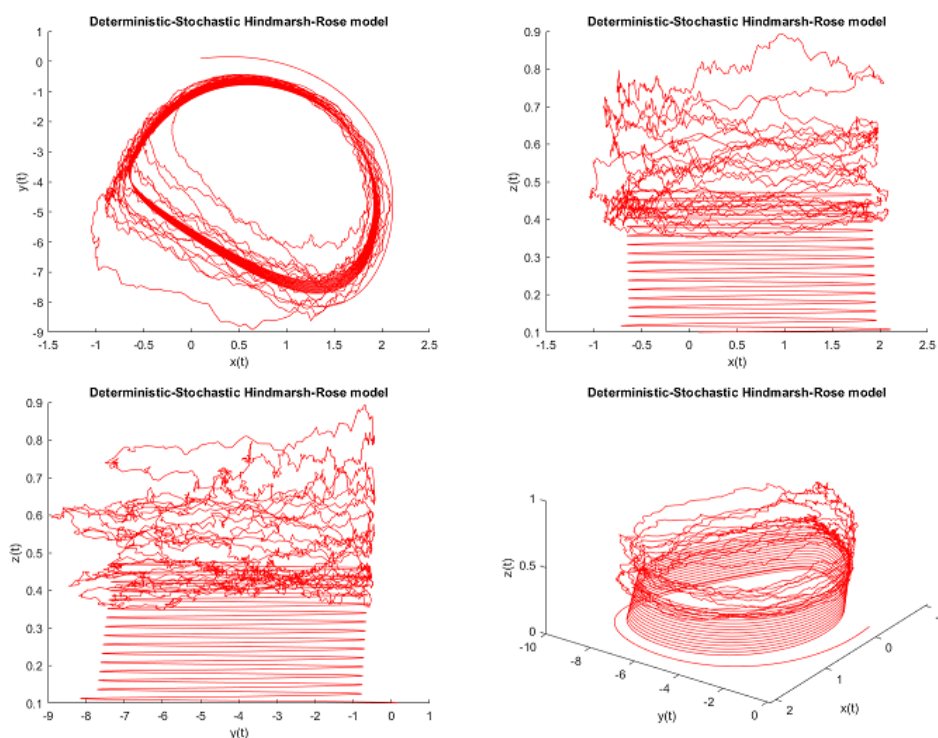


Figure 4. Numerical visualization of Hindmarsh-Rose model for $\alpha = 1, \sigma_1 = 0.01, \sigma_2 = 0.015, \sigma_3 = 0.012$.

4.9. Numerical scheme for stochastic-deterministic model of Hindmarsh-Rose model

In this section, we deal with the Hindmarsh-Rose model[13] where the first part is stochastic and the second part is global derivative. The stochastic-deterministic model of Hindmarsh-Rose model is represented by

$$\begin{cases} dx = (y - ax^3 + bx^2 - z + I)dt + \sigma_1 x dB_1(t) \\ dy = (c - dx^2 - y)dt + \sigma_2 y dB_2(t) \\ dz = (rs(x - x_R) - rz)dt + \sigma_3 z dB_3(t) \\ x(0) = x_0, y(0) = y_0, z(0) = z_0 \end{cases} \quad \text{if } 0 \leq t \leq t_1 \quad (4.39)$$

$$\begin{cases} D_g^\alpha x = (y - ax^3 + bx^2 - z + I) \\ D_g^\alpha y = (c - dx^2 - y) \\ D_g^\alpha z = (rs(x - x_R) - rz) \\ x(t_1) = x_1, y(t_1) = y_1, z(t_1) = z_1 \end{cases} \quad \text{if } t_1 \leq t \leq T. \quad (4.40)$$

Using the numerical scheme presented in Case 5, we can have the following scheme for the stochastic-deterministic Hindmarsh-Rose model

$$x(t_{n+1}) = \begin{cases} \left[\begin{aligned} & x^0 + \sum_{k=2}^i \left\{ \begin{aligned} & \frac{5}{12} (y^{k-2} - ax^{3k-6} + bx^{2k-4} - z^{k-2} + I) \Delta t \\ & - \frac{4}{3} (y^{k-1} - ax^{3k-3} + bx^{2k-2} - z^{k-1} + I) \Delta t \\ & + \frac{23}{12} (y^k - ax^{3k} + bx^{2k} - z^k + I) \Delta t \end{aligned} \right\} \\ & + \sum_{k=2}^i \left\{ \begin{aligned} & \frac{5}{12} (B_1(t_{k-1}) - B_1(t_{k-2})) \sigma_1 x^{k-2} \\ & - \frac{4}{3} (B_1(t_k) - B_1(t_{k-1})) \sigma_1 x^{k-1} \\ & + \frac{23}{12} (B_1(t_{k+1}) - B_1(t_k)) \sigma_1 x^k \end{aligned} \right\} \end{aligned} \right], \text{ if } 0 < t \leq t_1 \\ x^1 + \sum_{k=i+3}^n \left\{ \begin{aligned} & \frac{5}{12} (y^{k-2} - ax^{3k-6} + bx^{2k-4} - z^{k-2} + I) (g(t_{k-1}) - g(t_{k-2})) \\ & - \frac{4}{3} (y^{k-1} - ax^{3k-3} + bx^{2k-2} - z^{k-1} + I) (g(t_k) - g(t_{k-1})) \\ & + \frac{23}{12} (y^k - ax^{3k} + bx^{2k} - z^k + I) (g(t_{k+1}) - g(t_k)) \end{aligned} \right\}, \text{ if } t_1 \leq t \leq T \end{cases}, \quad (4.41)$$

$$y(t_{n+1}) = \begin{cases} \left[\begin{aligned} & y^0 + \sum_{k=2}^i \left\{ \begin{aligned} & \frac{5}{12} (c - dx^{2k-4} - y^{k-2}) \Delta t \\ & - \frac{4}{3} (c - dx^{2k-2} - y^{k-1}) \Delta t \\ & + \frac{23}{12} (c - dx^{2k} - y^k) \Delta t \end{aligned} \right\} \\ & + \sum_{k=2}^i \left\{ \begin{aligned} & \frac{5}{12} (B_2(t_{k-1}) - B_2(t_{k-2})) \sigma_2 y^{k-2} \\ & - \frac{4}{3} (B_2(t_k) - B_2(t_{k-1})) \sigma_2 y^{k-1} \\ & + \frac{23}{12} (B_2(t_{k+1}) - B_2(t_k)) \sigma_2 y^k \end{aligned} \right\} \end{aligned} \right], \text{ if } 0 < t \leq t_1 \\ y^1 + \sum_{k=i+3}^n \left\{ \begin{aligned} & \frac{5}{12} (c - dx^{2k-4} - y^{k-2}) (g(t_{k-1}) - g(t_{k-2})) \\ & - \frac{4}{3} (c - dx^{2k-2} - y^{k-1}) (g(t_k) - g(t_{k-1})) \\ & + \frac{23}{12} (c - dx^{2k} - y^k) (g(t_{k+1}) - g(t_k)) \end{aligned} \right\}, \text{ if } t_1 \leq t \leq T \end{cases}, \quad (4.42)$$

$$z(t_{n+1}) = \begin{cases} \left[\begin{aligned} & z^0 + \sum_{k=2}^i \left\{ \begin{aligned} & \frac{5}{12} (rs(x^{k-2} - x_R) - rz^{k-2}) \Delta t \\ & - \frac{4}{3} (rs(x^{k-1} - x_R) - rz^{k-1}) \Delta t \\ & + \frac{23}{12} (rs(x^k - x_R) - rz^k) \Delta t \end{aligned} \right\} \\ & + \sum_{k=2}^i \left\{ \begin{aligned} & \frac{5}{12} (B_3(t_{k-1}) - B_3(t_{k-2})) \sigma_3 z^{k-2} \\ & - \frac{4}{3} (B_3(t_k) - B_3(t_{k-1})) \sigma_3 z^{k-1} \\ & + \frac{23}{12} (B_3(t_{k+1}) - B_3(t_k)) \sigma_3 z^k \end{aligned} \right\} \end{aligned} \right], \text{ if } 0 < t \leq t_1 \\ z^1 + \sum_{k=i+3}^n \left\{ \begin{aligned} & \frac{5}{12} (rs(x^{k-2} - x_R) - rz^{k-2}) (g(t_{k-1}) - g(t_{k-2})) \\ & - \frac{4}{3} (rs(x^{k-1} - x_R) - rz^{k-1}) (g(t_k) - g(t_{k-1})) \\ & + \frac{23}{12} (rs(x^k - x_R) - rz^k) (g(t_{k+1}) - g(t_k)) \end{aligned} \right\}, \text{ if } t_1 \leq t \leq T \end{cases}. \quad (4.43)$$

$$a = 1, b = 3, c = 1, d = 5, s = 4, x_R = -8/5, r = 10^{-3}, I = [-10, 10].$$

4.10. Numerical simulation for stochastic-deterministic model of Hindmarsh-Rose model

In this subsection, we provide the numerical simulations for the following Hindmarsh-Rose model[13]

$$\begin{cases} dx = (y - ax^3 + bx^2 - z + I)dt + \sigma_1 x dB_1(t) \\ dy = (c - dx^2 - y)dt + \sigma_2 y dB_2(t) \\ dz = (rs(x - x_R) - rz)dt + \sigma_3 z dB_3(t) \\ x(0) = x_0, y(0) = y_0, z(0) = z_0 \end{cases} \quad \text{if } 0 \leq t \leq t_1 \quad (4.44)$$

$$\begin{cases} D_g^\alpha x = (y - ax^3 + bx^2 - z + I) \\ D_g^\alpha y = (c - dx^2 - y) \\ D_g^\alpha z = (rs(x - x_R) - rz) \\ x(t_1) = x_1, y(t_1) = y_1, z(t_1) = z_1 \end{cases} \quad \text{if } t_1 \leq t \leq T \quad (4.45)$$

where $g(t) = t$ and initial data is considered as

$$x(0) = 0.1, y(0) = 0.1, z(0) = 0.1. \quad (4.46)$$

In Figure 5, the numerical simulations are depicted with the parameters $a = 1, b = 3, c = 1, d = 5, s = 4, x_R = -8/5, r = 10^{-3}, I = 6$.

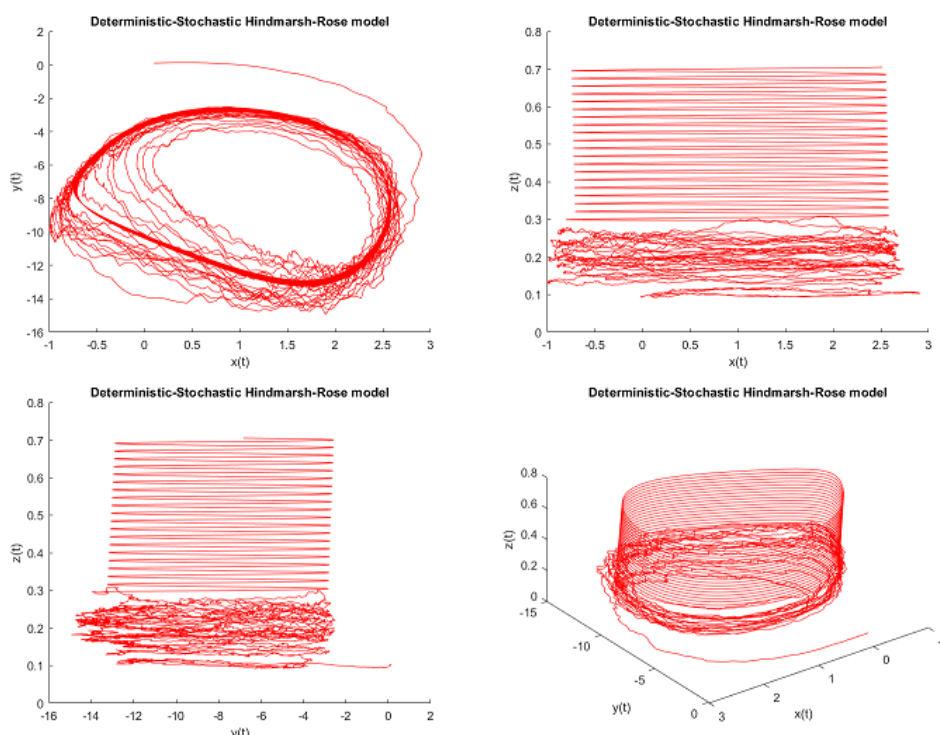


Figure 5. Numerical visualization of Hindmarsh-Rose model for $\alpha = 1, \sigma_1 = 0.01, \sigma_2 = 0.015, \sigma_3 = 0.012$.

5. Applications for stochastic-deterministic 4D Sinusoidally Driven Lorenz system model

In this section, we present illustrative examples for stochastic-deterministic 4D driven Lorenz system[14] by taking into account all cases presented in this work.

5.1. Numerical simulation for stochastic-deterministic 4D Sinusoidally Driven Lorenz system model

We present the numerical solution for 4D sinusoidally Driven Lorenz model[14] in piecewise where the first part is considered with classical and the second part is stochastic. Such model is represented by

$$\left\{ \begin{array}{l} \frac{dx}{dt} = -a(x - y - c \sin u) \\ \frac{dy}{dt} = bx - xz - y \\ \frac{dz}{dt} = xy - dz \\ \frac{du}{dt} = w \end{array} \right. \quad \text{if } 0 \leq t \leq t_1 \quad (5.1)$$

$$\left\{ \begin{array}{l} dx = (-a(x - y - c \sin u)) dt + \sigma_1 x dB_1(t) \\ dy = (bx - xz - y) dt + \sigma_2 y dB_2(t) \\ dz = (xy - dz) dt + \sigma_3 z dB_3(t) \\ du = w dt + \sigma_4 u dB_4(t) \end{array} \right. \quad \text{if } t_1 \leq t \leq T$$

where initial data is considered as

$$x(0) = 0.1, y(0) = 0.1, z(0) = 0.1, u(0) = 0.1. \quad (5.2)$$

For Case 1, the numerical simulations are performed in Figure 6. Here the parameters are chosen as $a = 10, b = 30, d = 8/3, c = 17, w = 0.5$.

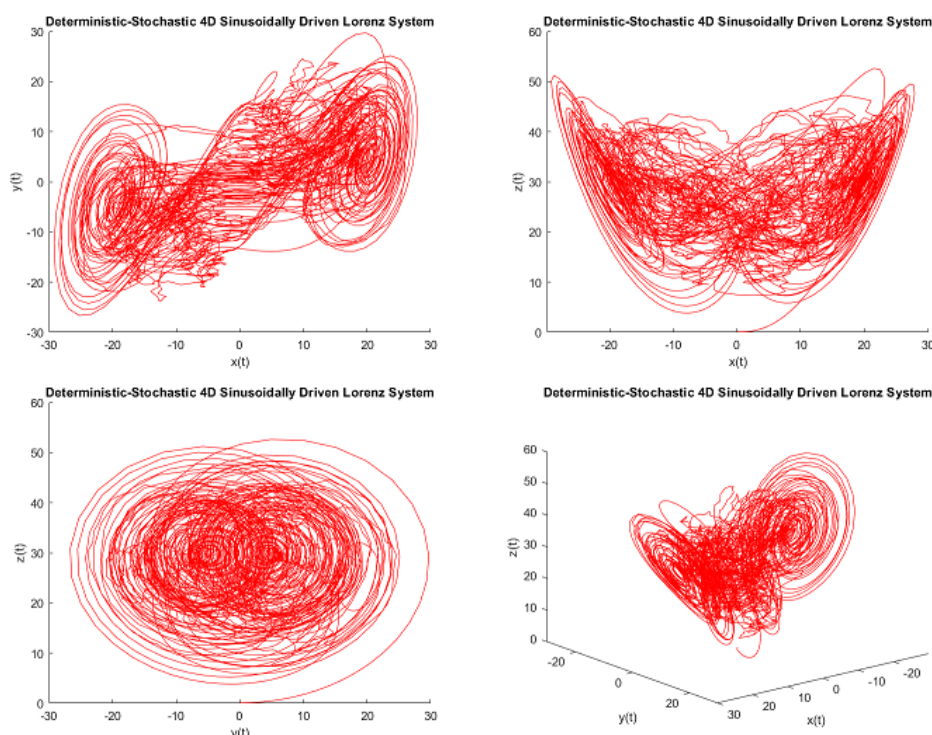


Figure 6. Numerical visualization of 4D Sinusoidally Driven Lorenz system for $\alpha = 1, \sigma_1 = 0.012, \sigma_2 = 0.013, \sigma_3 = 0.015, \sigma_4 = 0.014$.

5.2. Numerical simulation for stochastic-deterministic 4D Sinusoidally Driven Lorenz system model

In this subsection, numerical simulations for 4D sinusoidally Driven Lorenz model[14] with piecewise derivative are presented

$$\begin{cases} {}^C_0D_t^\alpha x = -a(x - y - c \sin u) \\ {}^C_0D_t^\alpha y = bx - xz - y \\ {}^C_0D_t^\alpha z = xy - dz \\ {}^C_0D_t^\alpha u = w \end{cases} \quad \text{if } 0 < t \leq t_1 \quad (5.3)$$

$$\begin{cases} dx = (-a(x - y - c \sin u)) dt + \sigma_1 x dB_1(t) \\ dy = (bx - xz - y) dt + \sigma_2 y dB_2(t) \\ dz = (xy - dz) dt + \sigma_3 z dB_3(t) \\ du = w dt + \sigma_4 u dB_4(t) \end{cases} \quad \text{if } t_1 \leq t \leq T$$

where initial conditions are as follows

$$x(0) = 0.1, y(0) = 0.1, z(0) = 0.1, u(0) = 0.1. \quad (5.4)$$

In Figure 7, the numerical simulations are depicted for the parameters $a = 10, b = 30, d = 8/3, c = 17, w = 0.5$.

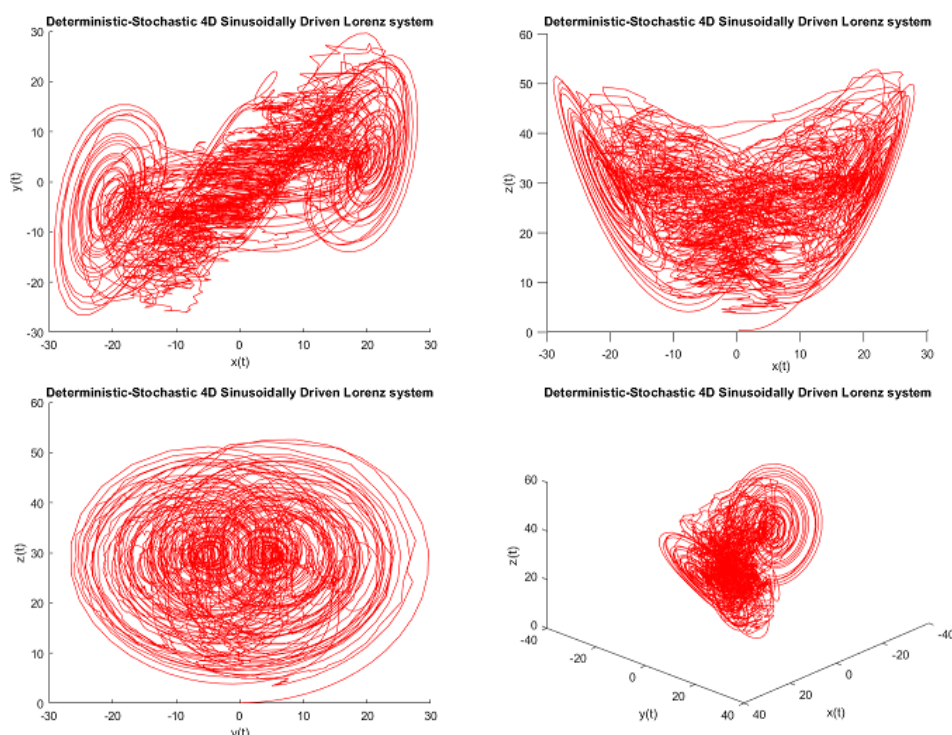


Figure 7. Numerical visualization of 4D Sinusoidally Driven Lorenz system for $\alpha = 1, \sigma_1 = 0.012, \sigma_2 = 0.013, \sigma_3 = 0.015, \sigma_4 = 0.014$.

5.3. Numerical simulation for stochastic-deterministic 4D Sinusoidally Driven Lorenz system model

To give numerical simulation for 4D sinusoidally Driven Lorenz model[14], we deal with the following problem

$$\left\{ \begin{array}{l} {}^CF_0^\alpha D_t^\alpha x = -a(x - y - c \sin u) \\ {}^CF_0^\alpha D_t^\alpha y = bx - xz - y \\ {}^CF_0^\alpha D_t^\alpha z = xy - dz \\ {}^CF_0^\alpha D_t^\alpha u = w \end{array} \right. \quad \text{if } 0 < t \leq t_1 \quad (5.5)$$

$$\left\{ \begin{array}{l} dx = (-a(x - y - c \sin u)) dt + \sigma_1 x dB_1(t) \\ dy = (bx - xz - y) dt + \sigma_2 y dB_2(t) \\ dz = (xy - dz) dt + \sigma_3 z dB_3(t) \\ du = w dt + \sigma_4 u dB_4(t) \end{array} \right. \quad \text{if } t_1 \leq t \leq T$$

where first part is with exponential decay kernel and the second part is stochastic. Initial conditions are as follows

$$x(0) = 0.1, y(0) = 0.1, z(0) = 0.1, u(0) = 0.1. \quad (5.6)$$

In Figure 8, we provide the numerical simulations for 4D Sinusoidally Driven Lorenz system with the parameters $a = 10, b = 30, d = 8/3, c = 17, w = 0.5$.

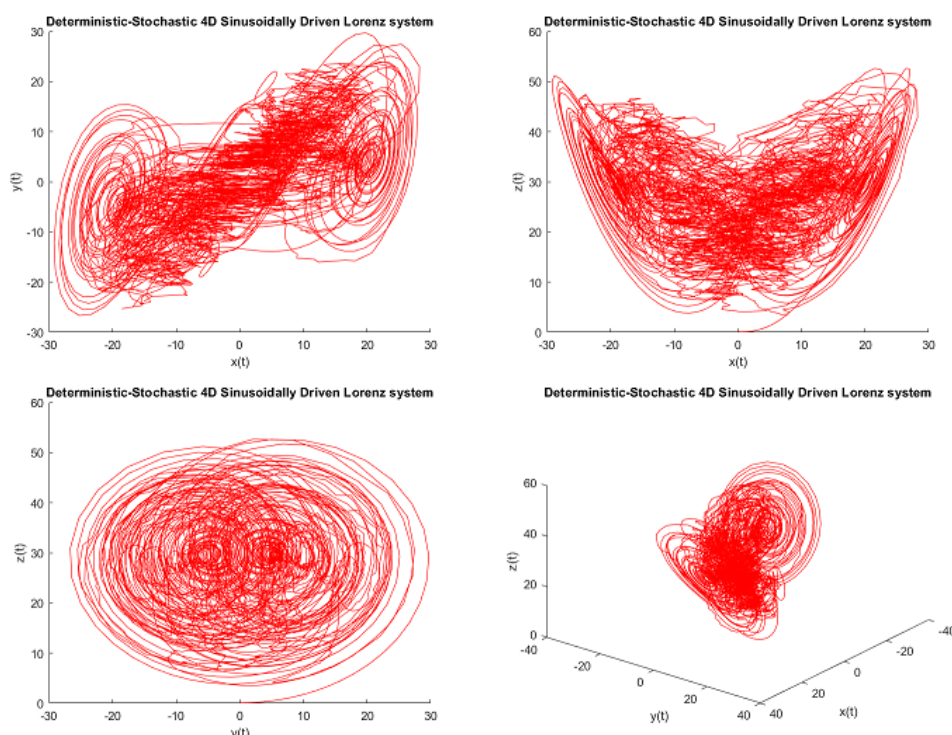


Figure 8. Numerical visualization of 4D Sinusoidally Driven Lorenz system for $\alpha = 1, \sigma_1 = 0.012, \sigma_2 = 0.013, \sigma_3 = 0.015, \sigma_4 = 0.014$.

5.4. Numerical simulation for stochastic-deterministic 4D Sinusoidally Driven Lorenz system model

In this subsection, we present the numerical simulation for the following 4D sinusoidally Driven Lorenz model[14]

$$\begin{cases} {}_0^{ABC}D_t^\alpha x = -a(x - y - c \sin u) \\ {}_0^{ABC}D_t^\alpha y = bx - xz - y \\ {}_0^{ABC}D_t^\alpha z = xy - dz \\ {}_0^{ABC}D_t^\alpha u = w \end{cases} \quad \text{if } 0 < t \leq t_1 \quad (5.7)$$

$$\begin{cases} dx = (-a(x - y - c \sin u)) dt + \sigma_1 x dB_1(t) \\ dy = (bx - xz - y) dt + \sigma_2 y dB_2(t) \\ dz = (xy - dz) dt + \sigma_3 z dB_3(t) \\ du = w dt + \sigma_4 u dB_4(t) \end{cases} \quad \text{if } t_1 \leq t \leq T$$

where initial conditions are as follows

$$x(0) = 0.1, y(0) = 0.1, z(0) = 0.1, u(0) = 0.1. \quad (5.8)$$

In Figure 9, the numerical simulations for the considered system are performed with the parameters $a = 10, b = 30, d = 8/3, c = 17, w = 0.5$.

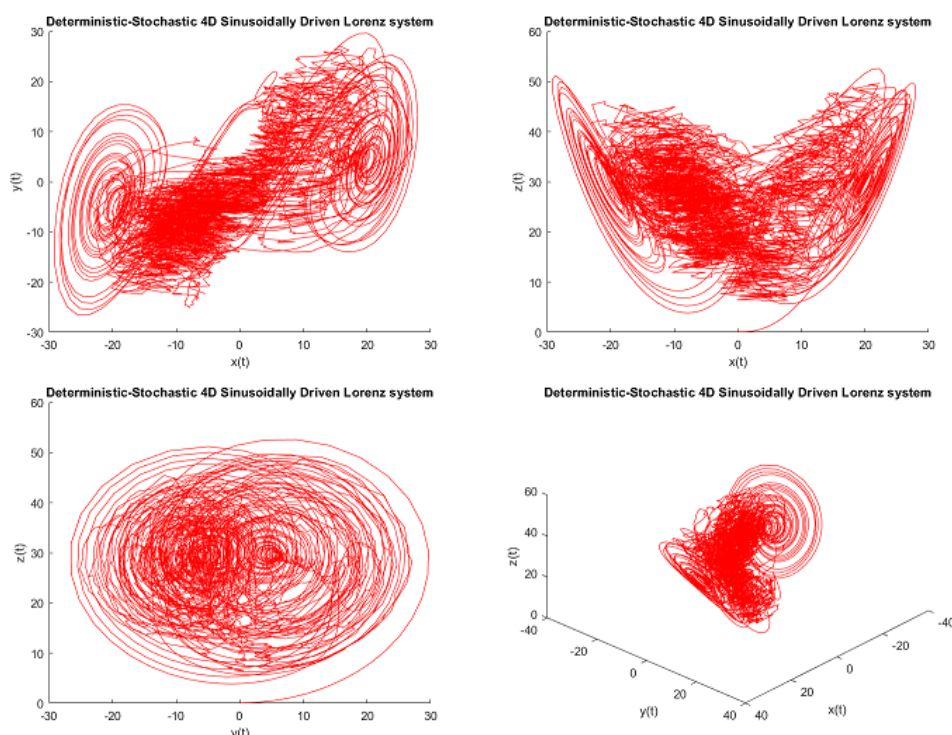


Figure 9. Numerical visualization of 4D Sinusoidally Driven Lorenz system for $\alpha = 1, \sigma_1 = 0.012, \sigma_2 = 0.013, \sigma_3 = 0.015, \sigma_4 = 0.014$.

5.5. Numerical simulation for stochastic-deterministic 4D Sinusoidally Driven Lorenz system model

We consider 4D sinusoidally Driven Lorenz model[14] with piecewise derivative where the first part is stochastic and the second part is global derivative

$$\left\{ \begin{array}{l} dx = (-a(x - y - c \sin u))dt + \sigma_1 x dB_1(t) \\ dy = (bx - xz - y)dt + \sigma_2 y dB_2(t) \\ dz = (xy - dz)dt + \sigma_3 z dB_3(t) \\ du = wdt + \sigma_4 u dB_4(t) \end{array} \right. \quad \text{if } 0 \leq t \leq t_1 \quad (5.9)$$

$$\left\{ \begin{array}{l} D_g^\alpha x = -a(x - y - c \sin u) \\ D_g^\alpha y = bx - xz - y \\ D_g^\alpha z = xy - dz \\ D_g^\alpha u = w \end{array} \right. \quad \text{if } t_1 \leq t \leq T$$

such that $g(t) = t$. Here initial conditions are as follows

$$x(0) = 0.1, y(0) = 0.1, z(0) = 0.1, u(0) = 0.1. \quad (5.10)$$

In Figure 10, the numerical simulations are depicted for the considered problem where the parameters are chosen as $a = 10, b = 30, d = 8/3, c = 17, w = 0.5$.

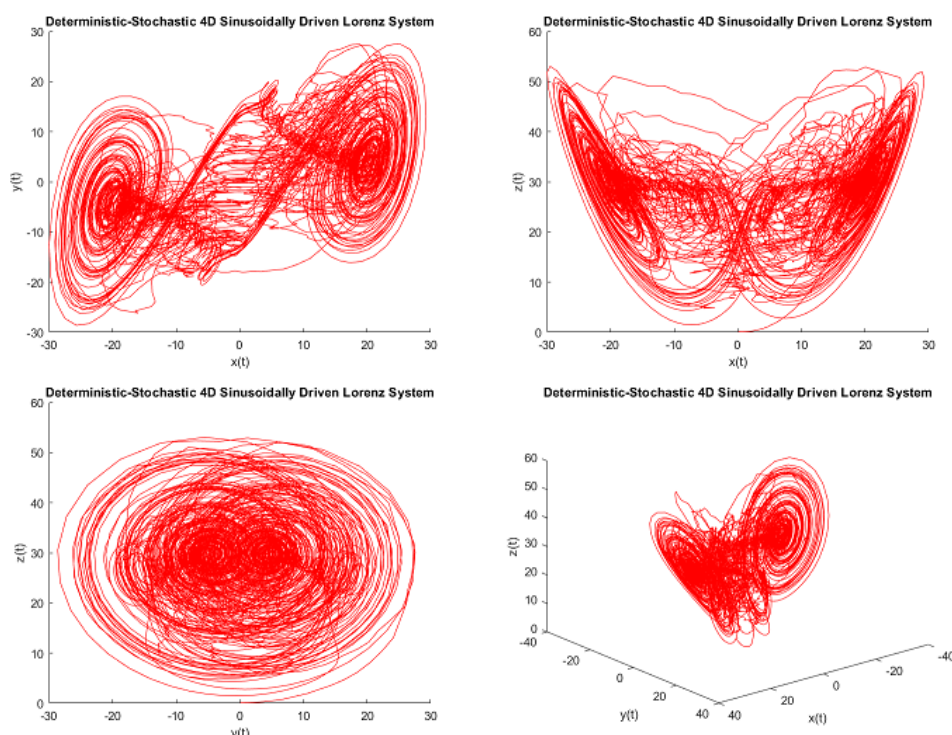


Figure 10. Numerical visualization of 4D Sinusoidally Driven Lorenz system for $\alpha = 1, \sigma_1 = 0.012, \sigma_2 = 0.013, \sigma_3 = 0.015, \sigma_4 = 0.014$.

6. Applications for stochastic-deterministic model of Bouali system

In this section, we present applications of stochastic-deterministic Bouali system for all cases [15].

6.1. Numerical simulation for stochastic-deterministic model of Bouali system

We consider the Bouali model[15] in piecewise where the first part is classical and the second part is stochastic as follows

$$\begin{cases} \frac{dx}{dt} = ax(1-y) - \beta z \\ \frac{dy}{dt} = -y(1-x^2) \\ \frac{dz}{dt} = \mu x \end{cases} \quad \text{if } 0 \leq t \leq t_1 \quad (6.1)$$

$$\begin{cases} dx = (ax(1-y) - \beta z)dt + \sigma_1 x dB_1(t) \\ dy = (-by(1-x^2))dt + \sigma_2 y dB_2(t) \\ dz = \mu x dt + \sigma_3 z dB_3(t) \end{cases} \quad \text{if } t_1 \leq t \leq T \quad (6.2)$$

where initial conditions are taken as

$$x(0) = 1, y(0) = 1, z(0) = -0.02. \quad (6.3)$$

In Figure 11 and 12, the numerical simulations are depicted for the parameters $a = 3, \beta = 2.2, b = 1, \mu = 0.001$.

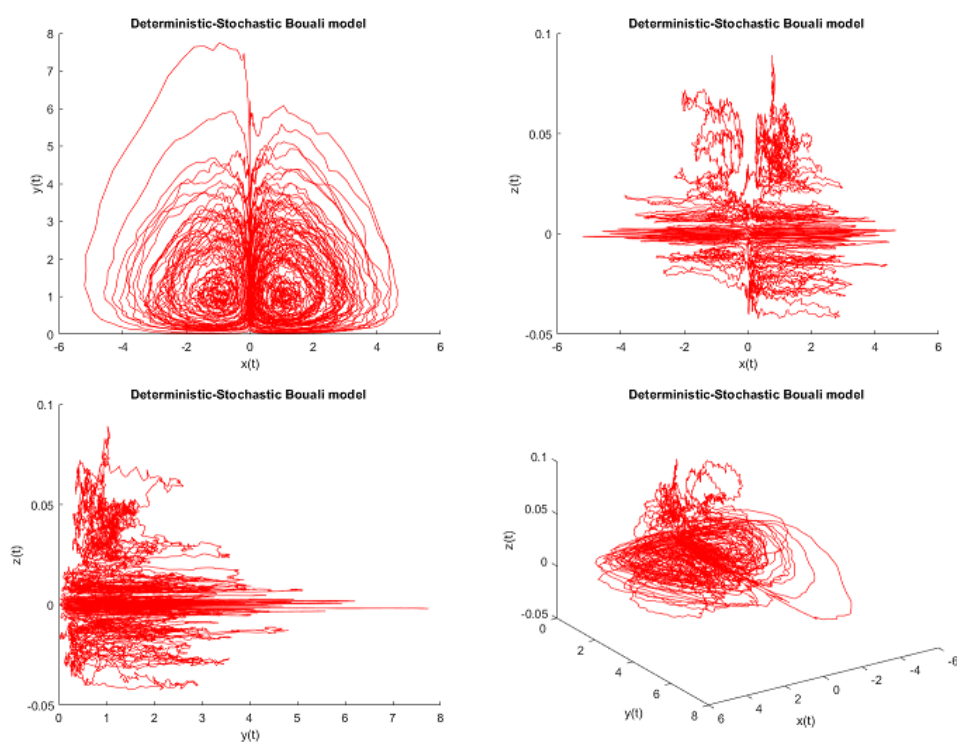


Figure 11. Numerical visualization of Bouali system for $\alpha = 1, \sigma_1 = 0.02, \sigma_2 = 0.022, \sigma_3 = 0.025$.

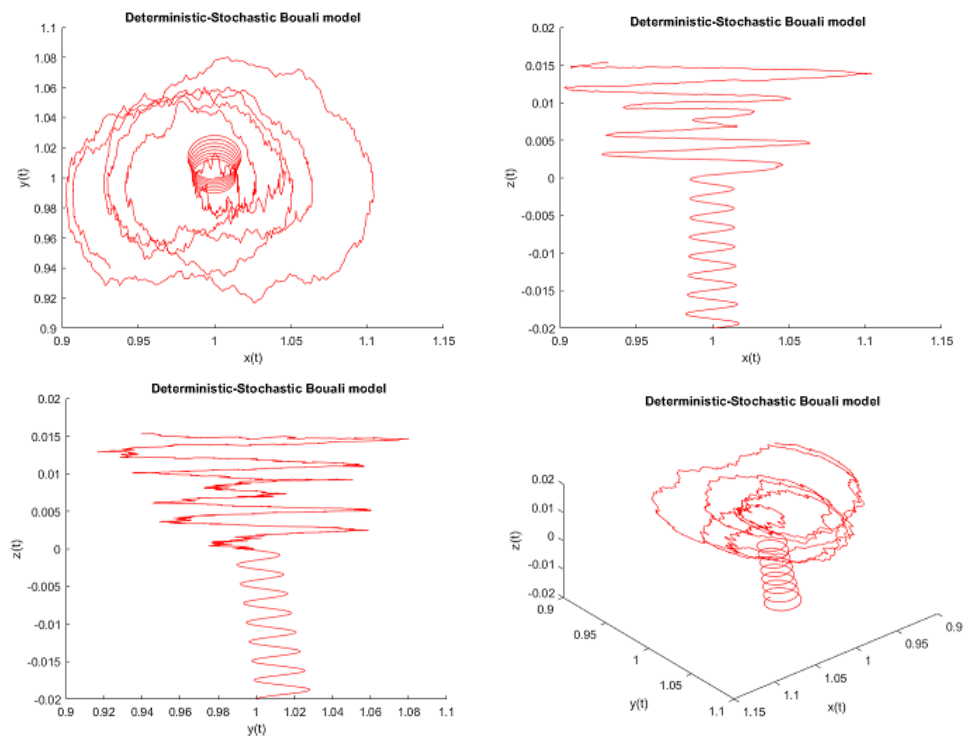


Figure 12. Numerical visualization of Bouali system for $\alpha = 1, \sigma_1 = 0.02, \sigma_2 = 0.022, \sigma_3 = 0.025$.

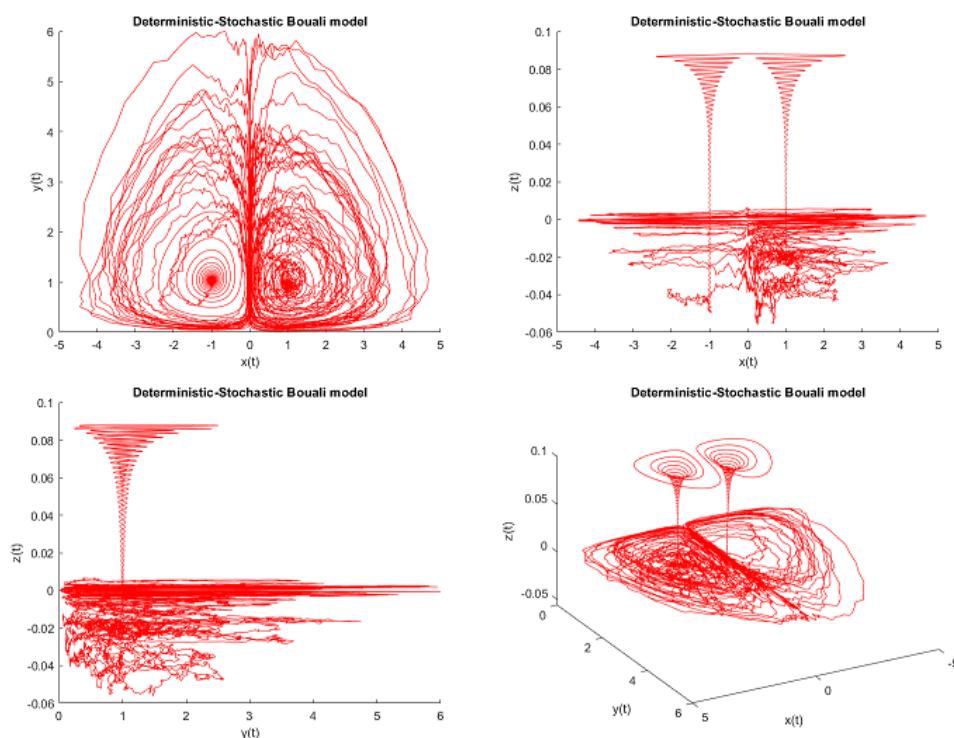


Figure 13. Numerical visualization of Bouali system for $\alpha = 1, \sigma_1 = 0.02, \sigma_2 = 0.022, \sigma_3 = 0.025$.

In Figure 14, we present the numerical simulations for stochastic-deterministic Bouali system for the parameters $a = 3, \beta = 2.2, b = 1, \mu = 1.51$.

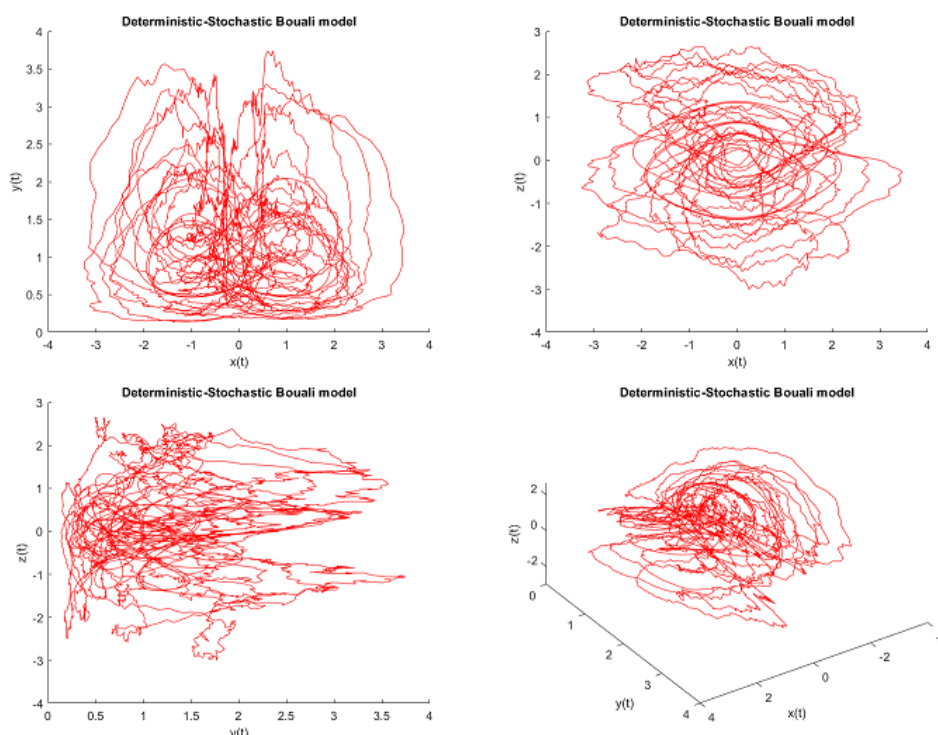


Figure 14. Numerical visualization of Bouali system for $\alpha = 1, \sigma_1 = 0.02, \sigma_2 = 0.022, \sigma_3 = 0.025$.

6.2. Numerical simulation for stochastic-deterministic model of Bouali system

We consider the Bouali model[15]

$$\begin{cases} {}^C_0D_t^\alpha x = ax(1-y) - \beta z \\ {}^C_0D_t^\alpha y = -y(1-x^2) \\ {}^C_0D_t^\alpha z = \mu x \end{cases} \quad \text{if } 0 < t \leq t_1 \quad (6.4)$$

$$\begin{cases} dx = (ax(1-y) - \beta z)dt + \sigma_1 x dB_1(t) \\ dy = (-by(1-x^2))dt + \sigma_2 y dB_2(t) \\ dz = \mu x dt + \sigma_3 z dB_3(t) \end{cases} \quad \text{if } t_1 \leq t \leq T \quad (6.5)$$

where the first part is Caputo and the second part is stochastic. Initial conditions are taken as

$$x(0) = 1, y(0) = 1, z(0) = -0.02. \quad (6.6)$$

In Figure 15, the numerical simulations are depicted for the parameters $a = 3, \beta = 2.2, b = 1, \mu = 0.001$.

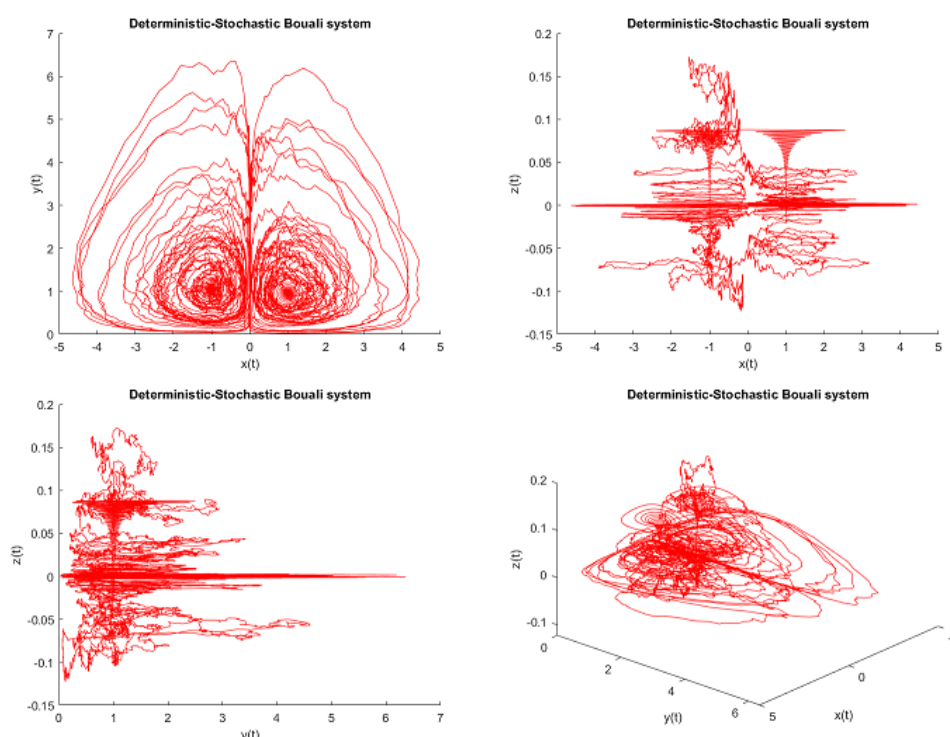


Figure 15. Numerical visualization of Bouali system for $\alpha = 1, \sigma_1 = 0.02, \sigma_2 = 0.022, \sigma_3 = 0.025$.

6.3. Numerical simulation for stochastic-deterministic model of Bouali system

We consider the Bouali model[15] with piecewise derivative

$$\begin{cases} {}^{\text{CF}}_0 D_t^\alpha x = ax(1-y) - \beta z \\ {}^{\text{CF}}_0 D_t^\alpha y = -y(1-x^2) \\ {}^{\text{CF}}_0 D_t^\alpha z = \mu x \end{cases} \quad \text{if } 0 < t \leq t_1 \quad (6.7)$$

$$\begin{cases} dx = (ax(1-y) - \beta z)dt + \sigma_1 x dB_1(t) \\ dy = (-by(1-x^2))dt + \sigma_2 y dB_2(t) \\ dz = \mu x dt + \sigma_3 z dB_3(t) \end{cases} \quad \text{if } t_1 \leq t \leq T. \quad (6.8)$$

Initial conditions are given by

$$x(0) = 1, y(0) = 1, z(0) = -0.02. \quad (6.9)$$

In Figure 16, the numerical simulations are provided for Bouali system where the parameters are taken as $a = 3, \beta = 2.2, b = 1, \mu = 0.001$.

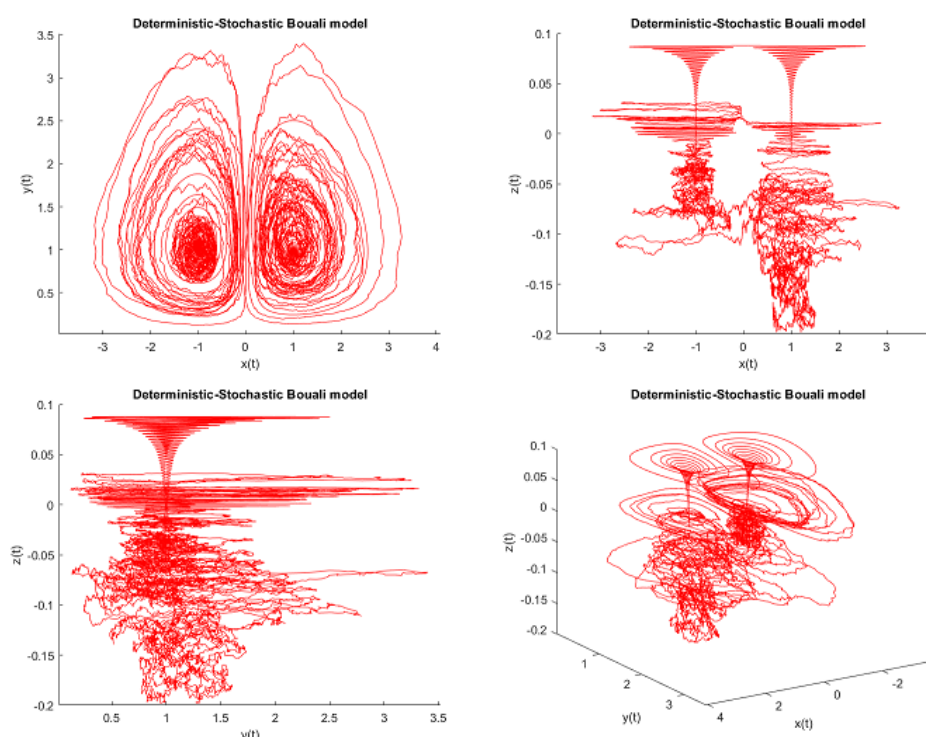


Figure 16. Numerical visualization of Bouali system for $\alpha = 1, \sigma_1 = 0.02, \sigma_2 = 0.022, \sigma_3 = 0.025$.

6.4. Numerical simulation for stochastic-deterministic model of Bouali system

We deal with the stochastic-deterministic Bouali model[15]

$$\begin{cases} {}_0^{AB}D_t^\alpha x = ax(1-y) - \beta z \\ {}_0^{AB}D_t^\alpha y = -y(1-x^2) \\ {}_0^{AB}D_t^\alpha z = \mu x \end{cases} \quad \text{if } 0 < t \leq t_1 \quad (6.10)$$

$$\begin{cases} dx = (ax(1-y) - \beta z)dt + \sigma_1 x dB_1(t) \\ dy = (-by(1-x^2))dt + \sigma_2 y dB_2(t) \\ dz = \mu x dt + \sigma_3 z dB_3(t) \end{cases} \quad \text{if } t_1 \leq t \leq T. \quad (6.11)$$

Here the first part is Atangana-Baleanu fractional derivative and the second part is stochastic. Also initial conditions are taken as

$$x(0) = 1, y(0) = 1, z(0) = -0.02. \quad (6.12)$$

In Figure 17, we provide simulations for Bouali system with the parameters $a = 3, \beta = 2.2, b = 1, \mu = 0.001$.

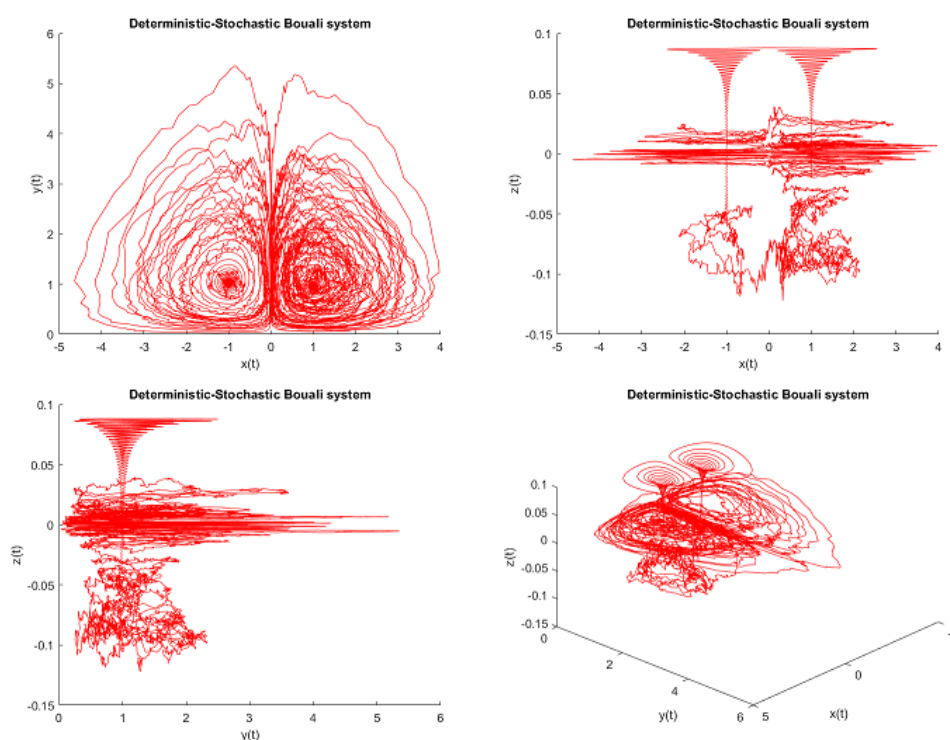


Figure 17. Numerical visualization of Bouali system for $\alpha = 1, \sigma_1 = 0.02, \sigma_2 = 0.022, \sigma_3 = 0.025$.

6.5. Numerical simulation for stochastic-deterministic model of Bouali system

We consider the Bouali model[15] with piecewise derivative

$$\begin{cases} dx = (ax(1-y) - \beta z)dt + \sigma_1 x dB_1(t) \\ dy = (-by(1-x^2))dt + \sigma_2 y dB_2(t) \\ dz = \mu x dt + \sigma_3 z dB_3(t) \end{cases} \quad \text{if } 0 \leq t \leq t_1 \quad (6.13)$$

$$\begin{cases} D_g^\alpha x = ax(1-y) - \beta z \\ D_g^\alpha y = -y(1-x^2) \\ D_g^\alpha z = \mu x \end{cases} \quad \text{if } t_1 \leq t \leq T \quad (6.14)$$

such that $g(t) = t$. Also initial conditions are as follows

$$x(0) = 1, y(0) = 1, z(0) = -0.02. \quad (6.15)$$

In Figure 18, the numerical simulations are presented for the parameters $a = 3, \beta = 2.2, b = 1, \mu = 0.001$.

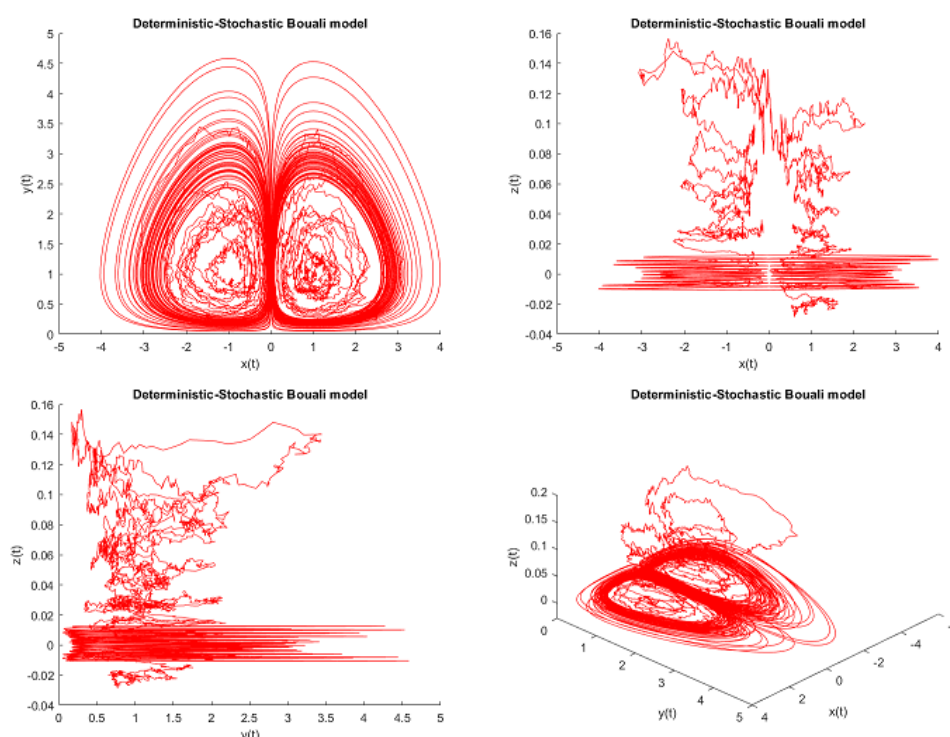


Figure 18. Numerical visualization of Bouali system for $\alpha = 1, \sigma_1 = 0.02, \sigma_2 = 0.022, \sigma_3 = 0.025$.

7. Conclusion

Using the concept of piecewise differentiation and integration suggested by Atangana and Seda recently, we suggested classes of nonlinear ordinary differential equations. These classes are called piecewise deterministic stochastic equations. We suggested five classes, of course more classes can be obtained depending on how the piecewise differential and integral operators are defined with given intervals. We informed that, of each class presented, a suitable method to prove existence and uniqueness of piecewise solution can be performed in suitable spaces. However, we also informed that such discussion was not presented in general in our paper. Nevertheless, we considered a few chaotic models and extended each one of the five classes suggested, for some we presented conditions under which existence and unique solutions are obtained. General numerical solutions using the Newton polynomial interpolation were obtained. Illustrative examples are presented together with simulations. The obtained solutions leave us no choice to conclude that, this approach is the future of modelling complex real world problems.

Conflict of interest

The authors declare there is no conflict of interest.

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