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Research article

Rhythmic behaviors of the human heart with piecewise derivative

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Abstract: It has been noticed that heartbeats can display different patterns according to situations faced by a human. It has been indicated that, those passages from one pattern to another cannot be modelled using a single differential operator, either classical, fractional, or stochastic. In 2021, alternative concepts were introduced and called piecewise differentiation and integration, these concepts were applied in several complex problems with great insight. It is strongly believed that such will be leading concepts to modelling real-world problems with crossover behaviors. Crossover behaviors have been observed in heart rhythm, therefore, in this paper, the well-known van Der Pol equation will be subjected to piecewise analysis. Several simulations will be obtained using a numerical scheme based on Newton polynomial interpolation. Obtained figures show real world behaviors of heart rhythm with piecewise patterns.

Keywords: crossover behaviors; piecewise derivative; piecewise integral; heart rhythm

1. Introduction

Mathematicians always aim at modelling real world behaviors that they observed in their daily activities. Several important biological problems have been modelled successfully using the concept of differentiation and integral in the last periods. Heartbeat rhythm as important phenomena taking place within every living biological being including animals and humans. One of the important aspects of this biological phenomenon is perhaps the rhythm understanding, as it is very useful to information for health care during the period of Emergency Medical Services. Indeed, the cyclic machine-driven pumping of the heart is vital for the nourishment of the physique function. The classification of cardiac rhythms has been listed in the literature. It has been documented that a normal cardiac should always have a normal sinus rhythm, they can be noticed by a ventricular rate ranging from 60 to 100 bpm with a regular rate step from 0.120 to 0.20 second. A sinus bradycardia was documented to be a regular rhythm, where the ventricular rate is between 40 to 60 bpm with a regular PR interval. Another was known as sinus tachycardia, which is a regular cardiac rhythm where the ventricular step is quicker between 100-160 bpm. Nevertheless, sinus arrhythmia was documented to be an irregular rhythm where a ventricular step range between 60 to100 normally, nevertheless a slow rhythm can be illustrious when the step is smaller than 60. Finally, sinus pause is known to be a regular rhythm nevertheless a sudden pause is detected in the rhythm which makes it miss a few beats. However, if rhythm resumes on time after the pause, then this was recorded to be a sinus block. However, if the rhythm does not resume time after the steady state, this was recorded as a sinus arrest. It was also documented that; cardiac disease is leading sources of indisposition and impermanence in the developed world. Several studies have been done in the last decades to understand precisely cardiac function during the period of heath and disease for both humans and animals. As modelling is the key on several occasions to better predict future behaviors, in this field also, advanced computational modelling of heart electromechanical function has become very important complementary methodology to experimental and clinical investigation. Baleanu et al, applied fractional differentiation of linear triatomic molecule [1], the model of control for immunogenic tumor was investigated under the framework of fractional differentiation [2], Atangana and Seda presented a concept of deterministic-stochastic approach to capture crossover behaviors [3]. A modified model of heart was presented in [4], while a model of Covid-19 spread in Turkey and South Africa was studied in [5], optimal control of Covid-19 spread model was investigated in [6]. Khan et al analyzed the rabies model using harmonic mean type of incident rate [7], while Baleanu et al investigated the behavior of hyperchaotic in [8]. Several authors have investigated heart rhythm using some mathematical models, for example Gokus et al, [9] studied this model for the simulation of electric fields, Balakrishnan et al, performed the simulation of cardiac arrhythmias [10], and a similar investigation was performed in [11] and [12]. Caputo Fabrizio fractional derivative was applied to a pneumococcal pneumonia infection model in [13]. The prediction of heart rhythmic using the Van Der Pol Oscillator model was presented in [13] and [14] and a similar investigation was performed in [15] and [16]. Some mathematical models of heart rhythm have been presented and studies in the literature, for example the well-known Van Der Pol oscillation modelled has been studied and was revealed to model several behaviors in electro-mechanic [17]. One of its applications was found to be heart rhythm. There are several emotions that make the cardiac rhythm to present different patterns, for example, a human that got a very bad new after having good time, his heart rhythm will have a passage from normal rhythm to irregular rhythm. In fact, on some occasion's humans got heart attack to due sudden change in heart rhythm. Indeed, the crossover behaviors have posed a great challenge to modelers, as some time the changes vary from nonlocal behaviors to classical behaviors, from normal to stochastic . A single mathematical differential operator cannot be used to account for crossover behaviors, this argument was presented by Atangana and Seda via the concept of piecewise where can be used singular and nonsigular kernels given in [18] and [19]. They suggested new calculus called piecewise differential and integral calculus in [20] and [21]. In this work, we shall model the heart rhythm using the well-known Van Der Pol model with piecewise differential operators.

2. Piecewise differentiation and integration

In this section, we present a piecewise derivative when the involved derivatives are the classical and fractional differential operators with singular and non-singular kernels introduced in [18] and [19].

Definition 1. Let the function f be differentiable. Thus, the piecewise derivative with classical and fractional derivatives with exponential kernel[21] is given as

$$\begin{cases} f\prime(t) \text{ if } 0 \le t \le t_1, \\ {}^{CF}_{t_1}D^{\alpha}_{t} f\ (t) \text{ if } t_1 \le t \le T \end{cases}. \tag{2.1}$$

Definition 2. Let the function f be continuous. A piecewise integral[21] of the function f is given as

$$\begin{cases} \int_0^t f\ (\tau)\, d\tau \text{ if } 0 \le t \le t_1, \\ \frac{1-\alpha}{M(\alpha)} f\ (t) + \frac{\alpha}{M(\alpha)} \int_{t_1}^t f\ (\tau)\, d\tau \text{ if } t_1 \le t \le T \end{cases}. \tag{2.2}$$

Definition 3. Assume that the function f is differentiable. The piecewise derivative with classical and Mittag-Leffler kernel[21] is given as

$$\begin{cases} f\prime(t) \text{ if } 0 \le t \le t_1, \\ {}^{ABC}_{t_1}D^{\alpha}_{t} f\ (t) \text{ if } t_1 \le t \le T \end{cases}. \tag{2.3}$$

Definition 4. Assume that the function f is continuous. The piecewise integral with Mittag-Leffler kernel[21] is given as

$$\begin{cases} \int_0^t f\ (\tau)\, d\tau \text{ if } 0 \le t \le t_1, \\ \frac{1-\alpha}{AB(\alpha)} f\ (t) + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \int_{t_1}^t f\ (\tau)\,(t-\tau)^{\alpha-1}\, d\tau \text{ if } t_1 \le t \le T \end{cases}. \tag{2.4}$$

Definition 5. Let the function f is differentiable. The piecewise derivative with classical and power-law kernel[21] is given as

$$\begin{cases} f\prime(t) \text{ if } 0 \le t \le t_1, \\ {}^{C}_{t_1}D^{\alpha}_{t} f\ (t) \text{ if } t_1 \le t \le T \end{cases}. \tag{2.5}$$

Definition 6. Let the function f is continuous. The piecewise integral with power-law kernel[21] is given as

$$\begin{cases} \int_0^t f\ (\tau)\, d\tau \text{ if } 0 \le t \le t_1, \\ \frac{1}{\Gamma(\alpha)} \int_{t_1}^t f\ (\tau)\,(t-\tau)^{\alpha-1}\, d\tau \text{ if } t_1 \le t \le T \end{cases}. \tag{2.6}$$

3. A mathematical model of heart rhythm model with piecewise modelling

To model heart rhythm with different patterns, we will use the Van Der Pol oscillator introduced by Balthasar Van Der Pol. This oscillator can be written in its two-dimensional form[14]

$$\begin{aligned} \dot{x} &= \mu\left(x - \frac{x^3}{3} - y\right), \\ \dot{y} &= \frac{1}{\mu}x \end{aligned} \tag{3.1}$$

where the parameter μ describes the strength of the damping.

To achieve our aim, we will consider some scenarios where human heart exhibits three processes: with irregular rhythm, regular rhythm and no rhythm. These scenarios will be implemented using piecewise differential and integral operators, where randomness and fractional differential operators with different kernels can be used.

Case 1: Piecewise heart rhythm model with randomness-power-law-classical

In this case, we assume a scenario where heart exhibits anomalous behavior from 0 to T_1, later heart depicts normal rhythm with power law process from T_1 to T_2 and then it dies from T_2 to T. A piecewise mathematical model depicting this scenario can be given as

$$\begin{cases} \begin{cases} dx = \mu\left(x - \frac{x^3}{3} - y\right)dt + \sigma_1 x dB_1(t) \\ dy = \frac{1}{\mu} x dt + \sigma_2 y dB_2(t) \end{cases}, \text{if } 0 \leq t \leq T_1, \\ x(0) = x_0, y(0) = y_0, \\ \begin{cases} {}_{T_1}^{C}D_t^{\alpha} x = \mu\left(x - \frac{x^3}{3} - y\right) \\ {}_{T_1}^{C}D_t^{\alpha} y = \frac{1}{\mu} x \end{cases} \text{if } T_1 \leq t \leq T_2, \\ x(T_1) = x_1, y(T_1) = y_1, 0 < \alpha \leq 1 \\ \begin{cases} \dot{x} = 0 \\ \dot{y} = 0 \end{cases}, \text{if } T_2 \leq t \leq T, \\ x(T_2) = x_2, y(T_2) = y_2 \end{cases} \tag{3.2}$$

where σ_1, σ_2 are densities of randomness and $B_1(t), B_2(t)$ are the functions of noise.

Case 2: Piecewise heart rhythm model with randomness-Mittag–Leffler law-classical

In this case, we assume a scenario where heart exhibits anomalous behavior performed by randomness within the interval $0 \leq t \leq T_1$, later heart depicts normal rhythm with Mittag-Leffler law within the interval $T_1 \leq t \leq T_2$ and then it dies within the interval $T_2 \leq t \leq T$. A piecewise model associated to the above scenarios can be defined by

$$\begin{cases} \begin{cases} dx = \mu\left(x - \frac{x^3}{3} - y\right)dt + \sigma_1 x dB_1(t) \\ dy = \frac{1}{\mu} x dt + \sigma_2 y dB_2(t) \end{cases}, \text{if } 0 \leq t \leq T_1, \\ x(0) = x_0, y(0) = y_0, \\ \begin{cases} {}_{T_1}^{ABC}D_t^{\alpha} x = \mu\left(x - \frac{x^3}{3} - y\right) \\ {}_{T_1}^{ABC}D_t^{\alpha} y = \frac{1}{\mu} x \end{cases} \text{if } T_1 \leq t \leq T_2, \\ x(T_1) = x_1, y(T_1) = y_1, , 0 < \alpha \leq 1 \\ \begin{cases} \dot{x} = 0 \\ \dot{y} = 0 \end{cases}, \text{if } T_2 \leq t \leq T, \\ x(T_2) = x_2, y(T_2) = y_2. \end{cases} \tag{3.3}$$

Case 3: Piecewise heart rhythm model with randomness-exponential decay law-classical

In this case, we assume that heart has a process anomalous behavior performed by randomness within the interval $0 \leq t \leq T_1$, later heart has normal rhythm with exponential decay law within the interval $T_1 \leq t \leq T_2$ and then it dies within the interval $T_2 \leq t \leq T$. A piecewise mathematical model associate to this can be given as

$$\begin{cases} \begin{cases} dx = \mu\left(x - \frac{x^3}{3} - y\right)dt + \sigma_1 x dB_1(t) \\ dy = \frac{1}{\mu} x dt + \sigma_2 y dB_2(t) \end{cases}, \text{if } 0 \le t \le T_1, \\ x(0) = x_0, y(0) = y_0, \\ \begin{cases} {}^{CF}_{T_1}D^{\alpha}_{t} x = \mu\left(x - \frac{x^3}{3} - y\right) \\ {}^{CF}_{T_1}D^{\alpha}_{t} y = \frac{1}{\mu} x \end{cases} \text{if } T_1 \le t \le T_2, \\ x(T_1) = x_1, y(T_1) = y_1, 0 < \alpha \le 1 \\ \begin{cases} \dot{x} = 0 \\ \dot{y} = 0 \end{cases}, \text{if } T_2 \le t \le T, \\ x(T_2) = x_2, y(T_2) = y_2. \end{cases} \tag{3.4}$$

4. Numerical solution of piecewise heart rhythm model

In this section, we present the numerical solution of piecewise heart rhythm model for three cases that were presented in the last section.

4.1. Numerical solution for Case 1:

In this subsection, the numerical scheme based on Newton polynomial[22] for heart rhythm model with Case 1 is presented.

$$\begin{cases} \begin{cases} dx = \mu\left(x - \frac{x^3}{3} - y\right)dt + \sigma_1 x dB_1(t) \\ dy = \frac{1}{\mu} x dt + \sigma_2 y dB_2(t) \end{cases}, \text{if } 0 \le t \le T_1, \\ x(0) = x_0, y(0) = y_0, \\ \begin{cases} {}^{C}_{T_1}D^{\alpha}_{t} x = \mu\left(x - \frac{x^3}{3} - y\right) \\ {}^{C}_{T_1}D^{\alpha}_{t} y = \frac{1}{\mu} x \end{cases} \text{if } T_1 \le t \le T_2, \\ x(T_1) = x_1, y(T_1) = y_1, 0 < \alpha \le 1 \\ \begin{cases} \dot{x} = 0 \\ \dot{y} = 0 \end{cases}, \text{if } T_2 \le t \le T, \\ x(T_2) = x_2, y(T_2) = y_2 \end{cases}. \tag{4.1}$$

We divide $[0, T]$ in three

$$\begin{aligned} 0 &\le t_1 \le t_1 \le \ldots \le t_{n_1} = T_1 \le t_{n_1+1} \le t_{n_1+2} \le \ldots \le t_{n_2} = T_2 \\ &\le t_{n_2+1} \le t_{n_2+2} \le \ldots \le t_{n_3} = T. \end{aligned} \tag{4.2}$$

The numerical solution can then be provided as

$$\begin{cases} x^{n_1} = x(0) + \sum_{j_1=2}^{n_1} \left[\begin{array}{c} \frac{23}{12}\mu\left(x_{j_1} - \frac{x_{j_1}^3}{3} - y_{j_1}\right) \\ +\sigma_1 x\left(B_1\left(t_{j_1+1}\right) - B_1\left(t_{j_1}\right)\right) \\ -\frac{4}{3}\mu\left(x_{j_1-1} - \frac{x_{j_1-1}^3}{3} - y_{j_1-1}\right) \\ +\sigma_1 x\left(B_1\left(t_{j_1}\right) - B_1\left(t_{j_1-1}\right)\right) \\ +\frac{5}{12}\mu\left(x_{j_1-2} - \frac{x_{j_1-2}^3}{3} - y_{j_1-2}\right) \\ +\sigma_1 x\left(B_1\left(t_{j_1-1}\right) - B_1\left(t_{j_1-2}\right)\right) \end{array} \right] \\ y^{n_1} = y(0) + \sum_{j_1=2}^{n_1} \left[\begin{array}{c} \frac{23}{12}\frac{1}{\mu}x_{j_1} + \sigma_2 y\left(B_2\left(t_{j_1+1}\right) - B_2\left(t_{j_1}\right)\right) \\ -\frac{4}{3}\frac{1}{\mu}x_{j_1-1} + \sigma_2 y\left(B_2\left(t_{j_1}\right) - B_2\left(t_{j_1-1}\right)\right) \\ +\frac{5}{12}\frac{1}{\mu}x_{j_1-2} + \sigma_2 y\left(B_2\left(t_{j_1-1}\right) - B_2\left(t_{j_1-2}\right)\right) \end{array} \right] \end{cases}, 0 \le t \le T_1, \tag{4.3}$$

$$\begin{cases} x^{n_2} = x(T_1) + \frac{(\Delta t)^\alpha}{\Gamma(\alpha+1)} \sum_{j_2=n_1+3}^{n_2} \mu\left(x_{j_2-2} - \frac{x_{j_2-2}^3}{3} - y_{j_2-2}\right) \\ \times\left\{(n_2 - j_2 + 1)^\alpha - (n_2 - j_2)^\alpha\right\} \\ +\frac{(\Delta t)^\alpha}{\Gamma(\alpha+2)} \sum_{j_2=n_1+3}^{n_2} \left[\begin{array}{c} \mu\left(x_{j_2-1} - \frac{x_{j_2-1}^3}{3} - y_{j_2-1}\right) \\ -\mu\left(x_{j_2-2} - \frac{x_{j_2-2}^3}{3} - y_{j_2-2}\right) \end{array} \right] \\ \times\left\{ \begin{array}{c} (n_2 - j_2 + 1)^\alpha (n_2 - j_2 + 3 + 2\alpha) \\ -(n_2 - j_2)^\alpha (n_2 - j_2 + 3 + 3\alpha) \end{array} \right\} \\ +\frac{(\Delta t)^\alpha}{2\Gamma(\alpha+3)} \sum_{j_2=n_1+3}^{n_2} \left[\begin{array}{c} \mu\left(x_{j_2} - \frac{x_{j_2}^3}{3} - y_{j_2}\right) \\ -2\mu\left(x_{j_2-1} - \frac{x_{j_2-1}^3}{3} - y_{j_2-1}\right) \\ +\mu\left(x_{j_2-2} - \frac{x_{j_2-2}^3}{3} - y_{j_2-2}\right) \end{array} \right] \end{cases} \tag{4.4}$$

$$\begin{cases} \times\left(\begin{array}{c} (n_2 - j_2 + 1)^\alpha \left[\begin{array}{c} 2(n_2 - j_2)^2 + (3\alpha + 10)(n_2 - j_2) \\ +2\alpha^2 + 9\alpha + 12 \end{array} \right] \\ -(n_2 - j_2)^\alpha \left[\begin{array}{c} 2(n_2 - j_2)^2 + (5\alpha + 10)(n_2 - j_2) \\ +6\alpha^2 + 18\alpha + 12 \end{array} \right] \end{array} \right) \end{cases}, T_1 \le t \le T_2, \tag{4.5}$$

$$\begin{cases} y^{n_2} = y(T_1) + \frac{(\Delta t)^\alpha}{\Gamma(\alpha+1)} \sum_{j_2=n_1+3}^{n_2} \frac{1}{\mu}x_{j_2-2} \\ \times\left\{(n_2 - j_2 + 1)^\alpha - (n_2 - j_2)^\alpha\right\} \\ +\frac{(\Delta t)^\alpha}{\Gamma(\alpha+2)} \sum_{j_2=n_1+3}^{n_2} \left[\frac{1}{\mu}x_{j_2-1} - \frac{1}{\mu}x_{j_2-2}\right] \\ \times\left\{ \begin{array}{c} (n_2 - j_2 + 1)^\alpha (n_2 - j_2 + 3 + 2\alpha) \\ -(n_2 - j_2)^\alpha (n_2 - j_2 + 3 + 3\alpha) \end{array} \right\} \\ +\frac{(\Delta t)^\alpha}{2\Gamma(\alpha+3)} \sum_{j_2=n_1+3}^{n_2} \left[\frac{1}{\mu}x_{j_2} - 2\frac{1}{\mu}x_{j_2-1} + \frac{1}{\mu}x_{j_2-2}\right] \end{cases} \tag{4.6}$$

$$\begin{cases} \times\left\{ \begin{array}{c} (n_2 - j_2 + 1)^\alpha \left[\begin{array}{c} 2(n_2 - j_2)^2 + (3\alpha + 10)(n_2 - j_2) \\ +2\alpha^2 + 9\alpha + 12 \end{array} \right] \\ -(n_2 - j_2)^\alpha \left[\begin{array}{c} 2(n_2 - j_2)^2 + (5\alpha + 10)(n_2 - j_2) \\ +6\alpha^2 + 18\alpha + 12 \end{array} \right] \end{array} \right\} \end{cases}, T_1 \le t \le T_2, \tag{4.7}$$

$$\begin{cases} x^{n_3} = x(T_2) \\ y^{n_3} = y(T_2) \end{cases}, T_2 \le t \le T. \tag{4.8}$$

4.2. Numerical method for Case 2:

In this subsection, the numerical solution of associated model for Case 2 is presented. To do this, we shall recall that our model

$$\begin{cases} \begin{cases} dx = \mu\left(x - \frac{x^3}{3} - y\right)dt + \sigma_1 x dB_1(t) \\ dy = \frac{1}{\mu} x dt + \sigma_2 y dB_2(t) \end{cases}, \text{if } 0 \le t \le T_1, \\ x(0) = x_0, y(0) = y_0, \\ \begin{cases} {}_{T_1}^{ABC}D_t^{\alpha} x = \mu\left(x - \frac{x^3}{3} - y\right) \\ {}_{T_1}^{ABC}D_t^{\alpha} y = \frac{1}{\mu} x \end{cases} \text{if } T_1 \le t \le T_2, \\ x(T_1) = x_1, y(T_1) = y_1, , 0 < \alpha \le 1 \\ \begin{cases} \dot{x} = 0 \\ \dot{y} = 0 \end{cases}, \text{if } T_2 \le t \le T, \\ x(T_2) = x_2, y(T_2) = y_2. \end{cases} \tag{4.9}$$

The numerical solution can then be provided as

$$\begin{cases} x^{n_1} = x(0) + \sum_{j_1=2}^{n_1} \begin{bmatrix} \frac{23}{12}\mu\left(x_{j_1} - \frac{x_{j_1}^3}{3} - y_{j_1}\right) \\ +\sigma_1 x\left(B_1\left(t_{j_1+1}\right) - B_1\left(t_{j_1}\right)\right) \\ -\frac{4}{3}\mu\left(x_{j_1-1} - \frac{x_{j_1-1}^3}{3} - y_{j_1-1}\right) \\ +\sigma_1 x\left(B_1\left(t_{j_1}\right) - B_1\left(t_{j_1-1}\right)\right) \\ +\frac{5}{12}\mu\left(x_{j_1-2} - \frac{x_{j_1-2}^3}{3} - y_{j_1-2}\right) \\ +\sigma_1 x\left(B_1\left(t_{j_1-1}\right) - B_1\left(t_{j_1-2}\right)\right) \end{bmatrix} \\ y^{n_1} = y(0) + \sum_{j_1=2}^{n_1} \begin{bmatrix} \frac{23}{12}\frac{1}{\mu} x_{j_1} + \sigma_2 y\left(B_2\left(t_{j_1+1}\right) - B_2\left(t_{j_1}\right)\right) \\ -\frac{4}{3}\frac{1}{\mu} x_{j_1-1} + \sigma_2 y\left(B_2\left(t_{j_1}\right) - B_2\left(t_{j_1-1}\right)\right) \\ +\frac{5}{12}\frac{1}{\mu} x_{j_1-2} + \sigma_2 y\left(B_2\left(t_{j_1-1}\right) - B_2\left(t_{j_1-2}\right)\right) \end{bmatrix} \end{cases}, 0 \le t \le T_1, \tag{4.10}$$

$$\left\{\begin{array}{c} x^{n_2} = x(T_1) + \frac{(1-\alpha)}{AB(\alpha)}\mu\left(x_{n_2-1} - \frac{x_{n_2-1}^3}{3} - y_{n_2-1}\right) \\ +\frac{\alpha(\Delta t)^\alpha}{AB(\alpha)\Gamma(\alpha+1)}\sum_{j_2=n_1+3}^{n_2}\mu\left(x_{j_2-2} - \frac{x_{j_2-2}^3}{3} - y_{j_2-2}\right) \\ \times\{(n_2 - j_2 + 1)^\alpha - (n_2 - j_2)^\alpha\} \\ +\frac{\alpha(\Delta t)^\alpha}{AB(\alpha)\Gamma(\alpha+2)}\sum_{j_2=n_1+3}^{n_2}\left[\begin{array}{c} \mu\left(x_{j_2-1} - \frac{x_{j_2-1}^3}{3} - y_{j_2-1}\right) \\ -\mu\left(x_{j_2-2} - \frac{x_{j_2-2}^3}{3} - y_{j_2-2}\right)\end{array}\right] \\ \times\left\{\begin{array}{c} (n_2 - j_2 + 1)^\alpha(n_2 - j_2 + 3 + 2\alpha) \\ -(n_2 - j_2)^\alpha(n_2 - j_2 + 3 + 3\alpha)\end{array}\right\} \\ +\frac{\alpha(\Delta t)^\alpha}{2AB(\alpha)\Gamma(\alpha+3)}\sum_{j_2=n_1+3}^{n_2}\left[\begin{array}{c} \mu\left(x_{j_2} - \frac{x_{j_2}^3}{3} - y_{j_2}\right) \\ -2\mu\left(x_{j_2-1} - \frac{x_{j_2-1}^3}{3} - y_{j_2-1}\right) \\ +\mu\left(x_{j_2-2} - \frac{x_{j_2-2}^3}{3} - y_{j_2-2}\right)\end{array}\right] \end{array}\right. \tag{4.11}$$

$$\left\{\times\left(\begin{array}{c} (n_2 - j_2 + 1)^\alpha\left[\begin{array}{c} 2(n_2 - j_2)^2 + (3\alpha + 10)(n_2 - j_2) \\ +2\alpha^2 + 9\alpha + 12\end{array}\right] \\ -(n_2 - j_2)^\alpha\left[\begin{array}{c} 2(n_2 - j_2)^2 + (5\alpha + 10)(n_2 - j_2) \\ +6\alpha^2 + 18\alpha + 12\end{array}\right]\end{array}\right)\right., T_1 \le t \le T_2, \tag{4.12}$$

$$\left\{\begin{array}{c} y^{n_2} = y(T_1) + \frac{(1-\alpha)}{AB(\alpha)}\frac{1}{\mu}x_{n_2} + \frac{\alpha(\Delta t)^\alpha}{AB(\alpha)\Gamma(\alpha+1)}\sum_{j_2=n_1+3}^{n_2}\frac{1}{\mu}x_{j_2-2} \\ \times\{(n_2 - j_2 + 1)^\alpha - (n_2 - j_2)^\alpha\} \\ +\frac{\alpha(\Delta t)^\alpha}{AB(\alpha)\Gamma(\alpha+2)}\sum_{j_2=n_1+3}^{n_2}\left[\frac{1}{\mu}x_{j_2-1} - \frac{1}{\mu}x_{j_2-2}\right] \\ \times\left\{\begin{array}{c} (n_2 - j_2 + 1)^\alpha(n_2 - j_2 + 3 + 2\alpha) \\ -(n_2 - j_2)^\alpha(n_2 - j_2 + 3 + 3\alpha)\end{array}\right\} \\ +\frac{\alpha(\Delta t)^\alpha}{2AB(\alpha)\Gamma(\alpha+3)}\sum_{j_2=n_1+3}^{n_2}\left[\frac{1}{\mu}x_{j_2} - 2\frac{1}{\mu}x_{j_2-1} + \frac{1}{\mu}x_{j_2-2}\right] \end{array}\right. \tag{4.13}$$

$$\left\{\times\left\{\begin{array}{c} (n_2 - j_2 + 1)^\alpha\left[\begin{array}{c} 2(n_2 - j_2)^2 + (3\alpha + 10)(n_2 - j_2) \\ +2\alpha^2 + 9\alpha + 12\end{array}\right] \\ -(n_2 - j_2)^\alpha\left[\begin{array}{c} 2(n_2 - j_2)^2 + (5\alpha + 10)(n_2 - j_2) \\ +6\alpha^2 + 18\alpha + 12\end{array}\right]\end{array}\right\}\right., T_1 \le t \le T_2, \tag{4.14}$$

$$\left\{\begin{array}{l} x^{n_3} = x(T_2) \\ y^{n_3} = y(T_2)\end{array}\right., T_2 \le t \le T. \tag{4.15}$$

4.3. Numerical method for Case 3:

Here, we present the numerical solution of heart rhythm model with the Case 3. We consider the following model

$$\left\{\begin{array}{c} \left\{\begin{array}{l} dx=\mu\left(x-\frac{x^3}{3}-y\right)dt+\sigma_1 x dB_1(t) \\ dy\ =\frac{1}{\mu}xdt+\sigma_2 y dB_2(t) \end{array}\right., \text{if } 0\le t\le T_1, \\ x(0)=x_0, y(0)=y_0, \\ \left\{\begin{array}{l} {}_{T_1}^{CF}D_t^{\alpha}x=\mu\left(x-\frac{x^3}{3}-y\right) \\ {}_{T_1}^{CF}D_t^{\alpha}y=\frac{1}{\mu}x \end{array}\right. \text{if } T_1\le t\le T_2, \\ x(T_1)=x_1, y(T_1)=y_1,, 0<\alpha\le 1 \\ \left\{\begin{array}{l} \dot{x}=0 \\ \dot{y}\ =0 \end{array}\right., \text{if } T_2\le t\le T, \\ x(T_2)=x_2, y(T_2)=y_2. \end{array}\right. \tag{4.16}$$

We divide $[0,T]$ in three

$$\begin{aligned} 0 &\le t_1\le t_1\le ...\le t_{n_1}=T_1\le t_{n_1+1}\le t_{n_1+2}\le ...\le t_{n_2}=T_2 \\ &\le t_{n_2+1}\le t_{n_2+2}\le ...\le t_{n_3}=T. \end{aligned} \tag{4.17}$$

The numerical solution can then be provided as

$$\left\{\begin{array}{l} x^{n_1}=x(0)+\sum_{j_1=2}^{n_1}\left[\begin{array}{c} \frac{23}{12}\mu\left(x_{j_1}-\frac{x_{j_1}^3}{3}-y_{j_1}\right) \\ +\sigma_1 x\left(B_1\left(t_{j_1+1}\right)-B_1\left(t_{j_1}\right)\right) \\ -\frac{4}{3}\mu\left(x_{j_1-1}-\frac{x_{j_1-1}^3}{3}-y_{j_1-1}\right) \\ +\sigma_1 x\left(B_1\left(t_{j_1}\right)-B_1\left(t_{j_1-1}\right)\right) \\ +\frac{5}{12}\mu\left(x_{j_1-2}-\frac{x_{j_1-2}^3}{3}-y_{j_1-2}\right) \\ +\sigma_1 x\left(B_1\left(t_{j_1-1}\right)-B_1\left(t_{j_1-2}\right)\right) \end{array}\right] \\ y^{n_1}=y(0)+\sum_{j_1=2}^{n_1}\left[\begin{array}{c} \frac{23}{12}\frac{1}{\mu}x_{j_1}+\sigma_2 y\left(B_2\left(t_{j_1+1}\right)-B_2\left(t_{j_1}\right)\right) \\ -\frac{4}{3}\frac{1}{\mu}x_{j_1-1}+\sigma_2 y\left(B_2\left(t_{j_1}\right)-B_2\left(t_{j_1-1}\right)\right) \\ +\frac{5}{12}\frac{1}{\mu}x_{j_1-2}+\sigma_2 y\left(B_2\left(t_{j_1-1}\right)-B_2\left(t_{j_1-2}\right)\right) \end{array}\right] \end{array}\right., 0\le t\le T_1, \tag{4.18}$$

$$\left\{\begin{array}{c} x^{n_2}=x(T_1)+\frac{(1-\alpha)}{M(\alpha)}\mu\left(x_{n_2}-\frac{x_{n_2}^3}{3}-y_{n_2}\right) \\ +\frac{\alpha\Delta t}{M(\alpha)}\sum_{j_2=n_1+3}^{n_2}\left[\begin{array}{c} \frac{23}{12}\mu\left(x_{j_2}-\frac{x_{j_2}^3}{3}-y_{j_2}\right) \\ -\frac{4}{3}\mu\left(x_{j_2-1}-\frac{x_{j_2-1}^3}{3}-y_{j_2-1}\right) \\ +\frac{5}{12}\mu\left(x_{j_2-1}-\frac{x_{j_2-1}^3}{3}-y_{j_2-1}\right) \end{array}\right] \\ y^{n_2}=y(T_1)+\frac{(1-\alpha)}{M(\alpha)}\frac{1}{\mu}x_{n_2} \\ +\frac{\alpha\Delta t}{M(\alpha)}\sum_{j_2=n_1+3}^{n_2}\left[\begin{array}{c} \frac{23}{12}\frac{1}{\mu}x_{j_2}-\frac{4}{3}\frac{1}{\mu}x_{j_2-1} \\ +\frac{5}{12}\frac{1}{\mu}x_{j_2-2} \end{array}\right] \end{array}\right., T_1\le t\le T_2, \tag{4.19}$$

$$\left\{\begin{array}{l} x^{n_3}=x(T_2) \\ y^{n_3}=y(T_2) \end{array}\right., T_2\le t\le T. \tag{4.20}$$

5. Applications of heart rhythm model with piecewise derivative

There is no doubt that the piecewise differential and integral operators are powerful in describing processes with different patterns. Therefore, in this section, we will give some examples where irregular or regular rhythmic behaviors is performed within different intervals.

Example 1. To capture heart rhythms with different patterns, we consider the following problem which is given by

$$\begin{cases} {}_{0}^{ABC}D_t^{\alpha} x = \mu\left(x - \frac{x^3}{3} - y\right) + \sigma_1 x B_1\prime(t) \\ {}_{0}^{ABC}D_t^{\alpha} y = \frac{1}{\mu}x + \sigma_2 y B_2\prime(t). \end{cases} \quad \text{if } 0 \le t \le T_1, \tag{5.1}$$

$$\begin{cases} {}_{T_1}^{ABC}D_t^{\alpha} x = \mu\left(x - \frac{x^3}{3} - y\right) \\ {}_{T_1}^{ABC}D_t^{\alpha} y = \frac{1}{\mu}x. \end{cases} \quad \text{if } T_1 \le t \le T_2,$$

$$\begin{cases} \frac{dx}{dt} = 0 \\ \frac{dy}{dt} = 0 \end{cases} \quad \text{if } T_2 \le t \le T. \tag{5.2}$$

The initial conditions are considered as

$$x(0) = 0.01, y(0) = 0.001,$$

and the parameters are

$$\mu = 2, \alpha = 0.85, \sigma_1 = 0.11, \sigma_2 = 0.23.$$

The numerical simulations for piecewise model are performed in Figure 1-6.

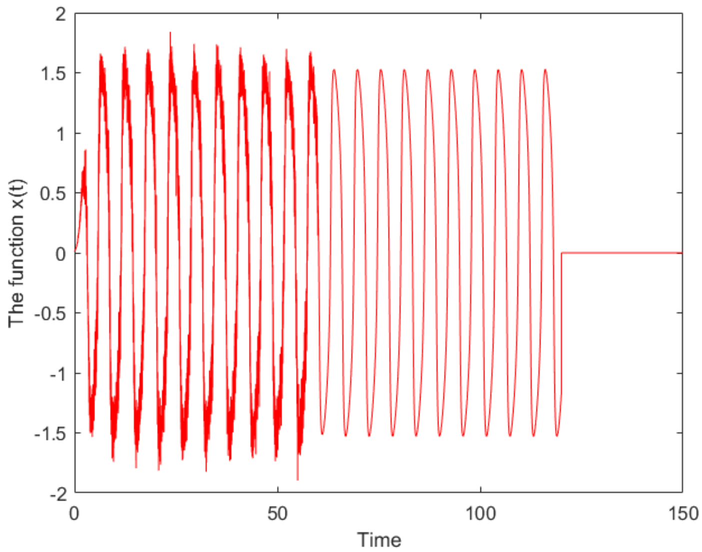

Figure 1. Numerical visualization for the function x.

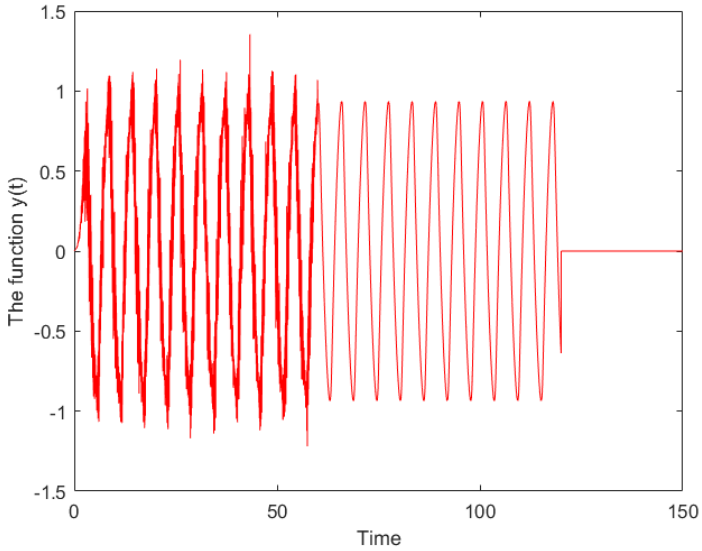

Figure 2. Numerical visualization for the function *y*.

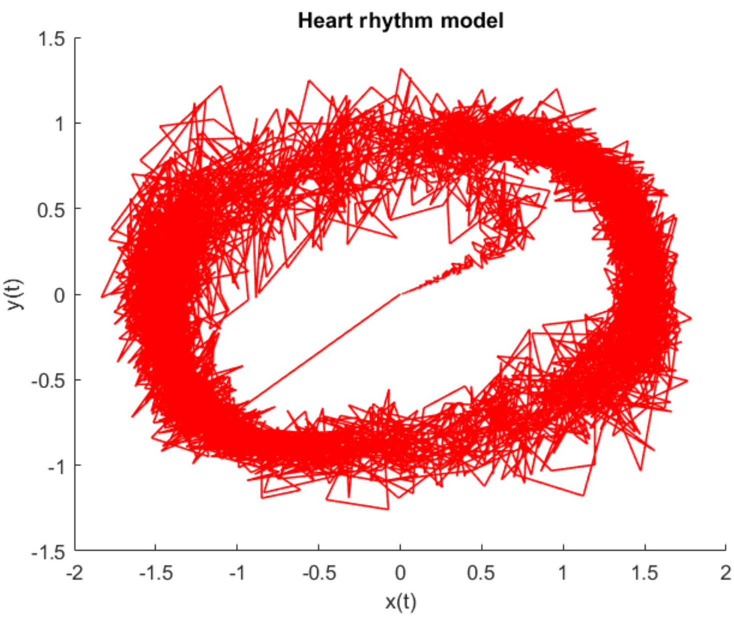

Figure 3. Numerical visualization for the heart rhythm model.

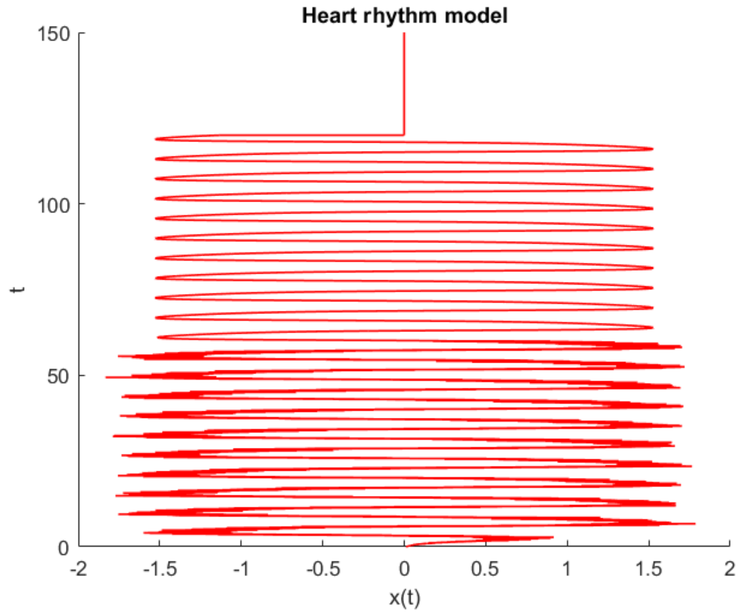

Figure 4. Numerical visualization for the heart rhythm model.

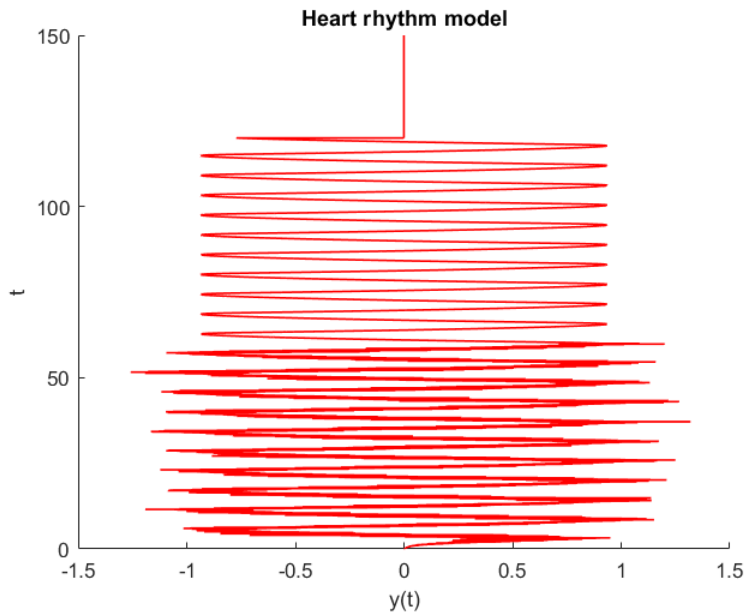

Figure 5. Numerical visualization for the heart rhythm model.

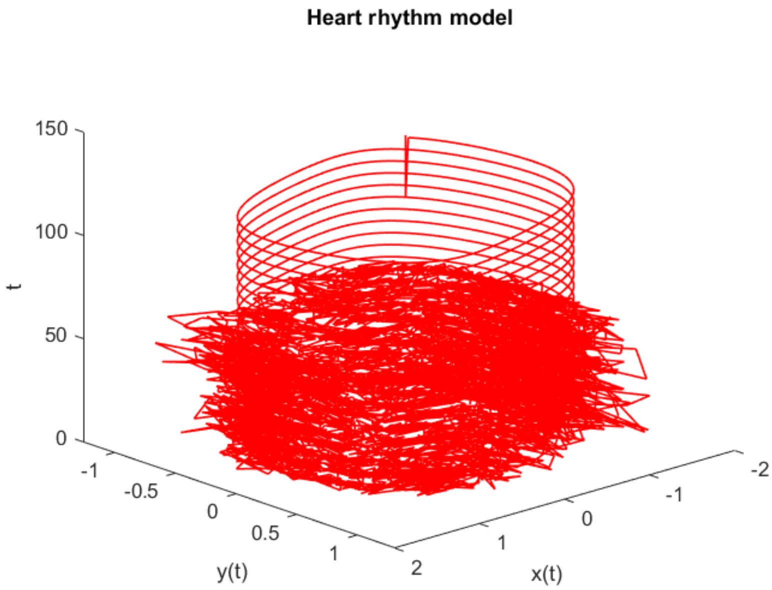

Figure 6. Numerical visualization for the heart rhythm model.

The numerical simulations are performed from Figure 1 to 6 for different fractional orders and densities of randomness. These simulations represent the following process: While a healthy human heart beats normally, his heart may exhibit an anomalous beat which is depicted by randomness after having emotions such as anger, excitement or happiness, later one day the heart beat stops and the individual dies.

Example 2. We next consider heart rhythm model with piecewise derivative

$$\left\{ \begin{array}{c} {}_{0}^{ABC}D_{t}^{\alpha}x = \mu\left(x - \frac{x^3}{3} - y\right) \\ {}_{0}^{ABC}D_{t}^{\alpha}y = \frac{1}{\mu}x. \end{array} \right. \text{ if } 0 \le t \le T_1, \tag{5.3}$$

$$\left\{ \begin{array}{c} {}_{T_1}^{ABC}D_{t}^{\alpha}x = \mu\left(x - \frac{x^3}{3} - y\right) + \sigma_1 xB\prime_1(t) \\ {}_{T_1}^{ABC}D_{t}^{\alpha}y = \frac{1}{\mu}x + \sigma_2 xB\prime_2(t). \end{array} \right. \text{ if } T_1 \le t \le T_2,$$

$$\left\{ \begin{array}{c} \frac{dx}{dt} = 0 \\ \frac{dy}{dt} = 0 \end{array} \right. \text{ if } T_2 \le t \le T. \tag{5.4}$$

The initial conditions are considered as

$$x(0) = 0.01, y(0) = 0.001,$$

and the parameters are

$$\mu = 2, \alpha = 0.9, \sigma_1 = 0.31, \sigma_2 = 0.25.$$

The numerical simulations for piecewise model are performed in Figure 7-12.

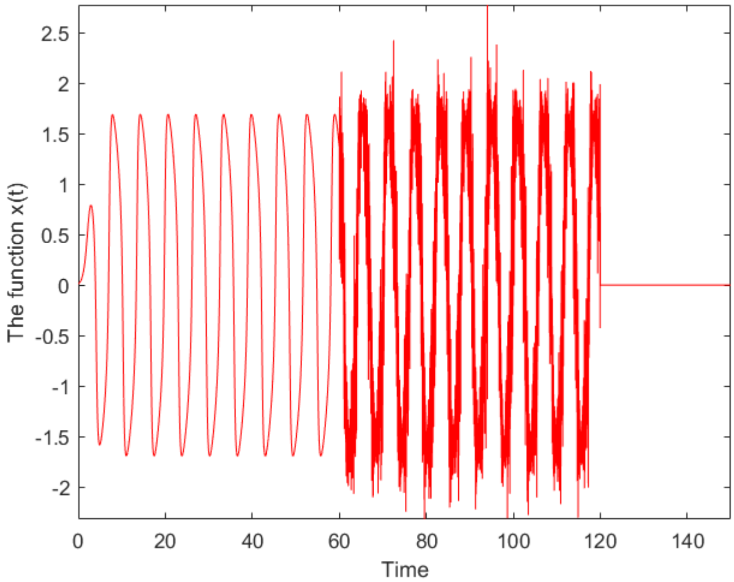

Figure 7. Numerical visualization for the function x.

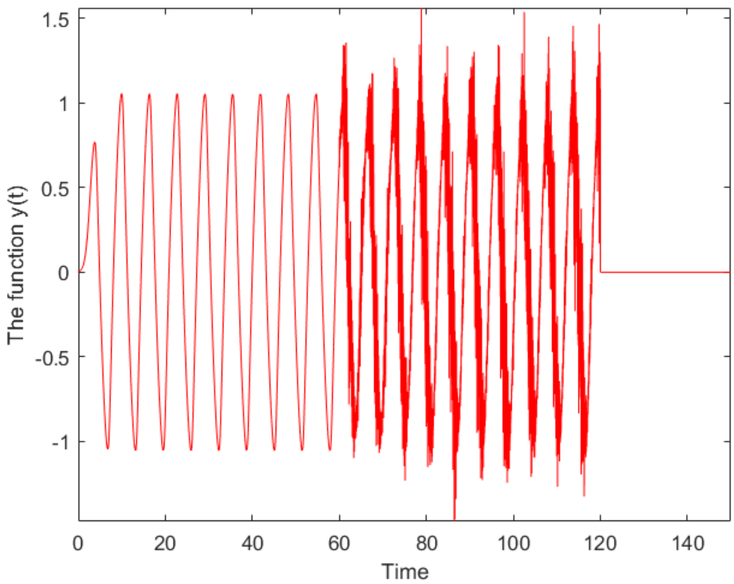

Figure 8. Numerical visualization for the function y.

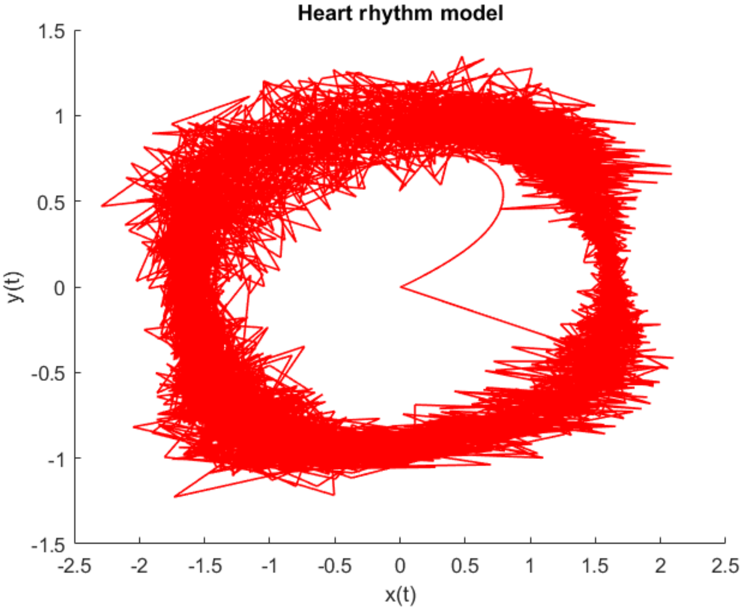

Figure 9. Numerical visualization for the heart rhythm model.

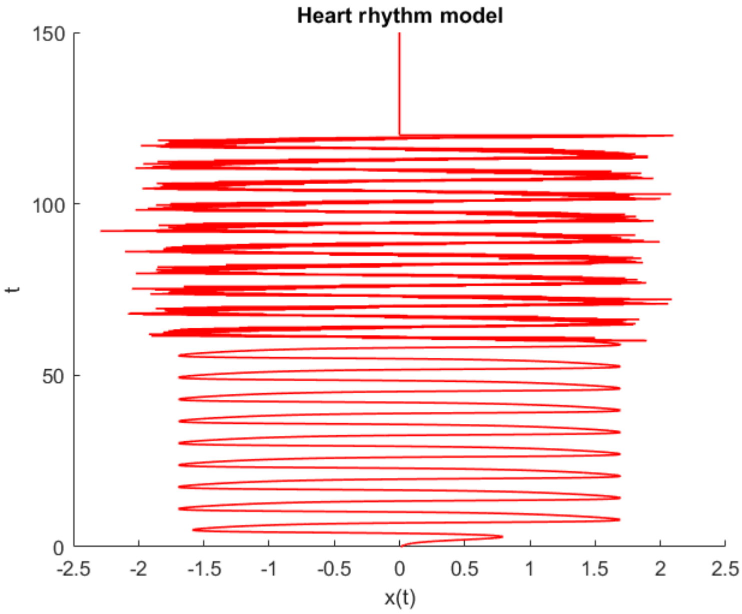

Figure 10. Numerical visualization for the heart rhythm model.

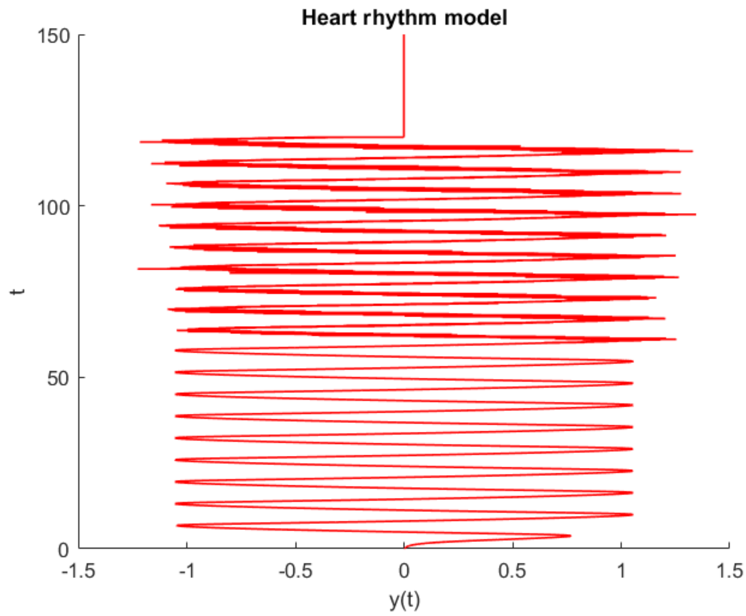

Figure 11. Numerical visualization for the heart rhythm model.

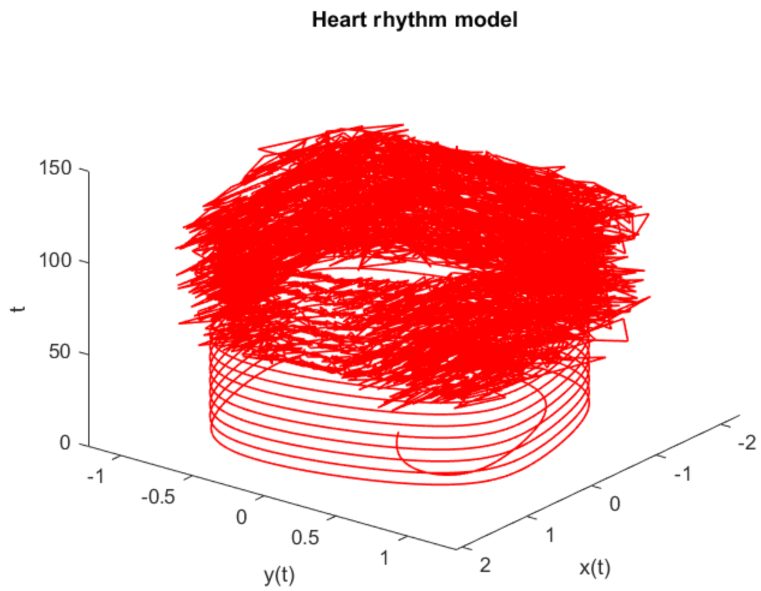

Figure 12. Numerical visualization for the heart rhythm model.

The numerical simulations are presented from Figure 7 to 12 for different fractional orders and densities of randomness. The results obtained from these simulations depict the process by which a healthy human has a normal heart beat later, after receiving bad or good news his heart performs an anomalous beat which is represented by randomness later the heart beat stops and the individual dies. Indeed, the original equation or system of equations was unable to capture such scenarios, however, using piecewise differential operators, these behaviors can be accurately replicated.

6. Conclusion

To accurately replicate crossover behaviours observed in nature, the concepts of piecewise differentiation and integration have been introduced very recently and have been recognized as powerful mathematical operators to accurately replicate non-localities observed in nature. The concepts have been applied in some important problems including epidemiology and chaos. This paper has attempted to further apply the concept to a nonlinear model known as the Van Der Pol equation. This equation has been used to model heartbeats. However, it can only replicate normal behaviour of heartbeat. When dealing with irregular behaviour, this original model cannot capture different crossover effects. In this paper, the concept of piecewise differentiation was then applied to capture crossover behaviours. We present numerical simulations with different outcomes. We were able to capture a situation of an individual who had a normal heartbeat, then he was informed of bad news, suddenly got irregular heartbeats and later died. We strongly believed that this concept is the future of modelling processes with crossover behaviours.

Conflict of interest

The authors declare there is no conflict of interest.

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