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Research article

A reduced distribution of the modified Weibull distribution and its applications to medical and engineering data

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Abstract: In this work, we suggest a reduced distribution with two parameters of the modified Weibull distribution to avoid some estimation difficulties. The hazard rate function of the reduced distribution exhibits decreasing, increasing or bathtub shape. The suggested reduced distribution can be applied to many problems of modelling lifetime data. Some statistical properties of the proposed distribution have been discussed. The maximum likelihood is employed to estimate the model parameters. The Fisher information matrix is derived and then applied to construct confidence intervals for parameters. A simulation is conducted to illustrate the performance of maximum likelihood estimation. Four sets of real data are tested to prove the proposed distribution advantages. According to the statistical criteria, the proposed distribution fits the tested data better than some well-known two-and three-parameter distributions.

Keywords: reduced modified Weibull distribution; modified Weibull distribution; reliability; goodness of fit; asymptotic confidence interval; lifetime data

1. Introduction

In the different fields of applied sciences, many lifetime distributions were applied for modelling several types of data, such as medical sciences, biological studies, engineering, environmental sciences, and economics. The selected models or distributions of probability define the quality of the statistical analysis procedures. As such, scholars have developed numerous classes of standard probability distributions [1–7] and their relevant statistical methodologies [8–14]. Lai et al. [3] introduced a modified Weibull (MW) distribution with three parameters with cumulative distribution function (CDF) given by

$$F(x) = 1 - e^{-a\,x^b\,e^{\lambda x}}, \quad x \geq 0, \quad a,\, \lambda > 0,\, b \geq 0. \tag{1.1}$$

It can be used to model bathtub-shaped failure rate life data. From this distribution, many authors have developed three- and five-parameter distributions, including log-modified Weibull distribution [5], generalized modified Weibull distribution [6], beta modified Weibull distribution [7], exponentiated modified Weibull extension distribution [8], a new modified Weibull distribution [9], Kumaraswamy modified Weibull distribution [10], the additive modified Weibull distribution [11], generalized modified Weibull power series distribution [12], among others.

Almalki and Yuan [9] proposed a new modified Weibull (NMW) distribution by combining Weibull distribution and MW distribution with CDF

$$F(x) = 1 - e^{-(\alpha\, x^{\theta} + \beta\, x^{\gamma}\, e^{\lambda x})}, \quad x > 0, \quad \alpha, \theta, \beta, \gamma > 0, \; \lambda \geq 0. \tag{1.2}$$

The hazard rate function of the NMW distribution can be decreasing, increasing or bathtub shaped. Almalki [15] suggested a reduced version of the NMW distribution by setting $\theta = \gamma = \frac{1}{2}$ in Eq (1.2). It is shown that the reduced version has the same desirable properties of the NMW distribution in spite of having two fewer parameters. He et al. [11] presented an additive modified Weibull (AMW) distribution with five-parameter. The CDF of the AMW distribution is given by

$$F(x) = 1 - e^{-\left(e^{\lambda x - \beta} - e^{-\beta} + \alpha\, x^{\theta}\, e^{\gamma x}\right)}, \quad x \geq 0, \quad \alpha, \theta, \beta > 0, \; \gamma, \lambda \geq 0. \tag{1.3}$$

The AMW distribution is decreasing, increasing and bathtub-shaped hazard function. Shakhatreh et al. [16] introduced a new lifetime distribution called the log-normal modified Weibull (LNMW5) distribution with CDF

$$R(x) = (1 - \Phi(z))\, e^{-\beta\, x^{\gamma}\, e^{\lambda x}}, \quad x > 0, \quad \beta, \gamma, \lambda > 0, \tag{1.4}$$

where $z(x; \mu, \sigma) = \frac{\ln(x) - \mu}{\sigma}$, $\mu \in \Re$, $\sigma > 0$ and $\Phi(.)$ is CDF of the standard normal distribution. The LNMW5 distribution is described and analyze monotone, upside-down bathtub, bathtub-shaped and modified bathtub-shaped failure rates. Shakhatreh et al. [16] obtained a special case of the LNMW5 Weibull distribution with four-parameter when setting $\gamma = \frac{1}{2}$ in Eq (1.4) and called the four-parameter log-normal modified Weibull (LNMW4) distribution. The LNMW5 and The LNMW4 distributions are flexible with respect to skewness and kurtosis. further, the LNMW4 distribution is simpler. Although the previous distributions are flexible having increasing, decreasing and bathtub-shaped hazard rate function, the increasing number of parameters is problematic for their estimations, particularly when the sample size is small. So, this article aims at reducing the MW distribution to a two-parameter distribution and preserving its flexibility for fitting data.

In this article, we present a reduced distribution of modified Weibull distribution with two parameters, which can be considered as a special case from MW distribution [3]. The motivations for the suggested distribution are to obtain a few parameters and hazard rate function with multiple shapes (decreasing, increasing or bathtub). The reduced modified Weibull distribution is characterized by its flexibility and importance in modelling medical and engineering applications (see Section 5). We can see that the reduced modified Weibull distribution provides better fits than other distributions.

The remainder of the current article is organized according to the following structure. Section 2 presents the reduced modified Weibull distribution and a discussion of some of its statistical properties. In Section 3, the maximum likelihood estimation is utilized to estimate the two parameters of the reduced modified Weibull distribution. In Section 4, we perform a simulation study to assess the stability of the parameters. The reduced modified Weibull distribution is applied to four real data sets as illustrated in Section 5. A conclusion is provided in Section 6.

2. A reduced modified Weibull distribution

2.1. Definition

Setting $a = \frac{\lambda}{\sigma} e^{-\sigma}$ and $b = 0.5$ in Eq (1.1), we get the CDF of the reduced modified Weibull (RMW) distribution given by

$$F(x) = 1 - e^{-\frac{\lambda}{\sigma} \sqrt{x} e^{\lambda x - \sigma}}, \qquad x \geq 0, \quad \lambda, \sigma > 0, \tag{2.1}$$

and its corresponding probability density function (PDF) of the RMW distribution given by

$$f(x) = \frac{\lambda}{2\sigma\sqrt{x}} (1 + 2\lambda x) e^{\lambda x - \sigma - \frac{\lambda}{\sigma} \sqrt{x} e^{\lambda x - \sigma}}. \tag{2.2}$$

The survival $S(x)$ and hazard rate $h(x)$ functions of the RMW distribution are, respectively,

$$S(x) = e^{-\frac{\lambda}{\sigma} \sqrt{x} e^{\lambda x - \sigma}}, \tag{2.3}$$

and

$$h(x) = \frac{\lambda}{2\sigma\sqrt{x}} (1 + 2\lambda x) e^{\lambda x - \sigma}. \tag{2.4}$$

To investigate the characterization of the hazard rate function, first take the derivative of $h(x)$ in Eq (2.4) with respect to x, we extract

$$\frac{dh(x)}{dx} = \frac{\lambda}{4\sigma} x^{\frac{-3}{2}} \Big(4\lambda x(\lambda x + 1) - 1\Big) e^{\lambda x - \sigma} \tag{2.5}$$

Solving Eq (2.5), a change point can be obtained as $x^* = \frac{\sqrt{2}-1}{2\lambda}$, we can see that $h(x)$ is increasing as $x^* > x$ and decreasing as $x^* < x$. Thus, $h(x)$ can be of bathtub shape as $x^* = x$. Figure 1 displays the PDF, CDF, $S(x)$ and $h(x)$ of the RMW distribution.

2.2. Statistical properties

2.2.1. Quantile function

The quantile x_q of the RMW distribution can be given as the solution Eq (2.6)

$$\lambda x_q + 0.5 \log(x_q) - \sigma - \log\Big(-\frac{\sigma}{\lambda} \log(1-q)\Big) = 0, \qquad 0 < q < 1. \tag{2.6}$$

Using Eq (2.6), the median of the RMW distribution can be given by setting $q = 0.5$.

2.2.2. Mode

The mode of the RMW distribution can be obtained as a non-negative solution Eq (2.7)

$$\sigma e^{\sigma} \Big(4\lambda x(1 + \lambda x) - 1\Big) - \lambda \sqrt{x}(1 + 2\lambda x)^2 e^{\lambda x} = 0. \tag{2.7}$$

The Eq (2.7) cannot be solved analytically and therefore they require some suitable numerical methods like fixed-point or Newton-Raphson.

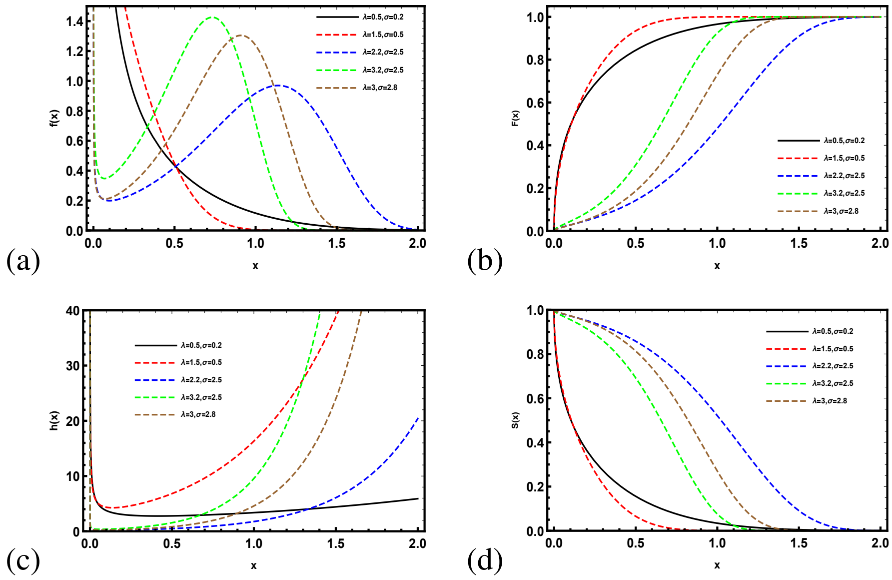

Figure 1. Plots of (a) PDF, (b) CDF, (c) $h(x)$, and (d) $S(x)$ for the RMW distribution at different parameters λ, σ.

2.2.3. Moments

Moments are essential for any statistical analysis, in general, and applications, in particular. They are applied to identify the distributions' most significent properties and characteristics e.g., dispersion, kurtosis and skewness. Using the definition of the r^{th} ordinary moment of X and substituting Eq (2.2), then

$$\mu_r'(\underline{\psi}) = \frac{\lambda}{2\,\sigma} \int_0^{+\infty} (x^{r-\frac{1}{2}} + 2\,\lambda\, x^{r+\frac{1}{2}})\, e^{\lambda\, x-\sigma-\frac{\lambda}{\sigma}\, \sqrt{x}\, e^{\lambda\, x-\sigma}}\, dx. \tag{2.8}$$

The integration in Eq (2.8) is computed numerically using the Wolfram Mathematica (see, Table 1). According to the first four ordinary moments of the RMW distribution, the measures of variance ($V(\underline{\psi})$), skewness ($S\,K(\underline{\psi})$), and kurtosis ($K(\underline{\psi})$) can obtained as

$$V(\underline{\psi}) = \mu_2'(\underline{\psi}) - (\mu_1'(\underline{\psi}))^2,$$

$$S\,K(\underline{\psi}) = \frac{\mu_3'(\underline{\psi}) - 3\,\mu_1'(\underline{\psi})\,\mu_2'(\underline{\psi}) + 2\,(\mu_1'(\underline{\psi}))^3}{(V(\underline{\psi}))^{\frac{3}{2}}},$$

and

$$K(\underline{\psi}) = \frac{\mu_4'(\underline{\psi}) - 4\,\mu_1'(\underline{\psi})\,\mu_3'(\underline{\psi}) + 6\,(\mu_1'(\underline{\psi}))^2\,\mu_2'(\underline{\psi}) - 3\,(\mu_1'(\underline{\psi}))^4}{(V(\underline{\psi}))^2}.$$

Table 1 lists the first four moments ($\mu_1'(\underline{\psi})$, $\mu_2'(\underline{\psi})$, $\mu_3'(\underline{\psi})$, $\mu_4'(\underline{\psi})$), $V(\underline{\psi})$, $S\,K(\underline{\psi})$ and $K(\underline{\psi})$ of the RMW distribution for different values of λ and σ. According to Table 1, we extract that:

- The mean and variance of the RMW distribution increase as σ increases
- The mean and variance of the RMW distribution decrease as λ increases.

- The skewness of the RMW distribution varies in the interval $(-1.13479, 3.62555)$.
- The kurtosis of the RMW distribution ranges in the interval $(2.25784, 21.711)$.
- The skewness of the RMW distribution can be negative or positive, so it is flexible for modelling skewed data.

Table 1. The first four ordinary moments, $V(\underline{\psi})$, $SK(\underline{\psi})$ and $K(\underline{\psi})$ of the RMW distribution for different λ, σ.

λ	σ	Mean(μ_1')	μ_2'	μ_3'	μ_4'	$V(\underline{\psi})$	$SK(\underline{\psi})$	$K(\underline{\psi})$
0.6	0.5	0.65756	0.79965	1.22129	2.13991	0.36727	0.95466	3.27163
0.6	1.5	2.53888	8.15243	29.1497	111.956	1.70649	-0.0958	**2.25784**
0.6	4	7.25742	56.0608	450.823	3737.34	3.3906	-0.84048	3.82892
0.6	10	17.9915	327.831	6037.75	112233	4.13785	-1.10446	5.19425
0.6	20	35.2484	1246.8	44244.4	1.57478×10^{6}	4.35238	-1.13201	5.36045
0.6	25	43.7726	1920.44	84436.9	3.72002×10^{6}	4.39488	**-1.13479**	5.37549
3	0.5	0.05729	0.0079	0.0015	0.00034	0.00462	1.63905	5.67889
9	0.5	0.00942	0.00027	0.00001	5.89211×10^{-7}	0.00018	2.27509	9.23468
15	0.5	0.00391	0.00005	1.02523×10^{-6}	2.68339×10^{-8}	0.00004	2.63091	11.8299
25	0.5	0.00158	9.15598×10^{-6}	8.75731×10^{-8}	1.1133×10^{-9}	6.6635×10^{-6}	3.02761	15.2823
35	0.5	0.00086	2.88627×10^{-6}	1.6603×10^{-8}	1.2967×10^{-10}	2.15027×10^{-6}	3.31018	18.1274
50	0.5	0.00044	8.28133×10^{-7}	2.74357×10^{-9}	1.26429×10^{-11}	6.304×10^{-7}	**3.62555**	**21.711**

3. Estimation and inference

In this section, the estimators of the parameters for RMW distribution are derived using the maximum-likelihood method in the case of complete data and censored data.

3.1. Complete data

Suppose $X_1, ..., X_n$ be a random sample of size n from the RMW distribution with parameters $\underline{\psi} = (\lambda, \sigma)$. From Eq (2.2), the log likelihood function $\ell(\underline{\psi})$ written as:

$$\ell(\underline{\psi}) = \Big(\log\big(\frac{\lambda}{2\sigma}\big) - \sigma\Big) n + \lambda \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \log\Big(\frac{1}{\sqrt{x_i}} + 2\lambda\sqrt{x_i}\Big) - \frac{\lambda}{\sigma} \sum_{i=1}^{n} \sqrt{x_i}\, e^{\lambda x_i - \sigma}. \tag{3.1}$$

The first partial derivatives of Eq (3.1) with respect to λ and σ are:

$$\frac{\partial \ell(\underline{\psi})}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n} x_i + 2\sum_{i=1}^{n} \frac{x_i}{1 + 2\lambda x_i} - \frac{1}{\sigma}\sum_{i=1}^{n} \sqrt{x_i}\, e^{\lambda x_i - \sigma} - \frac{\lambda}{\sigma}\sum_{i=1}^{n} x_i^{\frac{3}{2}}\, e^{\lambda x_i - \sigma}, \tag{3.2}$$

and

$$\frac{\partial \ell(\underline{\psi})}{\partial \sigma} = -(1 + \frac{1}{\sigma}) n + \frac{\lambda}{\sigma^2}(1+\sigma)\sum_{i=1}^{n} \sqrt{x_i}\, e^{\lambda x_i - \sigma}. \tag{3.3}$$

We set the nonlinear system of equations $\frac{\partial \ell(\underline{\psi})}{\partial \lambda} = 0$ and $\frac{\partial \ell(\underline{\psi})}{\partial \sigma} = 0$. The maximum likelihood estimates (MLEs) of the RMW distribution can be obtained by solving these equations numerically with the aid of software, such as Wolfram Mathematica.

For asymptotic interval of the estimation of $\underline{\psi}$, we obtain the observed Fisher information matrix. The local information matrix $\mathbf{F}(\hat{\underline{\psi}})$ is obtained as:

$$\mathbf{F}(\hat{\underline{\psi}}) = -\begin{pmatrix} \frac{\partial \ell^2(\underline{\psi})}{\partial \lambda^2} & \frac{\partial \ell^2(\underline{\psi})}{\partial \lambda \partial \sigma} \\ \frac{\partial \ell^2(\underline{\psi})}{\partial \lambda \partial \sigma} & \frac{\partial \ell^2(\underline{\psi})}{\partial \sigma^2} \end{pmatrix}_{\hat{\underline{\psi}}=(\hat{\lambda},\hat{\sigma})}, \tag{3.4}$$

where the elements of the observed Fisher information matrix in Eq (3.4) are as follows

$$\begin{aligned} \frac{\partial \ell^2(\underline{\psi})}{\partial \lambda^2} &= -\frac{n}{\lambda^2} - 4\sum_{i=1}^{n} \frac{x_i^2}{(1+2\lambda x_i)^2} - \frac{2}{\sigma}\sum_{i=1}^{n} x_i^{\frac{3}{2}} e^{\lambda x_i - \sigma} \\ &\quad -\frac{\lambda}{\sigma}\sum_{i=1}^{n} x_i^{\frac{5}{2}} e^{\lambda x_i - \sigma}, \end{aligned} \tag{3.5}$$

$$\frac{\partial \ell^2(\underline{\psi})}{\partial \lambda \partial \sigma} = \frac{1}{\sigma^2}(1+\sigma)\Big(\sum_{i=1}^{n} \sqrt{x_i}\, e^{\lambda x_i - \sigma} + \lambda \sum_{i=1}^{n} x_i^{\frac{3}{2}} e^{\lambda x_i - \sigma}\Big), \tag{3.6}$$

$$\frac{\partial \ell^2(\underline{\psi})}{\partial \sigma^2} = \frac{n}{\sigma^2} - \frac{\lambda}{\sigma^3}(2 + 2\sigma + \sigma^2)\sum_{i=1}^{n} \sqrt{x_i}\, e^{\lambda x_i - \sigma}. \tag{3.7}$$

Therefore, the approximate variances of $\underline{\psi} = (\lambda, \sigma)$ can be obtained as

$$\mathbf{F}^{-1}(\hat{\underline{\psi}}) = \begin{pmatrix} \widehat{\mathrm{Var}}(\hat{\lambda}) & \widehat{\mathrm{Cov}}(\hat{\lambda}, \hat{\sigma}) \\ \widehat{\mathrm{Cov}}(\hat{\lambda}, \hat{\sigma}) & \widehat{\mathrm{Var}}(\hat{\sigma}) \end{pmatrix}. \tag{3.8}$$

Thus, the asymptotic confidence intervals (CIs) for λ and σ with $100(1-\tau)\%$ confidence level are given by:

$$\hat{\lambda} \mp Z_{\frac{\tau}{2}} \sqrt{\widehat{\mathrm{Var}}(\hat{\lambda})}\,, \qquad \hat{\sigma} \mp Z_{\frac{\tau}{2}} \sqrt{\widehat{\mathrm{Var}}(\hat{\sigma})}.$$

$Z_{\frac{\tau}{2}}$ is the $100(1-\tau)^{\text{th}}$ percentile of the $N(0, 1)$ and τ is the significance level.

Though the parameters are positive, the CIs may result in a negative lower bound. To solve this problem, we replace the negative values by zero. We can also use the log-transformed MLE proposed by [17]. According to the normal approximation of log-transformed MLE, the $100(1-\tau)\%$ CIs for λ and σ are given by

$$\lambda \in \left[\hat{\lambda}\, e^{-\frac{1}{\hat{\lambda}} Z_{\frac{\tau}{2}} \sqrt{\widehat{\mathrm{Var}}(\hat{\lambda})}},\quad \hat{\lambda}\, e^{\frac{1}{\hat{\lambda}} Z_{\frac{\tau}{2}} \sqrt{\widehat{\mathrm{Var}}(\hat{\lambda})}}\right],$$

and

$$\sigma \in \left[\hat{\sigma}\, e^{-\frac{1}{\hat{\sigma}} Z_{\frac{\tau}{2}} \sqrt{\widehat{\mathrm{Var}}(\hat{\sigma})}},\quad \hat{\sigma}\, e^{\frac{1}{\hat{\sigma}} Z_{\frac{\tau}{2}} \sqrt{\widehat{\mathrm{Var}}(\hat{\sigma})}}\right],$$

respectively.

3.2. Censored data

This subsection is devoted to estimate the vector of two parameters $\underline{\psi} = (\lambda, \sigma)$ of the RMW distribution based on Type II censored samples. Suppose that $X_1, ..., X_r$ is a Type II censored sample of size r obtained from a life-test on n items whose lifetimes have the RMW distribution. Then the likelihood function is

$$\ell(\underline{\psi}) \propto \Big(\frac{\lambda\, e^{-\sigma}}{2\,\sigma}\Big)^r e^{\lambda \sum_{i=1}^{n} x_i} \prod_{i=1}^{r}\big(\frac{1}{\sqrt{x_i}} + 2\,\lambda\, \sqrt{x_i}\big)\, e^{-\frac{\lambda}{\sigma} \sum_{i=1}^{r} \sqrt{x_i}\, e^{\lambda x_i - \sigma}} \times e^{-(n-r)\frac{\lambda}{\sigma}\, \sqrt{x_r}\, e^{\lambda x_r - \sigma}} \tag{3.9}$$

The log likelihood function is

$$\ell(\underline{\psi}) = \Big(\log\big(\frac{\lambda}{2\,\sigma}\big) - \sigma\Big) r + \lambda \sum_{i=1}^{r} x_i + \sum_{i=1}^{r} \log\Big(\frac{1}{\sqrt{x_i}} + 2\,\lambda\, \sqrt{x_i}\Big) - \frac{\lambda}{\sigma} \sum_{i=1}^{r} \sqrt{x_i}\, e^{\lambda x_i - \sigma} - (n-r)\,\frac{\lambda}{\sigma}\, \sqrt{x_r}\, e^{\lambda x_r - \sigma}. \tag{3.10}$$

Differentiating the log likelihood function in Eq (3.10), with respect to λ and σ, one obtains

$$\frac{\partial \ell(\underline{\psi})}{\partial \lambda} = \frac{r}{\lambda} + \sum_{i=1}^{r} x_i + 2 \sum_{i=1}^{r} \frac{x_i}{1 + 2\,\lambda\, x_i} - \frac{1}{\sigma} \sum_{i=1}^{r} \sqrt{x_i}\, e^{\lambda x_i - \sigma} - \frac{\lambda}{\sigma} \sum_{i=1}^{r} x_i^{\frac{3}{2}}\, e^{\lambda x_i - \sigma} - \frac{1}{\sigma}\,(n-r)\, \sqrt{x_r}\,(1 + \lambda\, x_r)\, e^{\lambda x_r - \sigma}, \tag{3.11}$$

and

$$\frac{\partial \ell(\underline{\psi})}{\partial \sigma} = -(1 + \frac{1}{\sigma})\, r + \frac{\lambda}{\sigma^2}\,(1+\sigma) \sum_{i=1}^{r} \sqrt{x_i}\, e^{\lambda x_i - \sigma} + \frac{\lambda}{\sigma^2}\,(n-r)\,(1+\sigma)\, \sqrt{x_r}\, e^{\lambda x_r - \sigma}. \tag{3.12}$$

The ML estimators of the parameters λ and σ are derived by equating the two nonlinear Eqs (3.11) and (3.12) to zeros and solving numerically.

4. Simulations

We carry out a simulation study to examine the average values of the estimates (AVs) along with their corresponding average absolute biases (ABs) and average mean square errors (MSEs) for all sample sizes n and different parametric values λ and σ. We generated samples of sizes $n = 20, 50, 100, 200, 300$ and 500, and the simulations were repeated $N = 1000$ times from the RMW distribution with different parametric values $\lambda = 0.5, 0.8, 1.5, 2.3$ and $\sigma = 0.6, 1.2, 2.7$. Tables 2 and 3 give the AVs, ABs, and MSEs of the RMW distribution for different sample sizes. From Tables 2 and 3, we note that for all the parametric values of λ and σ, the biases decrease and the MSEs decay toward zero as the sample size n increases. A simulation study for measures of the goodness-of-fit AIC, AICc, BIC, and HQIC is reported in Table 4. The shapes of the failure rate function for the parameter values used in the simulation are shown in Figure 2.

Table 2. The AVs, ABs, and MSEs of the RMW distribution for $\lambda = 0.5, 0.8$ and $\sigma = 0.6, 1.2, 2.7$.

n	parametric values		AVs		ABs		MSEs	
	λ	σ	$\hat{\lambda}$	$\hat{\sigma}$	Bias($\hat{\lambda}$)	Bias($\hat{\sigma}$)	MSE($\hat{\lambda}$)	MSE($\hat{\sigma}$)
20	0.5	0.6	0.57018	0.7046	0.07018	0.1046	0.09668	0.19737
50			0.5347	0.65035	0.0347	0.05035	0.02088	0.23449
100			0.51371	0.61998	0.01371	0.01998	0.00612	0.01469
200			0.50685	0.60933	0.00685	0.00933	0.00308	0.00723
300			0.5037	0.60446	0.0037	0.00446	0.0019	0.00448
500			0.50294	0.60408	0.00294	0.00408	0.00111	0.0027
20	0.5	1.2	0.54582	1.33113	0.04582	0.13113	0.02391	0.24383
50			0.52195	1.26297	0.02195	0.06297	0.00749	0.07687
100			0.51326	1.23897	0.01326	0.03897	0.00324	0.03524
200			0.50678	1.22147	0.00678	0.02147	0.00173	0.01809
300			0.50363	1.21114	0.00363	0.01114	0.00105	0.01109
500			0.50132	1.20349	0.00132	0.00349	0.00065	0.00683
20	0.5	2.7	0.54288	2.98236	0.04288	0.28236	0.01761	0.84388
50			0.51518	2.79604	0.01518	0.09604	0.00469	0.23935
100			0.50871	2.75625	0.00871	0.05625	0.00231	0.11949
200			0.50467	2.73285	0.00467	0.03285	0.00126	0.06477
300			0.50215	2.71226	0.00215	0.01226	0.00074	0.0369
500			0.50036	2.70087	0.00036	0.00087	0.00042	0.02184
20	0.8	0.6	0.89091	0.6651	0.09091	0.0651	0.2774	0.20355
50			0.84495	0.63704	0.04495	0.03704	0.0517	0.04097
100			0.82972	0.62309	0.02972	0.02309	0.01896	0.01445
200			0.81489	0.61237	0.01489	0.01237	0.00911	0.0073
300			0.81177	0.60967	0.01177	0.00967	0.00565	0.00475
500			0.80751	0.60632	0.00751	0.00632	0.00351	0.00281
20	0.8	1.2	0.88209	1.34124	0.08209	0.14124	0.07398	0.25068
50			0.8386	1.2633	0.0386	0.0633	0.02111	0.07307
100			0.81941	1.2338	0.01941	0.0338	0.00897	0.03263
200			0.80481	1.20687	0.00481	0.00687	0.00434	0.01598
300			0.80234	1.20406	0.00234	0.00406	0.00266	0.00998
500			0.8017	1.20233	0.0017	0.00233	0.00167	0.0061
20	0.8	2.7	0.87092	2.98036	0.07092	0.28036	0.04261	0.76644
50			0.82596	2.80023	0.02596	0.10023	0.01278	0.23119
100			0.81331	2.75525	0.01331	0.05525	0.00611	0.11195
200			0.81005	2.73952	0.01005	0.03952	0.00284	0.05309
300			0.80666	2.7277	0.00666	0.0277	0.00204	0.0379
500			0.80184	2.7068	0.00184	0.0068	0.00112	0.0209

Table 3. The AVs, ABs, and MSEs of the RMW distribution for $\lambda = 1.5, 2.3$ and $\sigma = 0.6, 1.2, 2.7$.

	parametric values		AVs		ABs		MSEs	
n	λ	σ	$\hat{\lambda}$	$\hat{\sigma}$	Bias($\hat{\lambda}$)	Bias($\hat{\sigma}$)	MSE($\hat{\lambda}$)	MSE($\hat{\sigma}$)
20	1.5	0.6	1.78794	0.69732	0.28794	0.09732	1.09073	0.1657
50			1.62968	0.64816	0.12968	0.04816	0.21862	0.03981
100			1.56746	0.6269	0.06746	0.0269	0.08958	0.01662
200			1.52694	0.61107	0.02694	0.01107	0.04036	0.00757
300			1.5112	0.60429	0.0112	0.00429	0.02449	0.00455
500			1.50921	0.60326	0.00921	0.00326	0.01505	0.00285
20	1.5	1.2	1.67236	1.33827	0.17236	0.13827	0.28284	0.23297
50			1.56914	1.2545	0.06914	0.0545	0.08513	0.07185
100			1.52931	1.22584	0.02931	0.02584	0.03541	0.03067
200			1.51707	1.2137	0.01707	0.0137	0.01847	0.01622
300			1.51347	1.21042	0.01347	0.01042	0.01186	0.01083
500			1.5072	1.20513	0.0072	0.00513	0.0064	0.00574
20	1.5	2.7	1.64136	2.98072	0.14136	0.28072	0.14685	0.65663
50			1.55882	2.81039	0.05882	0.11039	0.04531	0.20731
100			1.51626	2.72998	0.01626	0.02998	0.02076	0.09681
200			1.51426	2.72823	0.01426	0.02823	0.01058	0.04971
300			1.50556	2.70878	0.00556	0.00878	0.00723	0.03444
500			1.50237	2.70294	0.00237	0.00294	0.00435	0.02013
20	2.3	0.6	2.88695	0.70689	0.58695	0.10689	3.15289	0.18738
50			2.5037	0.64022	0.2037	0.04022	0.62892	0.04334
100			2.40557	0.62239	0.10557	0.02239	0.26724	0.01703
200			2.35136	0.61126	0.05136	0.01126	0.12594	0.00873
300			2.33342	0.60772	0.03342	0.00772	0.07838	0.00544
500			2.31351	0.6017	0.01351	0.0017	0.04507	0.00299
20	2.3	1.2	2.63334	1.35922	0.33334	0.15922	0.7114	0.20117
50			2.39313	1.24136	0.09313	0.04136	0.21536	0.06809
100			2.34999	1.22545	0.04999	0.02545	0.09553	0.03107
200			2.33358	1.21735	0.03358	0.01735	0.04344	0.01384
300			2.31686	1.20897	0.01686	0.00897	0.03068	0.01006
500			2.31411	1.20685	0.01411	0.00685	0.01833	0.0061
20	2.3	2.7	2.53743	2.98484	0.23743	0.28484	0.40218	0.66928
50			2.3717	2.78268	0.0717	0.08268	0.11331	0.20173
100			2.34187	2.74949	0.04187	0.04949	0.0559	0.10075
200			2.3203	2.72464	0.0203	0.02464	0.02432	0.04439
300			2.31093	2.71311	0.01093	0.01311	0.01709	0.03143
500			2.30701	2.70904	0.00701	0.00904	0.00974	0.01763

Table 4. Formal goodness of fit statistics AIC, AICc, BIC, and HQIC for the simulation data.

n	λ	σ	AIC	AICc	BIC	HQIC
20	0.8	0.6	16.4862	17.192	18.4776	16.8749
50			32.3528	32.6082	36.1769	33.8091
100			62.443	62.5667	67.6534	64.5517
200			118.611	118.672	125.208	121.281
300			177.373	177.413	184.781	180.338
500			296.064	296.088	304.493	299.372
20	0.8	1.2	45.7841	46.49	47.7756	46.1728
50			111.5	111.755	115.324	112.956
100			220.691	220.814	225.901	222.799
200			439.06	439.121	445.656	441.729
300			656.76	656.83	664.197	659.754
500			1094.43	1094.45	1102.86	1097.74
20	1.5	1.2	15.8385	16.5444	17.83	16.2273
50			34.6043	34.8597	38.4284	36.0606
100			66.8281	66.9519	72.0385	68.9369
200			131.796	131.856	138.392	134.465
300			196.036	196.077	203.444	199.001
500			325.02	325.044	333.449	328.328
20	1.5	2.7	38.6728	39.3787	40.6643	39.0616
50			93.2538	93.5091	97.0779	94.71
100			185.289	185.413	190.5	187.398
200			368.378	368.439	374.975	371.048
300			550.668	550.709	558.076	553.633
500			917.178	917.202	925.607	920.485
20	2.3	2.7	20.4232	21.1291	22.4147	20.812
50			48.0273	48.2827	51.8514	49.4836
100			94.2936	94.4174	99.504	96.4024
200			187.543	187.604	194.139	190.212
300			278.741	278.781	286.148	281.705
500			464.115	464.139	472.544	467.422

5. Four applications to lifetime data

To study the performance of RMW distribution in practice, four real data sets are studied. The first data set is leukemia data [18], the second is refractory lining data [19], the third is turbochargers data [20], and the fourth is Aarset data [21]. We compare the fitting of the RMW distribution to the four real data sets and other distributions with two and three parameters. The two-parameter distributions

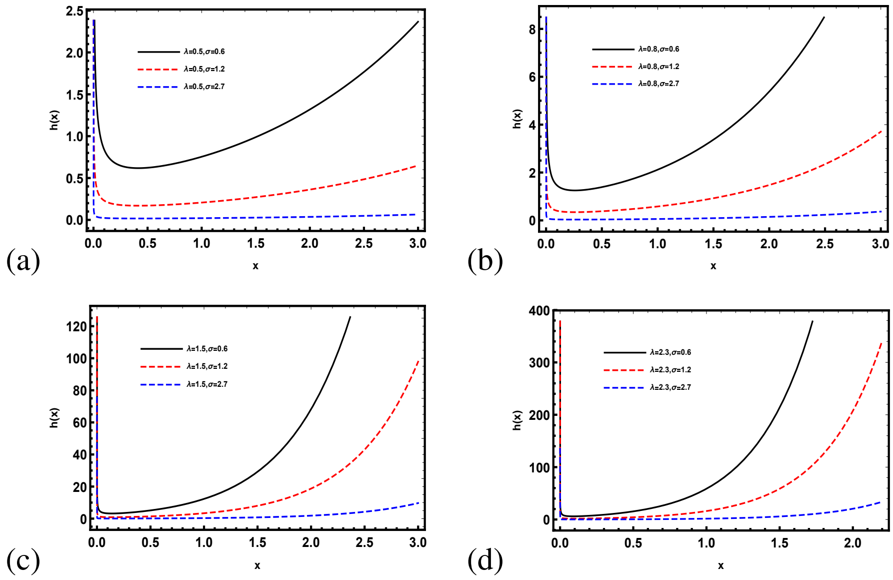

Figure 2. The shapes of the failure rate function for the parameter values used in the simulation.

are Weibull (W) [1], exponentiated exponential (EE) [22], exponentiated Rayleigh (ER) [23], flexible Weibull extension (FW) [4] and power Lindley (PL) [24] distributions. The distributions with three parameters are modified Weibull extension (MWE) [2] and new exponential-type (NET) [25] distributions. To achieve this objective, we consider the various measures of the goodness-of-fit including the maximum of the $\ell(\hat{\underline{\psi}})$, Akaike information criterion (AIC) [26], second order Akaike information criterion (AICc) [27], Bayesian information criterion (BIC) [28], Hannan-Quinn information criterion (HQIC) [29], Kolmogorov-Smirnov (K-S) statistics and the corresponding p-values, Cramér-von misses (W^*), and Anderson-Darling (A^*). The details about the W^*, A^*, and K-S statistics are discussed in [30]. Generally, the smaller these statistics are, the better is the fit to the data set. All the computations were carried out using Wolfram Mathematica 12 software.

5.1. Censored data

5.1.1. Leukemia data

Data in Table 5 display the lifetime by days of $n = 43$ patients with leukemia from one of the Ministry of Health Hospitals in Saudi Arabia [18]. To demonstrate the likelihood equations, we plot the profiles of the $\ell(\hat{\underline{\psi}})$ of $\underline{\psi} = (\lambda, \sigma)$ in Figure 3. Table 6 gives the MLE of parameters of the RMW as well as the W, EE, ER, FW, PL, MWE and NET distributions and the observed K-S test statistic values for fitting to leukemia data. From the measure of fit values given in Tables 6 and 7, we can see that our proposed RMW distribution has the largest $\ell(\hat{\underline{\psi}})$ values and has the smallest K-S, W^*, A^*, AIC, AICc, BIC, and HQIC values. It means that it provides the best fit to leukemia data in comparison with the W, EE, ER, FW, PL, MWE and NET distributions. The empirical and fitted scaled total time on test transform (TTT-transform) [31] plot of the RMW distribution for leukemia data is illustrated in Figure 4. Figure 5 shows the estimated PDFs, survival and hazard rate functions of the RMW, W, EE, ER, FW, PL, MWE and NET distributions for fitting to leukemia data. From these plots, we can observe that the RMW distribution is very close to the empirical line more than other distributions.

The estimated variance-covariance matrix of the RMW distribution for leukemia data is given by

$$\mathbf{F}(\underline{\hat{\psi}}) = \begin{pmatrix} 8.57695 \times 10^{-8} & 0.0000682747 \\ 0.0000682747 & 0.0576365 \end{pmatrix}.$$

Thus, the approximate 95% confidence intervals for λ and σ are respectively, $[0.00143,\ 0.0026]$ and $[0.27601,\ 1.31568]$.

Table 5. Lifetimes of 43 patients (by days) suffering from leukemia.

115	181	255	418	441	461	516	739	743
789	807	865	824	983	1025	1062	1063	1165
1191	1222	1222	1251	1277	1290	1357	1369	1408
1455	1478	1519	1578	1578	1599	1603	1605	1696
1735	1799	1815	1852	1899	1925	1965		

Table 6. The MLE of parameters, Log-likelihood, K-S values and the associated p-values for leukemia data.

Distribution	CDF	MLE of the parameters	$\ell(\underline{\hat{\psi}})$	K–S	p-value
RMW	$1 - e^{-\frac{\lambda}{\sigma}\sqrt{x}\, e^{\lambda x - \sigma}}$	$\hat{\lambda} = 0.00193,\ \hat{\sigma} = 0.60261$	**-325.624**	**0.06871**	**0.98723**
W	$1 - e^{-(\frac{x}{\lambda})^{\sigma}}$	$\hat{\lambda} = 1334.47,\ \hat{\sigma} = 2.55681$	-329.424	0.11135	0.6606
EE	$(1 - e^{-\lambda x})^{\sigma}$	$\hat{\lambda} = 0.00164,\ \hat{\sigma} = 3.64823$	-335.45	0.16369	0.19945
ER	$(1 - e^{-\lambda x^2})^{\sigma}$	$\hat{\lambda} = 6.71395 \times 10^{-7},\ \hat{\sigma} = 1.20911$	-330.601	0.14166	0.35405
FW	$1 - e^{-e^{\lambda x - \frac{\sigma}{x}}}$	$\hat{\lambda} = 0.0008,\ \hat{\sigma} = 1133.23$	-335.756	0.22361	0.02713
PL	$1 - (1 + \frac{\sigma}{1+\sigma} x^{\lambda}) e^{-\sigma x^{\lambda}}$	$\hat{\lambda} = 1.57014,\ \hat{\sigma} = 0.00003$	-331.899	0.13623	0.40202
MWE	$1 - e^{\lambda\theta(1 - e^{(\frac{x}{\lambda})^{\sigma}})}$	$\hat{\lambda} = 9037.79,\ \hat{\sigma} = 2.53074,\ \hat{\theta} = 0.01397$	-329.36	0.11254	0.64744
NET	$(1 - e^{1-(1+\lambda x)^{\sigma}})^{\beta}$	$\hat{\lambda} = 0.00005,\ \hat{\sigma} = 15.7307,\ \hat{\beta} = 2.61276$	-329.741	0.12328	0.53054

Table 7. Formal goodness of fit statistics for leukemia data.

Distribution	W^*	A^*	AIC	AICc	BIC	HQIC
RMW	**0.02428**	**0.20412**	**655.248**	**655.548**	**658.771**	**656.547**
W	0.12324	0.90727	662.847	663.147	666.37	664.146
EE	0.3158	1.83953	674.9	675.2	678.422	676.199
ER	0.19977	1.15737	665.202	665.502	668.725	666.501
FW	0.64862	3.30038	675.513	675.813	679.035	676.812
PL	0.18507	1.26609	667.798	668.098	671.32	669.097
MWE	0.12472	0.89962	664.72	665.335	670.003	666.668
NET	0.14877	0.99046	665.481	666.097	670.765	667.43

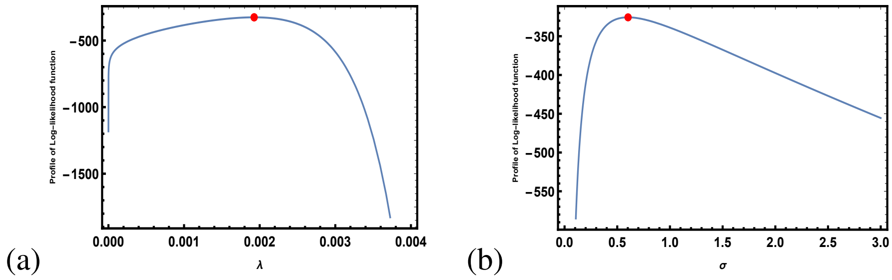

Figure 3. The profile of the $l(\hat{\underline{\psi}})$ of λ and σ for leukemia data.

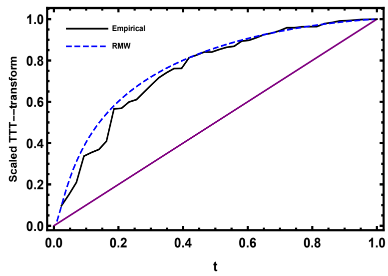

Figure 4. TTT-transform plot of the RMW distribution of fitting to leukemia data.

5.1.2. Refractory lining data

Data in Table 8 represent hot repair time duration by minutes as the time unit [19]. Figure 6 displays the plot of the profiles of the $\ell(\hat{\underline{\psi}})$ and $\underline{\psi} = (\lambda, \sigma)$. Table 9 illustrates the MLE of the parameters of the RMW as well as the W, EE, ER, FW, PL, MWE and NET distributions and the observed K-S test statistic values for fitting to refractory lining data. In Tables 9 and 10, we compare the fits of the RMW distribution with the W, EE, ER, FW, PL, MWE and the NET distributions. The values in Tables 9 and 10 show that the RMW distribution has a close fit for the refractory lining data among the competing distributions. Figure 7 presents the empirical and fitted scaled TTT-transform plot of the RMW distribution for refractory lining data. Figure 8 illustrates the estimated PDFs, survival and hazard rate functions of the RMW, W, EE, ER, FW, PL, MWE and NET distributions for fitting to the data set. From these plots, we can observe that the RMW distribution is very close to the empirical line more than other distributions.

The estimated variance-covariance matrix of the RMW distribution for the data set is given by

$$\mathbf{F}(\hat{\underline{\psi}}) = \begin{pmatrix} 0.0000183263 & 0.00135663 \\ 0.00135663 & 0.107961 \end{pmatrix}.$$

Thus, the approximate 95% confidence intervals for λ and σ are respectively, [0.01329, 0.03056] and [0.3311, 1.77592].

5.1.3. Turbochargers data

Data in Table 11 shows the original test records of the time-to-failure data for $n = 40$ suits of turbochargers [20]. Figure 9 shows the plot of the profiles of the $\ell(\hat{\underline{\psi}})$ and $\underline{\psi} = (\lambda, \sigma)$. The MLE of the parameters of the RMW as well as the W, EE, ER, FW, PL, MWE and NET distributions and the observed K-S test statistic values for fitting to turbochargers data are provided in Table 12. From the

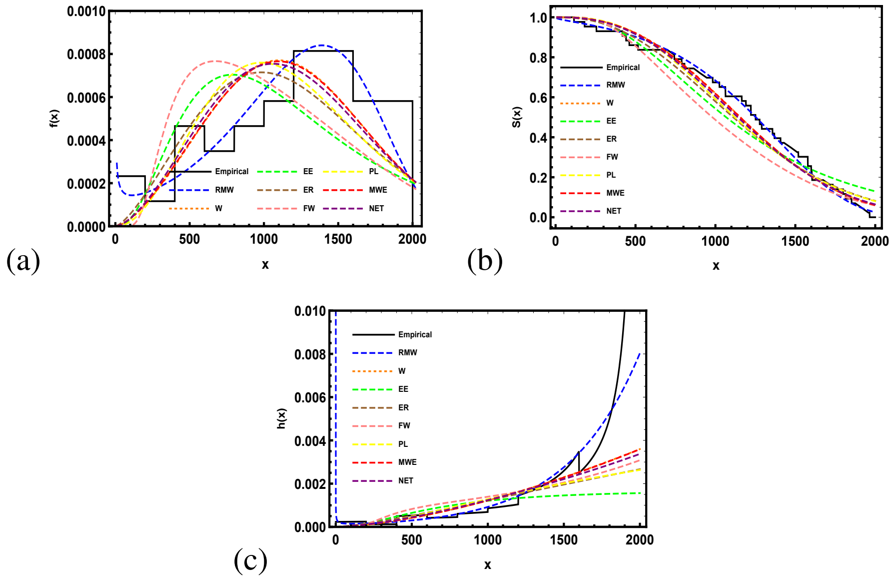

Figure 5. (a) Empirical and estimated PDFs; (b) Empirical and estimated $S(x)$ and (c) Empirical and estimated $h(x)$ of the RMW, W, EE, ER, FW, PL, MWE and NET distributions for fitting leukemia data.

Table 8. Hot repair time duration (by minutes) per unit time.

7	22	25	25	35	45	47	57	63
65	86	90	90	95	97	116	117	118
118	118	123	125	144	145	157		

Table 9. The MLE of parameters, Log-likelihood, K-S values and the associated p-values for refractory lining data.

Distribution	CDF	MLE of the parameters	$\ell(\underline{\hat{\psi}})$	K-S	p-value
RMW	$1-e^{-\frac{\lambda}{\sigma}\sqrt{x}e^{\lambda x-\sigma}}$	$\hat{\lambda}=0.02016$, $\hat{\sigma}=0.76682$	**-127.713**	**0.14398**	**0.6779**
W	$1-e^{-(\frac{x}{\lambda})^{\sigma}}$	$\hat{\lambda}=95.6232$, $\hat{\sigma}=2.04644$	-129.384	0.17347	0.43935
EE	$(1-e^{-\lambda x})^{\sigma}$	$\hat{\lambda}=0.02006$, $\hat{\sigma}=2.67257$	-131.523	0.19195	0.31566
ER	$(1-e^{-\lambda x^{2}})^{\sigma}$	$\hat{\lambda}=1.04012\times 10^{-4}$, $\hat{\sigma}=0.91186$	-129.325	0.17235	0.44765
FW	$1-e^{-e^{\lambda x-\frac{\sigma}{x}}}$	$\hat{\lambda}=0.00913$, $\hat{\sigma}=58.1831$	-132.558	0.27196	0.04954
PL	$1-(1+\frac{\sigma}{1+\sigma}x^{\lambda})e^{-\sigma x^{\lambda}}$	$\hat{\lambda}=1.27328$, $\hat{\sigma}=0.00663$	-130.477	0.17534	0.42569
MWE	$1-e^{\lambda\theta(1-e^{(\frac{x}{\lambda})^{\sigma}})}$	$\hat{\lambda}=15058.6$, $\hat{\sigma}=2.04637$, $\hat{\theta}=2.08224$	-129.384	0.17346	0.43942
NET	$(1-e^{1-(1+\lambda x)^{\sigma}})^{\beta}$	$\hat{\lambda}=0.00039$, $\hat{\sigma}=24.3519$, $\hat{\beta}=1.66453$	-128.817	0.17533	0.42576

measure of fit values given in Tables 12 and 13, our proposed RMW distribution has the largest $\ell(\underline{\hat{\psi}})$ values and has the smallest K-S, W^*, A^*, AIC, AICc, BIC, and HQIC values which mean that it provides the best fit to turbochargers data in comparison with other distributions. Figure 10 demonstrates the empirical and fitted scaled TTT-transform plot of the RMW distribution for turbochargers data. Figure 11 shows the estimated PDFs, survival and hazard rate functions of the RMW, W, EE, ER, FW, PL,

Table 10. Formal goodness of fit statistics for refractory lining data.

Distribution	W^*	A^*	AIC	AICc	BIC	HQIC
RMW	**0.0611**	**0.3656**	**259.426**	**259.972**	**261.864**	**260.102**
W	0.12298	0.747	262.769	263.314	265.207	263.445
EE	0.16448	0.93951	267.046	267.592	269.484	267.722
ER	0.12761	0.70593	262.651	263.196	265.088	263.327
FW	0.42318	2.18821	269.117	269.662	271.554	269.793
PL	0.14526	0.86959	264.954	265.499	267.392	265.63
MWE	0.12297	0.74694	264.769	265.912	268.425	265.783
NET	0.12706	0.6738	263.634	264.777	267.291	264.649

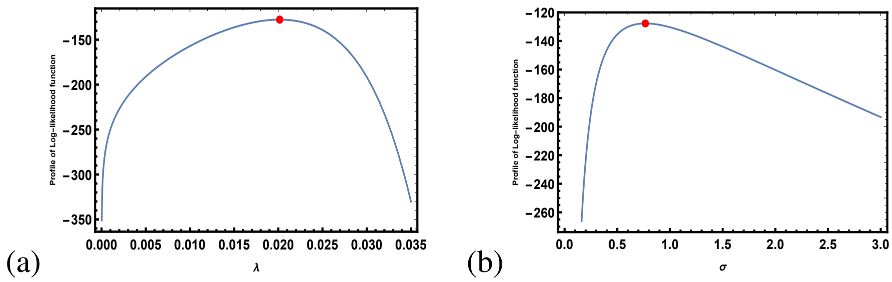

Figure 6. The profile of the $l(\hat{\psi})$ of λ and σ for refractory lining data.

MWE and NET distributions for fitting to turbochargers data. From these plots, we can observe that the RMW distribution is very close to the empirical line more than other distributions.

The estimated variance-covariance matrix of the RMW distribution for turbochargers data is given by

$$\mathbf{F}(\underline{\hat{\psi}}) = \begin{pmatrix} 0.00713665 & 0.0520858 \\ 0.0520858 & 0.394893 \end{pmatrix}.$$

Thus, the approximate 95% confidence intervals for λ and σ are respectively, [0.42979, 0.76555] and [2.28482, 4.80524].

Table 11. Turbochargers failure data.

1.6	2	2.6	3	3.5	3.9	4.5	4.6	4.8
5	5.1	5.3	5.4	5.6	5.8	6	6	6.1
6.3	6.5	6.5	6.7	7	7.1	7.3	7.3	7.3
7.7	7.7	7.8	7.9	8	8.1	8.3	8.4	8.4
8.5	8.7	8.8	9					

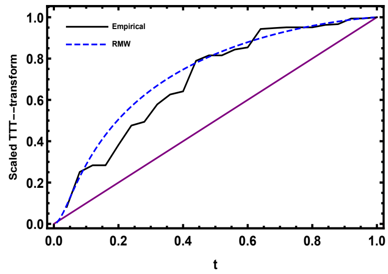

Figure 7. TTT-transform plot of the RMW distribution of fitting to refractory lining data.

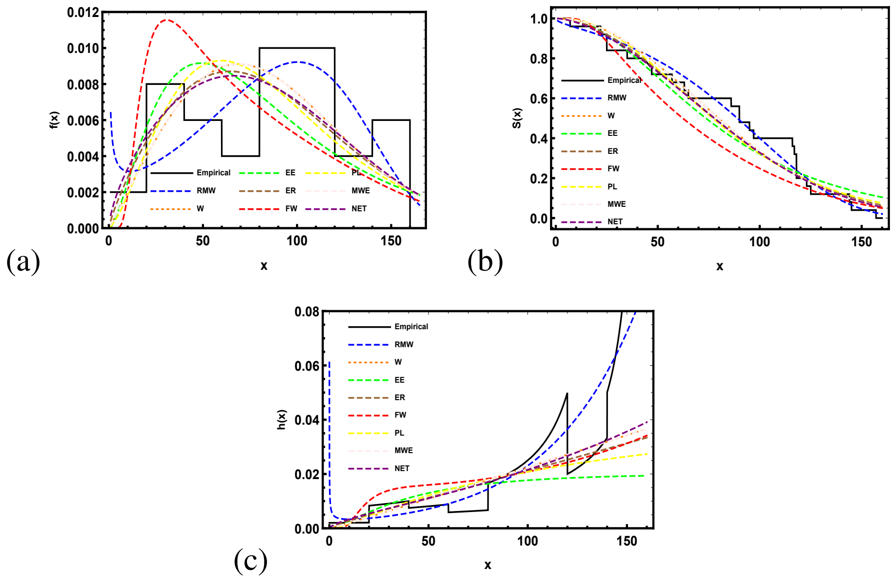

Figure 8. (a) Empirical and estimated PDFs; (b) Empirical and estimated $S(x)$ and (c) Empirical and estimated $h(x)$ of the RMW, W, EE, ER, FW, PL, MWE and NET distributions for fitting refractory lining data.

Table 12. The MLE of parameters, Log-likelihood, K-S values and the associated p-values for the turbochargers data.

Distribution	CDF	MLE of the parameters	$\ell(\underline{\hat{\psi}})$	K–S	p-value
RMW	$1-e^{-\frac{\lambda}{\sigma}\sqrt{x}e^{\lambda x-\sigma}}$	$\hat{\lambda}=0.5736,\ \hat{\sigma}=3.31345$	**-80.1525**	**0.08993**	**0.90275**
W	$1-e^{-(\frac{x}{\lambda})^{\sigma}}$	$\hat{\lambda}=6.92003,\ \hat{\sigma}=3.87251$	-82.4755	0.1077	0.74234
EE	$(1-e^{-\lambda x})^{\sigma}$	$\hat{\lambda}=0.44984,\ \hat{\sigma}=9.51462$	-90.1427	0.1542	0.29748
ER	$(1-e^{-\lambda x^{2}})^{\sigma}$	$\hat{\lambda}=0.03775,\ \hat{\sigma}=2.38479$	-85.7963	0.11757	0.63799
FW	$1-e^{-e^{\lambda x-\frac{\sigma}{x}}}$	$\hat{\lambda}=0.25812,\ \hat{\sigma}=11.8174$	-83.9757	0.12855	0.52311
PL	$1-(1+\frac{\sigma}{1+\sigma}x^{\lambda})e^{-\sigma x^{\lambda}}$	$\hat{\lambda}=2.44077,\ \hat{\sigma}=0.01945$	-84.1515	0.11435	0.67233
MWE	$1-e^{\lambda\theta(1-e^{(\frac{x}{\lambda})^{\sigma}})}$	$\hat{\lambda}=30.2464,\ \hat{\sigma}=3.87105,\ \hat{\theta}=9.95773$	-82.4452	0.10772	0.74213
NET	$(1-e^{1-(1+\lambda x)^{\sigma}})^{\beta}$	$\hat{\lambda}=0.00633,\ \hat{\sigma}=28.7796,\ \hat{\beta}=4.65619$	-84.5712	0.11556	0.65943

Table 13. Formal goodness of fit statistics for the turbochargers data.

Distribution	W^*	A^*	AIC	AICc	BIC	HQIC
RMW	**0.04093**	**0.2876**	**164.305**	**164.629**	**167.683**	**165.526**
W	0.08147	0.65841	168.951	169.275	172.329	170.172
EE	0.28005	1.7478	184.285	184.61	187.663	185.507
ER	0.17577	1.12615	175.593	175.917	178.97	176.814
FW	0.16464	1.09544	171.951	172.276	175.329	173.173
PL	0.11462	0.85888	172.303	172.627	175.681	173.524
MWE	0.08136	0.65746	170.904	171.571	175.971	172.736
NET	0.14231	0.96048	175.142	175.809	180.209	176.974

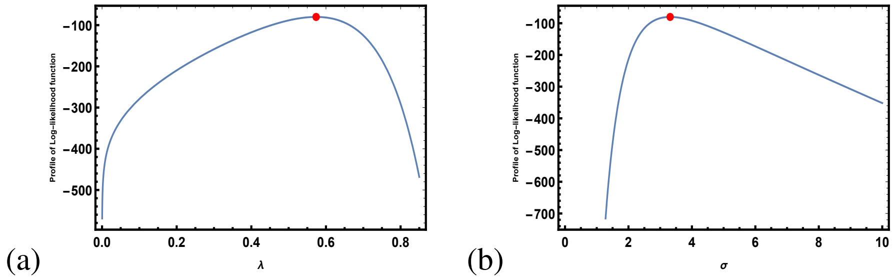

Figure 9. The profile of the $l(\hat{\psi})$ of λ and σ for the turbochargers data.

5.2. Censored data

5.2.1. Aarset data

This data represents the lifetimes of fifty devices [21]. According to Figure 12, the Aarset data demonstrates first convex and then concave shape indicating a bathtub-shaped hazard rate. Figure 12 shows the empirical and fitted scaled TTT-transform plot of the RMW distribution for Aarset data. The MLE of the parameters of the RMW as well as competing distributions and the observed K-S test statistic values for fitting to the Aarset data are provided in Table 14. From the measure of fit values given in Tables 14 and 15, our proposed RMW distribution has the largest $\ell(\hat{\underline{\psi}})$ values and has the smallest K-S, W^*, A^*, AIC, AICc, BIC, and HQIC values which mean that it provides the best fit to the Aarset data in comparison with competing distributions.

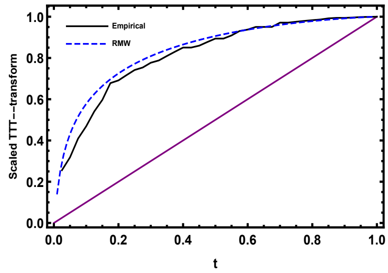

Figure 10. TTT-transform plot of the RMW distribution of fitting to turbochargers data.

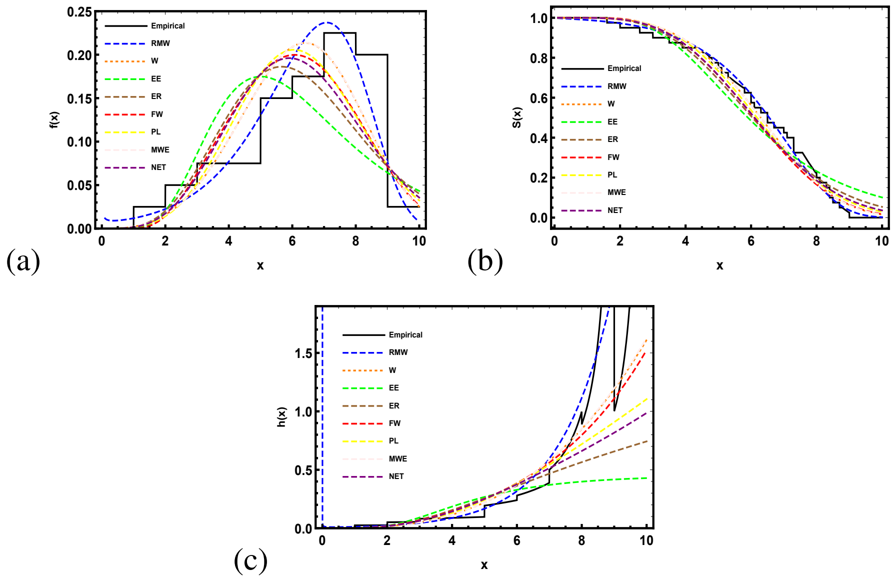

Figure 11. (a) Empirical and estimated PDFs; (b) Empirical and estimated $S(x)$ and (c) Empirical and estimated $h(x)$ of the RMW, W, EE, ER, FW, PL, MWE and NET distributions for fitting turbochargers data.

Table 14. The MLE of parameters, Log-likelihood, K-S values and the associated p-values for the Aarset data.

Distribution	CDF	MLE of the parameters	$\ell(\underline{\hat{\psi}})$	K-S	p-value
RMW	$1-e^{-\frac{\lambda}{\sigma}\sqrt{x}\,e^{\lambda x-\sigma}}$	$\hat{\lambda}=0.01174,\ \hat{\sigma}=0.16012$	**-195.024**	**0.20391**	**0.03128**
W	$1-e^{-(\frac{x}{\lambda})^{\sigma}}$	$\hat{\lambda}=55.515,\ \hat{\sigma}=0.75877$	-199.374	0.2481	0.00424
EE	$(1-e^{-\lambda x})^{\sigma}$	$\hat{\lambda}=0.0123,\ \hat{\sigma}=0.65105$	-198.455	0.24246	0.0056
ER	$(1-e^{-\lambda x^{2}})^{\sigma}$	$\hat{\lambda}=0.00008,\ \hat{\sigma}=0.30018$	-334.032	0.21492	0.01972
FW	$1-e^{-e^{\lambda x-\frac{\sigma}{x}}}$	$\hat{\lambda}=0.0095,\ \hat{\sigma}=0.72887$	-214.393	0.42842	0.00001
PL	$1-(1+\frac{\sigma}{1+\sigma}x^{\lambda})e^{-\sigma x^{\lambda}}$	$\hat{\lambda}=0.55464,\ \hat{\sigma}=0.21117$	-199.728	0.2519	0.00351
MWE	$1-e^{\lambda\theta(1-e^{(\frac{x}{\lambda})^{\sigma}})}$	$\hat{\lambda}=0.17817,\ \hat{\sigma}=0.24304,\ \hat{\theta}=0.09541$	-197.351	0.22361	0.01347
NET	$(1-e^{1-(1+\lambda x)^{\sigma}})^{\beta}$	$\hat{\lambda}=4.36086,\ \hat{\sigma}=0.18705,\ \hat{\beta}=2.42506$	-206.294	0.2888	0.00048

6. Conclusions

This article introduces a reduced distribution of the modified Weibull distribution, which is characterized by its flexibility and importance in modelling data. We conclude some of its basic

Table 15. Formal goodness of fit statistics for the Aarset data.

Distribution	W^*	A^*	AIC	AICc	BIC	HQIC
RMW	**0.22305**	**1.72576**	**394.049**	**394.304**	**397.873**	**395.505**
W	0.42445	2.8213	402.748	403.003	406.572	404.204
EE	0.40172	2.60829	400.91	401.165	404.734	402.366
ER	0.32043	2.15996	672.063	672.318	675.887	673.519
FW	3.50723	16.8347	432.785	433.041	436.609	434.242
PL	0.43267	2.83759	403.456	403.711	407.28	404.912
MWE	0.29471	2.11742	400.702	401.223	406.438	402.886
NET	0.77397	4.35649	418.588	419.11	424.324	420.773

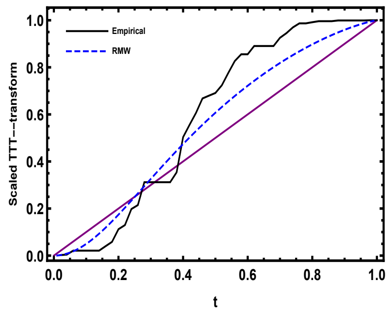

Figure 12. TTT-transform plot of the RMW distribution of fitting to Aarset data.

statistical properties. We introduce a simulation discussion to show the accuracy and performance of the maximum likelihood estimates. We prove by means of four applications to real data that the reduced modified Weibull distribution can yield better fits than some other distributions.

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Conflict of interest

The authors declare there is no conflict of interest.

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