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Research article

Attractivity criterion on a delayed tick population dynamics equation with a reproductive function $f(u) = ru^{\gamma}e^{-\sigma u}$

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Abstract: The aim of this article is to analyze the delay influence on the attraction for a scalar tick population dynamics equation accompanying two disparate delays. Taking advantage of the fluctuation lemma and some dynamic inequalities, we derive a criterion to assure the persistence and positiveness on the considered model. Furthermore, a time-lag-dependent condition is proposed to insure the global attractivity for the addressed model. Besides, we give some simulation diagrams to substantiate the validity of the theoretical outcomes.

Keywords: tick population; delay; equilibrium; attractivity

1. Introduction

Recently, the following delay scalar equation

$$E'(t) = -\delta E(t) + f((1-\alpha)\rho E(t-\tau_1(t)) + \alpha\rho E(t-\tau_2(t))), \tag{1.1}$$

has been presented by Zhang and Wu [1] and Zhang et al. [2] to show the qualitative properties on tick population in a fixed region. Here $E(t)$ describes the spawning density at time t, $\rho \in (0, 1)$ designates the survival likelihood, and $\delta > 0$ is the death rate. The reproductive function can be expressed by

$$f(u) = ru^{\gamma}e^{-\sigma u},\ u \geq 0,\ \gamma \in (0,\ 1], \tag{1.2}$$

where u is the spawning density, $r > 0$ denotes the maximal egg production rate per unit time, γ is a proxy for measuring returns, and $\sigma > 0$ reflects the strength of density dependence. Suppose $1-\alpha \in [0,\ 1]$ is a proportion of eggs that undergo development by standard development delay $\tau_1(t) \geq 0$, and the remaining portion α with respect to the so-called "diapause-induced delay" $\tau_2(t) \geq 0$. For more details we direct the reader to Zhang and Wu [1].

When more disparate delays are considered in the reproductive function (1.1), the authors in [1,2] have constructed some examples to show that chaotic oscillations may appear. The authors in [1,2] also point out that it is very difficult to explore the dynamics of (1.1) with different delay functions and two disparate delays rather than one can cause sustainable oscillations. Especially, under the following assumptions,

$$\tau_1(t) \equiv \tau, \quad \tau_2(t) \equiv 2\tau, \gamma = 1, \text{where } \tau > 0 \text{ is a constant,} \tag{1.3}$$

the Hopf bifurcations have been respectively studied in [1] and [2] when τ and α are the bifurcation parameters. As is known, differential equations with multiple delays have been handled as proper applications in the fields of economics, physiology and epidemiology and engineering [3–7]. When the delay functions take two different positive constants, some dynamic analysis is usually carried out by using dynamic system theory [8,9]. On the other hand, the population and ecology models with time-varying delays are interpreted as non-autonomous delay differential equations (DDEs) which generally do not generate a semi-flow and cannot be analyzed by the methods for DDEs accompanying constant delays in [8,9]. Moreover, the obtained findings on population and ecology models accompanying two disparate time-varying delays in [10–14] have not studied to the global attractivity of the non-autonomous tick equation with $f(u) = ru^{\gamma}e^{-\sigma u}$ and $\gamma \in (0,\ 1)$. Consequently, it is essential to further research the dynamic characteristics of (1.1) without considering (1.3).

It has been pointed out that time delay may lead to oscillations or instability in many biomathematics systems, and hence it is of great importance to discover the effect of time delay on the long-term dynamic characteristics of population equations [15–20]. Nevertheless, there have been no results on the delay-dependent attractiveness exploration of the tick population equation (1.1).

Motivated by the above considerations, assume that the reproductive function is

$$f(u) = ru^{\gamma}e^{-\sigma u} \ \text{ with } \ \gamma \in (0,\ 1), \tag{1.4}$$

the goal of this work is to derive some delay-dependent conditions for ensuring the global attractivity on the equilibrium point of (1.1) without assumption (1.3). Our result is an extension of [1–3] and its effectiveness will be verified by some numerical simulations.

Label $Q(t) = \sigma\rho E(t)$, then we can write (1.1) involving the reproductive function (1.4) as

$$\begin{aligned} Q'(t) &= -\delta Q(t) + r\rho\sigma^{1-\gamma}e^{-[(1-\alpha)Q(t-\tau_1(t))+\alpha Q(t-\tau_2(t))]} \\ &\quad \times[(1-\alpha)Q(t-\tau_1(t)) + \alpha Q(t-\tau_2(t))]^{\gamma},\ t \geq t_0,\ \gamma \in (0,\ 1). \end{aligned} \tag{1.5}$$

Obviously, Q^* obeying

$$-\delta(Q^*)^{1-\gamma} + r\rho\sigma^{1-\gamma}e^{-Q^*} = 0, \tag{1.6}$$

is a unique positive equilibrium point of equation (1.5). And we will show that the zero equilibrium point is not stable in the following analysis. For $i = 1, 2$, let

$$\tau_i^+ = \sup_{t\in[t_0,+\infty)} \tau_i(t), \quad \overline{\sigma} = \max_{j\in\{1,2\}} \tau_j^+ > 0 \text{ and } C_+ = C([-\overline{\sigma},\ 0], [0, +\infty)). \tag{1.7}$$

GIven $Q(t;t_0,\varphi)(Q_t(t_0,\varphi))$ as a solution of the system (1.5) incorporating the initial value conditions:

$$Q(t_0+\theta)=\varphi(\theta),\ -\overline{\sigma}\le\theta\le 0,\ \varphi\in C_+\ \text{ with }\ \varphi(0)>0, \tag{1.8}$$

we let $[t_0,\ \eta(\varphi))$ denote the right saturation interval.

This research is arranged as follows. In Section 2, we demonstrate the positiveness, boundedness and persistence of the addressed model. In Section 3, the main result involving delay-dependent attractivity is stated and guaranteed. In Sections 4 and 5, some numerical cases and conclusions are proposed, separately.

2. Positiveness and persistence

Lemma 2.1. $Q(t)=Q(t;t_0,\varphi)$ $(Q_t=Q_t(t_0,\varphi))$ possesses positiveness and boundedness on $[t_0,\ +\infty)$.

Proof. According to Theorem 5.2.1 in [21] , we gain that $Q_t\in C_+$ for each $t\in[t_0,\ \eta(\varphi))$. Since $Q(t_0)=\varphi(0)>0$, it follows that for arbitrary $t\in[t_0,\ \eta(\varphi))$,

$$\begin{aligned} Q(t) &= e^{-\delta(t-t_0)}Q(t_0)+\int_{t_0}^{t}e^{-\delta(t-s)}r\rho\sigma^{1-\gamma}[(1-\alpha)Q(s-\tau_1(s))+\alpha Q(s-\tau_2(s))]^{\gamma}\\ &\quad \times e^{-[(1-\alpha)Q(s-\tau_1(s))+\alpha Q(s-\tau_2(s))]}ds\\ &> 0, \end{aligned}$$

which, together with $\sup\limits_{x\in[0,+\infty)}x^{\gamma}e^{-x}=\frac{\gamma^{\gamma}}{e^{\gamma}}$ and (1.5), indicates that for all $t\in[t_0,\ \eta(\varphi))$,

$$\begin{aligned} Q'(t) &= -\delta Q(t)+r\rho\sigma^{1-\gamma}[(1-\alpha)Q(t-\tau_1(t))+\alpha Q(t-\tau_2(t))]^{\gamma}e^{-[(1-\alpha)Q(t-\tau_1(t))+\alpha Q(t-\tau_2(t))]}\\ &\le -\delta Q(t)+\frac{r\rho\sigma^{1-\gamma}\gamma^{\gamma}}{e^{\gamma}}, \end{aligned}$$

and

$$Q(t)\le\frac{r\rho\sigma^{1-\gamma}\gamma^{\gamma}}{\delta e^{\gamma}}(1-\frac{1}{e^{\delta(t-t_0)}})+\varphi(0)\frac{1}{e^{\delta(t-t_0)}}.$$

Consequently, $Q(t)$ has boundedness on $[t_0,\eta(\varphi))$. On the basis of Theorem 2.3.1 in [22] , we acquire $\eta(\varphi)=+\infty$. This furnishes the proof of Lemma 2.1.

Lemma 2.2. $l:=\liminf\limits_{t\to+\infty}Q(t)=\liminf\limits_{t\to+\infty}Q(t;t_0,\varphi)>0$.

Proof. Conversely, $\liminf\limits_{t\to+\infty}Q(t)=0$. For any $t\ge t_0$, define

$$\beta(t)=\max\left\{\varrho\in[t_0-\overline{\sigma},\ t]|Q(\varrho)=\min_{t_0-\overline{\sigma}\le s\le t}Q(s)\right\},$$

one can obtain from $\liminf\limits_{t\to+\infty}Q(t)=0$ that

$$\beta(t)\to+\infty\ \text{ as }\ t\to+\infty\ \ \text{with}\ \lim_{t\to+\infty}Q(\beta(t))=0.$$

Set $T_*\ge t_0$ accompany that

$$\beta(t)\ge t_0+\overline{\sigma}\ \text{ for arbitrary }\ t\in[T_*-\overline{\sigma},\ +\infty). \tag{2.1}$$

Clearly, (1.5) results in

$$\begin{aligned} 0 &\geq Q'(\beta(t)) \\ &= -\delta Q(\beta(t)) + r\rho\sigma^{1-\gamma}[(1-\alpha)Q(\beta(t)-\tau_1(\beta(t))) + \alpha Q(\beta(t)-\tau_2(\beta(t)))]^{\gamma} \\ &\quad \times e^{-[(1-\alpha)Q(\beta(t)-\tau_1(\beta(t)))+\alpha Q(\beta(t)-\tau_2(\beta(t)))]}, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} & r\rho\sigma^{1-\gamma}[(1-\alpha)Q(\beta(t)-\tau_1(\beta(t))) + \alpha Q(\beta(t)-\tau_2(\beta(t)))]^{\gamma} \\ & \times e^{-[(1-\alpha)Q(\beta(t)-\tau_1(\beta(t)))+\alpha\rho Q(\beta(t)-\tau_2(\beta(t)))]} \\ \leq\ & \delta Q(\beta(t)), \text{ where } \ t \geq T_*, \end{aligned}$$

which, combined with $\lim\limits_{t\to+\infty} Q(\beta(t)) = 0$, give us

$$\lim_{t\to+\infty}[(1-\alpha)Q(\beta(t)-\tau_1(\beta(t))) + \alpha Q(\beta(t)-\tau_2(\beta(t)))] = 0. \tag{2.3}$$

According to (2.3),

$$\begin{aligned} 0 &\geq Q'(\beta(t)) \\ &\geq Q^{\gamma}(\beta(t))\{-\delta Q^{1-\gamma}(\beta(t)) \\ &\quad + r\rho\sigma^{1-\gamma}e^{-[(1-\alpha)Q(\beta(t)-\tau_1(\beta(t)))+\alpha Q(\beta(t)-\tau_2(\beta(t)))]}\} \quad \text{for all } \ t \geq T_*, \end{aligned}$$

and

$$\frac{r\rho\sigma^{1-\gamma}}{\delta}e^{-[(1-\alpha)Q(\beta(t)-\tau_1(\beta(t)))+\alpha Q(\beta(t)-\tau_2(\beta(t)))]} \leq Q^{1-\gamma}(\beta(t)), \tag{2.4}$$

letting $t \to +\infty$ leads to

$$0 \geq \frac{r\rho\sigma^{1-\gamma}}{\delta} > 0,$$

which is absurd and accomplishes the proof.

Remark 2.1. Obviously, Lemma 2.2 shows that the zero equilibrium point of (1.5) has instability.

3. Attractivity analysis on the tick population dynamics equation

Now, label $Q(t) = Q(t; t_0, \varphi)$, we shall derive the global attraction on Q^*.

Proposition 3.1. Presume that $\eta(t) = Q(t) - Q^*$ is ultimately nonnegative, then

$$\lim_{t\to+\infty} Q(t) = Q^*.$$

Proof. Clearly, there is $T > t_0$ satisfying

$$\eta(t) = Q(t) - Q^* \geq 0 \text{ for all } t \geq T.$$

We claim that $\limsup\limits_{t\to+\infty} \eta(t) = 0$. Assume that, contrary to our claim,

$$\limsup_{t\to+\infty} \eta(t) > 0.$$

According to Lemma A.1 in [23], fluctuation lemma, it is easy to find a sequence $\{t_h\}_{h\geq 1}$ that obeys

$$t_k \to +\infty, \quad \eta(t_h) \to \limsup_{t\to+\infty} \eta(t), \quad \eta'(t_h) \to 0 \quad \text{as } h \to +\infty.$$

From (1.5), we get

$$\begin{aligned} \eta'(t_h) &= -\delta Q(t_h) + r\rho\sigma^{1-\gamma}[(1-\alpha)Q(t_h - \tau_1(t_h)) + \alpha Q(t_h - \tau_2(t_h))]^\gamma \\ &\quad \times e^{-[(1-\alpha)Q(t_h-\tau_1(t_h))+\alpha Q(t_h-\tau_2(t_h))]}, \ t_h \in (T + \overline{\sigma}, \ +\infty). \end{aligned} \tag{3.1}$$

More generally, we can take a subsequence of $\{t_h\}$ (still denoted $\{t_h\}$) accompanying that $\lim_{h\to+\infty} Q(t_h - \tau_j(t_h))$ exists for arbitrary $j \in \Omega := \{1, 2\}$. Manifestly,

$$Q^* \leq \lim_{h\to+\infty} Q(t_h - \tau_j(t_h)) \leq \limsup_{t\to+\infty} \eta(t) + Q^* \text{ for arbitrary } j \in \Omega.$$

and one of the following two cases must hold.

Case I: $\lim_{h\to+\infty} [(1-\alpha)Q(t_h - \tau_1(t_h)) + \alpha Q(t_h - \tau_2(t_h))] = Q^*$;

Case II: $Q^* < \lim_{h\to+\infty} [(1-\alpha)Q(t_h - \tau_1(t_h)) + \alpha Q(t_h - \tau_2(t_h))]$.

Considering that **Case I** occurs, by (1.6) and (3.1), taking limits yields

$$0 = -\delta(\limsup_{t\to+\infty} \eta(t) + Q^*) + r\rho\sigma^{1-\gamma}(Q^*)^\gamma e^{-Q^*} < -\delta Q^* + \delta Q^* = 0,$$

an impossible result. Consequently, $\limsup_{t\to+\infty} \eta(t) = 0$.

If **Case II** occurs, then (1.6) and (3.1) derive

$$\begin{aligned} 0 &< -\delta(\limsup_{t\to+\infty} \eta(t) + Q^*) + r\rho\sigma^{1-\gamma}(\limsup_{t\to+\infty} \eta(t) + Q^*)^\gamma e^{-Q^*} \\ &= (\limsup_{t\to+\infty} \eta(t) + Q^*)^\gamma[-\delta(\limsup_{t\to+\infty} \eta(t) + Q^*)^{1-\gamma} + r\rho\sigma^{1-\gamma}e^{-Q^*}] \\ &< (\limsup_{t\to+\infty} \eta(t) + Q^*)^\gamma[-\delta(Q^*)^{1-\gamma} + r\rho\sigma^{1-\gamma}e^{-Q^*}] \\ &= 0. \end{aligned}$$

This yields a contradiction and verifies Proposition 3.1.

Proposition 3.2. Provided that $\eta(t) = Q(t) - Q^*$ is ultimately non-positive, we acquire

$$\lim_{t\to+\infty} Q(t) = Q^*.$$

Proof. Noticeably, there is $T > t_0$ satisfying that

$$\eta(t) = Q(t) - Q^* \leq 0 \text{ for arbitrary } t \geq T.$$

We now verify that $\liminf_{t\to+\infty} \eta(t) = 0$. First we assume the opposite,

$$\liminf_{t\to+\infty} \eta(t) < 0.$$

By Lemma A.1 of [23] , one can pick a sequence $\{\bar{t}_h\}_{h\geq 1}$ agreeing with

$$\bar{t}_h \to +\infty, \quad \eta(\bar{t}_h) \to \liminf_{t\to+\infty} \eta(t), \quad \eta'(\bar{t}_h) \to 0 \quad \text{as } h \to +\infty.$$

Then (1.5) leads to

$$\begin{aligned} \eta'(\bar{t}_h) &= -\delta Q(\bar{t}_h) + r\rho\sigma^{1-\gamma}[(1-\alpha)Q(\bar{t}_h - \tau_1(\bar{t}_h)) + \alpha Q(\bar{t}_h - \tau_2(\bar{t}_h))]^\gamma \\ &\quad \times e^{-[(1-\alpha)Q(\bar{t}_h-\tau_1(\bar{t}_h))+\alpha Q(\bar{t}_h-\tau_2(\bar{t}_h))]}, \bar{t}_h > T + \overline{\sigma}. \end{aligned} \tag{3.2}$$

Without loss of generality, for all $j \in \Omega$, let $\lim_{h\to+\infty} Q(\bar{t}_h - \tau_j(\bar{t}_h))$ exist. It can be derived from Lemma 2.2 that for all $j \in \Omega$,

$$0 < Q^* + \liminf_{t\to+\infty} \eta(t) \leq \lim_{h\to+\infty} Q(\bar{t}_h - \tau_j(\bar{t}_h)) \leq Q^*,$$

and one of the following two situations must appear:

Case (1): $\lim_{h\to+\infty} [(1-\alpha)Q(\bar{t}_h - \tau_1(\bar{t}_h)) + \alpha Q(\bar{t}_h - \tau_2(\bar{t}_h))] = Q^*$;

Case (2): $Q^* > \lim_{h\to+\infty} [(1-\alpha)Q(\bar{t}_h - \tau_1(\bar{t}_h)) + \alpha Q(\bar{t}_h - \tau_2(\bar{t}_h))]$.

When **Case (1)** occurs, from (1.6) and (3.2), we obtain

$$0 = -\delta(\liminf_{t\to+\infty} \eta(t) + Q^*) + r\rho\sigma^{1-\gamma}(Q^*)^\gamma e^{-Q^*} > -\delta Q^* + \delta Q^* = 0,$$

which is impossible and shows that $\liminf_{t\to+\infty} \eta(t) = 0$.

If **Case (2)** emerges, then (1.6) and (3.2) imply that

$$\begin{aligned} 0 &> -\delta(\liminf_{t\to+\infty} \eta(t) + Q^*) + r\rho\sigma^{1-\gamma}(\liminf_{t\to+\infty} \eta(t) + Q^*)^\gamma e^{-Q^*} \\ &= (\liminf_{t\to+\infty} \eta(t) + Q^*)^\gamma[-\delta(\liminf_{t\to+\infty} \eta(t) + Q^*)^{1-\gamma} + r\rho\sigma^{1-\gamma}e^{-Q^*}] \\ &> (\liminf_{t\to+\infty} \eta(t) + Q^*)^\gamma[-\delta(Q^*)^{1-\gamma} + r\rho\sigma^{1-\gamma}e^{-Q^*}] \\ &= 0, \end{aligned}$$

which leads to an impossible result and substantiates Proposition 3.2.

Next, we consider the case of $Q(t)$ with respect to oscillations about Q^*.

Proposition 3.3. Assume that

$$e(1 - e^{-\delta\overline{\sigma}}) < 1, \;\; Q^*(e^{\delta\overline{\sigma}} - 1) < 1, \;\; \frac{Q^*(1 - e^{-\delta\overline{\sigma}})}{1 - e(1 - e^{-\delta\overline{\sigma}})} < 1, \tag{3.3}$$

holds, and $\eta(t) = Q(t) - Q^*$ oscillates about zero. Then $\lim_{t\to+\infty} Q(t) = Q^*$.

Proof. It follows from (1.5) that

$$\begin{aligned} &\eta'(t) + \delta\eta(t) + \delta Q^* \\ = \;&\frac{r\rho\sigma^{1-\gamma}[(1-\alpha)(\eta(t-\tau_1(t)) + Q^*) + \alpha(\eta(t-\tau_2(t)) + Q^*)]^\gamma}{e^{[(1-\alpha)(\eta(t-\tau_1(t))+Q^*)+\alpha(\eta(t-\tau_2(t))+Q^*)]}}, \; t > t_0. \end{aligned} \tag{3.4}$$

Label

$$\lambda = \liminf_{t\to+\infty} \eta(t), \;\; \mu = \limsup_{t\to+\infty} \eta(t). \tag{3.5}$$

Note that if $w(t)$ is oscillating about zero, we have

$$\lambda \leq 0 \leq \mu.$$

Hence, all that remains is to prove that

$$\lambda = \mu = 0.$$

In view of the oscillation of $\eta(t)$, one can pick a sequence $\{k_n\}_{n\geq 1}$ which is strictly monotonically increasing and obeys that

$$k_n > \overline{\sigma}, \lim_{n\to+\infty} k_n = +\infty, \eta(k_n) = 0 \text{ for all } n = 1, 2, \cdots,$$

and $\eta(t)$ has both negative and positive values in every interval $(k_n,\ k_{n+1})$. For each positive integer n, select $t_n, s_n \in (k_n,\ k_{n+1})$ which satisfy that

$$\eta(t_n) = \max_{t\in[k_n,\ k_{n+1}]} \eta(t) > 0, \quad \eta(s_n) = \min_{t\in[k_n,\ k_{n+1}]} \eta(t) < 0,$$

which suggests that

$$\eta'(t_n) = \eta'(s_n) = 0, n \in \{1, 2, \cdots, \}, \tag{3.6}$$

and

$$\liminf_{n\to+\infty} \eta(s_n) = \lambda = \liminf_{t\to+\infty} \eta(t), \quad \limsup_{n\to+\infty} \eta(t_n) = \mu = \limsup_{t\to+\infty} \eta(t). \tag{3.7}$$

Next, we claim that, for every positive integer n, one can select $T_n \in [t_n - \overline{\sigma}, t_n) \cap [k_n, t_n)$ accompanying that

$$\eta(T_n) = 0 \text{ and } \eta(t) > 0 \text{ for arbitrary } t \in (T_n, t_n). \tag{3.8}$$

Suppose the contrary and choose a positive integer n satisfying that for arbitrary $t \in [t_n - \overline{\sigma}, t_n)$,

$$k_n < t_n - \overline{\sigma} < k_{n+1} \text{ and } \eta(t) > 0,$$

which, combined with (3.4)–(3.6), yields

$$\begin{aligned}
0 &= -\delta[Q^* + \eta(t_n)] + r\rho\sigma^{1-\gamma}[(1-\alpha)(\eta(t_n - \tau_1(t_n)) + Q^*) + \alpha(\eta(t_n - \tau_2(t_n)) + Q^*)]^{\gamma} \\
&\quad \times e^{-[(1-\alpha)(\eta(t_n-\tau_1(t_n))+Q^*)+\alpha(\eta(t_n-\tau_2(t_n))+Q^*)]} \\
&< -\delta[Q^* + \eta(t_n)] + r\rho\sigma^{1-\gamma}[Q^* + \eta(t_n)]^{\gamma} e^{-Q^*} \\
&< [Q^* + \eta(t_n)]^{\gamma}[-\delta(Q^*)^{1-\gamma} + r\rho\sigma^{1-\gamma} e^{-Q^*}] \\
&= 0.
\end{aligned}$$

This is impossible and proves (3.8).

Similarly, it can be shown that for arbitrary positive integer $n > 0$, one can pick $S_n \in [s_n - \overline{\sigma}, s_n) \cap [k_n, s_n)$ that obeys

$$\eta(S_n) = 0 \text{ and } \eta(t) < 0 \text{ for arbitrary } t \in (S_n, s_n). \tag{3.9}$$

For all $\varepsilon > 0$, (3.7) shows that there exists a positive integer n^* that satisfies for $t \in (\min\{t_{n^*}, s_{n^*}\} - \overline{\sigma},\ +\infty)$,

$$\lambda - \varepsilon < \eta(t) < \mu + \varepsilon. \tag{3.10}$$

Furthermore, we can assume that

$$0 < \varepsilon < Q^* + \lambda \text{ with } 0 < Q^* + (\lambda - \varepsilon), \tag{3.11}$$

because $0 < \lim_{n\to+\infty} Q(s_n) = Q^* + \lim_{n\to+\infty} \eta(s_n) = Q^* + \lambda$.

Moreover, from (1.5), (3.9), (3.10) and

$$\begin{aligned}[\eta(t)e^{\delta t}]' &= -\delta Q^* e^{\delta t} + r\rho\sigma^{1-\gamma}[(1-\alpha)(\eta(t-\tau_1(t)) + Q^*) + \alpha(\eta(t-\tau_2(t)) + Q^*)]^{\gamma} \\ &\quad \times \frac{e^{\delta t}}{e^{[(1-\alpha)(\eta(t-\tau_1(t))+Q^*)+\alpha(\eta(t-\tau_2(t))+Q^*)]}}, \ t > t_0, \end{aligned} \tag{3.12}$$

we get

$$\begin{aligned}\eta(s_n)e^{\delta s_n} &= -Q^*(e^{\delta s_n} - e^{\delta S_n}) \\ &\quad + r\rho\sigma^{1-\gamma}\int_{S_n}^{s_n} [(1-\alpha)(\eta(t-\tau_1(t)) + Q^*) + \alpha(\eta(t-\tau_2(t)) + Q^*)]^{\gamma} \\ &\quad \times \frac{e^{\delta t}}{e^{[(1-\alpha)(\eta(t-\tau_1(t))+Q^*)+\alpha(\eta(t-\tau_2(t))+Q^*)]}} dt \\ &> -Q^*(e^{\delta s_n} - e^{\delta S_n}) \\ &\quad + r\rho\sigma^{1-\gamma}\int_{S_n}^{s_n} [(1-\alpha)((\lambda-\varepsilon) + Q^*) + \alpha((\lambda-\varepsilon) + Q^*)]^{\gamma} \\ &\quad \times e^{-[(1-\alpha)((\mu+\varepsilon)+Q^*)+\alpha((\mu+\varepsilon)+Q^*)]} e^{\delta t} dt \\ &= -Q^*(e^{\delta s_n} - e^{\delta S_n}) \\ &\quad + r\rho\sigma^{1-\gamma}\int_{S_n}^{s_n} (Q^*)^{\gamma}\left(\frac{(\lambda-\varepsilon) + Q^*}{Q^*}\right)^{\gamma} e^{-((\mu+\varepsilon)+Q^*)} e^{\delta t} dt \\ &> -Q^*(e^{\delta s_n} - e^{\delta S_n}) \\ &\quad + r\rho\sigma^{1-\gamma}\int_{S_n}^{s_n} (Q^*)^{\gamma}\frac{(\lambda-\varepsilon) + Q^*}{Q^*} e^{-((\mu+\varepsilon)+Q^*)} e^{\delta t} dt \\ &= -Q^*(e^{\delta s_n} - e^{\delta S_n}) + r\rho\sigma^{1-\gamma}(Q^*)^{\gamma-1}((\lambda-\varepsilon) + Q^*)e^{-((\mu+\varepsilon)+Q^*)}\frac{e^{\delta s_n} - e^{\delta S_n}}{\delta} \\ &= Q^*(e^{\delta s_n} - e^{\delta S_n})\left[\frac{r\rho\sigma^{1-\gamma}(Q^*)^{\gamma-1}e^{-Q^*}e^{-(\mu+\varepsilon)}}{\delta} - 1\right] \\ &\quad + r\rho\sigma^{1-\gamma}(Q^*)^{\gamma-1}e^{-Q^*}e^{-(\mu+\varepsilon)}\frac{e^{\delta s_n} - e^{\delta S_n}}{\delta}(\lambda-\varepsilon) \\ &> (\lambda-\varepsilon)(e^{\delta s_n} - e^{\delta S_n}) + [e^{-(\mu+\varepsilon)} - 1]Q^*(e^{\delta s_n} - e^{\delta S_n}),\end{aligned}$$

and

$$\begin{aligned}\eta(s_n) + (e^{-\delta\overline{\sigma}} - 1)(\lambda-\varepsilon) &\geq \eta(s_n) + (\lambda-\varepsilon)(e^{\delta(S_n - s_n)} - 1) \\ &> Q^*[e^{-(\mu+\varepsilon)} - 1](1 - e^{\delta(S_n - s_n)}) \\ &\geq [e^{-(\mu+\varepsilon)} - 1]Q^*(1 - e^{-\delta\overline{\sigma}}),\end{aligned} \tag{3.13}$$

where $n > n^*$.

Allowing $n \to \infty$ and $\varepsilon \to 0^+$, one can obtain from (3.3) and (3.13) that

$$\lambda \geq (e^{-\mu} - 1)Q^*(e^{\delta\overline{\sigma}} - 1) \geq e^{-\mu} - 1 \geq -1. \tag{3.14}$$

From (1.5) and (3.9)–(3.12), it can be seen that

$$\begin{aligned}
\eta(t_n)e^{\delta t_n} &= -Q^*(e^{\delta t_n} - e^{\delta T_n}) \\
&\quad + r\rho\sigma^{1-\gamma}\int_{T_n}^{t_n}[(1-\alpha)(\eta(t-\tau_1(t)) + Q^*) + \alpha(\eta(t-\tau_2(t)) + Q^*)]^{\gamma} \\
&\quad \times \frac{e^{\delta t}}{e^{[(1-\alpha)(\eta(t-\tau_1(t))+Q^*)+\alpha(\eta(t-\tau_2(t))+Q^*)]}}dt \\
&< -Q^*(e^{\delta t_n} - e^{\delta T_n}) \\
&\quad + r\rho\sigma^{1-\gamma}\int_{T_n}^{t_n}[Q^* + (\mu+\varepsilon)]^{\gamma}e^{-Q^*}e^{-(\lambda-\varepsilon)}e^{\delta t}dt \\
&= -Q^*(e^{\delta t_n} - e^{\delta T_n}) \\
&\quad + r\rho\sigma^{1-\gamma}\int_{T_n}^{t_n}(Q^*)^{\gamma}[\frac{Q^* + (\mu+\varepsilon)}{Q^*}]^{\gamma}e^{-Q^*}e^{-(\lambda-\varepsilon)}e^{\delta t}dt \\
&< -Q^*(e^{\delta t_n} - e^{\delta T_n}) \\
&\quad + r\rho\sigma^{1-\gamma}\int_{T_n}^{t_n}(Q^*)^{\gamma}\frac{Q^* + (\mu+\varepsilon)}{Q^*}e^{-Q^*}e^{-(\lambda-\varepsilon)}e^{\delta t}dt \\
&= -Q^*(e^{\delta t_n} - e^{\delta T_n}) \\
&\quad + r\rho\sigma^{1-\gamma}[(Q^*)^{\gamma-1}e^{-Q^*}e^{-(\lambda-\varepsilon)}(Q^* + (\mu+\varepsilon))]\frac{e^{\delta t_n} - e^{\delta T_n}}{\delta} \\
&= Q^*(e^{\delta t_n} - e^{\delta T_n})[\frac{r\rho\sigma^{1-\gamma}(Q^*)^{\gamma-1}e^{-Q^*}e^{-(\lambda-\varepsilon)}}{\delta} - 1] \\
&\quad + r\rho\sigma^{1-\gamma}(Q^*)^{\gamma-1}e^{-Q^*}\frac{e^{\delta t_n} - e^{\delta T_n}}{\delta}(\mu+\varepsilon)e^{-(\lambda-\varepsilon)} \\
&\leq [e^{-(\lambda-\varepsilon)} - 1]Q^*(e^{\delta t_n} - e^{\delta T_n}) + e^{1+\varepsilon}(e^{\delta t_n} - e^{\delta T_n})(\mu+\varepsilon), \text{ where } n > n^*,
\end{aligned}$$

and

$$\begin{aligned}
\eta(t_n) &< [e^{-(\lambda-\varepsilon)} - 1]Q^*(1 - e^{\delta(T_n - t_n)}) + (\mu+\varepsilon)e^{1+\varepsilon}(1 - e^{\delta(T_n - t_n)}) \\
&\leq [e^{-(\lambda-\varepsilon)} - 1]Q^*(1 - e^{-\delta\overline{\sigma}}) + e^{1+\varepsilon}(1 - e^{-\delta\overline{\sigma}})(\mu+\varepsilon), \text{ where } n > n^*.
\end{aligned} \tag{3.15}$$

By (3.3) and (3.15), permitting $n \to \infty$ and $\varepsilon \to 0^+$ lead to

$$\mu \leq (e^{-\lambda} - 1)\frac{Q^*(1 - e^{-\delta\overline{\sigma}})}{1 - e(1 - e^{-\delta\overline{\sigma}})} \leq e^{-\lambda} - 1. \tag{3.16}$$

Hence, (3.14) and (3.16) give us

$$e^{-\lambda} - 1 \geq \mu \geq \lambda \geq e^{-\mu} - 1. \tag{3.17}$$

Define $B(x) = e^{1-e^{-x}} - 1 - x$ on $[0, +\infty)$, we can easily check its monotonicity on $[0, +\infty)$. Then, (3.17) yields

$$B(0) = 0,\ 0 \leq e^{-\lambda} - 1 - \mu \leq B(\mu), \text{ and } B'(x) = e^{1-x-e^{-x}} - 1 < 0 \text{ for all } x \in (0, +\infty),$$

which indicates that $\mu = 0$ and $\lambda = 0$. The proof is complete.

According to Propositions 3.1–3.3, we get the main theorem on the global attractivity of Q^*.

Theorem 3.1. Under (3.3), Q^* is a global attractor of (1.5) and (1.8).

Remark 3.2. It can be discovered from Propositions 3.1 and 3.2 that the delays $\tau_1(t)$ and $\tau_2(t)$ have no influence on the attractivity of the non-oscillatory solutions about N^*. From the facts that

$$\lim_{\overline{\sigma}\to 0^+} Q^*(e^{\delta\overline{\sigma}} - 1) = 0 \text{ and } \lim_{\overline{\sigma}\to 0^+} \frac{Q^*(1 - e^{-\delta\overline{\sigma}})}{1 - e(1 - e^{-\delta\overline{\sigma}})} = 0,$$

we know that condition (3.3) is naturally valid when the delays are small enough and hence Q^* is a global attractor of (1.5) accompanying small lags. However, the attractivity of the oscillatory solutions about Q^* is closely related to two time-varying delays. In other words, the two disparate delays undertake a significant role in characterizing the attractivity of (1.5). Furthermore,

$$\lim_{\overline{\sigma}\to +\infty} Q^*(e^{\delta\overline{\sigma}} - 1) = +\infty$$

suggests that condition (3.3) is not satisfied when the delays are large enough in (1.5).

4. A numerical case

In order to validate the effectiveness of the results of qualitative analysis model (1.1), we also carried out numerical simulation via MATLAB. Consider a scalar tick population equation involving two disparate time-varying delays,

$$\begin{aligned} E'(t) &= -0.09E(t) + e^{\frac{3}{2}}[(1-0.4)\times 0.9E(t-\tau_1(t)) + 0.4\times 0.9E(t-\tau_2(t))]^{0.5} \\ &\quad \times e^{-0.7[(1-0.4)\times 0.9E(t-\tau_1(t))+0.4\times 0.9E(t-\tau_2(t))]}, \ t \geq t_0 = 0, \end{aligned} \tag{4.1}$$

where $r = e^{\frac{3}{2}}$, $\alpha = 0.4$, $\delta = 0.09$, $\rho = 0.9$, $\gamma = 0.5$ and $\sigma = 0.7$.

Let

$$\tau_1(t) = 0.05, \ \tau_2(t) = 0.1, \tag{4.2}$$

we can easily obtain that (3.3) is obeyed. From Theorem 3.1, $E^* = \frac{1}{\sigma\rho}Q^* \approx 4.8571$ has global attractivity, which is strongly substantiated by Figure 1. Here $\{\varphi \in C([-\frac{1}{10}, 0], \mathbb{R}^+)|\varphi(0) > 0\}$ is the attraction domain.

Nevertheless, if we pick

$$\tau_1(t) = 100, \ \tau_2(t) = 200, \tag{4.3}$$

then (3.3) is not satisfied. Here, the assumptions adopted in Theorem 3.1 can not be obeyed. Figure 2 shows that the positive equilibrium point does not possesses attractivity.

Remark 4.1. According to the above MATLAB simulations, the following observations can be obtained. The positive equilibrium point is attractive involving small delays. But big delays will cause complex dynamic behavior. Moreover, the assumption mentioned in (1.3) does not hold in (4.1), and the global attractivity of the non-autonomous tick equation accompanying two disparate time-varying delays has not been considered in [1, 2, 8–10, 24–29]. Consequently, our results in this paper are essentially novel and complement previous studies to some extent.

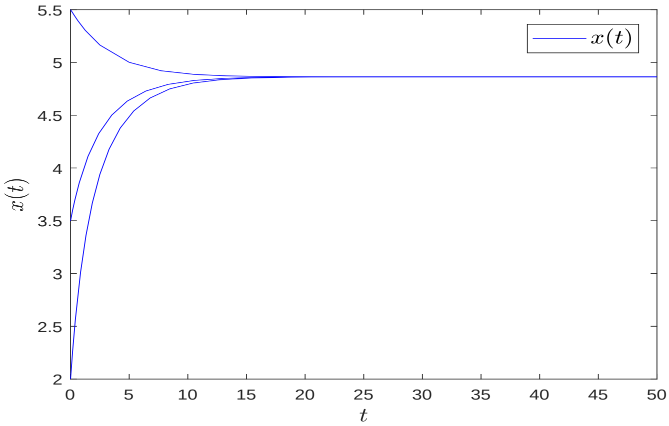

Figure 1. Numerical curves of (4.1) incorporating time lags (4.2) and different initial values, and $x(t) = E(t)$.

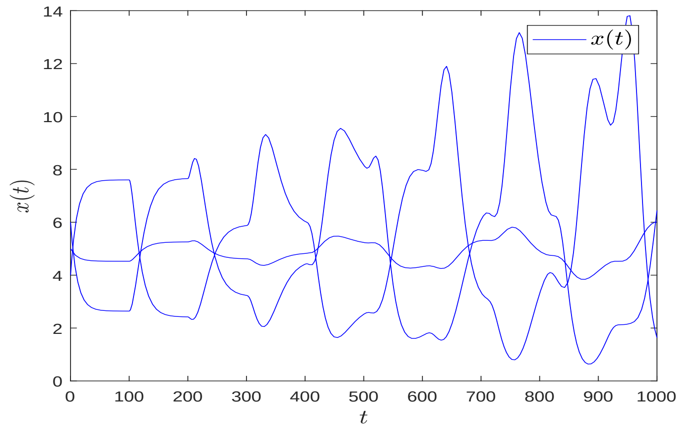

Figure 2. Numerical curves of (4.1) incorporating time lags (4.3) and different initial values, and $x(t) = E(t)$.

5. Conclusions

This article investigated a non-autonomous tick population dynamics equation involving two different time-varying lags. Combining fluctuation lemma and novel techniques of differential inequality, the positiveness, persistence and the global attractivity of all solutions have firstly been derived for the addressed equation. The established outcomes show that if the delays in the development are small enough, the global attractiveness of the positive equilibrium can be ensured by controlling the survival probability, mortality, maximum spawning number, the strength of density dependence and the proxy of measuring return. The strategy proposed in this paper can also be applied to other dynamic problems on delayed population models incorporating two or more disparate delays in the identical breeding function.

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Conflict of interest

The authors declare that they have no competing interests.

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