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Research article

Predator-prey systems with defense switching and density-suppressed dispersal strategy

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Abstract: In this paper, we consider the following predator-prey system with defense switching mechanism and density-suppressed dispersal strategy

$$\begin{cases} u_t = \Delta(d_1(w)u) + \frac{\beta_1 uvw}{u+v} - \alpha_1 u, & x \in \Omega, \quad t > 0, \\ v_t = \Delta(d_2(w)v) + \frac{\beta_2 uvw}{u+v} - \alpha_2 v, & x \in \Omega, \quad t > 0, \\ w_t = \Delta w - \frac{\beta_3 uvw}{u+v} + \sigma w\left(1 - \frac{w}{K}\right), & x \in \Omega, \quad t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega, \quad t > 0, \\ (u, v, w)(x, 0) = (u_0, v_0, w_0)(x), & x \in \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary. Based on the method of energy estimates and Moser iteration, we establish the existence of global classical solutions with uniform-in-time boundedness. We further prove the global stability of co-existence equilibrium by using the Lyapunov functionals and LaSalle's invariant principle. Finally we conduct linear stability analysis and perform numerical simulations to illustrate that the density-suppressed dispersal may trigger the pattern formation.

Keywords: prey-predator system; defense switching; density-suppressed diffusion; global stability; pattern formation

1. Introduction

Defense switching means that prey species pay more attention on guarding against the relatively more abundant population predator [1], certain fish species in Lake Tanganyika against two phenotypes (dextral and sinistral) of cichlid *Perissodus microlepis* [2] is a typical example. The dextral and sinistral phenotypes attack the prey fishes from the left-side and right-side, respectively. Pretend that the population of dextral individuals is more abundant, then prey fishes tend to be more defensive against the attacks from left-side, which leads to greater hunting success for sinistral individuals (relatively rare population). Based on a simple Lotka-Volterra equations of a two-predator one-prey system, Saleem et al. proposed a defensive switching model [1]

$$\begin{cases} u_t = u(-\alpha_1 + f_1 w), \\ v_t = v(-\alpha_2 + f_2 w), \\ w_t = w(\sigma - f_1 u - f_2 v), \end{cases} \tag{1.1}$$

with the predatory rates functions

$$\begin{cases} f_1(u,v) = \frac{\beta_1}{1+\frac{u}{v}}, \\ f_2(u,v) = \beta_2(1 - \frac{f_1}{\beta_1}), \end{cases} \tag{1.2}$$

where $u := u(x,t)$ and $v := v(x,t)$ denote the density of two predators while $w := w(x,t)$ is the prey density. The parameters α_j $(j = 1,2)$ account the death rate, $\beta_j > 0$ $(j = 1,2)$ are the predation coefficients and $\sigma > 0$ accounts for the growth rate of the prey species. Moreover, the predatory rates f_1 and f_2, also called "defensive switching functions", possess a characteristic property that the rate of the prey attacked by a predator will decrease if this predator population becomes much more abundant than the population of another predator. Specifically, when a predator population becomes large, the prey species guards against it more vigilantly and switches to another predator, which is in relatively small population, to keep its individual from being hunted too much. Such prey behaviors result in less successful hunting for abundant population predator species and more successful hunting for relatively rare one [1].

In light of the defensive switching model Eqs (1.1) and (1.2) in [1], Pang and Wang [3] considered the following reaction-diffusion system by introducing the random movements of species and the intra-specific interaction between the prey in Eq (1.1)

$$\begin{cases} u_t = d_1 \Delta u + \frac{\beta_1 uvw}{u+v} - \alpha_1 u, \\ v_t = d_2 \Delta v + \frac{\beta_2 uvw}{u+v} - \alpha_2 v, \\ w_t = d_3 \Delta w - \frac{(\beta_1+\beta_2)uvw}{u+v} + \sigma w\left(1 - \frac{w}{K}\right), \end{cases} \tag{1.3}$$

where constants $d_j > 0$ $(j = 1,2,3)$ account for diffusion coefficients and the positive parameter K represents the environmental carrying capacity to the prey species. We note that the interaction mechanism between the predators and prey in the defensive switching model is substantially different from that in the ratio-dependent predator-prey system [4] though they seem to have some similar structures.

In [1], Saleem et al. proved that the co-existence steady state is globally asymptotically stable except the case where the two predators have the same mortality rates; Otherwise, the system has a periodic solution. Pang and Wang [3] proved the co-existence steady states is globally asymptotically stable no matter α_1 and α_2 are equal or not and hence no pattern formation will arise from the system (1.3). If the term $d_1 \Delta u$ is replaced by a cross diffusion term $\Delta(d_1 u + \frac{ku}{\epsilon+v^2})$ $(k, \epsilon > 0)$, however, they showed that the cross diffusion between predators can drive stationary patterns by using

Leray-Schauder degree theory. Recently, the system (1.3) with prey-taxis (the cross diffusion between the predators and prey) was considered by Wang and Guo [5]. They established the existence of globally bounded solutions and global stability of constant steady states for small prey-tactic coefficient and numerically demonstrated that strong prey-taxis can induce the pattern formation if the two predators have different mortality rates. Subsequently, the existence of nonconstant steady states was obtained for some range of repulsive prey-taxis coefficient with more general functional response functions in [6]. From the above results, we find that the dispersal strategies (like cross-diffusion or prey-taxis) play important roles in determining the population distribution profiles.

In this paper, we shall consider a different dispersal strategy - density-suppressed diffusion, which was first used in [7] to describe the directed movement of predators in the predator-prey systems to fit the experimental observations. This type of diffusion assumes that the predator's diffusion decreasingly depends on the density distribution of the prey. As shown in [8], the density-suppressed diffusion can explain the heterogenous population distribution observed in the field experiment of [7] while random diffusion can not. Hence density-suppressed diffusion is a dispersal strategy employed in the predator-prey system. By taking into account the density-suppressed diffusion, the defensive switching model reads

$$\begin{cases} u_t = \Delta(d_1(w)u) + \frac{\beta_1 uvw}{u+v} - \alpha_1 u, \\ v_t = \Delta(d_2(w)v) + \frac{\beta_2 uvw}{u+v} - \alpha_2 v, \\ w_t = d_3 \Delta w - \frac{(\beta_1+\beta_2)uvw}{u+v} + \sigma w\left(1 - \frac{w}{K}\right), \end{cases} \tag{1.4}$$

where $d_j(w) > 0$ and $d_j'(w) < 0$ $(j = 1, 2)$. Note that the property $d_j'(w) < 0$ means that predators will decrease their random diffusion rates at higher density of the prey species in order for predation. If we expand the density-suppressed diffusion

$$\Delta(d_j(w)\phi) = \nabla \cdot (d_j(w)\nabla\phi + \phi d_j'(w)\nabla w), \;\; \phi = u, v,$$

we find that the density-suppressed diffusion indeed intrinsically includes both random diffusion and advection (prey-taxis) components. The differences from the spatial models considered in [3,5,6] is that here both diffusion and advection coefficient are not constant but functions of the prey density. It was shown that the density-suppressed diffusion in the predator-prey systems may generate spatially non-homogeneous patterns that the random diffusion can not do [8] and may bring substantially different dynamics [9].

Beyond the predator-prey systems, the density-suppressed diffusion has already been commonly used in the modeling of other biological processes such as the chemotaxis [10, 11], bacterial movement [12, 13] and so on. Since the possible degeneracy caused by the density-suppressed diffusion brings considerable challenges for analysis, the studies of these biological models with density-suppressed diffusion have been increasingly attracting attentions and produced many interesting analytical results [14–28] alongside rich numerical simulations demonstrating complex dynamics and patterns [29–31].

The purpose of this paper is to study the global dynamics of the defensive switching model with density-suppressed diffusion including global existence and asymptotic behavior of solutions as well as pattern formations. Specifically we consider the following problem

$$\begin{cases} u_t = \Delta(d_1(w)u) + \frac{\beta_1 uvw}{u+v} - \alpha_1 u, & x \in \Omega, \quad t > 0, \\ v_t = \Delta(d_2(w)v) + \frac{\beta_2 uvw}{u+v} - \alpha_2 v, & x \in \Omega, \quad t > 0, \\ w_t = \Delta w - \frac{\beta_3 uvw}{u+v} + \sigma w\left(1 - \frac{w}{K}\right), & x \in \Omega, \quad t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega, \quad t > 0, \\ (u, v, w)(x, 0) = (u_0, v_0, w_0)(x), & x \in \Omega, \end{cases} \tag{1.5}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary. The parameters α_j $(j = 1, 2)$, β_j $(j = 1, 2, 3)$, σ and K are all positive constants. Note that we have assumed $d_3 = 1$ without loss of generality and consider more general constant β_3 compared to Eq (1.4). We suppose the density-suppressed diffusion coefficients $d_j(w)$ satisfy the following conditions:

(H_0) $d_j(w) \in C^2([0, \infty))$ with $d_j(w) > 0$ and $d_j'(w) \le 0$ for all $w \ge 0$, $j = 1, 2$.

We further suppose the initial data satisfy

$$(u_0, v_0, w_0) \in [W^{1,\infty}(\Omega)]^3 \textit{ with } u_0, v_0 > 0, w_0 \ge 0. \tag{1.6}$$

Then our first result on the global existence of solutions is stated in the following theorem.

Theorem 1.1 (Global boundedness). *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Assume the assumption (H_0) holds and the initial data (u_0, v_0, w_0) satisfy Eq (1.6). Then there exists a uniquely determined triple (u, v, w) of nonnegative functions which solves Eq (1.5) classically in $\Omega \times (0, \infty)$ and satisfies*

$$\|u(\cdot, t)\|_{L^\infty} + \|v(\cdot, t)\|_{L^\infty} + \|w(\cdot, t)\|_{W^{1,\infty}} \le M_0 \quad \text{for all } t > 0,$$

where $M_0 > 0$ is a constant independent of t. Particularly, one has

$$0 < w(x, t) \le K_* := \max\{\|w_0\|_{L^\infty}, K\} \quad \textit{for all} \quad (x, t) \in \Omega \times (0, \infty).$$

Remark 1.2. Note that in works [5, 6] considering the defensive switching model with prey-taxis, the smallness of prey-taxis coefficient was required to ensure the global boundedness of solutions. Here we obtain the existence of global classical solutions with uniform-in-time boundedness without any smallness assumptions on the parameters.

Next, we shall show the global stability of constant steady states. In fact it is straightforward to find that a positive constant steady state (u_*, v_*, w_*) exists if and only if $K > \frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2}$, where $u_*, v_*, w_* > 0$, satisfying

$$\alpha_1 = \frac{\beta_1 v_* w_*}{u_* + v_*}, \quad \alpha_2 = \frac{\beta_2 u_* w_*}{u_* + v_*}, \quad \sigma\left(1 - \frac{w_*}{K}\right) = \frac{\beta_3 u_* v_*}{u_* + v_*}, \tag{1.7}$$

which can be solved to obtain

$$u_* = \frac{\sigma}{\beta_3}\left(1 - \frac{w_*}{K}\right)\left(1 + \frac{\alpha_2\beta_1}{\alpha_1\beta_2}\right), \quad v_* = \frac{\sigma}{\beta_3}\left(1 - \frac{w_*}{K}\right)\left(1 + \frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right), \quad w_* = \frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2}. \tag{1.8}$$

Then we can show that (u_*, v_*, w_*) is globally asymptotically stable under certain conditions, as stated in the following theorem.

Theorem 1.3 (Global stabilization)**.** *Suppose the conditions in Theorem 1.1 hold and let (u, v, w) be the solution obtained in Theorem 1.1. If $K > \frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2}$ and*

$$\frac{1}{\sigma} > \max_{0<w\leq K}\left(\frac{|d_1'(w)|^2 w^2}{\beta_1 d_1(w)} + \frac{|d_2'(w)|^2 w^2}{\beta_2 d_2(w)}\right)\frac{K - w_*}{4Kw_*}, \tag{1.9}$$

then the co-existence steady state (u_*, v_*, w_*) is globally asymptotically stable.

The rest of this paper is organized as follows. In Section 2, we present the local existence theorem of solutions and establish some preliminary results. Then in Section 3, we prove Theorem 1.1. We prove Theorem 1.3 in Section 4 and explore the pattern formation in Section 5.

2. Local existence and preliminaries

In the sequel, the integral $\int_\Omega f(x)dx$ and $\|f\|_{L^p(\Omega)}$ will be abbreviated as $\int_\Omega f$ and $\|f\|_{L^p}$, respectively. The generic constants c_j or K_j for $j = 1, 2, \cdots$, are independent of t and will vary in the context. Below we present the local existence result of Eq (1.5), which can be proved in a similar way as [8] by applying Amann's theorem [36, 37], and we omit the details for brevity.

Lemma 2.1 (Local existence)**.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Assume that the initial data (u_0, v_0, w_0) satisfies (1.6) and suppose the hypothesis (H_0) holds. Then there exists $T_{\max} \in (0, \infty]$ such that the system (1.5) admits a unique classical solution $(u, v, w) \in [C(\bar{\Omega}\times[0, T_{\max}))\cap C^{2,1}(\bar{\Omega} \times (0, T_{\max}))]^3$ satisfying $u, v, w > 0$ for all $t > 0$. Moreover, if $T_{\max} < \infty$, then*

$$\lim_{t\nearrow T_{\max}} (\|u(\cdot, t)\|_{L^\infty} + \|v(\cdot, t)\|_{L^\infty} + \|w(\cdot, t)\|_{W^{1,\infty}}) = \infty.$$

Using the similar argument as in [38, Lemma 2.2], we obtain the global boundedness of w immediately.

Lemma 2.2. *Let (u, v, w) be the classical solution of Eq (1.5) obtained in Lemma 2.1. Then it holds that*

$$0 < w(x, t) \leq K_* := \max\{\|w_0\|_{L^\infty}, K\}, \text{ for all } x \in \Omega \text{ and } t \in (0, T_{\max}). \tag{2.1}$$

Furthermore, one has

$$\limsup_{t\to\infty} w(x, t) \leq K \quad \textit{for all } \ x \in \bar{\Omega}. \tag{2.2}$$

Lemma 2.3. *Let (u, v, w) be the classical solution of (1.5) obtained in Lemma 2.1. Assume there is a constant $c_1 > 0$ such that*

$$\|u(\cdot, t)\|_{L^r} \leq c_1 \text{ for all } t \in (0, T_{\max}), \tag{2.3}$$

then one has

$$\|w(\cdot, t)\|_{W^{1,q}} \leq c_2 \text{ for all } t \in (0, T_{\max}), \tag{2.4}$$

with

$$q \in \begin{cases} [1, \frac{nr}{n-r}), & \text{if } r < n, \\ [1, \infty), & \text{if } r = n, \\ [1, \infty], & \text{if } r > n. \end{cases} \tag{2.5}$$

Proof. From the third equation of (1.5), we have

$$w_t = \Delta w - w + g(u, v, w) \text{ in } \Omega, \quad \frac{\partial w}{\partial \nu} = 0,$$

with $g(u, v, w) := w - \frac{\beta_3 uvw}{u+v} + \sigma w\left(1 - \frac{w}{K}\right)$. Since $0 < w \le K_*$ (see Eq (2.1)) and $u, v > 0$, we have

$$\begin{aligned} |g(u, v, w)| &\le K_* + \beta_3 K_* u + \sigma K_* + \frac{\sigma K_*^2}{K} \\ &= K_*\left(1 + \beta_3 u + \sigma + \frac{\sigma K_*}{K}\right) \\ &\le K_*\left(1 + \beta_3 + \sigma + \frac{\sigma K_*}{K}\right)(u + 1), \end{aligned}$$

which, together with Eq (2.3), gives

$$\|g(u, v, w)\|_{L^r} \le c_3. \tag{2.6}$$

With Eq (2.6), we use the results in [39, Lemma 1] to obtain Eq (2.4) with Eq (2.5) directly. □

Now we will show some basic boundedness properties of the solution (u, v, w) obtained in Lemma 2.1.

Lemma 2.4. *Let (u, v, w) be the classical solution of (1.5) obtained in Lemma 2.1. Then it holds that*

$$\int_\Omega u + \int_\Omega v \le K_1 \text{ for all } t \in (0, T_{\max}), \tag{2.7}$$

and

$$\int_t^{t+\tau} \int_\Omega u^2 + \int_t^{t+\tau} \int_\Omega v^2 \le K_2 \text{ for all } t \in (0, T_{\max} - \tau), \tag{2.8}$$

where the constants K_1, $K_2 > 0$ are independent of t and $\tau := \min\{1, T_{\max}/2\}$.

Proof. Multiplying the first and second equations of (1.5) by β_3, and multiplying the third equation of (1.5) by $(\beta_1 + \beta_2)$, then adding the resulting equations, one obtains

$$\begin{aligned} [\beta_3(u + v) + (\beta_1 + \beta_2)w]_t =& \Delta[\beta_3(d_1(w)u + d_2(w)v) + (\beta_1 + \beta_2)w] \\ & - \beta_3(\alpha_1 u + \alpha_2 v) + \sigma(\beta_1 + \beta_2)w\left(1 - \frac{w}{K}\right). \end{aligned} \tag{2.9}$$

Thus, integrating Eq (2.9) with respect to x over Ω, we have

$$\frac{d}{dt}\int_\Omega [\beta_3(u + v) + (\beta_1 + \beta_2)w] + \beta_3 \int_\Omega (\alpha_1 u + \alpha_2 v) + \frac{\sigma(\beta_1 + \beta_2)}{K}\int_\Omega w^2 = \sigma(\beta_1 + \beta_2)\int_\Omega w. \tag{2.10}$$

Using Young's inequality, one has

$$(\sigma + 1)(\beta_1 + \beta_2)\int_\Omega w \le \frac{\sigma(\beta_1 + \beta_2)}{K}\int_\Omega w^2 + \frac{(\sigma + 1)^2(\beta_1 + \beta_2)K|\Omega|}{4\sigma},$$

which, substituted into Eq (2.10), gives

$$\frac{d}{dt}\int_\Omega [\beta_3 u + \beta_3 v + (\beta_1 + \beta_2)w] + \gamma_0 \int_\Omega [\beta_3 u + \beta_3 v + (\beta_1 + \beta_2)w] \le c_1, \tag{2.11}$$

where $\gamma_0 := \min\{\alpha_1, \alpha_2, 1\}$ and $c_1 := \frac{(\sigma+1)^2(\beta_1+\beta_2)K|\Omega|}{4\sigma}$. Then, applying the Grönwall's inequality to Eq (2.11) gives

$$\int_\Omega (u+v) \le \int_\Omega (u_0+v_0) + \frac{(\beta_1+\beta_2)}{\beta_3}\int_\Omega w_0 + \frac{c_1}{\gamma_0\beta_3},$$

which yields Eq (2.7).

Next, we shall show Eq (2.8) holds based on some ideas in [8]. Under the homogeneous Neumann boundary conditions, we define a shifted Laplacian operator $\mathcal{B} = -\Delta + \gamma_1$ with $\gamma_1 := \min\left\{\frac{\alpha_1}{2d_1(0)}, \frac{\alpha_2}{2d_2(0)}\right\} > 0$. Then $\mathcal{B}$ is a sectorial operator in $L^p(\Omega)$ for all $p \in (1,\infty)$ [32], and one can easily show that its inverse $\mathcal{B}^{-1}$ satisfies

$$\|\mathcal{B}^{-1}\phi\|_{L^2} \le c_2\|\phi\|_{L^2} \text{ for all } \phi \in L^2(\Omega), \tag{2.12}$$

and

$$\|\mathcal{B}^{-\frac{1}{2}}\phi\|_{L^2}^2 \le c_2\|\phi\|_{L^2}^2 \text{ for all } \phi \in L^2(\Omega) \tag{2.13}$$

for some constant $c_2 > 0$. Using the assumptions in (H_0) and Eq (2.1), we can derive that there exist two positive constants δ_1 and δ_2 independent of t such for all $j = 1, 2$ one has

$$0 < \delta_1 \le |d_j'(w)| \le \delta_2, \tag{2.14}$$

and

$$0 < d_j(K_*) \le d_j(w) \le d_j(0). \tag{2.15}$$

Using the definition of $\mathcal{B}$, we can rewrite Eq (2.9) as

$$\begin{aligned}
&[\beta_3(u+v) + (\beta_1+\beta_2)w]_t + \mathcal{B}[\beta_3(d_1(w)u + d_2(w)v) + (\beta_1+\beta_2)w] \\
&= (\gamma_1 d_1(w) - \alpha_1)\beta_3 u + (\gamma_1 d_2(w) - \alpha_2)\beta_3 v + (\gamma_1+\sigma)(\beta_1+\beta_2)w - \frac{\sigma(\beta_1+\beta_2)}{K}w^2 \\
&=: f(u,v,w).
\end{aligned} \tag{2.16}$$

Then applying the facts $\gamma_1 := \min\left\{\frac{\alpha_1}{2d_1(0)}, \frac{\alpha_2}{2d_2(0)}\right\} > 0$ and Eq (2.15) as well as $0 < w \le K_*$ (see Eq (2.1)), we can derive that

$$f(u,v,w) \le (\gamma_1 d_1(0) - \alpha_1)\beta_3 u + (\gamma_1 d_2(0) - \alpha_2)\beta_3 v + (\gamma_1+\sigma)(\beta_1+\beta_2)K_* \le c_3,$$

where $c_3 := (\gamma_1+\sigma)(\beta_1+\beta_2)K_* > 0$.

We multiply Eq (2.16) by $\mathcal{B}^{-1}[\beta_3(u+v) + (\beta_1+\beta_2)w] \ge 0$ and integrate the result with respect to x over Ω to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \left|\mathcal{B}^{-\frac{1}{2}}[\beta_3(u+v) + (\beta_1+\beta_2)w]\right|^2 \\
&\quad + \int_\Omega [\beta_3(d_1(w)u + d_2(w)v) + (\beta_1+\beta_2)w] \cdot [\beta_3(u+v) + (\beta_1+\beta_2)w] \\
&\le c_3 \int_\Omega \mathcal{B}^{-1}[\beta_3(u+v) + (\beta_1+\beta_2)w],
\end{aligned}$$

which combined with Eq (2.15) enables us to find a positive constant $\gamma_2 := \min\{d_1(K_*), d_2(K_*), 1\}$ such that

$$\begin{aligned}&\frac{d}{dt}\int_\Omega \left|\mathcal{B}^{-\frac{1}{2}}[\beta_3(u+v)+(\beta_1+\beta_2)w]\right|^2 + 2\gamma_2\int_\Omega[\beta_3(u+v)+(\beta_1+\beta_2)w]^2\\ \le& 2c_3\int_\Omega \mathcal{B}^{-1}[\beta_3(u+v)+(\beta_1+\beta_2)w].\end{aligned} \tag{2.17}$$

On one hand, using Eq (2.12), the Hölder inequality and Young's inequality, one has

$$\begin{aligned}2c_3\int_\Omega \mathcal{B}^{-1}[\beta_3(u+v)+(\beta_1+\beta_2)w] &\le 2c_3|\Omega|^{\frac{1}{2}}\left\|\mathcal{B}^{-1}[\beta_3(u+v)+(\beta_1+\beta_2)w]\right\|_{L^2}\\ &\le 2c_2c_3|\Omega|^{\frac{1}{2}}\|\beta_3(u+v)+(\beta_1+\beta_2)w\|_{L^2}\\ &\le \frac{\gamma_2}{2}\int_\Omega[\beta_3(u+v)+(\beta_1+\beta_2)w]^2 + \frac{2c_2^2c_3^2|\Omega|}{\gamma_2}.\end{aligned} \tag{2.18}$$

On the other hand, using Eq (2.13), we get

$$\frac{\gamma_2}{2c_2}\int_\Omega |\mathcal{B}^{-\frac{1}{2}}[\beta_3(u+v)+(\beta_1+\beta_2)w]|^2 \le \frac{\gamma_2}{2}\int_\Omega[\beta_3(u+v)+(\beta_1+\beta_2)w]^2. \tag{2.19}$$

Defining $z(t) := \int_\Omega |\mathcal{B}^{-\frac{1}{2}}[\beta_3(u+v)+(\beta_1+\beta_2)w]|^2$, and substituting Eqs (2.18) and (2.19) into Eq (2.17), we obtain

$$z'(t) + \frac{\gamma_2}{2c_2}z(t) + \gamma_2\int_\Omega[\beta_3(u+v)+(\beta_1+\beta_2)w]^2 \le \frac{2c_2^2c_3^2|\Omega|}{\gamma_2}. \tag{2.20}$$

Then applying the Grönwall's inequality to Eq (2.20) alongside Eq (2.13), one has

$$\begin{aligned}z(t) &\le \frac{4c_2^3c_3^2|\Omega|}{\gamma_2^2} + \int_\Omega \left|\mathcal{B}^{-\frac{1}{2}}[\beta_3(u_0+v_0)+(\beta_1+\beta_2)w_0]\right|^2\\ &\le \frac{4c_2^3c_3^2|\Omega|}{\gamma_2^2} + c_2\int_\Omega |\beta_3(u_0+v_0)+(\beta_1+\beta_2)w_0|^2 \le c_4.\end{aligned} \tag{2.21}$$

Integrating Eq (2.20) over $(t, t+\tau)$ for $\tau = \min\{1, T_{\max}/2\}$ and applying Eq (2.21), we get

$$\begin{aligned}\gamma_2\beta_3^2\int_t^{t+\tau}\int_\Omega(u^2+v^2) &\le \gamma_2\int_t^{t+\tau}\int_\Omega[\beta_3(u+v)+(\beta_1+\beta_2)w]^2\\ &\le \frac{2c_2^2c_3^2|\Omega|\tau}{\gamma_2} + z(t)\\ &\le \frac{2c_2^2c_3^2|\Omega|}{\gamma_2} + c_4,\end{aligned}$$

which gives Eq (2.8). Then we complete the proof of Lemma 2.4. □

Lemma 2.5. *Let (u, v, w) be the classical solution of (1.5) and $\tau = \min\{1, T_{\max}/2\}$. Then it holds that*

$$\int_t^{t+\tau}\int_\Omega |\nabla w|^2 \le K_3 \text{ for all } t \in (0, T_{\max}-\tau), \tag{2.22}$$

and

$$\|\nabla w\|_{L^2} \le K_4 \ \text{ for all } t \in (0, T_{\max}), \tag{2.23}$$

as well as

$$\int_t^{t+\tau}\int_\Omega |\Delta w|^2 \le K_5 \ \text{ for all } t \in (0,, T_{\max}-\tau), \tag{2.24}$$

where the positive constants K_3, K_4 and K_5 are independent of t.

Proof. Multiplying the third equation of (1.5) by w, integrating the resulting equation by parts and using $0 < w \le K_*$ in Eq (2.1), we obtain

$$\frac{d}{dt}\int_\Omega w^2 + 2\int_\Omega |\nabla w|^2 + 2\beta_3\int_\Omega \frac{uvw^2}{u+v} + \frac{2\sigma}{K}\int_\Omega w^3 = 2\sigma\int_\Omega w^2 \le 2\sigma K_*^2|\Omega|. \tag{2.25}$$

Integrating Eq (2.25) over $(t, t+\tau)$ and using the facts $0 < \tau \le 1$ and Eq (2.1), one has Eq (2.22) directly.

Next, we multiply the third equation of (1.5) by $-\Delta w$, use the integration by parts to the resulting equation, and apply Young's inequality as well as Eq (2.1) to get

$$\begin{aligned}
\frac{d}{dt}\int_\Omega |\nabla w|^2 + 2\int_\Omega |\Delta w|^2 &= 2\beta_3\int_\Omega \frac{uvw}{u+v}\Delta w + \frac{2\sigma}{K}\int_\Omega w^2\Delta w - 2\sigma\int_\Omega w\Delta w \\
&\le \int_\Omega |\Delta w|^2 + 3\beta_3^2\int_\Omega\left(\frac{uvw}{u+v}\right)^2 + \frac{3\sigma^2}{K^2}\int_\Omega w^4 + 3\sigma^2\int_\Omega w^2 \\
&\le \int_\Omega |\Delta w|^2 + 3\beta_3^2K_*^2\int_\Omega u^2 + \frac{3\sigma^2K_*^4|\Omega|}{K^2} + 3\sigma^2K_*^2|\Omega|,
\end{aligned}$$

which entails

$$\frac{d}{dt}\int_\Omega |\nabla w|^2 + \int_\Omega |\Delta w|^2 \le 3\beta_3^2K_*^2\int_\Omega u^2 + c_1, \tag{2.26}$$

where $c_1 := \frac{3\sigma^2K_*^4|\Omega|}{K^2} + 3\sigma^2K_*^2|\Omega|$. Applying a Gagliardo-Nirenberg type inequality derived in [16, Lemma 2.5] and noting the fact $\|w\|_{L^2} \le K_*|\Omega|^{\frac{1}{2}}$, we have

$$\begin{aligned}
\int_\Omega |\nabla w|^2 &\le c_2(\|\Delta w\|_{L^2}\|w\|_{L^2} + \|w\|_{L^2}^2) \\
&\le c_2K_*|\Omega|^{\frac{1}{2}}\left(\|\Delta w\|_{L^2} + K_*|\Omega|^{\frac{1}{2}}\right) \\
&\le \frac{1}{2}\int_\Omega |\Delta w|^2 + \frac{(2+c_2)c_2K_*^2|\Omega|}{2},
\end{aligned}$$

which substituted into Eq (2.26) gives

$$\frac{d}{dt}\int_\Omega |\nabla w|^2 + \int_\Omega |\nabla w|^2 + \frac{1}{2}\int_\Omega |\Delta w|^2 \le c_3\left(\int_\Omega u^2 + 1\right), \tag{2.27}$$

with $c_3 := 3\beta_3^2K_*^2 + c_1 + \frac{(2+c_2)c_2K_*^2|\Omega|}{2}$. For any $t \in (0, T_{\max})$, by Eq (2.22) there exists a $t_0 := t_0(t) \in ((t-\tau)_+, t)$ such that $t_0 \ge 0$ and

$$\int_\Omega |\nabla w(x, t_0)|^2 dx \le c_4. \tag{2.28}$$

Moreover, from Eq (2.8), we know that

$$\int_{t_0}^{t_0+\tau}\int_\Omega u^2(x,s)dxds \le K_2 \text{ for all } t_0 \in (0, T_{\max}-\tau). \tag{2.29}$$

Multiplying Eq (2.27) by e^t and integrating the results with respect to t over (t_0, t), and using the facts Eqs (2.28) and (2.29) and $t_0 < t \le t_0 + \tau \le t_0 + 1$, one has

$$\begin{aligned}\|\nabla w(\cdot,t)\|_{L^2}^2 &\le e^{t_0-t}\|\nabla w(\cdot,t_0)\|_{L^2}^2 + e^{-t}c_3\int_{t_0}^t e^s\left(\|u(\cdot,s)\|_{L^2}^2+1\right)\\ &\le c_4 + c_3\int_{t_0}^t\left(\|u(\cdot,s)\|_{L^2}^2+1\right)\\ &\le c_4 + c_3\int_{t_0}^{t_0+\tau}\left(\|u(\cdot,s)\|_{L^2}^2+1\right)\\ &\le c_4 + c_3K_2 + c_3.\end{aligned}$$

Hence Eq (2.23) holds. On the other hand, integrating Eq (2.27) along with Eqs (2.8) and (2.23), it holds that

$$\begin{aligned}\int_t^{t+\tau}\int_\Omega |\Delta w|^2 &\le 2c_3\int_t^{t+\tau}\int_\Omega u^2 + 2c_3\tau + 2\int_\Omega |\nabla w(\cdot,t)|^2\\ &\le 2c_3(K_2+1) + 2K_4^2,\end{aligned}$$

which gives Eq (2.24). Then we complete the proof of Lemma 2.5. □

3. Global existence: Proof of Theorem 1.1

In this section, we shall show the existence of global classical solution based on some ideas in [8]. To this end, we first establish the *a priori* L^2-estimates of u and v.

Lemma 3.1 (L^2-Boundedness)**.** *Suppose that the assumptions in Lemma 2.1 hold and let* (u, v, w) *be the classical solution of (1.5) which is defined on its maximal existence time interval* $[0, T_{\max})$*. Then the solution of (1.5) satisfies*

$$\|u(\cdot,t)\|_{L^2} + \|v(\cdot,t)\|_{L^2} \le K_6 \text{ for all } t \in (0, T_{\max}), \tag{3.1}$$

where the constant $K_6 > 0$ *is independent of* t*.*

Proof. Multiplying the first equation of (1.5) by u, integrating the results with respect to x over Ω by parts and using the fact $u, v > 0$ for all $t \in (0, T_{\max})$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 + \alpha_1\int_\Omega u^2 + \int_\Omega d_1(w)|\nabla u|^2 &= \beta_1\int_\Omega \frac{uvw}{u+v}u - \int_\Omega d_1'(w)u\nabla w\cdot\nabla u\\ &\le \beta_1K_*\int_\Omega u^2 + \int_\Omega u|d_1'(w)||\nabla w||\nabla u|,\end{aligned}$$

which along with Eqs (2.14) and (2.15) indicates

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 + \alpha_1\int_\Omega u^2 + d_1(K_*)\int_\Omega |\nabla u|^2 \le \beta_1K_*\int_\Omega u^2 + \delta_2\int_\Omega u|\nabla w||\nabla u|. \tag{3.2}$$

Applying Young's inequality and the Hölder inequality, one gets

$$\delta_2 \int_\Omega u|\nabla w||\nabla u| \le \frac{d_1(K_*)}{2}\int_\Omega |\nabla u|^2 + \frac{\delta_2^2}{2d_1(K_*)}\int_\Omega u^2|\nabla w|^2$$
$$\le \frac{d_1(K_*)}{2}\int_\Omega |\nabla u|^2 + \frac{\delta_2^2}{2d_1(K_*)}||u||_{L^4}^2||\nabla w||_{L^4}^2,$$

which updates Eq (3.2) as

$$\frac{d}{dt}||u||_{L^2}^2 + 2\alpha_1||u||_{L^2}^2 + d_1(K_*)||\nabla u||_{L^2}^2 \le 2\beta_1 K_*||u||_{L^2}^2 + \frac{\delta_2^2}{d_1(K_*)}||u||_{L^4}^2||\nabla w||_{L^4}^2. \tag{3.3}$$

Using the Gagliardo-Nirenberg inequality, the following inequality [16, Lemma 2.5]

$$||\nabla w||_{L^4}^2 \le c_1(||\Delta w||_{L^2}||\nabla w||_{L^2} + ||\nabla w||_{L^2}^2),$$

and Eq (2.23), we obtain

$$\begin{aligned}\frac{\delta_2^2}{d_1(K_*)}||u||_{L^4}^2||\nabla w||_{L^4}^2 &\le c_2(||\nabla u||_{L^2}||u||_{L^2} + ||u||_{L^2}^2)(||\Delta w||_{L^2}K_4 + K_4^2)\\ &\le c_2K_4||\nabla u||_{L^2}||u||_{L^2}||\Delta w||_{L^2} + c_2K_4||u||_{L^2}^2||\Delta w||_{L^2}\\ &\quad + c_2K_4^2||\nabla u||_{L^2}||u||_{L^2} + c_2K_4^2||u||_{L^2}^2\\ &\le d_1(K_*)||\nabla u||_{L^2}^2 + c_3||u||_{L^2}^2||\Delta w||_{L^2}^2 + c_4||u||_{L^2}^2,\end{aligned} \tag{3.4}$$

where $c_3 := \frac{c_2^2K_4^2}{d_1(K_*)}$ and $c_4 := \frac{c_2^2K_4^4}{2d_1(K_*)} + \frac{d_1(K_*)}{2} + c_2K_4^2$.

Then substituting Eq (3.4) into Eq (3.3), we obtain

$$\begin{aligned}\frac{d}{dt}||u||_{L^2}^2 &\le (2\beta_1K_* + c_4)||u||_{L^2}^2 + c_3||u||_{L^2}^2||\Delta w||_{L^2}^2\\ &\le c_5||u||_{L^2}^2(1 + ||\Delta w||_{L^2}^2),\end{aligned} \tag{3.5}$$

where $c_5 := 2\beta_1K_* + c_4 + c_3$. For any $t \in (0, T_{\max})$, by Eq (2.8) there exists a $t_0 := t_0(t) \in ((t-\tau)_+, t)$ such that $t_0 \ge 0$ and

$$\int_\Omega u^2(x, t_0)dx \le c_6. \tag{3.6}$$

On the other hand, from Eq (2.24), we know that

$$\int_{t_0}^{t_0+\tau}\int_\Omega |\Delta w(x,s)|^2dxds \le K_5, \text{ for all } t \in (0, T_{max} - \tau). \tag{3.7}$$

Then integrating Eq (3.5) with respect to t over (t_0, t), and using the Eqs (3.6) and (3.7) and $t \le t_0 + \tau \le t_0 + 1$, one has

$$\begin{aligned}||u(\cdot,t)||_{L^2}^2 &\le ||u(\cdot,t_0)||_{L^2}^2 \cdot e^{c_5\int_{t_0}^t(1+||\Delta w(\cdot,s)||_{L^2}^2)ds}\\ &\le ||u(\cdot,t_0)||_{L^2}^2 \cdot e^{c_5\int_{t_0}^{t_0+\tau}(1+||\Delta w(\cdot,s)||_{L^2}^2)ds}\\ &\le c_6e^{c_5(1+K_5)}.\end{aligned} \tag{3.8}$$

Similarly, we can show that

$$||v(\cdot,t)||_{L^2}^2 \le c_7. \tag{3.9}$$

Then the combination of Eqs (3.8) and (3.9) gives Eq (3.1). The proof of Lemma 3.1 is completed. □

Lemma 3.2 (L^∞-Boundedness)**.** *Suppose that the assumptions in Lemma 2.1 hold and let* (u, v, w) *be the classical solution of (1.5) defined on its maximal existence time interval* $[0, T_{\max})$*. Then it holds that*

$$\|u(\cdot, t)\|_{L^\infty} + \|v(\cdot, t)\|_{L^\infty} \le K_7 \ \text{ for all } t \in (0, T_{\max}), \tag{3.10}$$

where the constant $K_7 > 0$ *is independent of* t*.*

Proof. Multiplying the first equation of (1.5) by u^{p-1} with $p > 2$ and integrating the resulting equation over Ω by parts, one has

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + (p-1)\int_\Omega u^{p-2} d_1(w)|\nabla u|^2 + \alpha_1 \int_\Omega u^p \\ &\quad = -(p-1)\int_\Omega u^{p-1} d_1'(w)\nabla u \cdot \nabla w + \beta_1 \int_\Omega \frac{vw}{u+v} u^p. \end{aligned} \tag{3.11}$$

Using Eqs (2.14) and (2.15) and $0 < \frac{vw}{u+v} \le w \le K_*$ which is satisfied by the fact $u, v > 0$ for all $t \in (0, T_{\max})$, from Eq (3.11) we get

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + (p-1)d_1(K_*)\int_\Omega u^{p-2}|\nabla u|^2 + \alpha_1 \int_\Omega u^p \\ &\le (p-1)\delta_2 \int_\Omega u^{p-1}|\nabla u||\nabla w| + \beta_1 K_* \int_\Omega u^p \\ &\le \frac{(p-1)d_1(K_*)}{2}\int_\Omega u^{p-2}|\nabla u|^2 + \frac{\delta_2^2(p-1)}{2d_1(K_*)}\int_\Omega u^p|\nabla w|^2 + \beta_1 K_* \int_\Omega u^p, \end{aligned}$$

which together with the fact

$$\frac{2(p-1)d_1(K_*)}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 = \frac{p(p-1)d_1(K_*)}{2}\int_\Omega u^{p-2}|\nabla u|^2,$$

gives

$$\begin{aligned} &\frac{d}{dt}\int_\Omega u^p + \frac{2(p-1)d_1(K_*)}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + \alpha_1 p \int_\Omega u^p \\ &\quad \le \frac{p(p-1)\delta_2^2}{2d_1(K_*)}\int_\Omega u^p|\nabla w|^2 + p\beta_1 K_* \int_\Omega u^p. \end{aligned} \tag{3.12}$$

Noting the fact $\|u\|_{L^2} \le K_6$ in Eq (3.1), and using Lemma 2.3, we can find a constant $c_1 > 0$ such that

$$\|\nabla w\|_{L^4} \le c_1. \tag{3.13}$$

Then we can use the Hölder inequality, Gagliardo-Nirenberg inequality and Young's inequality as well as Eq (3.13) to obtain

$$
\begin{aligned}
\frac{p(p-1)\delta_2^2}{2d_1(K_*)}\int_\Omega u^p|\nabla w|^2 &\leq \frac{p(p-1)\delta_2^2}{2d_1(K_*)}\left(\int_\Omega u^{2p}\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla w|^4\right)^{\frac{1}{2}} \\
&\leq \frac{p(p-1)c_1^2\delta_2^2}{2d_1(K_*)}\left(\int_\Omega u^{2p}\right)^{\frac{1}{2}} \\
&\leq \frac{p(p-1)c_1^2\delta_2^2}{2d_1(K_*)}\|u^{\frac{p}{2}}\|_{L^4}^2 \\
&\leq c_2\left(\|\nabla u^{\frac{p}{2}}\|_{L^2}^{2(1-\frac{1}{p})}\|u^{\frac{p}{2}}\|_{L^{\frac{4}{p}}}^{\frac{2}{p}} + \|u^{\frac{p}{2}}\|_{L^{\frac{4}{p}}}^2\right) \\
&\leq c_2K_6\|\nabla u^{\frac{p}{2}}\|_{L^2}^{2(1-\frac{1}{p})} + c_2K_6^p \\
&\leq \frac{(p-1)d_1(K_*)}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + \left(\frac{c_2K_6}{d_1(K_*)}\right)^p\frac{d_1(K_*)}{p} + c_2K_6^p,
\end{aligned}
\tag{3.14}
$$

where we have used the fact that

$$
\|u^{\frac{p}{2}}\|_{L^{\frac{4}{p}}} = \|u\|_{L^2}^{\frac{p}{2}} \leq K_6^{\frac{p}{2}}. \tag{3.15}
$$

Moreover, using the Gagliardo-Nirenberg inequality and Eq (3.15) again, we obtain

$$
\begin{aligned}
p\beta_1K_*\int_\Omega u^p &\leq p\beta_1K_*\|u^{\frac{p}{2}}\|_{L^2}^2 \\
&\leq c_3\left(\|\nabla u^{\frac{p}{2}}\|_{L^2}^{2(1-\frac{2}{p})}\|u^{\frac{p}{2}}\|_{L^{\frac{4}{p}}}^{\frac{4}{p}} + \|u^{\frac{p}{2}}\|_{L^{\frac{4}{p}}}^2\right) \\
&\leq c_3K_6^2\|\nabla u^{\frac{p}{2}}\|_{L^2}^{2(1-\frac{2}{p})} + c_3K_6^p \\
&\leq \frac{(p-1)d_1(K_*)}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + \left(\frac{c_3K_6^2}{d_1(K_*)}\right)^{\frac{p}{2}}\frac{2d_1(K_*)}{p} + c_3K_6^p.
\end{aligned}
\tag{3.16}
$$

Then substituting Eqs (3.14) and (3.16) into Eq (3.12) yields

$$
\frac{d}{dt}\int_\Omega u^p + \alpha_1 p\int_\Omega u^p \leq c_4(p), \tag{3.17}
$$

where

$$
c_4(p) := \left(\frac{c_2K_6}{d_1(K_*)}\right)^p\frac{d_1(K_*)}{p} + \left(\frac{c_3K_6^2}{d_1(K_*)}\right)^{\frac{p}{2}}\frac{2d_1(K_*)}{p} + (c_2+c_3)K_6^p.
$$

Then applying the Grönwall's inequality to Eq (3.17), we get

$$
\|u(\cdot,t)\|_{L^p}^p \leq \|u_0\|_{L^p}^p + \frac{c_4(p)}{p\alpha_1}. \tag{3.18}
$$

Taking $p = 4$ in Eq (3.18), one enables $\|u(\cdot,t)\|_{L^4} \leq c_5$, which together with Lemma 2.3 gives

$$
\|\nabla w(\cdot,t)\|_{L^\infty} \leq c_6,
$$

where the constant $c_6 > 0$ is independent of t and p. Then using the Moser iteration process [33], there exists a constant $c_7 > 0$ independent of t such that

$$\|u(\cdot,t)\|_{L^\infty} \leq c_7. \tag{3.19}$$

Similarly, we can find a constant $c_8 > 0$ such that

$$\|v(\cdot,t)\|_{L^\infty} \leq c_8. \tag{3.20}$$

Then the combination of Eqs (3.19) and (3.20) gives Eq (3.10). The proof of Lemma 3.2 is completed. □

Proof of Theorem 1.1. From Lemma 3.2, we know there exists a constant $c_1 > 0$ independent of t such that

$$\|u(\cdot,t)\|_{L^\infty} + \|v(\cdot,t)\|_{L^\infty} + \|\nabla w(\cdot,t)\|_{L^\infty} \leq c_1,$$

which along with the extensibility criterion in Lemma 2.1 and Lemma 2.2 gives Theorem 1.1. □

4. Global stabilization: Proof of Theorem 1.3

Under the condition $K > \frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2}$, the system (1.5) has only one positive constant steady states (u_*, v_*, w_*) which is defined in Eq (1.8). In what follows, we shall show that the co-existence steady state (u_*, v_*, w_*) is globally asymptotically stable based on the following energy functional:

$$\mathcal{E}(t) := \mathcal{E}(u,v,w) = \beta_2 v_* \mathcal{J}_u(t) + \beta_1 u_* \mathcal{J}_v(t) + \frac{\beta_1\beta_2(u_* + v_*)}{\beta_3}\mathcal{J}_w(t), \tag{4.1}$$

where

$$\mathcal{J}_z(t) = \int_\Omega \left(z - z_* - z_* \ln\frac{z}{z_*}\right), \quad z = u, v, w.$$

Then we have the following results.

Lemma 4.1. *Let (u,v,w) be the solution of system (1.5) obtained in Theorem 1.1. If $K > \frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2}$ and*

$$\frac{1}{\sigma} > \max_{0<w\leq K}\left(\frac{|d_1'(w)|^2 w^2}{\beta_1 d_1(w)} + \frac{|d_2'(w)|^2 w^2}{\beta_2 d_2(w)}\right)\frac{K - w_*}{4Kw_*}, \tag{4.2}$$

then the co-existence steady states (u_*, v_*, w_*) is globally asymptotically stable.

Proof. Letting $\psi(f) = f - f_* \ln f$ and applying Taylor's expansion, for all positive f_* and f, one has

$$\begin{aligned} f - f_* - f_* \ln\frac{f}{f_*} = \psi(f) - \psi(f_*) &= \psi'(f_*)(f - f_*) + \frac{1}{2}\psi''(\xi)(f - f_*)^2 \\ &= \frac{f_*}{2\xi^2}(f - f_*)^2, \end{aligned} \tag{4.3}$$

where ξ is between f and f_*. Then choosing $f = u$ and $f_* = u_*$, we obtain from Eq (4.3) that there exists ξ_1 is between u and u_* such that

$$\mathcal{J}_u(t) = \int_\Omega \left(u - u_* - u_* \ln\frac{u}{u_*}\right) = \int_\Omega \frac{u_*}{2\xi_1^2}(u - u_*)^2 \geq 0,$$

and $\mathcal{J}_u(t) = 0$ iff $u = u_*$. Using the similar way, one has $\mathcal{J}_v(t),\ \mathcal{J}_w(t) \geq 0$ and "=" holds iff $v = v_*,\ w = w_*$. Hence according to the definition of $\mathcal{E}(t)$ in Eq (4.1), we obtain that $\mathcal{E}(u,v,w) > 0$ for all $(u,v,w) \neq (u_*,v_*,w_*)$ and moreover $\mathcal{E}(u_*,v_*,w_*) = 0$.

Next, we shall show that $\frac{d}{dt}\mathcal{E}(t) \leq 0$ under the conditions Eq (4.2) and $K > \frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2}$. In fact, using the equations in Eq (1.5), one has

$$\begin{aligned}
\frac{d\mathcal{E}(t)}{dt} &= \beta_2 v_* \int_\Omega \Big(\frac{u-u_*}{u}\Big) u_t + \beta_1 u_* \int_\Omega \Big(\frac{v-v_*}{v}\Big) v_t + \frac{\beta_1\beta_2(u_*+v_*)}{\beta_3} \int_\Omega \Big(\frac{w-w_*}{w}\Big) w_t \\
&= -\beta_2 v_* u_* \int_\Omega \frac{d_1(w)|\nabla u|^2}{u^2} - \beta_2 v_* u_* \int_\Omega \frac{d_1'(w)\nabla u \cdot \nabla w}{u} + \beta_2 v_* \int_\Omega (u-u_*)\Big(\frac{\beta_1 v w}{u+v} - \alpha_1\Big) \\
&\quad - \beta_1 u_* v_* \int_\Omega \frac{d_2(w)|\nabla v|^2}{v^2} - \beta_1 u_* v_* \int_\Omega \frac{d_2'(w)\nabla v \cdot \nabla w}{v} + \beta_1 u_* \int_\Omega (v-v_*)\Big(\frac{\beta_2 u w}{u+v} - \alpha_2\Big) \\
&\quad - \frac{\beta_1\beta_2(u_*+v_*)w_*}{\beta_3} \int_\Omega \frac{|\nabla w|^2}{w^2} + \frac{\beta_1\beta_2(u_*+v_*)}{\beta_3} \int_\Omega (w-w_*)\Big(\sigma - \frac{\sigma w}{K} - \frac{\beta_3 u v}{u+v}\Big),
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
\frac{d\mathcal{E}(t)}{dt} &= -\int_\Omega X^T B X + \beta_2 v_* \int_\Omega (u-u_*)\Big(\frac{\beta_1 v w}{u+v} - \alpha_1\Big) \\
&\quad + \beta_1 u_* \int_\Omega (v-v_*)\Big(\frac{\beta_2 u w}{u+v} - \alpha_2\Big) \\
&\quad + \frac{\beta_1\beta_2(u_*+v_*)}{\beta_3} \int_\Omega (w-w_*)\Big(\sigma - \frac{\sigma w}{K} - \frac{\beta_3 u v}{u+v}\Big) \\
&=: -\int_\Omega X^T B X + J_1 + J_2 + J_3,
\end{aligned} \tag{4.4}$$

where

$$X = \begin{pmatrix} \frac{\nabla u}{u} \\ \frac{\nabla v}{v} \\ \frac{\nabla w}{w} \end{pmatrix} \text{ and } B = \begin{pmatrix} v_* u_* d_1(w)\beta_2 & 0 & \frac{\beta_2 v_* u_* d_1'(w) w}{2} \\ 0 & \beta_1 u_* v_* d_2(w) & \frac{\beta_1 u_* v_* d_2'(w) w}{2} \\ \frac{\beta_2 v_* u_* d_1'(w) w}{2} & \frac{\beta_1 u_* v_* d_2'(w) w}{2} & \frac{\beta_1\beta_2(u_*+v_*)w_*}{\beta_3} \end{pmatrix}.$$

After some calculations, we can check that the matrix B is positive definite provided

$$\frac{4(u_*+v_*)w_*}{\beta_3 u_* v_*} > \frac{|d_1'(w)|^2 w^2}{\beta_1 d_1(w)} + \frac{|d_2'(w)|^2 w^2}{\beta_2 d_2(w)},$$

which, together with the fact $\frac{\beta_3 u_* v_*}{u_*+v_*} = \sigma(1 - \frac{w_*}{K})$ (see Eq (1.7)), gives

$$\frac{1}{\sigma} > \frac{K - w_*}{4Kw_*} F(w), \tag{4.5}$$

where the function

$$F(w) := \frac{|d_1'(w)|^2 w^2}{\beta_1 d_1(w)} + \frac{|d_2'(w)|^2 w^2}{\beta_2 d_2(w)}.$$

If $\|w_0\|_{L^\infty} \leq K$, one can easily check that Eq (4.5) holds provided Eq (4.2) is true. Now, we consider the cases $\|w_0\|_{L^\infty} \geq K$. In fact if Eq (4.2) holds, then there exists a small $\varepsilon_0 > 0$ such that

$$\frac{1}{\sigma} > \frac{K - w_*}{4Kw_*} \max_{0<w\leq K} F(w) + \varepsilon_0. \tag{4.6}$$

From the assumptions (H_0), we know that $F(w)$ belongs to $C^1([0,\infty))$, which combined with the fact Eq (2.2), implies that

$$\limsup_{t\to\infty} \frac{K - w_*}{4Kw_*} F(w) = \frac{K - w_*}{4Kw_*} \limsup_{t\to\infty} F(w) \leq \frac{K - w_*}{4Kw_*} \max_{0<w\leq K} F(w),$$

and thus for the chosen ε_0 in Eq (4.6), there exists a $T_* > 0$ such that

$$\frac{K - w_*}{4Kw_*} F(w) \leq \frac{K - w_*}{4Kw_*} \max_{0<w\leq K} F(w) + \varepsilon_0 \text{ for all } (x,t) \in \bar{\Omega} \times [T_*, \infty). \tag{4.7}$$

Then the combination of Eqs (4.6) and (4.7) gives

$$\frac{1}{\sigma} > \frac{K - w_*}{4Kw_*} F(w), \quad \text{for all } (x,t) \in \bar{\Omega} \times [T_*, \infty).$$

Hence the matrix B is positive and there is a constant $c_1 > 0$ such that

$$-\int_\Omega X^T B X \leq -c_1 \int_\Omega \left(\frac{|\nabla u|^2}{u^2} + \frac{|\nabla v|^2}{v^2} + \frac{|\nabla w|^2}{w^2} \right) \text{ for all } t \geq T_*. \tag{4.8}$$

On the other hand, using the fact $\frac{\alpha_1}{\beta_1} = \frac{v_* w_*}{u_* + v_*}$ (see Eq (1.7)), one gets

$$\begin{aligned}
J_1 :&= \beta_2 v_* \int_\Omega (u - u_*) \left(\frac{\beta_1 v w}{u + v} - \alpha_1 \right) \\
&= \beta_1 \beta_2 v_* \int_\Omega (u - u_*) \left(\frac{vw}{u + v} - \frac{\alpha_1}{\beta_1} \right) \\
&= \beta_1 \beta_2 v_* \int_\Omega (u - u_*) \left(\frac{vw}{u + v} - \frac{v_* w_*}{u_* + v_*} \right) \\
&= \frac{\beta_1 \beta_2 v_*}{u_* + v_*} \int_\Omega (u - u_*) \frac{vwu_* + vwv_* - uv_* w_* - vv_* w_*}{u + v},
\end{aligned}$$

which together with the fact

$$\begin{aligned}
vwu_* + vwv_* - uv_* w_* - vv_* w_* &= vv_*(w - w_*) + vwu_* - uv_* w_* \\
&= vv_*(w - w_*) + vu_*(w - w_*) + vw_* u_* - uv_* w_* \\
&= v(u_* + v_*)(w - w_*) + vw_* u_* - vw_* u + vw_* u - uv_* w_* \\
&= v(u_* + v_*)(w - w_*) - vw_*(u - u_*) + uw_*(v - v_*),
\end{aligned}$$

gives

$$\begin{aligned}
J_1 =& \beta_1 \beta_2 v_* \int_\Omega \frac{v(u - u_*)(w - w_*)}{u + v} + \frac{\beta_1 \beta_2 v_* w_*}{u_* + v_*} \int_\Omega \frac{u(u - u_*)(v - v_*)}{u + v} \\
&- \frac{\beta_1 \beta_2 v_* w_*}{u_* + v_*} \int_\Omega \frac{v(u - u_*)^2}{u + v}.
\end{aligned} \tag{4.9}$$

Similarly, we can use the fact $\frac{\alpha_2}{\beta_2} = \frac{u_* w_*}{u_* + v_*}$ (see Eq (1.7)) to obtain

$$
\begin{aligned}
J_2 :&= \beta_1 u_* \int_\Omega (v - v_*)\left(\frac{\beta_2 uw}{u+v} - \alpha_2\right) \\
&= \beta_1\beta_2 u_* \int_\Omega \frac{u(v-v_*)(w-w_*)}{u+v} + \frac{\beta_1\beta_2 u_* w_*}{u_*+v_*}\int_\Omega \frac{v(u-u_*)(v-v_*)}{u+v} \\
&\quad - \frac{\beta_1\beta_2 u_* w_*}{u_*+v_*}\int_\Omega \frac{u(v-v_*)^2}{u+v}.
\end{aligned}
\tag{4.10}
$$

At last, using the fact $\sigma = \frac{\beta_3 u_* v_*}{u_* + v_*} + \frac{\sigma w_*}{K}$ (see Eq (1.7)), we have

$$
\begin{aligned}
J_3 :&= \frac{\beta_1\beta_2(u_*+v_*)}{\beta_3}\int_\Omega (w-w_*)\left(\sigma - \frac{\sigma w}{K} - \frac{\beta_3 uv}{u+v}\right) \\
&= \frac{\beta_1\beta_2(u_*+v_*)}{\beta_3}\int_\Omega (w-w_*)\left(\frac{\beta_3 u_* v_*}{u_*+v_*} - \frac{\beta_3 uv}{u+v}\right) - \frac{\beta_1\beta_2(u_*+v_*)\sigma}{\beta_3 K}\int_\Omega (w-w_*)^2 \\
&= \beta_1\beta_2(u_*+v_*)\int_\Omega (w-w_*)\frac{u_*v_*u + u_*v_*v - uvu_* - uvv_*}{(u+v)(u_*+v_*)} \\
&\quad - \frac{\beta_1\beta_2(u_*+v_*)\sigma}{\beta_3 K}\int_\Omega (w-w_*)^2 \\
&= -\beta_1\beta_2 u_* \int_\Omega \frac{u(v-v_*)(w-w_*)}{u+v} - \beta_1\beta_2 v_* \int_\Omega \frac{v(u-u_*)(w-w_*)}{u+v} \\
&\quad - \frac{\beta_1\beta_2(u_*+v_*)\sigma}{\beta_3 K}\int_\Omega (w-w_*)^2.
\end{aligned}
\tag{4.11}
$$

Combining Eqs (4.9) and (4.10) with Eq (4.11), we have

$$
\begin{aligned}
\sum_{j=1}^{3} J_j &= \frac{\beta_1\beta_2 w_*}{u_*+v_*}\int_\Omega \frac{(v_*u + u_*v)(u-u_*)(v-v_*)}{u+v} \\
&\quad - \frac{\beta_1\beta_2 w_*}{u_*+v_*}\int_\Omega \frac{vv_*(u-u_*)^2 + uu_*(v-v_*)^2}{u+v} - \frac{\beta_1\beta_2(u_*+v_*)\sigma}{\beta_3 K}\int_\Omega (w-w_*)^2 \\
&= \frac{\beta_1\beta_2 w_*}{u_*+v_*}\int_\Omega \frac{f(u,v)}{u+v} - \frac{\beta_1\beta_2(u_*+v_*)\sigma}{\beta_3 K}\int_\Omega (w-w_*)^2,
\end{aligned}
\tag{4.12}
$$

where

$$
\begin{aligned}
f(u,v) &= (v_*u + u_*v)(u-u_*)(v-v_*) - vv_*(u-u_*)^2 - uu_*(v-v_*)^2 \\
&= v_*(u-u_*)(u-u_*)(v-v_*) + v_*u_*(u-u_*)(v-v_*) \\
&\quad + u_*(v-v_*)(u-u_*)(v-v_*) + u_*v_*(u-u_*)(v-v_*) \\
&\quad - vv_*(u-u_*)^2 - uu_*(v-v_*)^2 \\
&= v_*(u-u_*)^2(v-v_*) + 2v_*u_*(u-u_*)(v-v_*) \\
&\quad + u_*(u-u_*)(v-v_*)^2 - vv_*(u-u_*)^2 - uu_*(v-v_*)^2 \\
&= -v_*^2(u-u_*)^2 - u_*^2(v-v_*)^2 + 2u_*v_*(u-u_*)(v-v_*) \\
&= -(v_*u - u_*v)^2.
\end{aligned}
$$

Then we can update Eq (4.12) as

$$\sum_{j=1}^{3} J_j = -\frac{\beta_1\beta_2 w_*}{u_* + v_*}\int_\Omega \frac{(v_* u - u_* v)^2}{u+v} - \frac{\beta_1\beta_2(u_*+v_*)\sigma}{\beta_3 K}\int_\Omega (w-w_*)^2. \tag{4.13}$$

Substituting Eqs (4.8) and (4.13) into Eq (4.4) gives for all $t \geq T_*$

$$\begin{aligned}\frac{d\mathcal{E}(t)}{dt} \leq &-\frac{\beta_1\beta_2 w_*}{u_* + v_*}\int_\Omega \frac{(v_* u - u_* v)^2}{u+v} - \frac{\beta_1\beta_2(u_*+v_*)\sigma}{\beta_3 K}\int_\Omega (w-w_*)^2 \\ &- c_1\int_\Omega\left(\frac{|\nabla u|^2}{u^2} + \frac{|\nabla v|^2}{v^2} + \frac{|\nabla w|^2}{w}\right),\end{aligned}$$

which indicates $\frac{d\mathcal{E}(t)}{dt} \leq 0$ for all u, v, w. Moreover, one can check that if $\frac{d\mathcal{E}(t)}{dt} = 0$, then $\nabla u = \nabla v = 0$ and $w = w_*$, which implies that

$$u = \tilde{u}_*, \quad v = \tilde{v}_*$$

with $\tilde{u}_*$ and $\tilde{v}_*$ are some positive constants. Since $(u, v, w) = (\tilde{u}_*, \tilde{v}_*, w_*)$ is a solution of (1.5), then one has

$$\begin{cases}\frac{\beta_1\tilde{v}_* w_*}{\tilde{u}_*+\tilde{v}_*} - \alpha_1 = 0,\\ \frac{\beta_2\tilde{u}_* w_*}{\tilde{u}_*+\tilde{v}_*} - \alpha_2 = 0,\\ \frac{\beta_3\tilde{u}_*\tilde{v}_*}{\tilde{u}_*+\tilde{v}_*} - \sigma + \frac{\sigma w_*}{K} = 0,\end{cases}$$

which gives

$$\tilde{u}_* = \frac{\sigma}{\beta_3}\left(1 - \frac{w_*}{K}\right)\left(1 + \frac{\alpha_2\beta_1}{\alpha_1\beta_2}\right) = u_*, \quad \tilde{v}_* = \frac{\sigma}{\beta_3}\left(1 - \frac{w_*}{K}\right)\left(1 + \frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right) = v_*,$$

Thus $\frac{d\mathcal{E}(t)}{dt} = 0$ if and only if $(u, v, w) = (u_*, v_*, w_*)$. By using the LaSalle's invariant principle (cf. [41, Theorem 5.24]), we get that the co-existence steady state (u_*, v_*, w_*) is globally asymptotically stable. □

Proof of Theorem 1.3. Theorem 1.3 is a consequence of Lemma 4.1. □

5. Spatio-temporal patterns

In this section, we assume $K > \frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2}$ and will study the effect of density-suppressed dispersals on the dynamics of the system (1.5).

5.1. Linear instability analysis

Denote $d_j := d_j(w_*), d_j' := d_j'(w_*)$ $(j = 1, 2)$ and linearize (1.5) at (u_*, v_*, w_*) to obtain

$$\begin{cases}\Phi_t = A\Delta\Phi + B\Phi, & x \in \Omega, \quad t > 0,\\ (\nu\cdot\nabla)\Phi = 0, & x \in \partial\Omega, \quad t > 0,\\ \Phi(x, 0) = (u_0 - u_*, v_0 - v_*, w_0 - w_*)^{\mathcal{T}}, & x \in \Omega,\end{cases} \tag{5.1}$$

where $\mathcal{T}$ denotes the transpose and

$$\Phi = \begin{pmatrix} u-u_* \\ v-v_* \\ w-w_* \end{pmatrix}, A = \begin{pmatrix} d_1 & 0 & d_1^{'}u_* \\ 0 & d_2 & d_2^{'}v_* \\ 0 & 0 & 1 \end{pmatrix}, \quad B = (B_{ij})_{3\times 3}$$

with

$$\begin{aligned} B_{11} &= -\frac{\alpha_1 u_*}{u_*+v_*} < 0, \quad B_{12} = \frac{\alpha_2\beta_1 u_*}{\beta_2(u_*+v_*)} > 0, \quad B_{13} = \frac{\beta_1 u_* v_*}{u_*+v_*} > 0, \\ B_{21} &= \frac{\alpha_1\beta_2 v_*}{\beta_1(u_*+v_*)} > 0, \quad B_{22} = -\frac{\alpha_2 v_*}{u_*+v_*} < 0, \quad B_{23} = \frac{\beta_2 u_* v_*}{u_*+v_*} > 0, \\ B_{31} &= -\frac{\alpha_1\beta_3 v_*}{\beta_1(u_*+v_*)} < 0, \ B_{32} = -\frac{\alpha_2\beta_3 u_*}{\beta_2(u_*+v_*)} < 0, \ B_{33} = -\frac{\sigma w_*}{K} < 0. \end{aligned}$$

Then by the standard linear stability principle, the linear stability of (u_*, v_*, w_*) is determined by the eigenvalues of the matrix $(-\mu_k A + B)$ where the sequence $\{\mu_k\}_{k=0}^{\infty}$ with $0 = \mu_0 < \mu_1 \le \mu_2 \le \mu_3 \le \ldots$ denotes the eigenvalues of $-\Delta$ under the Neumann boundary condition. The characteristic equation of $(-\mu_k A + B)$ is

$$\rho^3 + D_1(\mu_k)\rho^2 + D_2(\mu_k, |d_j'|)\rho + D_3(\mu_k, |d_j'|) = 0, \tag{5.2}$$

where

$$D_1(\mu_k) := \mu_k(d_1+d_2+1) + \frac{\sigma w_*}{K} + \frac{\alpha_1 u_* + \alpha_2 v_*}{u_*+v_*} > 0,$$

$$\begin{aligned} D_2(\mu_k, |d_j'|) :=& \mu_k^2(d_1d_2+d_1+d_2) + \mu_k\left\{(d_1+d_2)\frac{\sigma w_*}{K} + (d_1+1)\frac{\alpha_2 v_*}{u_*+v_*} + (d_2+1)\frac{\alpha_1 u_*}{u_*+v_*}\right\} \\ &+ \mu_k\left\{\frac{\beta_3 u_* v_*}{u_*+v_*}\left(-\frac{\alpha_1 d_1'}{\beta_1} - \frac{\alpha_2 d_2'}{\beta_2}\right)\right\} \\ &+ \frac{\beta_3 u_* v_*(\alpha_1 v_* + \alpha_2 u_*)}{(u_*+v_*)^2} + \frac{\sigma w_*(\alpha_1 u_* + \alpha_2 v_*)}{K(u_*+v_*)} > 0, \end{aligned}$$

$$\begin{aligned} D_3(\mu_k, |d_j'|) :=& \mu_k^3 d_1 d_2 + \mu_k^2\left\{d_1d_2\frac{\sigma w_*}{K} + d_1\frac{\alpha_2 v_*}{u_*+v_*} + d_2\frac{\alpha_1 u_*}{u_*+v_*} + \frac{\beta_3 u_* v_*}{u_*+v_*}\left(-\frac{\alpha_1 d_1' d_2}{\beta_1} - \frac{\alpha_2 d_1 d_2'}{\beta_2}\right)\right\} \\ &+ \mu_k\left\{d_1\frac{\alpha_2 v_*}{u_*+v_*}\left(\frac{\sigma w_*}{K} + \frac{\beta_3 u_*^2}{u_*+v_*}\right) + d_2\frac{\alpha_1 u_*}{u_*+v_*}\left(\frac{\sigma w_*}{K} + \frac{\beta_3 v_*^2}{u_*+v_*}\right)\right\} \\ &+ \mu_k\left\{\frac{\alpha_1\alpha_2\beta_3 u_* v_*}{u_*+v_*}\left(-\frac{d_1'}{\beta_1} - \frac{d_2'}{\beta_2}\right)\right\} \\ &+ \frac{\alpha_1\alpha_2\beta_3 u_* v_*}{u_*+v_*} > 0. \end{aligned}$$

Hence, by direct calculations, one has

$$\begin{aligned} H(\mu_k, |d_j'|) :&= D_1(\mu_k)D_2(\mu_k, |d_j'|) - D_3(\mu_k, |d_j'|) \\ &= \gamma_1\mu_k^3 + \gamma_2\mu_k^2 + \gamma_3\mu_k + A_1A_2 - A_3 + A_0(\mu_k, |d_j'|) + A_4(\mu_k, |d_j'|), \end{aligned} \tag{5.3}$$

where $\gamma_1, \gamma_2, \gamma_3$ are positive constants and $A_1A_2 - A_3 > 0$ are independent of d_j' ($j = 1, 2$) and μ_k (see details in the Appendix for the precise definitions of γ_i and A_i, $i = 1, 2, 3$), and

$$A_0(\mu_k, |d_j'|) := \mu_k^2\frac{\beta_3 u_* v_*}{u_*+v_*}\left\{-\frac{(d_1+1)\alpha_1 d_1'}{\beta_1} - \frac{(d_2+1)\alpha_2 d_2'}{\beta_2}\right\} \ge 0,$$

as well as

$$A_4(\mu_k, |d_j'|) := \mu_k \frac{\beta_3 u_* v_*}{(u_* + v_*)} \left\{ \frac{(-d_1')\alpha_1}{\beta_1} \left(\frac{\sigma w_*}{K} + \frac{(\alpha_1 - \alpha_2)u_*}{u_* + v_*} \right) + \frac{(-d_2')\alpha_2}{\beta_2} \left(\frac{\sigma w_*}{K} + \frac{(\alpha_2 - \alpha_1)v_*}{u_* + v_*} \right) \right\}.$$

Then using the Routh-Hurwitz criterion, we have the results stated as below.

Proposition 5.1. *The (u_*, v_*, w_*) is linearly stable if one of the following conditions holds:*

1) $d_1(w), d_2(w)$ are constants with $\sigma > 0$;
2) $d_j'(w) < 0$ with $\sigma > 0$ if $\alpha_1 = \alpha_2$ or $\sigma \geq \frac{K\beta_1\beta_2 f(\beta)|\alpha_1-\alpha_2|}{(\alpha_1\beta_2+\alpha_2\beta_1)^2}$ if $\alpha_1 \neq \alpha_2$, where

$$f(\beta) := \begin{cases} \alpha_2\beta_1, & \text{if } \alpha_1 < \alpha_2, \\ \alpha_1\beta_2, & \text{if } \alpha_1 > \alpha_2, \end{cases} \tag{5.4}$$

and $j = 1, 2$.

Proof. In the case that $d_j(w)$ $(j = 1, 2)$ are constants, one has

$$A_0(\mu_k, |d_j'|) = A_4(\mu_k, |d_j'|) = 0.$$

On the other hand, considering the case $d_j'(w) < 0$. If $\alpha_1 = \alpha_2$, we get

$$A_0(\mu_k, |d_j'|),\ A_4(\mu_k, |d_j'|) > 0.$$

As for $\alpha_1 \neq \alpha_2$ with

$$\sigma \geq \frac{K\beta_1\beta_2 f(\beta)|\alpha_1 - \alpha_2|}{(\alpha_1\beta_2 + \alpha_2\beta_1)^2},$$

we can calculate to get

$$A_0(\mu_k, |d_j'|) > 0,\ A_4(\mu_k, |d_j'|) \geq 0.$$

All cases discussed above indicate $D_1(\mu_k)D_2(\mu_k, |d_j'|) - D_3(\mu_k, |d_j'|) > 0$ for all $k \in \mathbb{N}$. Then by the Routh-Hurwitz criterion [34], we know (u_*, v_*, w_*) is linearly stable. □

Hence, we are left to study the possible patterns bifurcating from the co-existence steady state (u_*, v_*, w_*) in the following range of parameters:

(H_1) $\alpha_1 \neq \alpha_2$, $0 < \sigma < \frac{K\beta_1\beta_2 f(\beta)|\alpha_1-\alpha_2|}{(\alpha_1\beta_2+\alpha_2\beta_1)^2}$ and $d_i'(w) < 0$, where $i = 1$ if $\alpha_1 < \alpha_2$ or $i = 2$ if $\alpha_1 > \alpha_2$ and $f(\beta)$ is defined in (5.4).

Proposition 5.2. *Let $\sigma, K, \alpha_1, \alpha_2, \beta_1, \beta_2$ and β_3 be fixed positive parameters, and suppose the assumption (H_1) holds. Then (u_*, v_*, w_*) is linearly unstable provided that $|d_1'(w_*)|$ is large enough if $\alpha_1 < \alpha_2$ (or $|d_2'(w_*)|$ is large enough if $\alpha_2 < \alpha_1$) and there is some $k_0 \geq 1$ such that*

$$0 < \mu_{k_0} < \begin{cases} \frac{1}{1+d_1} \left| \frac{\sigma w_*}{K} + \frac{(\alpha_1-\alpha_2)u_*}{u_*+v_*} \right|, & \text{if } \alpha_1 < \alpha_2, \\ \frac{1}{1+d_2} \left| \frac{\sigma w_*}{K} + \frac{(\alpha_2-\alpha_1)v_*}{u_*+v_*} \right|, & \text{if } \alpha_1 > \alpha_2. \end{cases} \tag{5.5}$$

Proof. Noting that $H(\mu_k, |d'_j|)$ can be rewritten as

$$\begin{aligned} H(\mu_k, |d'_j|) = & \gamma_1\mu_k^3 + \gamma_2\mu_k^2 + \gamma_3\mu_k + A_1A_2 - A_3 \\ & + \frac{(-d'_1)\alpha_1\beta_3 u_* v_*}{\beta_1(u_* + v_*)}\left\{(d_1 + 1)\mu_k^2 + \left(\frac{\sigma w_*}{K} + \frac{(\alpha_1 - \alpha_2)u_*}{u_* + v_*}\right)\mu_k\right\} \\ & + \frac{(-d'_2)\alpha_2\beta_3 u_* v_*}{\beta_2(u_* + v_*)}\left\{(d_2 + 1)\mu_k^2 + \left(\frac{\sigma w_*}{K} + \frac{(\alpha_2 - \alpha_1)v_*}{u_* + v_*}\right)\mu_k\right\}. \end{aligned}$$

Since (H_1) holds, then for some fixed $\mu_{k_0} > 0$ satisfying Eq (5.5), we have

$$H(\mu_k, |d'_j|) := D_1(\mu_k)D_2(\mu_k, |d'_j|) - D_3(\mu_k, |d'_j|) < 0$$

when $|d'_1(w_*)|$ is large enough if $\alpha_1 < \alpha_2$ (or $|d'_2(w_*)|$ is large enough if $\alpha_2 < \alpha_1$) and (u_*, v_*, w_*) is linearly unstable by the Routh-Hurwitz criterion [40]. □

Remark 5.3. *The above results imply that the density-suppressed diffusion can induce the instability of (u_*, v_*, w_*) and trigger the pattern formation.*

5.2. Numerical simulations

In this subsection, we aim to give some numerical simulations to complement the previous analysis in Section 5.1. Without loss of generality, we assume $\alpha_1 < \alpha_2$, and define the set

$$\mathcal{H} = \{(|d'_1|, \eta) \in \mathbb{R}_+^2 : \quad H(|d'_1|, \eta) = 0\}$$

as the bifurcation curve [35], where the linearized system (5.1) admits an eigenvalue with zero real part and the curve $\mathcal{H}$ is the graph of the function

$$\begin{aligned} d_{\mathcal{H}}(\eta) = & \frac{\beta_1(u_* + v_*)}{\alpha_1\beta_3 u_* v_*}\left(-(d_1 + 1)\eta - \frac{\sigma w_*}{K} - \frac{(\alpha_1 - \alpha_2)u_*}{u_* + v_*}\right)^{-1} \cdot \left\{\left(\eta^2\gamma_1 + \eta\gamma_2 + \gamma_3 + \frac{A_1A_2 - A_3}{\eta}\right) \right. \\ & \left. + \frac{(-d'_2)\alpha_2\beta_3 u_* v_*}{\beta_2(u_* + v_*)}\left((d_2 + 1)\eta + \frac{\sigma w_*}{K} + \frac{(\alpha_2 - \alpha_1)v_*}{u_* + v_*}\right)\right\}, \end{aligned} \tag{5.6}$$

where the constants $\gamma_j, A_j > 0$ ($j = 1, 2, 3$) are given in the Appendix. Then the characteristic Eq (5.2) has a pair of purely imaginary eigenvalues if $|d'_1(w_*)| = d_{\mathcal{H}}(\mu_{k_0})$ for some $k_0 \in \mathbb{N}$ and μ_{k_0} satisfying Eq (5.5) and the Hopf bifurcation may emerge.

We first fix parameter values as follows:

$$\beta_1 = 0.3, \beta_2 = 1, \beta_3 = 1.3, \alpha_1 = 0.1, \alpha_2 = 1, K = 2. \tag{5.7}$$

Then by the assumption (H_1), one has

$$0 < \sigma < \frac{81}{80}.$$

Hence, we fix $\sigma = \frac{1}{10}$ in what follows without loss of generality. By Eq (1.8) we have

$$u_* = \frac{4}{39},\ v_* = \frac{4}{117},\ w_* = \frac{4}{3}. \tag{5.8}$$

In addition, we choose the motility function $d_1(w)$ as

$$d_1(w) = e^{-D(w-\frac{4}{3})},$$

where the constant $D > 0$ and $d_1'(w) = -De^{-D(w-\frac{4}{3})}$. Then Eq (5.5) is equivalent to

$$0 < \mu_{k_0} < \frac{73}{240}.$$

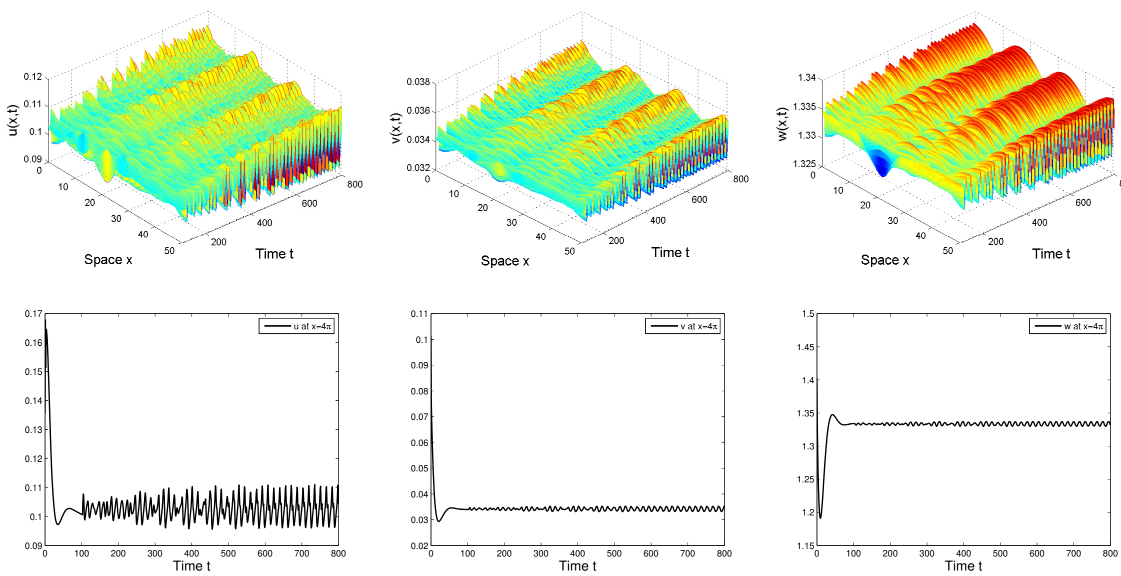

Figure 1. Numerical simulation of spatio-temporal patterns (first panel) and time evolutionary profiles (second panel) generated by the model (1.5), where $d_1(w) = e^{-D(w-\frac{4}{3})}$ with $D = 180$ and $d_2(w) = 1$, other parameter values are given in Eq (5.7) with $\sigma = 0.1$, and the initial value (u_0, v_0, w_0) is chosen as a small random perturbation of (u_*, v_*, w_*) given in Eq (5.8).

Using Theorem 1.3 and Proposition 5.2, the co-existence steady state $(\frac{4}{39}, \frac{4}{117}, \frac{4}{3})$ may lose its stability when the parameter $D > 0$ satisfies

$$\frac{10D^2}{3} \max_{0<w\le K_*} e^{-D(w-\frac{4}{3})}w^2 + \max_{0<w\le K_*} \frac{|d_2'(w)|^2 w^2}{d_2(w)} \ge 160, \tag{5.9}$$

and

$$\left| d_1'\left(\frac{4}{3}\right)\right| = D \ge d_{\mathcal{H}}(\eta) \tag{5.10}$$

with $0 < \eta < \frac{73}{240}$.

In our numerical simulation, we set the initial value (u_0, v_0, w_0) as a small random perturbation of $(\frac{4}{39}, \frac{4}{117}, \frac{4}{3})$ and hence $K_* = 2$. Moreover, we perform numerical simulations in one dimension and set $\Omega = (0, 16\pi)$. Choosing $D > 0$ satisfies (5.9) and (5.10) with $\eta = (k/16)^2 < \frac{73}{240}$, which indicates

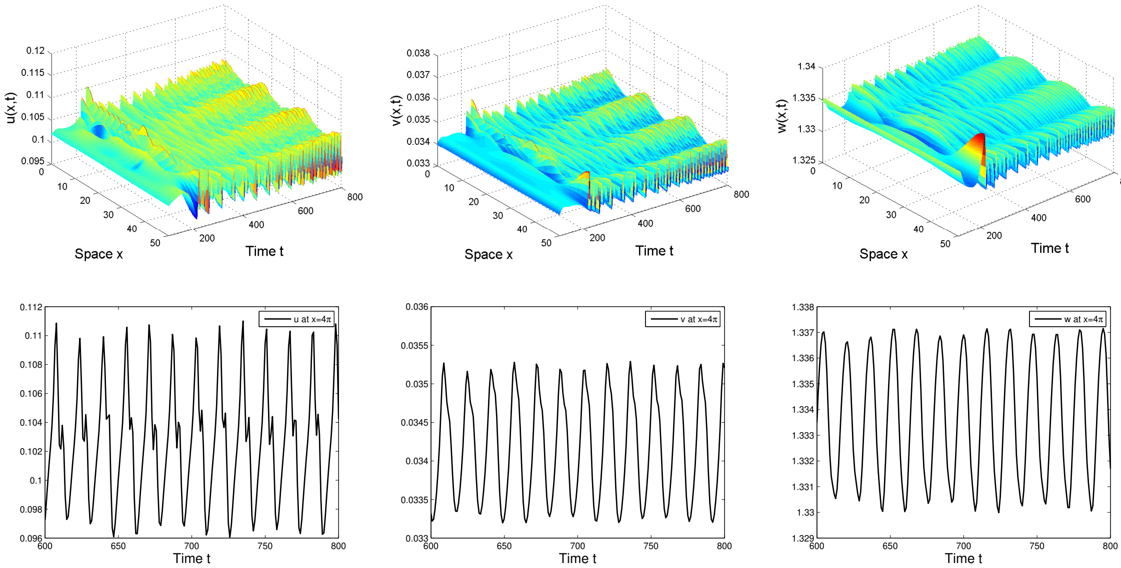

Figure 2. Numerical simulation of spatio-temporal patterns (first panel) and oscillatory time-evolutionary profiles (second panel) generated by the model (1.5), where $d_1(w) = e^{-D(w-\frac{4}{3})}$ with $D = 268$ and $d_2(w) = e^{-50(w-\frac{4}{3})}$, other parameter values are given in Eq (5.7) with $\sigma = 0.1$, and the initial value is chosen as a small random perturbation of (u_*, v_*, w_*) given in Eq (5.8).

allowable unstable modes are $k = 1, 2, 3, 4, 5, 6, 7, 8$. Next, we will consider the following two cases for $d_2(w)$.

Case 1: $d_2(w) = 1$. In this case, $d_2' = d_2'(w_*) = 0$ and according to the definition of $d_{\mathcal{H}}(\eta)$ in Eq (5.6), we can calculate to obtain that

$$d_{\mathcal{H}}(\eta) = 90\left(\frac{73}{120} - 2\eta\right)^{-1}\left(8\eta^2 + \frac{47}{15}\eta + \frac{2893}{7200} + \frac{733}{48000\eta}\right),$$

and Eq (5.9) can be updated as

$$e^{\frac{4}{3}D} \geq 12e^2$$

with $D > 4$. Then after some calculations, one gets $d_{\mathcal{H}}(\frac{1}{256}) \approx 647.963$, $d_{\mathcal{H}}(\frac{4}{256}) \approx 223.026$, $d_{\mathcal{H}}(\frac{9}{256}) \approx 159.956$, $d_{\mathcal{H}}(\frac{16}{256}) \approx 162.600$, $d_{\mathcal{H}}(\frac{25}{256}) \approx 204.933$, $d_{\mathcal{H}}(\frac{36}{256}) \approx 305.214$, $d_{\mathcal{H}}(\frac{49}{256}) \approx 548.498$ and $d_{\mathcal{H}}(\frac{64}{256}) \approx 1450.71$. Hence, $d_{\mathcal{H}}(\frac{9}{256}) \approx 159.956$ is the critical value for possible pattern formation. In our numerical simulations shown in Figure 1, we choose $D = 180$ and find the spatio-temporal pattern. In particular the time evolutionary profiles of solutions are oscillatory, which implies the bifurcation might be Hopf bifurcation.

Case 2: $d_2(w) = e^{-50(w-\frac{4}{3})}$. For this case, $d_2' = d_2'(w_*) = -50$ and hence

$$d_{\mathcal{H}}(\eta) = 90\left(\frac{73}{120} - 2\eta\right)^{-1}\left(8\eta^2 + \frac{97}{15}\eta + \frac{2131}{2400} + \frac{733}{48000\eta}\right),$$

and Eq (5.9) can be simplified as

$$e^{\frac{200}{3}} + \frac{10}{3}e^{\frac{4}{3}D} \geq 40e^2.$$

One also can calculate to get that $d_{\mathcal{H}}(\frac{1}{256}) \approx 722.767$, $d_{\mathcal{H}}(\frac{4}{256}) \approx 306.961$, $d_{\mathcal{H}}(\frac{9}{256}) \approx 260.876$, $d_{\mathcal{H}}(\frac{16}{256}) \approx 291.910$, $d_{\mathcal{H}}(\frac{25}{256}) \approx 381.793$, $d_{\mathcal{H}}(\frac{36}{256}) \approx 567.953$, $d_{\mathcal{H}}(\frac{49}{256}) \approx 997.112$ and $d_{\mathcal{H}}(\frac{64}{256}) \approx 2546.860$. Therefore, $d_{\mathcal{H}}(\frac{9}{256}) \approx 260.876$ is the critical value for possible pattern formation. In our numerical simulations, we choose $D = 268$ and find the spatio-temporal pattern shown in Figure 2. Again oscillatory time evolutionary profiles indicate the solution bifurcating from the co-existence steady state may undergo a Hopf bifurcation.

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Conflict of interest

The authors declare there is no conflict of interest.

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Appendix

Below we show the details for the parameters $\gamma_1, \gamma_2, \gamma_3 > 0$ and $A_1A_2 - A_3 > 0$ present in Eq (5.3). By direct calculations, we obtain

$$\begin{aligned}\gamma_1 &= (d_1 + d_2 + 1)(d_1d_2 + d_1 + d_2) - d_1d_2 \\ &= (d_1 + d_2)(d_1d_2 + d_1 + d_2) + (d_1 + d_2) \\ &= (d_1d_2 + d_1 + d_2 + 1)(d_1 + d_2) > 0.\end{aligned}$$

As for

$$\begin{aligned}\gamma_2 &= (d_1 + d_2 + 1)\left\{(d_1 + d_2)\frac{\sigma w_*}{K} + (d_1 + 1)\frac{\alpha_2 v_*}{u_* + v_*} + (d_2 + 1)\frac{\alpha_1 u_*}{u_* + v_*}\right\} \\ &\quad + (d_1d_2 + d_1 + d_2)\left(\frac{\sigma w_*}{K} + \frac{\alpha_1 u_* + \alpha_2 v_*}{u_* + v_*}\right) - \left(d_1d_2\frac{\sigma w_*}{K} + d_1\frac{\alpha_2 v_*}{u_* + v_*} + d_2\frac{\alpha_1 u_*}{u_* + v_*}\right) \\ &= (d_1 + d_2 + 1)\left\{(d_1 + d_2)\frac{\sigma w_*}{K} + (d_1 + 1)\frac{\alpha_2 v_*}{u_* + v_*} + (d_2 + 1)\frac{\alpha_1 u_*}{u_* + v_*}\right\} \\ &\quad + \frac{d_1d_2(\alpha_1 u_* + \alpha_2 v_*)}{u_* + v_*} + (d_1 + d_2)\frac{\sigma w_*}{K} + \frac{d_1\alpha_1 u_* + d_2\alpha_2 v_*}{u_* + v_*},\end{aligned}$$

which implies $\gamma_2 > 0$ and is independent of d_j' $(j = 1, 2)$ and μ_k. Now, we calculate the other parameters:

$$\begin{aligned}\gamma_3 &= \left(\frac{\sigma w_*}{K} + \frac{\alpha_1 u_* + \alpha_2 v_*}{u_* + v_*}\right)\left\{(d_1 + d_2)\frac{\sigma w_*}{K} + (d_1 + 1)\frac{\alpha_2 v_*}{u_* + v_*} + (d_2 + 1)\frac{\alpha_1 u_*}{u_* + v_*}\right\} \\ &\quad + (d_1 + d_2 + 1)\left\{\frac{\beta_3 u_* v_*(\alpha_1 v_* + \alpha_2 u_*)}{(u_* + v_*)^2} + \frac{\sigma w_*(\alpha_1 u_* + \alpha_2 v_*)}{K(u_* + v_*)}\right\} \\ &\quad - \frac{d_1\alpha_2 v_*}{u_* + v_*}\left(\frac{\sigma w_*}{K} + \frac{\beta_3 u_*^2}{u_* + v_*}\right) - \frac{d_2\alpha_1 u_*}{u_* + v_*}\left(\frac{\sigma w_*}{K} + \frac{\beta_3 v_*^2}{u_* + v_*}\right) \\ &= \frac{d_2\alpha_2\beta_3 u_*^2 v_*}{(u_* + v_*)^2} + \frac{d_1\alpha_1\beta_3 u_* v_*^2}{(u_* + v_*)^2} + \frac{\beta_3 u_* v_*(\alpha_1 v_* + \alpha_2 u_*)}{(u_* + v_*)^2} \\ &\quad + 2(d_1 + d_2 + 1)\frac{\sigma w_*(\alpha_1 u_* + \alpha_2 v_*)}{K(u_* + v_*)} + (d_1 + d_2)\left(\frac{\sigma w_*}{K}\right)^2 \\ &\quad + ((d_1 + 1)\alpha_2 v_* + (d_2 + 1)\alpha_1 u_*)\,\frac{\alpha_1 u_* + \alpha_2 v_*}{(u_* + v_*)^2} \\ &> 0,\end{aligned}$$

and

$$A_1 := \frac{\alpha_1 u_* + \alpha_2 v_*}{u_* + v_*} + \frac{\sigma w_*}{K} > 0,$$
$$A_2 := \frac{\beta_3 u_* v_* (\alpha_1 v_* + \alpha_2 u_*)}{(u_* + v_*)^2} + \frac{\sigma w_* (\alpha_1 u_* + \alpha_2 v_*)}{K(u_* + v_*)} > 0,$$
$$A_3 := \frac{\alpha_1 \alpha_2 \beta_3 u_* v_*}{u_* + v_*} > 0.$$

Hence, a direct calculation gives

$$\begin{aligned} A_1A_2 - A_3 &> \frac{(\alpha_1 u_* + \alpha_2 v_*)(\alpha_1 v_* + \alpha_2 u_*)\beta_3 u_* v_*}{(u_* + v_*)^3} \\ &\quad - \frac{\alpha_1 \alpha_2 \beta_3 (u_* v_*^3 + u_*^3 v_*) + 2\alpha_1 \alpha_2 \beta_3 u_*^2 v_*^2}{(u_* + v_*)^3} \\ &= \frac{(\alpha_1 - \alpha_2)^2 \beta_3 u_*^2 v_*^2}{(u_* + v_*)^3} \geq 0. \end{aligned}$$