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Research article

Existence and continuous dependence of solutions for equilibrium configurations of cantilever beam

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Abstract: This article explores the equilibrium configurations of a cantilever beam described by the minimizer of a generalized total energy functional. We reformulate the problem as a boundary value problem using the Euler-Lagrange condition and investigate the existence and uniqueness of minimizers. Furthermore, we discuss the dependence of solutions on the parameters of the boundary value problems. In addition, the Adomian decomposition method is derived for approximating the solution in terms of series. Finally, numerical results for the equilibrium configurations of cantilever beams are presented to support our theoretical analysis.

Keywords: equilibrium configuration; existence and uniqueness of solution; Euler-Lagrange theorem; Adomian decomposition method

1. Introduction

Cantilever beams are often used in civil engineering as construction elements such as bridges, roofs, traffic signal poles and traveling cranes. They are also applicable in the field of biomechanics, such as orthodontics [1], spinal implants, [2] and models of the rat whisker [3]. For two or three dimensional structures, cantilever plates incorporation of composite materials such as piezo composite hybrid laminate plates are also studied [4, 5]. These structural components are of interest since they can be used in numerous applications.

Nomenclature	
L	Length of beam.
θ	Angle between tangent to the base curve and the x-axis.
$\mathscr{J}$	Total potential energy functional for a cantilever beam.
$\mathscr{T}$	A contraction on $C(J,\mathbb{R})$.
s	Arclength of the beam.
α	Angle with respect to the x-axis.
μ	Load factor acting at an angle α .
$\mathscr{L}$	Linear operator.
$\mathscr{N}$	Nonlinear operator.
$\Phi_n(s)$	n th-stage approximation functions of $\theta(s)$.
H,G	Green function.

Large deflection and stability analysis of cantilever beams have been investigated since the seventeenth century. Bernoulli formulated the problem in the form of general variational formula. Later, Euler derived the differential equation for the deformed shape and obtained the Bernoulli-Euler theory of beams [6, 7]. Many researchers analyzed the equilibrium configuration to determine the equilibrium shape of the cantilever beam under the various applied load in both theoretical [8–11] and computational approaches [12–15]. One of the most classical problems is to minimize the total potential energy of the beam. Various researchers have analyzed the equilibrium configuration and stability of the cantilever beam through the stationary condition for the total energy functional [9, 10, 16].

Consider the total potential energy functional for a cantilever beam of length L, derived based on the Euler-Bernoulli beam theory where the effect of transverse shear deformation is not considered, as follows:

$$\mathscr{J}(\theta) = \int_0^L \frac{1}{2}|\theta'(s)|^2 ds + \mu \int_0^L (1-\cos(\theta+\alpha))\, ds,$$

where θ is the tangent angle, s is the arclength of the beam and $\mu > 0$ is the load factor acting at an angle $0 \leq \alpha \leq \pi$ with respect to the $x-$axis, as shown in Figure 1.

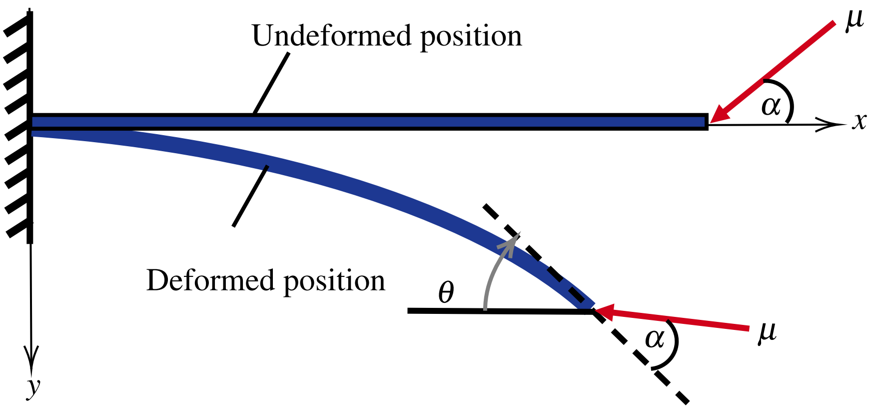

Figure 1. The undeformed and deformed positions of a cantilever beam.

When $\alpha = 0$, this functional can be used to analyze the problem of the cantilever beam subjected to a compressive load, as shown in Figure 2. When $\alpha = \frac{\pi}{2}$, it can be used for analyzing the problem of the cantilever beam subjected to a concentrated load as shown in Figure 3.

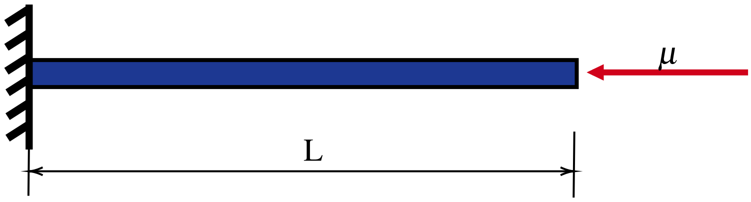

Figure 2. A cantilever beam subjected to a compressive load at the free end.

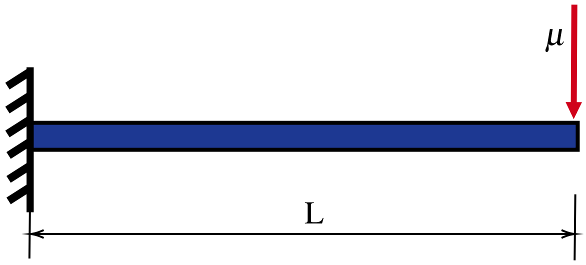

Figure 3. A cantilever beam subjected to a concentrated load at the free end.

In 2017, Della et al. [9] studied the equilibrium shapes of a clamped-end elastica with the uniformly distributed load as shown in Figure 4. The sufficient conditions for stability and instability of the solution were proven using the total energy functional of the problem which can be given by

$$\mathscr{J}(\theta) = \int_0^L \frac{1}{2}|\theta'(s)|^2 ds - \mu \int_0^L (L-s)\sin(\theta) ds,$$

where $\mu > 0$.

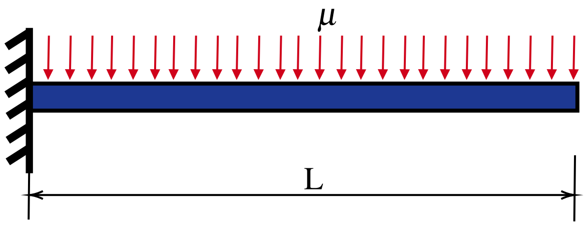

Figure 4. A cantilever beam subjected to uniformly distributed load.

Motivated by the previous works, we consider the generalized total energy functional $\mathscr{J}(\theta)$ of the cantilever beam which can be written in the abstract form

$$\mathscr{J}(\theta) = \int_{x_0}^L \frac{1}{2}|\theta'(s)|^2 ds + \mu \int_{x_0}^L f(s,\theta) ds \tag{1.1}$$

where $0 \leq x_0 \leq L$, $\mu \in \mathbb{R}$ and f is a continuously differentiable function. In particular, the functional (1.1) can be applied to the different load types for the cantilever beam.

To understand the equilibrium configurations of the cantilever beam, we first investigate the condition of existence and uniqueness of the minimizer for the generalized total potential energy functional of the cantilever beam. The well-known direct approach of variational calculus is employed. General theorems based on Tonelli's work [17, 18] assure the existence of this minimizer for the energy functional.

In experiment, there are many factors that cannot be controlled. Even though these factors may not affect the energy functional significantly. The energy functional for this model is an approximation of the actual situation. The dependence of the solution on a parameter can answer how a solution of the energy functional changes if these parameters change slightly. Therefore, the dependency of solutions on the parameters for the equilibrium configurations is also of interest. To investigate our theoretical analysis for the equilibrium configurations of the cantilever beam, we need to find the analytical solutions for the nonlinear boundary value problem which is often an arduous task. However, it can be solved in terms of the numerical solution of this type of problem. One of the methods for solving the nonlinear boundary value problem is the Adomain decomposition approach, as described in [19–22]. Motivated by [20], the Adomian decomposition method is used for accurate analytical solutions in series forms for the generalized total potential energy functional of the cantilever beam.

Our main contribution is that we establish the existence and uniqueness of the minimizer when the total energy functional associated with the cantilever beam is given in the abstract form. Hence, it can be applied to different load types in the literature, including the cantilever beam subject to compressive load [23–25], concentrated load, [26–29] and uniformly distributed load [9, 27]. Moreover, we prove the continuous dependence of solutions upon a small perturbation of the initial condition and the nonlinear term. We also give the form of analytical solutions in terms of series approximation. These results are complementary to the literature where numerical methods are used.

The remaining structure of this paper is as follows. Section 2 explains the theory and basic knowledge used in this paper. In Section 3, we establish the condition of the existence and uniqueness for the minimizer of the generalized total potential energy functional for a cantilever beam by using the Banach fixed point theorem and Schaefer's fixed point theorem. The following section provides the dependence of the solutions on the parameters. Section 5 is devoted to the approximation of the analytical solution by using the Adomian decomposition method. The final section provides examples of the generalized total energy functional for different load types.

2. Preliminaries

In this section, we give some notions and lemmas, which are important in order to state our results.

The deformations of an elastic beam in an equilibrium state can be described by the Euler-Lagrange equation of (1.1) as

$$\theta''(s) = \mu f_\theta(s, \theta). \tag{2.1}$$

Considering the solution of $\theta(s)$, satisfying the boundary conditions for each type

$$\text{Dirichlet-Dirichlet case:} \qquad \theta(x_0) = a, \quad \text{and} \quad \theta(L) = b, \tag{2.2}$$

Dirichlet-Neumann case: $\theta(x_0) = a, \quad \text{and} \quad \theta'(L) = c.$ (2.3)

We denote $C(J,\mathbb{R})$ as the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|\theta\|_\infty := \sup_{s\in J}\{|\theta(s)|\}, \quad J = [x_0, L].$$

Lemma 2.1. *Assume that $h : J \to \mathbb{R}$ is continuous and the function $\theta(s)$ is a solution of the boundary value problems*

$$\begin{cases} \theta''(s) = h(s), & s \in J, \\ \theta(x_0) = a, & \theta(L) = b. \end{cases} \tag{2.4}$$

Then, the function $\theta(s)$ is a solution of the integral equation

$$\theta(s) = a + \frac{s - x_0}{L - x_0}(b - a) - \int_{x_0}^{L} G(s,\xi)h(\xi)d\xi, \tag{2.5}$$

where

$$G(s,\xi) = \begin{cases} (\xi - x_0)\left(\dfrac{L-s}{L-x_0}\right), & x_0 \le s \le \xi \le L, \\ (s - x_0)\left(\dfrac{L-\xi}{L-x_0}\right), & x_0 \le \xi \le s \le L. \end{cases}$$

Proof. Integrating the first equation of (2.1) twice, we obtain

$$\begin{aligned} \theta(s) &= \theta(x_0) + \theta'(L)(s - x_0) + \int_{x_0}^{s}\int_{L}^{\xi} h(\xi)d\eta d\xi \\ &= a + \theta'(L)(s - x_0) + \int_{x_0}^{s}\int_{L}^{\xi} h(\xi)d\eta d\xi. \end{aligned} \tag{2.6}$$

Integration by parts of the integral with respect to ξ in Eq (2.6) gives

$$\begin{aligned} \theta(s) &= a + \theta'(L)(s - x_0) + x_0\int_{x_0}^{L} h(\xi)d\xi - s\int_{s}^{L} h(\xi)d\xi - \int_{x_0}^{s} \xi h(\xi)d\xi \\ &= a + \theta'(L)(s - x_0) - (s - x_0)\int_{s}^{L} h(\xi)d\xi - \int_{x_0}^{s} (\xi - x_0)h(\xi)d\xi. \end{aligned} \tag{2.7}$$

We determine $\theta'(L)$ from the boundary conditions to obtain

$$\begin{aligned} \theta(L) &= a + \theta'(L)(L - x_0) - \int_{x_0}^{L} (\xi - x_0)h(\xi)d\xi \\ \theta'(L) &= \frac{1}{L - x_0}\left(b - a + \int_{x_0}^{L} (\xi - x_0)h(\xi)d\xi\right). \end{aligned} \tag{2.8}$$

Substituting Eq (2.8) into Eq (2.7), we obtain

$$
\begin{aligned}
\theta(s) &= a + \frac{s-x_0}{L-x_0}\left(b - a + \int_{x_0}^{L} (\xi - x_0)\, h(\xi) d\xi\right) \\
&\quad - (s-x_0)\int_s^L h(\xi) d\xi - \mu \int_{x_0}^{s} (\xi - x_0)\, h(\xi) d\xi \\
&= a + \frac{s-x_0}{L-x_0}(b-a) + \frac{s-x_0}{L-x_0}\int_{x_0}^{L} (\xi - x_0)\, h(\xi) d\xi \\
&\quad - (s-x_0)\int_s^L h(\xi) d\xi - \int_{x_0}^{s} (\xi - x_0)\, h(\xi) d\xi \\
&= a + \frac{s-x_0}{L-x_0}(b-a) + \frac{s-x_0}{L-x_0}\int_{x_0}^{s} (\xi - x_0)\, h(\xi) d\xi - \int_{x_0}^{s} (\xi - x_0)\, h(\xi) d\xi \\
&\quad + \frac{s-x_0}{L-x_0}\int_{s}^{L} (\xi - x_0)\, h(\xi) d\xi - (s-x_0)\int_s^L h(\xi) d\xi \\
&= a + \frac{s-x_0}{L-x_0}(b-a) \\
&\quad - \left(\int_{x_0}^{s} (\xi - x_0)\left(\frac{L-s}{L-x_0}\right) h(\xi) d\xi + \int_{s}^{L} (s - x_0)\left(\frac{L-\xi}{L-x_0}\right) h(\xi) d\xi\right).
\end{aligned}
$$

Corollary 2.2. *The solution for the boundary value problems*

$$
\begin{cases}
\theta''(s) = h(s), & s \in J, \\
\theta(x_0) = a, & \theta'(L) = c,
\end{cases}
\tag{2.9}
$$

is given by

$$
\theta(s) = a + c(s - x_0) - \int_{x_0}^{L} H(s,\xi) h(\xi) d\xi, \tag{2.10}
$$

where

$$
H(s,\xi) = \begin{cases}
\xi - x_0, & x_0 \le s \le \xi \le L, \\
s - x_0, & x_0 \le \xi \le s \le L.
\end{cases}
$$

Proof. Using a similar argument as in the proof of Lemma 2.1, we apply Eq (2.7) to construct the solution of the problem (2.9).

To establish our uniqueness results, we need the following results concerning the kernel of (2.5). The proofs of Lemmas 2.3 and 2.4 are straightforward and hence omitted.

Lemma 2.3. *The Green's function*

$$
G(s,\xi) = \begin{cases}
(\xi - x_0)\left(\dfrac{L-s}{L-x_0}\right), & x_0 \le s \le \xi \le L, \\[2ex]
(s - x_0)\left(\dfrac{L-\xi}{L-x_0}\right), & x_0 \le \xi \le s \le L,
\end{cases}
$$

satisfies

$$|G(s,\xi)| \leq |G(s,s)|, \quad s,\xi \in J,$$

and

$$\int_{x_0}^{L} |G(s,s)|ds = \frac{1}{6}(L-x_0)^2.$$

Lemma 2.4. *The Green's function*

$$H(s,\xi) = \begin{cases} \xi - x_0, & x_0 \leq s \leq \xi \leq L, \\ s - x_0, & x_0 \leq \xi \leq s \leq L, \end{cases}$$

satisfies

$$|H(s,\xi)| \leq |H(s,s)|, \quad s,\xi \in J$$

and

$$\int_{x_0}^{L} |H(s,s)|ds = \frac{1}{2}(L-x_0)^2.$$

Theorem 2.5 (Banach's fixed point theorem, [30])**.** *Let $C(J,\mathbb{R})$ be a complete metric space, and $\mathscr{T}$ be a contraction on $C(J,\mathbb{R})$. Then there exists a unique $\theta \in C(J,\mathbb{R})$ such that $\mathscr{T}(\theta) = \theta$.*

Theorem 2.6 (Schaefer's fixed point theorem, [31])**.** *Let $\mathscr{T} : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ be a completely continuous operator. Suppose that the set $\varepsilon = \{\theta \in C(J,\mathbb{R}) : \theta = \lambda(\mathscr{T}\theta)$, for some $0 < \lambda < 1\}$ is bounded, then $\mathscr{T}$ has a fixed point.*

3. Existence and uniqueness of solutions

To investigate the existence and uniqueness of solutions for the problem (2.1) subject to the boundary conditions (2.2) and (2.3), we apply Lemma 2.1 and Corollary 2.2 to define the operators

$$\mathscr{T} : C(J,\mathbb{R}) \to C(J,\mathbb{R}) \quad \text{and} \quad \mathscr{S} : C(J,\mathbb{R}) \to C(J,\mathbb{R})$$

as

$$(\mathscr{T}\theta)(s) = a + \frac{s-x_0}{L-x_0}(b-a) - \int_{x_0}^{L} G(s,\xi)\mu f_\theta(\xi,\theta)d\xi \tag{3.1}$$

and

$$(\mathscr{S}\theta)(s) = a + c(s-x_0) - \int_{x_0}^{L} H(s,\xi)\mu f_\theta(\xi,\theta)d\xi. \tag{3.2}$$

Also, we have

$$(\mathscr{T}\theta)'(s) = \frac{b-a}{L-x_0} + \frac{\mu}{L-x_0}\left(\int_{x_0}^{s}(\xi-x_0)f_\theta(\xi,\theta)d\xi - \int_{s}^{L}(L-\xi)f_\theta(\xi,\theta)d\xi\right),$$

$$(\mathscr{S}\theta)'(s) = c - \mu\left(\int_s^L f_\theta(\xi,\theta)d\xi\right),$$

and

$$(\mathscr{T}\theta)''(s) = (\mathscr{S}\theta)''(s) = \mu f_\theta(s,\theta).$$

Note that $(\mathscr{T}\theta)(x_0) = (\mathscr{S}\theta)(x_0) = a$, $(\mathscr{T}\theta)(L) = b$ and $(\mathscr{S}\theta)'(L) = c$.

Firstly, we investigate the uniqueness of solution for the problem (2.1) subject to the boundary conditions (2.2) and (2.3), respectively.

Theorem 3.1. *Suppose that $f_\theta(s,\theta)$ is continuous on $J\times\mathbb{R}$ and there is a positive constant λ such that*

$$|f_\theta(s,\theta) - f_{\tilde\theta}(s,\tilde\theta)| \le \lambda|\theta-\tilde\theta|, \quad \textit{for all } \theta,\tilde\theta\in\mathbb{R}, s\in J.$$

Then, the boundary-value problems (2.1) *and* (2.2) *has a unique solution on J if*

$$\frac{\lambda\mu}{6}(L-x_0)^2 < 1.$$

Proof. The Banach contraction principle is used to prove that $\mathscr{T}$ has a fixed point. Let $\theta(s),\tilde\theta(s)\in C(J,\mathbb{R})$. Then, for all $s\in J$, we have

$$\begin{aligned}\left|\mathscr{T}\theta(s) - \mathscr{T}\tilde\theta(s)\right| &= \left|\int_{x_0}^L G(s,\xi)\mu f_\theta(\xi,\theta)d\xi - \int_{x_0}^L G(s,\xi)\mu f_{\tilde\theta}(\xi,\theta)d\xi\right| \\ &\le \mu\int_{x_0}^L |G(s,\xi)|\left|f_\theta(\xi,\theta) - f_{\tilde\theta}(\xi,\tilde\theta)\right|d\xi \\ &\le \lambda\mu\|\theta-\tilde\theta\|_\infty\int_{x_0}^L |G(s,\xi)|\,d\xi.\end{aligned}$$

It follows

$$\|\mathscr{T}\theta - \mathscr{T}\tilde\theta\|_\infty \le \frac{\lambda\mu}{6}(L-x_0)^2\|\theta-\tilde\theta\|_\infty < \|\theta-\tilde\theta\|_\infty.$$

Consequently, $\mathscr{T}$ is a contraction operator. From the Banach contraction mapping theorem, $\mathscr{T}$ has a unique fixed point, that is, there exists a unique $\theta\in C(J,\mathbb{R})$ such that $\mathscr{T}\theta = \theta$. This fixed point θ is a unique solution of this problem.

Theorem 3.2. *Suppose that the condition of Theorem 3.1 holds. The problem* (2.1) *with the boundary condition* (2.3) *has a unique solution on J provided that*

$$\frac{\lambda\mu}{2}(L-x_0)^2 < 1.$$

Proof. The proof is essentially similar to the proof of Theorem 3.1.

Next, the existence results will be proved by means of Schaefer's fixed point theorem. In Theorems 3.1 and 3.2, if the Lipschitz condition is weakened, the later result can be obtained, which is a more general existence result. The following assumption will be used to prove this general existence result.

(H1) The function $f_\theta : J \times \mathbb{R} \to \mathbb{R}$ is continuous.

(H2) There exists a constant $M > 0$, such that

$$|\mu f(s,\theta(s))| \leq M \quad \text{for each } s \in J \text{ and } \forall \theta(s) \in \mathbb{R}.$$

Theorem 3.3. *Assume (H1) and (H2) hold. Then, the problems* (2.1) *and* (2.2) *has at least one solution on J.*

Proof. Based on Schaefer's fixed point theorem, we need to prove that $\mathscr{T}$ defined by (3.1) has a fixed point. The proof will be divided into several steps

Step 1: $\mathscr{T}$ is continuous.

Suppose that $\theta_m \to \theta$ is a convergent sequence in $C(J,\mathbb{R})$. Then, for each $s \in J$, we obtain from Lemma (2.3) that

$$\begin{aligned} |(\mathscr{T}\theta_m)(s) - (\mathscr{T}\theta)(s)| &\leq \mu \int_{x_0}^{L} |G(s,\xi)|\,|f_\theta(\xi,\theta(\xi)) - f_{\theta_m}(\xi,\theta_m(\xi))|\,d\xi \\ &\leq \mu \sup_{\xi\in J} |f_\theta(\xi,\theta(\xi)) - f_{\theta_m}(\xi,\theta_m(\xi))| \int_{x_0}^{L} |G(s,\xi)|\,d\xi \\ &\leq \frac{\mu}{6}(L-x_0)^2 \sup_{\xi\in J} |f_\theta(\xi,\theta(\xi)) - f_{\theta_m}(\xi,\theta_m(\xi))|. \end{aligned}$$

As f is a continuously differentiable function, it follows that

$$\|\mathscr{T}\theta_m - \mathscr{T}\theta\|_\infty \leq \frac{\mu}{6}(L-x_0)^2 \|f_\theta(\cdot,\theta(\cdot)) - f_{\theta_m}(\cdot,\theta_m(\cdot))\|_\infty,$$

which implies

$$\|\mathscr{T}\theta_m - \mathscr{T}\theta\|_\infty \to 0 \quad \text{as} \quad m \to \infty.$$

Step 2: $\mathscr{T}$ maps bounded sets into bounded sets in $C(J,\mathbb{R})$.

We want to show that there exists $\ell > 0$ such that $\forall \theta \in B_{\eta^*} = \{\theta \in C(J,\mathbb{R}) : \|\theta\|_\infty \leq \eta^*\}$, and $\|\mathscr{T}\theta\|_\infty \leq \ell$ for $\eta^* > 0$. In fact, $\forall s \in J$, by (3.1) and (H2)

$$\begin{aligned} |(\mathscr{T}\theta)(s)| &\leq \left| a + \frac{s-x_0}{L-x_0}(b-a) \right| + \int_{x_0}^{L} |G(s,\xi)\mu f_\theta(\xi,\theta)|\,d\xi \\ &\leq |a| + \left| \frac{s-x_0}{L-x_0}(b-a) \right| + M \int_{x_0}^{L} |G(s,\xi)|\,d\xi. \end{aligned}$$

Thus

$$\|\mathscr{T}\theta\|_\infty \leq \ell,$$

where

$$\ell = |a| + |b-a| + \frac{M}{6}(L-x_0)^2.$$

Step 3: $\mathscr{T}$ maps bounded sets into equicontinuous sets of $C(J,\mathbb{R})$.

Let $s_1, s_2 \in J, s_1 < s_2, B_{\eta^*}$ be a bounded set of $C(J,\mathbb{R})$ as above, and $\theta \in B_\eta$. We have

$$\begin{aligned}
|(\mathscr{T}\theta)(s_2) - (\mathscr{T}\theta)(s_1)| &\leq \mu \int_{x_0}^{L} |(G(s_2,\xi) - G(s_1,\xi)) f_\theta(\xi,\theta(\xi))| d\xi \\
&\leq M\mu \int_{x_0}^{L} |G(s_2,\xi) - G(s_1,\xi)| d\xi \\
&= M\mu \left(\int_{x_0}^{s_1} |G(s_2,\xi) - G(s_1,\xi)| d\xi + \int_{s_1}^{L} |G(s_2,\xi) - G(s_1,\xi)| d\xi \right) \\
&= M\mu \left(\int_{x_0}^{s_1} \left| (s_2 - s_1) \left(\frac{L-\xi}{L-x_0} \right) \right| d\xi + \int_{s_1}^{s_2} |-G(s_1,\xi)| d\xi \right).
\end{aligned}$$

As $s_1 \to s_2$, the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3 together with the Arzelá-Ascoli theorem, $\mathscr{T} : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ is completely continuous.

Step 4: $\mathscr{T}$ is priori bounded.

Consider the set

$$\varepsilon = \{\theta \in C(J,\mathbb{R}) : \theta = \lambda\,(\mathscr{T}\theta) \text{ for some } 0 < \lambda < 1\}.$$

It is enough to show that the set ε is bounded.

Let $\theta \in \varepsilon$, then $\theta = \lambda\,(\mathscr{T}\theta)$ for some $0 < \lambda < 1$. Thus $\forall s \in J$,

$$\theta(s) = \lambda \left(a + \frac{s-x_0}{L-x_0}(b-a) - \int_{x_0}^{L} G(s,\xi)\mu f_\theta(\xi,\theta) d\xi \right).$$

By the condition (H2) and Step 2,

$$|\theta(s)| \leq |a| + \left| \frac{s-x_0}{L-x_0}(b-a) \right| + M \int_{x_0}^{L} |G(s,\xi)| d\xi.$$

Thus for every $\forall s \in J$,

$$\|\theta\|_\infty \leq |a| + |b-a| + \frac{M}{6}(L-x_0)^2 := R.$$

Therefore, the set ε is bounded. As a consequence of Schaefer's fixed point theorem, the operator $\mathscr{T}$ has at least one fixed point. This means that the problems (2.1) and (2.2) has at least one solution on J.

Theorem 3.4. *Assume (H1) and (H2) hold. Then, the problem* (2.1) *with the boundary condition* (2.3) *has at least one solution on J.*

Proof. The proof is essentially similar to the proof of Theorem 3.3.

4. Dependence of solutions on the parameters

In this section, we analyze how the solution is influenced under small perturbations of the given parameters, namely, of the initial function and the nonlinear term in (2.1).

Theorem 4.1. *Suppose that the conditions of Theorem 3.1 hold. If θ_1 is the solution of BVP* (2.1)–(2.2) *and θ_2 is the solution of the following problem:*

$$\begin{cases} \theta''(s) = \mu f_\theta(s,\theta,\alpha) \\ \theta(x_0) = a+\varepsilon_1, \qquad \theta(L) = b+\varepsilon_2. \end{cases}$$

Then,

$$\|\theta_1 - \theta_2\|_\infty = \mathscr{O}(\varepsilon_1,\varepsilon_2).$$

Proof. We have

$$\begin{aligned} |\theta_1(s)-\theta_2(s)| &= \left| \varepsilon_1 - \varepsilon_2\left(\frac{s-x_0}{L-x_0}\right) + \int_{x_0}^{L} G(s,\xi)\left(f_{\theta_1}(\xi,\theta_1) - f_{\theta_2}(\xi,\theta_2)\right) d\xi \right| \\ &\le \varepsilon_1 + \varepsilon_2\left(\frac{s-x_0}{L-x_0}\right) + \int_{x_0}^{L} |G(s,\xi)| \left|f_{\theta_1}(\xi,\theta_1) - f_{\theta_2}(\xi,\theta_2)\right| d\xi \\ &\le \varepsilon_1 + \varepsilon_2\left(\frac{s-x_0}{L-x_0}\right) + \frac{\lambda\mu}{6}(L-x_0)^2 \|\theta_1-\theta_2\|_\infty \\ &= \mathscr{O}(\varepsilon_1,\varepsilon_2). \end{aligned}$$

This gives the desired result.

Theorem 4.2. *Suppose that $f_\theta(s,\theta,\alpha)$ is continuous on $[x_0,L]\times\mathbb{R}$ and that there is a continuous function $\kappa : [x_0,L] \to \mathbb{R}^+$such that*

$$|f_\theta(s,\theta,\alpha) - f_\theta(s,\theta,\eta)| \le \kappa(s)\,|\alpha-\beta|\,, \forall \alpha,\beta \in \mathbb{R}^+, s\in[x_0,L].$$

If θ_1 is the solution of BVP (2.1)–(2.2) *and θ_2 is the solution of the following problem:*

$$\begin{cases} \theta''(s) = \mu f_\theta(s,\theta,\alpha) \\ \theta(x_0) = a, \qquad \theta(L) = b. \end{cases}$$

Then,

$$\|\theta_1-\theta_2\|_\infty \le \frac{\alpha\lambda\mu}{6}(L-x_0)^2 \|\kappa\|_\infty .$$

Proof. From Lemma 2.1, we have

$$\begin{aligned} |\theta_1(s)-\theta_2(s)| &= \left| \int_{x_0}^{L} G(s,\xi)\mu\left(f_{\theta_1}(\xi,\theta_1) - f_{\theta_2}(\xi,\theta_2,\alpha)\right) d\xi \right| \\ &\le \int_{x_0}^{L} |G(s,\xi)| \left|\mu\left(f_{\theta_1}(\xi,\theta_1) - f_{\theta_2}(\xi,\theta_2,\alpha)\right)\right| d\xi \\ &\le \alpha\mu \int_{x_0}^{L} \kappa(\xi)\,|G(s,\xi)|\, d\xi \\ &\le \frac{\alpha\lambda\mu}{6}(L-x_0)^2 \|\kappa\|_\infty . \end{aligned}$$

This completes the proof.

Theorem 4.3. *Suppose that the conditions of Theorem 3.1 hold. If θ_1 is the solution of* (2.1) *with the boundary condition* (2.3) *and θ_2 is the solution of the following problem:*

$$\begin{cases} \theta''(s) = \mu f_\theta(s,\theta,\alpha) \\ \theta(x_0) = a + \eta_1, \qquad \theta'(L) = c + \eta_2. \end{cases}$$

Then,

$$\|\theta_1 - \theta_2\|_\infty = \mathcal{O}(\eta_1, \eta_2).$$

Proof. We have

$$\begin{aligned} |\theta_1(s) - \theta_2(s)| &= \left| \eta_1 - \eta_2(s - x_0) + \int_{x_0}^{L} H(s,\xi)\,(f_{\theta_1}(\xi,\theta_1) - f_{\theta_2}(\xi,\theta_2))\,d\xi \right| \\ &\le \eta_1 + \eta_2(s - x_0) + \int_{x_0}^{L} |H(s,\xi)|\,\left|f_{\theta_1}(\xi,\theta_1) - f_{\theta_2}(\xi,\theta_2)\right|d\xi \\ &\le \eta_1 + \eta_2(s - x_0) + \frac{\lambda\mu}{2}\,(L - x_0)^2\,\|\theta_1 - \theta_2\|_\infty \\ &= \mathcal{O}(\eta_1, \eta_2). \end{aligned}$$

This gives the desired result.

Theorem 4.4. *Suppose that the assumption of Theorem 4.2 holds. If θ_1 is the solution of* (2.1) *with the boundary condition* (2.3) *and θ_2 is the solution of the following problem:*

$$\begin{cases} \theta''(s) = \mu f_\theta(s,\theta,\alpha) \\ \theta(x_0) = a, \qquad \theta'(L) = c. \end{cases}$$

Then,

$$\|\theta_1 - \theta_2\|_\infty \le \frac{\alpha\lambda\mu}{2}\,(L - x_0)^2\,\|\kappa\|_\infty.$$

Proof. From Lemma 2.1, we have

$$\begin{aligned} |\theta_1(s) - \theta_2(s)| &= \left| \int_{x_0}^{L} H(s,\xi)\mu\,(f_{\theta_1}(\xi,\theta_1) - f_{\theta_2}(\xi,\theta_2,\alpha))\,d\xi \right| \\ &\le \int_{x_0}^{L} |H(s,\xi)|\,\left|\mu\,(f_{\theta_1}(\xi,\theta_1) - f_{\theta_2}(\xi,\theta_2,\alpha))\right|d\xi \\ &\le \alpha\mu \int_{x_0}^{L} \kappa(\xi)\,|G(s,\xi)|\,d\xi \\ &\le \frac{\alpha\lambda\mu}{2}\,(L - x_0)^2\,\|\kappa\|_\infty. \end{aligned}$$

This completes the proof.

5. Approximate analytical solution by the Adomian decomposition method

We propose here to solve this boundary value problem by the Adomian decomposition method (ADM).

We rewrite the problem (2.1) in the general functional equation as

$$\mathscr{L}\theta = \mathscr{N}\theta, \tag{5.1}$$

where $\mathscr{L} = \dfrac{d^2}{ds^2}$ is the linear operator and $\mathscr{N}\theta = \mu f_\theta(s,\theta)$ is a nonlinear operator. Consequently, by Lemma 2.1 we have

$$\theta(s) = a + \frac{s-x_0}{L-x_0}(b-a) + \int_{x_0}^{L} G(s,\xi)\mathscr{N}\theta d\xi. \tag{5.2}$$

Define the solutions θ by its respective infinite series of components in the form

$$\theta(s) = \sum_{n=0}^{\infty} \theta_n(s) \tag{5.3}$$

and decomposing the nonlinear operator $\mathscr{N}$ as

$$\mathscr{N}\theta = \sum_{n=0}^{\infty} A_n, \tag{5.4}$$

where A_n are polynomials (called Adomian polynomials) of $\theta_0, \theta_1, \cdots, \theta_n$, which can be obtained from the definitional formula

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\mathscr{N}\left(\sum_{i=0}^{n}\lambda^i\theta_i\right)\right]_{\lambda=0}, n = 0,1,2,\ldots$$

Substituting (5.3) and (5.4) into (5.2) yields

$$\sum_{n=0}^{\infty}\theta_n(s) = a + \frac{s-x_0}{L-x_0}(b-a) + \int_{x_0}^{L} G(s,\xi)\sum_{n=0}^{\infty} A_n d\xi. \tag{5.5}$$

From (5.5), the iterates defined using the standard Adomian method are determined in the following recursive way

$$\begin{cases} \theta_0 = a + \dfrac{s-x_0}{L-x_0}(b-a) \\ \theta_{n+1} = \displaystyle\int_{x_0}^{L} G(s,\xi)A_n d\xi, \quad n = 0,1,2,\ldots. \end{cases} \tag{5.6}$$

Let $\Phi_n(s) = \sum_{i=0}^{n-1}\theta_i(s)$ be the n th-stage approximation functions of $\theta(s)$ by the ADM for the nonlinear Eq (5.1). By substitution the recursion scheme (5.6) into this sum, we obtain

$$\begin{cases} \Phi_0(s) = a + \dfrac{s-x_0}{L-x_0}(b-a) \\ \Phi_n(s) = \displaystyle\int_{x_0}^{L} G(s,\xi)\mu f_\theta\left(\Phi_{n-1}(\xi)\right) d\xi, \quad n \geq 1. \end{cases} \tag{5.7}$$

Similarly, for Eq (2.2), we obtain

$$\begin{cases} \tilde{\Phi}_0(s) = a + c(s - x_0) \\ \tilde{\Phi}_n(s) = \int_{x_0}^{L} H(s,\xi)\mu f_\theta\left(\tilde{\Phi}_{n-1}(\xi)\right)d\xi, \quad n \geq 1, \end{cases} \tag{5.8}$$

where $\tilde{\Phi}_n(s) = \sum_{i=0}^{n-1} \theta_i(s)$ and

$$\begin{cases} \theta_0 = a + c(s - x_0) \\ \theta_{n+1} = \int_{x_0}^{L} H(s,\xi)A_n d\xi, \quad n = 0, 1, 2, \ldots. \end{cases}$$

Let us prove the following results on the convergence of the Adomian decomposition method.

Theorem 5.1. *Let (Φ_n) be a sequence defined by (5.7). Then $\lim_{n\to\infty} \Phi_n(s) = \theta(s)$ and*

$$\|\Phi_n - \theta\|_\infty \leq c^n \frac{1}{1-c} \|\Phi_1 - \Phi_0\|_\infty, \tag{5.9}$$

where $\theta(s)$ satisfies the integral equation

$$\theta(s) = a + \frac{s - x_0}{L - x_0}(b - a) + \int_0^L G(s,\xi)\mu f_\theta(\theta(\xi))d\xi. \tag{5.10}$$

Proof. We show that (Φ_n) is a contractive sequence.

$$\begin{aligned} \|\Phi_{n+2} - \Phi_{n+1}\|_\infty &= \sup_{s\in J}\left|\int_{x_0}^{L} \mu G(s,\xi)\left[f_\theta\left(\Phi_{n+1}(\xi)\right) - f_\theta\left(\Phi_n(\xi)\right)\right]d\xi\right| \\ &\leq \sup_{s\in J}\int_{x_0}^{L} |\mu G(s,\xi)|\left|f_\theta\left(\Phi_{n+1}(\xi)\right) - f_\theta\left(\Phi_n(\xi)\right)\right|d\xi \\ &\leq \lambda\mu \|\Phi_{n+1} - \Phi_n\|_\infty \sup_{s\in J}\int_{x_0}^{L} |G(s,\xi)|\,d\xi \\ &\leq \frac{\lambda\mu}{6}(L - x_0)^2 \|\Phi_{n+1} - \Phi_n\|_\infty \\ &:= c\|\Phi_{n+1} - \Phi_n\|_\infty. \end{aligned}$$

If $m > n$, say $m = n + p, p = 1, 2, \ldots$, then

$$\begin{aligned} \left\|\Phi_{n+p} - \Phi_n\right\|_\infty \leq &\|\Phi_n - \Phi_{n-1}\|_\infty + \|\Phi_{n-1} - \Phi_{n-2}\|_\infty \\ &+ \cdots + \|\Phi_{n+1} - \Phi_n\|_\infty \\ \leq &c^n\left(1 + c + c^2 + \cdots + c^{p-1} + c^p + \cdots\right)\|\Phi_1 - \Phi_0\|_\infty \end{aligned}$$

Thus

$$\left\|\Phi_{n+p} - \Phi_n\right\|_\infty \leq c^n \frac{1}{1-c} \|\Phi_1 - \Phi_0\|_\infty. \tag{5.11}$$

As $n, m = n + p \to \infty$, we see that $\left\|\Phi_{n+p} - \Phi_n\right\|_\infty \to 0$, that is (Φ_n) is a Cauchy sequence in the Banach space $(C(J), \|\cdot\|_\infty)$. Hence, it must be convergent, say $\lim\limits_{n\to\infty} \Phi_n(s) = \theta(s)$. By taking the limit in (5.11) as $p \to \infty$, we obtain the desired inequality (Eq 5.9). It remains to verify that θ is a solution

$$\begin{aligned}&\left\|\theta - \int_{x_0}^{L} \mu G(s,\xi) f_\theta(\theta(\xi)) d\xi\right\|_\infty \\ &\leq \|\theta - \Phi_n\|_\infty + \left\|\Phi_n - \int_{x_0}^{L} \mu G(s,\xi) f_\theta\left(\Phi_{n-1}(\xi)\right) d\xi\right\|_\infty \\ &+ \left\|\int_{x_0}^{L} \mu G(s,\xi)\left[f_\theta(\theta(\xi)) - f_\theta\left(\Phi_{n-1}(\xi)\right)\right] d\xi\right\|_\infty .\end{aligned}$$

Since $\lim\limits_{n\to\infty} \Phi_n(s) = \theta(s)$, we conclude that θ satisfies Eq (5.10).

6. Examples

In order to illustrate our results, we have also considered the equilibrium configurations of a cantilever beam, that is,

$$\frac{dx}{ds} = \cos\theta \qquad \text{and} \qquad \frac{dy}{ds} = \sin\theta,$$

where x and y are the coordinates of a point on a cantilever beam.

Example 6.1. *Consider the functional $\mathscr{J}(\theta)$ which can be written as*

$$\begin{cases} \mathscr{J}(\theta) = \int_0^1 \left(\frac{1}{2}(\theta')^2 - 1.23\sin\theta\right) ds \\ \theta(0) = \theta'(1) = 0. \end{cases} \tag{6.1}$$

In this case, the cantilever beam is subjected to a concentrated load $\mu = 1.23$ as shown in Figure 2. Applying the Euler-Lagrange, we have

$$\theta''(s) + 1.23\cos(\theta) = 0.$$

By Corollary 2.2, the problem (6.1) can be written as

$$\theta(s) = -\int_0^1 H(s,\xi) 1.23\cos(\theta) d\xi.$$

Here, we have $\mu = 1.23$, $f_\theta = -\cos(\theta)$, $\lambda = 1$ and $\theta(0) = \theta'(1) = 0$. Then, we obtain

$$-1.23 \leq \mu f_\theta(s,\theta) \leq 1.23$$

and

$$\frac{\lambda\mu}{2}(L - x_0)^2 = 0.6150 < 1.$$

Then, by Theorem 3.2 the boundary value problem (6.1) has a unique solution on $J = [0,1]$. Moreover, in view of Theorem 3.4, the problem (6.1) has at least one solution on J because the conditions (H1) and (H2) are satisfied.

Applying the Adomian decomposition method, the recursion scheme (5.6) produces a rapidly convergent series as

$$\theta_0 = 0,$$
$$\theta_k = \int_0^s \xi A_{k-1} d\xi + \int_s^1 s A_{k-1} d\xi, \quad \text{for } k \geq 1,$$

where the Adomian polynomials A_k for the nonlinear term $\mathscr{N}(\theta) = -1.23\cos(\theta)$ are given as

$$A_0 = -1.23\cos(\theta_0)$$
$$A_1 = -1.23\theta_1 \sin(\theta_0)$$
$$A_2 = -1.23\left(\theta_2 \sin(\theta_0) + \frac{1}{2}\theta_1^2 \cos(\theta_0)\right)$$
$$A_3 = -1.23\left(\theta_3 \sin(\theta_0) + \theta_1\theta_2 \cos(\theta_0) - \frac{1}{3!}\theta_1^3 \sin(\theta_0)\right)$$
$$A_4 = -1.23\left(\theta_4 \sin(\theta_0) + \left[\frac{1}{2}\theta_2^2 - \theta_1\theta_3\right]\cos(\theta_0) - \frac{1}{2}\theta_1^2\theta_2 \sin(\theta_0) + \frac{1}{4!}\theta_1^4 \cos(\theta_0)\right)$$
$$A_5 = -1.23\left(\theta_5 \sin(\theta_0) + \left(\theta_1\theta_4 + \theta_2\theta_3 - \frac{1}{6}\theta_1^3\theta_2\right)\cos(\theta_0) - \frac{1}{2}\left(\theta_1^2\theta_3 + \theta_1\theta_2^2\right)\sin(\theta_0) + \frac{1}{5!}\theta_1^5 \sin(\theta_0)\right)$$
$$\vdots$$

This in turn gives the following solution components

$$\theta_0 = 0$$
$$\theta_1 = -0.615s(s-2)$$
$$\theta_2 = 0$$
$$\theta_3 = 0.46521675s(s-2)$$
$$\theta_4 = 0$$
$$\theta_5 = -0.71555686s(s-2)$$
$$\vdots$$

The sixth-stage approximate solution is therefore given as

$$\Phi_6(s) = \sum_{i=0}^{5} \theta_i(s) \approx -0.8653s(s-2),$$

which is shown in Figure 5.

Furthermore, those equilibrium configurations subjected to a concentrated load in Figure 6, which is in correspondence to [16] showing the deflection due to a point load.

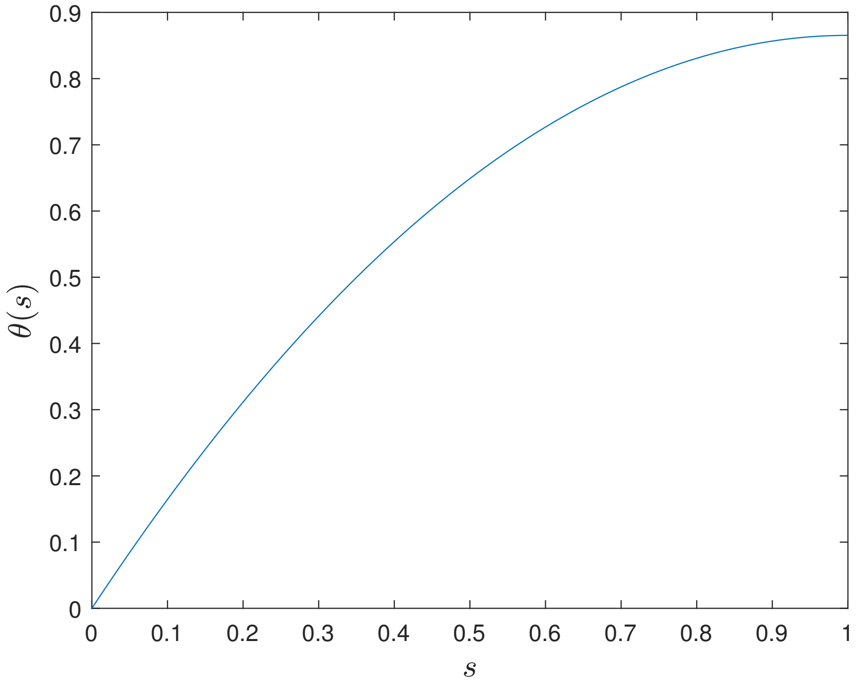

Figure 5. The approximated solution for tangent angle θ of a cantilever beam subjected to a concentrated load $\mu = 1.23$.

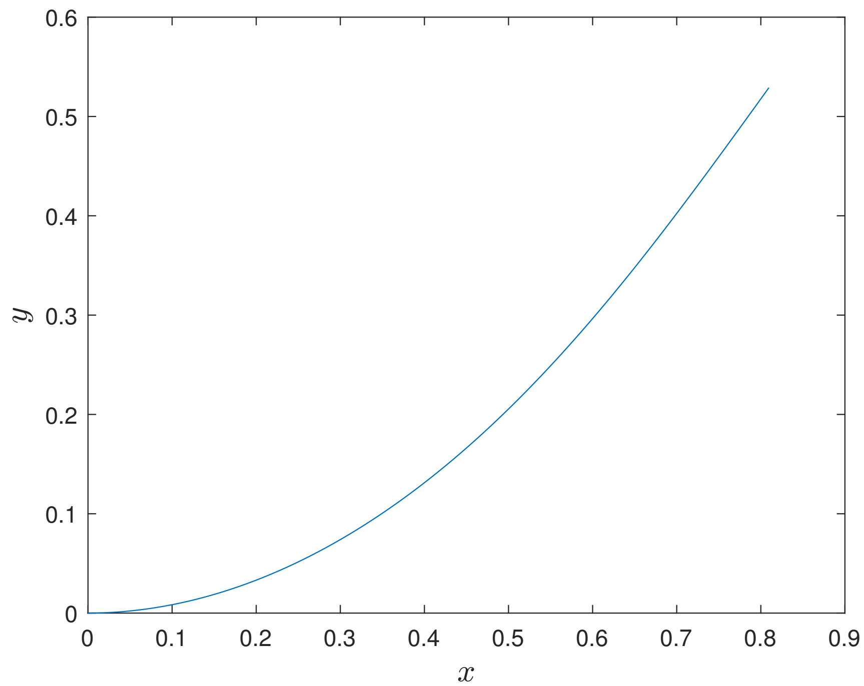

Figure 6. Equilibrium configuration of a cantilever beam subjected to a concentrated load $\mu = 1.23$.

Example 6.2. *Consider the functional* $\mathscr{J}(\theta)$ *which can be written as*

$$\begin{cases} \mathscr{J}(\theta) = \int_0^1 \left(\frac{1}{2} (\theta')^2 + 0.5 \cos\theta \right) ds \\ \theta(0) = 0, \qquad \theta(1) = 0.9. \end{cases} \tag{6.2}$$

In this case, the cantilever beam is subjected to a compressive load $\mu = 0.5$ as shown in Figure 3. Applying the Euler-Lagrange, we have

$$\theta''(s) = -0.5 \sin(\theta).$$

By Lemma 2.1, the problem (6.2) can be written as

$$\theta(s) = 0.9s - \int_0^1 G(s,\xi) 0.5 \sin(\theta) d\xi.$$

Here, we have $\mu = 0.5$, $f_\theta = -\sin(\theta)$, $\lambda = 1$ and $\theta(0) = \theta(1) = 0$. Then, we obtain

$$|\mu f_\theta(s,\theta)| \le 0.5$$

and

$$\frac{\lambda \mu}{6} (L - x_0)^2 = 0.0833 < 1.$$

Then, by Theorem 3.1 the boundary value problem (6.2) has a unique solution on $J = [0,1]$. Moreover, in view of Theorem 3.3, the problem (6.2) has at least one solution on J because the conditions (H1) and (H2) are satisfied.

Applying the Adomian decomposition method, the recursion scheme (5.6) produces a rapidly convergent series as

$$\theta_0 = 0.9s,$$
$$\theta_k = \int_0^s \xi (1-s) A_{k-1} d\xi + \int_s^1 s(1-\xi) A_{k-1} d\xi, \quad \text{for } k \ge 1,$$

where the Adomian polynomials A_k, for the nonlinear term $\mathscr{N}(\theta) = -0.5 \sin(\theta)$ are given as

$$A_0 = -0.5 \sin(\theta_0)$$
$$A_1 = -0.5 \theta_1 \cos(\theta_0)$$
$$A_2 = -0.5 \left(\theta_2 \cos(\theta_0) - \frac{1}{2} \theta_1^2 \sin(\theta_0) \right)$$
$$A_3 = -0.5 \left(\theta_3 \cos(\theta_0) - \theta_1 \theta_2 \sin(\theta_0) - \frac{1}{3!} \theta_1^3 \cos(\theta_0) \right)$$
$$A_4 = -0.5 \left(\theta_4 \cos(\theta_0) - \left[\frac{1}{2} \theta_2^2 + \theta_1 \theta_3 \right] \sin(\theta_0) - \frac{1}{2} \theta_1^2 \theta_2 \cos(\theta_0) + \frac{1}{4!} \theta_1^4 \sin(\theta_0) \right)$$
$$A_5 = -0.5 \left(\theta_5 \cos(\theta_0) - \left(\theta_1 \theta_4 + \theta_2 \theta_3 - \frac{1}{6} \theta_1^3 \theta_2 \right) \sin(\theta_0) - \frac{1}{2} \left(\theta_1^2 \theta_3 + \theta_1 \theta_2^2 \right) \cos(\theta_0) + \frac{1}{5!} \theta_1^5 \cos(\theta_0) \right)$$
$$\vdots$$

The sixth-stage approximate solution is shown in Figure 7. Furthermore, those equilibrium configurations subjected to a compressive load in Figure 8, which is in correspondence to [16] showing the deflection due to a compressive load. Moreover, the bucking of the cantilever beam with one fixed end and the other being free under a compressive load is analyzed in [32] with the critical load is $\frac{\pi^2}{4}$.

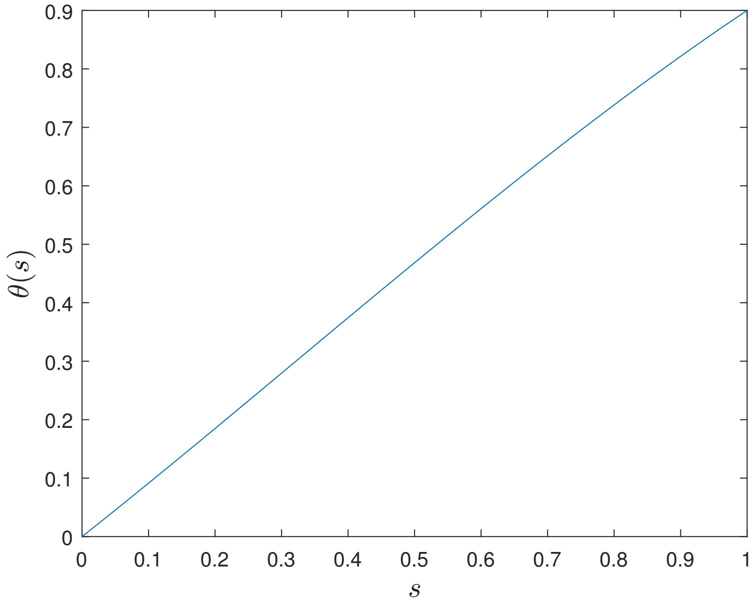

Figure 7. The approximated solution for tangent angle θ of a cantilever beam subjected to a compressive load $\mu = 0.5$.

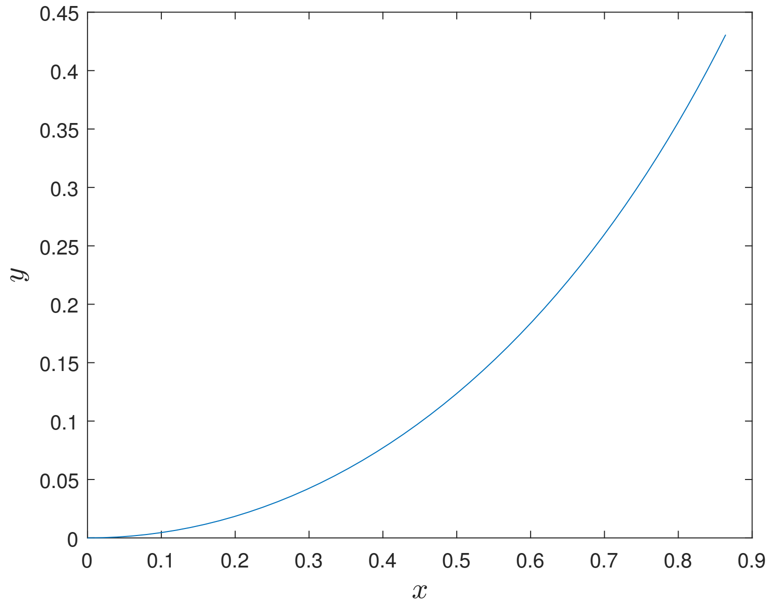

Figure 8. Equilibrium configuration of a cantilever beam subjected to a compressive load $\mu = 0.5$.

Example 6.3. *Consider the functional $\mathscr{J}(\theta)$ which can be written as*

$$\begin{cases} \mathscr{J}(\theta) = \int_0^1 \left(\frac{1}{2} (\theta')^2 - 1.84(1-s)\sin\theta \right) ds \\ \theta(0) = \theta'(1) = 0. \end{cases} \tag{6.3}$$

In this case, the cantilever beam is subjected to a uniformly distributed load $\mu = 1.84$ as shown in Figure 4. Applying the Euler-Lagrange, we have

$$\theta''(s) = -1.84(1-s)\cos(\theta).$$

By Corollary 2.2, the problem (6.3) can be written as

$$\theta(s) = -\int_0^1 H(s,\xi)1.84(1-\xi)\cos(\theta)d\xi.$$

Here, we have $\mu = 1.84$, $f_\theta = -(1-s)\cos(\theta)$, $\lambda = 1$ and $\theta(0) = \theta'(1) = 0$. Then, we obtain

$$|\mu f_\theta(s,\theta)| \leq 1.84$$

and

$$\frac{\lambda\mu}{2}(L - x_0)^2 = 0.92 < 1.$$

Then, by Theorem 3.2 the boundary value problem (6.3) has a unique solution on $J = [0,1]$. Moreover, in view of Theorem 3.4, the problem (6.3) has at least one solution on J because the conditions (H1) and (H2) are satisfied.

Applying the Adomian decomposition method, the recursion scheme (5.6) produces a rapidly convergent series as

$$\begin{aligned} &\theta_0 = 0, \\ &\theta_k = \int_0^s \xi A_{k-1} d\xi + \int_s^1 s A_{k-1} d\xi, \quad \text{for } k \geq 1, \end{aligned}$$

where the Adomian polynomials A_k with the nonlinear term $\mathscr{N}(\theta) = -1.84(1-s)\cos(\theta)$.

The sixth-stage approximate solution is shown in Figure 9. Furthermore, those equilibrium configurations subjected to a uniformly distributed load in Figure 10, which is in correspondence to [9] showing the deflection due to a uniformly distributed load.

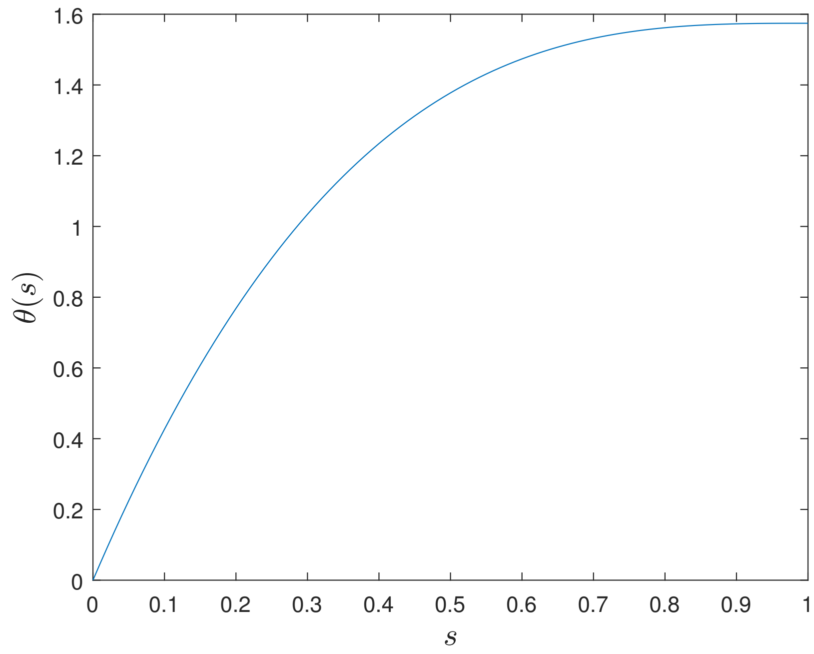

Figure 9. The approximated solution for tangent angle θ a cantilever beam subjected to a uniformly distributed load $\mu = 1.84$.

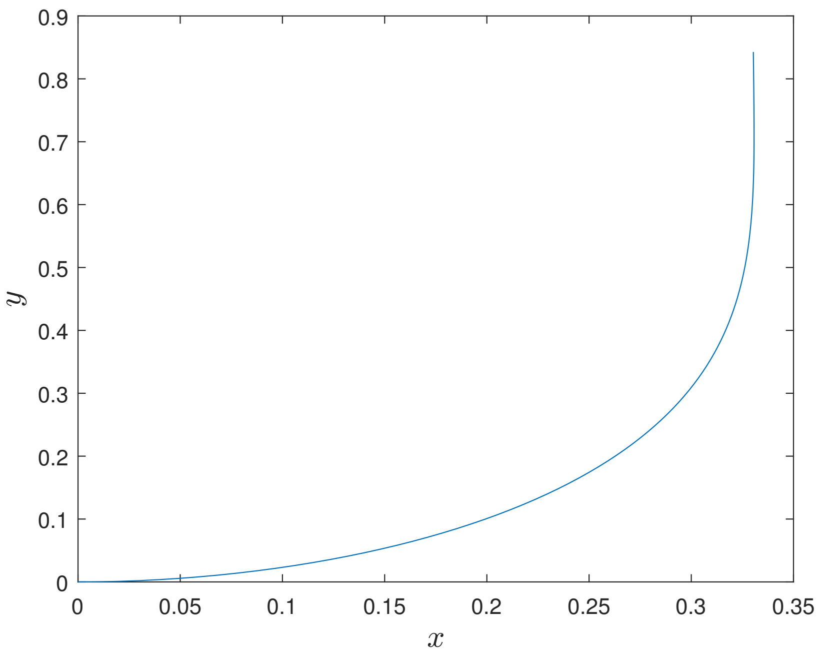

Figure 10. Equilibrium configuration of a cantilever beam subjected to a uniformly distributed load $\mu = 1.84$.

7. Conclusions

The Euler-Lagrange condition was applied to minimize the generalized total energy functional describing equilibrium configurations of cantilever beam. Based on the boundary value problems, we construct the solutions for this problem and prove the existence and uniqueness of solutions using the Banach fixed point theorem and Schaefer's fixed point theorem. Consequently, the continuous dependence on the parameters is investigated which assigns the parameter for the initial conditions and nonlinear terms. Furthermore, the boundary value problem is devoted to the approximation of the analytical solution by using Adomian decomposition method. The examples are presented the equilibrium configurations of cantilever beams on different load types.

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Conflict of interest

The authors declare there is no conflict of interest.

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