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Research article

On the solution and Ulam-Hyers-Rassias stability of a Caputo fractional boundary value problem

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Abstract: In this paper, we investigate a class of boundary value problems involving Caputo fractional derivative ${}^C\mathcal{D}_a^\alpha$ of order $\alpha \in (2,3)$, and the usual derivative, of the form

$$({}^C\mathcal{D}_a^\alpha x)(t) + p(t)x'(t) + q(t)x(t) = g(t), \quad a \le t \le b,$$

for an unknown x with $x(a) = x'(a) = x(b) = 0$, and $p,\ q,\ g \in C^2([a,b])$. The proposed method uses certain integral inequalities, Banach's Contraction Principle and Krasnoselskii's Fixed Point Theorem to identify conditions that guarantee the existence and uniqueness of the solution (for the problem under study) and that allow the deduction of Ulam-Hyers and Ulam-Hyers-Rassias stabilities.

Keywords: boundary value problem; Caputo fractional derivative; fixed point; Ulam-Hyers stability; Ulam-Hyers-Rassias stability

1. Introduction

In recent decades, fractional calculus has gained considerable popularity and importance. This is mainly due to its wide range of applications in different areas of engineering and other scientific fields such as biology, chemistry, economics, physics, image and signal processing, etc. (cf., for example, [1–6]). In fact, several studies have shown that fractional derivation allows different occurrences – such as complex long memory and hereditary properties of many processes – to be described in a much more satisfactory way when compared to models that consider only classical integer-order derivation (see, for example, [7, 8]).

Within this scope, different aspects and properties of fractional boundary value problems (FBVP) have been studied, with special emphasis on the analysis of the existence and uniqueness of solutions, as well as on different types of stabilities (cf., for example, [9–14]).

In the present work, we will focus on two important types of stabilities: the Ulam-Hyers and Ulam-Hyers-Rassias stabilities. In historical terms, it was Ulam who, as far back as 1940, questioned for the first time the stability of functional equations relating to group homomorphisms (cf. [15]). The question was initially answered the following year by Hyers in the context of Banach spaces for additive mappings (cf. [16]). This first result of Hyers was later generalized by T. Aoki [17] for additive mapping. Much later, in 1978, a generalization of the Ulam-Hyers stability was then proposed by Rassias [18], for linear mappings. In this case, the Cauchy differences were allowed to be unlimited, giving rise to the so-called Ulam-Hyers-Rassias stability. Since then, these types of stabilities, their properties and consequences, have attracted the attention of many mathematicians, as well as researchers from other more applied areas (cf. [10, 12, 19–24]). Note that if a system is stable in the Ulam-Hyers or Ulam-Hyers-Rassias sense, then significant properties hold around the exact solution. In this way, awareness of the existence of such types of stability constitutes an important tool in many applications in different areas, such as numerical analysis, optimization, biology or even economics (e.g., specially when determining an exact solution is sometimes quite difficult).

Taking into account [25], we address the study of the Ulam-Hyers and the Ulam-Hyers-Rassias stabilities for the following Caputo fractional boundary value problem (which also includes the usual derivative):

$$({}^{C}\mathcal{D}_a^{\alpha}x)(t) + p(t)x'(t) + q(t)x(t) = g(t), \quad a < t < b, \;\; 2 < \alpha < 3, \tag{1.1}$$

with $x(a) = x'(a) = x(b) = 0$, where p, q and $g \in C^2([a, b])$.

To the best of our knowledge, there is no results dealing with the Ulam-Hyers and Ulam-Hyers-Rassias stabilities of such fractional boundary value problem (FBVP).

The paper is organized as follows: Section 2 contains the necessary definitions from fractional calculus and the fundamental tools that are used throughout the paper; in Section 3, we focus on questions about the existence of solutions for the FBVP (1.1), identifying conditions for the existence of solutions and also for there to be only one solution; in Section 4, we discuss the Ulam-Hyers and the Ulam-Hyers-Rassias stabilities and introduce conditions for their existence. Finally, examples are given in Section 5 to illustrate the theoretical results.

2. Preliminaries and background material

In this section, just to have as self-contained work as possible, with the consequent benefit of the reader in mind, we recall some useful definitions and properties of the theory of fractional calculus [6] and necessary results in our future proofs.

We denote by $C^n([a, b]) := (C^n([a, b]), \|\cdot\|_{C^n})$ the space of functions x which are n-times continuously differentiable on $[a, b]$ endowed with the norm $\|x\|_{C^n} = \sum_{k=0}^{n} \sup_{t\in[a,b]} |x^{(k)}(t)|$. It is well-known that $C^n([a, b])$ is a Banach space.

Definition 1. [8] *The Riemann-Liouville fractional integral of order $\alpha \in \mathbb{R}^+$ of a function u is defined by*

$$I_a^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha-1} u(s) \mathrm{d}s,$$

provided the right-hand side is pointwise defined on (a, ∞), and where Γ is the well-known Euler Gamma function (given by $\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} \mathrm{d}t, \;\alpha > 0$).

Definition 2. [8] *The Caputo fractional derivative of order $\alpha > 0$ of a continuous function u is given by*

$$ {}^{C}\mathcal{D}_a^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{u^{(n)}(s)}{(t-s)^{\alpha-n+1}} \mathrm{d}s $$

provided that the right-hand side is pointwise defined on (a, ∞), and where $n \in \mathbb{N}$ is such that $n - 1 < \alpha < n$.

It is clear that if $\alpha \in \mathbb{N}$, then ${}^{C}\mathcal{D}_a^{\alpha} u(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\alpha} u(t)$.

Proposition 1. [8, Lemma 2.22] *Let $n - 1 < \alpha < n$, $n \in \mathbb{N}$. If $f \in C^{n-1}([a, b])$ (or $f \in AC^{n-1}([a, b])$), then the following relation holds true:*

$$ (I_a^{\alpha}\, {}^{C}\mathcal{D}_a^{\alpha} f)(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(t-a)^k. \tag{2.1} $$

As explained above, there are some classic and essential results that we will use in this work. We will recall them here, stating the *Banach Contraction Principle*, the *Krasnoselski Fixed Point Theorem* and the *Arzelà-Ascoli Theorem*.

Theorem 1. (Banach Contraction Principle) *Let (X, d) be a generalized complete metric space, and consider a mapping $T : X \to X$ which is a strictly contractive operator, that is,*

$$ d(Tx, Ty) \leq Ld(x, y), \quad \forall x, y \in X $$

for some constant $0 \leq L < 1$. Then

(a) the mapping T has a unique fixed point $x^* = Tx^*$;
(b) the fixed point x^* is globally attractive, in the sense that for any starting point $x \in X$, the following identity holds true:

$$ \lim_{n\to\infty} T^n x = x^*; $$

(c) we have the following inequalities:

$$ \begin{aligned} d(T^n x, x^*) &\leq L^n d(x, x^*), \quad n \geq 0,\ x \in X; \\ d(T^n x, x^*) &\leq \frac{1}{1-L} d(T^n x, T^{n+1} x), \quad n \geq 0,\ x \in X; \\ d(x, x^*) &\leq \frac{1}{1-L} d(x, Tx), \quad x \in X. \end{aligned} $$

Theorem 2. [26] (Krasnoselskii's Fixed Point Theorem) *Let M be a closed, bounded, convex and nonempty subset of a Banach space X. Let A and B be operators such that*

(i) $Ax + By \in M$ whenever $x, y \in M$;
(ii) A is compact and continuous;
(iii) B is a contraction mapping.

Then, there exists $z \in M$ such that $z = Az + Bz$.

Theorem 3. (Arzelà-Ascoli) *Let (X, d) be a compact metric space. A set of functions F in $C(X)$ is relatively compact if and only if it is bounded and equicontinuous.*

3. Existence and uniqueness of solutions

In this section, we derive the existence and uniqueness of solutions of the FBVP (1.1). To that purpose, let us introduce some notation and three important results about the solutions of the FBVP under study (see [25] for related techniques in this context).

Proposition 2. *A function $x \in C^2([a, b])$ is a solution of the boundary value problem* (1.1) *if and only if x satisfies the integral equation*

$$\begin{aligned} x(t) &= \frac{(t-a)^2}{(b-a)^2\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1} \left(p(s)x'(s) + q(s)x(s) - g(s)\right) \mathrm{d}s \\ &\quad - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \left(p(s)x'(s) + q(s)x(s) - g(s)\right) \mathrm{d}s. \end{aligned}$$

Proof. From Proposition 1, we can reduce the equation in the problem (1.1) to the following equivalent integral equation:

$$x(t) = c_0 + c_1(t-a) + c_2(t-a)^2 - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \left(p(s)x'(s) + q(s)x(s) - g(s)\right) \mathrm{d}s.$$

Having in mind the boundary conditions, we conclude that $c_0 = x(a) = 0$ and $c_1 = x'(a) = 0$. Thus, using the condition $x(b) = 0$, one also obtains

$$c_2 = \frac{1}{(b-a)^2\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1} \left(p(s)x'(s) + q(s)x(s) - g(s)\right) \mathrm{d}s.$$

Consequently, we have that

$$\begin{aligned} x(t) &= \frac{(t-a)^2}{(b-a)^2\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1} \left(p(s)x'(s) + q(s)x(s) - g(s)\right) \mathrm{d}s \\ &\quad - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \left(p(s)x'(s) + q(s)x(s) - g(s)\right) \mathrm{d}s \end{aligned}$$

and the proof is complete.

In what follows, we will use the notation

$$\mu := \max_{t\in[a,b]}\{|p(t)|, |q(t)|\}, \qquad \sup_{t\in[a,b]} |g(t)| := \beta, \tag{3.1}$$

$$M_1 := \frac{(b-a)^\alpha}{\Gamma(\alpha+1)} + \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(b-a)^{\alpha-2}}{\Gamma(\alpha-1)}, \tag{3.2}$$

$$M_2 := \frac{(b-a)^\alpha}{\Gamma(\alpha+1)} + 2\frac{(b-a)^{\alpha-1}}{\Gamma(\alpha+1)} + 2\frac{(b-a)^{\alpha-2}}{\Gamma(\alpha+1)}. \tag{3.3}$$

Theorem 4. *If $\mu(M_1 + M_2) < 1$, then the FBVP* (1.1) *has at least one solution in $C^2([a, b])$.*

Proof. From Proposition 2, we know that $x \in C^2([a,b])$ is a solution of the FBVP (1.1) if and only if

$$\begin{aligned} x(t) &= \frac{(t-a)^2}{(b-a)^2\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}\left(p(s)x'(s)+q(s)x(s)-g(s)\right)\mathrm{d}s \\ &\quad -\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\left(p(s)x'(s)+q(s)x(s)-g(s)\right)\mathrm{d}s. \end{aligned}$$

Let us choose a suitable constant R such $R \geq \frac{(M_1+M_2)\beta}{1-(M_1+M_2)\mu}$ and consider the set $B_R = \{x \in C^2([a,b]) : \|x\|_{C^2} \leq R\}$. Then, B_R is a nonempty bounded closed convex subset in $C^2([a,b])$. Now, we will define operators P and Q, on B_R, as follows:

$$\begin{aligned} (Px)(t) &:= -\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\left(p(s)x'(s)+q(s)x(s)-g(s)\right)\mathrm{d}s, \\ (Qx)(t) &:= \frac{(t-a)^2}{(b-a)^2\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}\left(p(s)x'(s)+q(s)x(s)-g(s)\right)\mathrm{d}s, \end{aligned}$$

for each $t \in [a,b]$.

For any $x, y \in B_R$, $t \in [a,b]$, one has

$$\begin{aligned} |(Px)(t)| &\leq \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\left(|p(s)||x'(s)|+|q(s)||x(s)|\right)\mathrm{d}s + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}|g(s)|\mathrm{d}s \\ &\leq \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\mu(|x'(s)|+|x(s)|)\mathrm{d}s + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}|g(s)|\mathrm{d}s \\ &\leq \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\mu(|x'(s)|+|x(s)|+|x''(s)|)\mathrm{d}s + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}|g(s)|\mathrm{d}s \\ &\leq \frac{\mu\|x\|_{C^2}}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\mathrm{d}s + \frac{\beta}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\mathrm{d}s \\ &\leq \frac{(b-a)^\alpha}{\Gamma(\alpha+1)}(\mu R+\beta), \end{aligned}$$

$$\begin{aligned} |(Px)'(t)| &\leq \frac{\alpha-1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-2}\left(|p(s)||x'(s)|+|q(s)||x(s)|\right)\mathrm{d}s + \frac{\alpha-1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-2}|g(s)|\mathrm{d}s \\ &\leq \frac{\alpha-1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-2}\mu(|x'(s)|+|x(s)|)\mathrm{d}s + \frac{\alpha-1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-2}|g(s)|\mathrm{d}s \\ &\leq \frac{\alpha-1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-2}\mu(|x'(s)|+|x(s)|+|x''(s)|)\mathrm{d}s + \frac{\alpha-1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-2}|g(s)|\mathrm{d}s \\ &\leq \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)}(\mu R+\beta), \end{aligned}$$

and

$$
\begin{aligned}
|(Px)''(t)| &\leq \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}\left(\int_a^t (t-s)^{\alpha-3}\left(|p(s)||x'(s)|+|q(s)||x(s)|\right)\mathrm{d}s+\int_a^t (t-s)^{\alpha-3}|g(s)|\mathrm{d}s\right)\\
&\leq \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}\left(\int_a^t (t-s)^{\alpha-3}\mu(|x'(s)|+|x(s)|)\mathrm{d}s+\int_a^t (t-s)^{\alpha-3}|g(s)|\mathrm{d}s\right)\\
&\leq \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}\left(\int_a^t (t-s)^{\alpha-3}\mu(|x'(s)|+|x(s)|+|x''(s)|)\mathrm{d}s+\int_a^t (t-s)^{\alpha-3}|g(s)|\mathrm{d}s\right)\\
&\leq \frac{(b-a)^{\alpha-2}}{\Gamma(\alpha-1)}(\mu R+\beta).
\end{aligned}
$$

Thus, we conclude that

$$
\|Px\|_{C^2} = \sup_{t\in[a,b]}|(Px)(t)| + \sup_{t\in[a,b]}|(Px)'(t)| + \sup_{t\in[a,b]}|(Px)''(t)| \leq M_1(\mu R+\beta).
$$

In the same way, we get

$$
\begin{aligned}
|(Qx)(t)| &\leq \frac{(t-a)^2}{(b-a)^2\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}\left(|p(s)||x'(s)|+|q(s)||x(s)|\right)\mathrm{d}s\\
&\quad+\frac{(t-a)^2}{(b-a)^2\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}|g(s)|\mathrm{d}s\\
&\leq \frac{1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}\mu(|x'(s)|+|x(s)|)\mathrm{d}s+\frac{1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}|g(s)|\mathrm{d}s\\
&\leq \frac{\mu\|x\|_{C^2}}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}\mathrm{d}s+\frac{\beta}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}\mathrm{d}s\\
&\leq \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}(\mu R+\beta),
\end{aligned}
$$

$$
\begin{aligned}
|(Qx)'(t)| &\leq \frac{2|t-a|}{(b-a)^2\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}\left(|p(s)||x'(s)|+|q(s)||x(s)|\right)\mathrm{d}s\\
&\quad+\frac{2|t-a|}{(b-a)^2\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}|g(s)|\mathrm{d}s\\
&\leq \frac{2}{(b-a)\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}\mu(|x'(s)|+|x(s)|)\mathrm{d}s+\frac{2}{(b-a)\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}|g(s)|\mathrm{d}s\\
&\leq \frac{2(b-a)^{\alpha-1}}{\Gamma(\alpha+1)}(\mu R+\beta),
\end{aligned}
$$

and

$$
\begin{aligned}
|(Qx)''(t)| &\leq \frac{2}{(b-a)^2\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}\left(|p(s)||x'(s)|+|q(s)||x(s)|\right)\mathrm{d}s\\
&\quad+\frac{2}{(b-a)^2\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}|g(s)|\mathrm{d}s\\
&\leq \frac{2(b-a)^{\alpha-2}}{\Gamma(\alpha+1)}(\mu R+\beta).
\end{aligned}
$$

Thus, we conclude that

$$\|Qy\|_{C^2} = \sup_{t\in[a,b]} |(Qy)(t)| + \sup_{t\in[a,b]} |(Qy)'(t)| + \sup_{t\in[a,b]} |(Qy)''(t)| \leq M_2(\mu R + \beta).$$

It follows that, for $R \geq \frac{(M_1+M_2)\beta}{1-(M_1+M_2)\mu}$,

$$\|Px + Qy\|_{C^2} \leq \|Px\|_{C^2} + \|Qy\|_{C^2} \leq (M_1 + M_2)(\mu R + \beta) \leq R,$$

and we conclude that $Px + Qy \in B_R$, for $x, y \in B_R$.

Let us show that P is a contraction. For every $x, y \in B_R$, we have

$$\begin{aligned} \|Px - Py\|_{C^2} &= \sup_{t\in[a,b]} |(Px)(t) - (Py)(t)| + \sup_{t\in[a,b]} |(Px)'(t) - (Py)'(t)| + \sup_{t\in[a,b]} |(Px)''(t) - (Py)''(t)| \\ &\leq M_1\mu\|x - y\|_{C^2}. \end{aligned}$$

Since $M_1\mu < 1$, we conclude that P is a contraction.

Since $c\frac{(t-a)^2}{(b-a)^2} \in C^2([a, b])$ for any $c \in \mathbb{R}$, we have that $Qx \in C^2([a, b])$. Moreover, for any bounded subset B_R of $C^2([a, b])$ and $x \in B_R$, we have that

$$\|Qx\|_{C^2} \leq M_2(\mu R + \beta)$$

which shows that the operator Q is uniformly bounded on B_R.

Let us prove that Q is a compact operator on B_R. Take $t_1, t_2 \in [a, b]$ with $t_2 \geq t_1$. One has

$$\begin{aligned} |(Qx)(t_2) - (Qx)(t_1)| &= \left| \frac{(t_2 - a)^2 - (t_1 - a)^2}{(b - a)^2\Gamma(\alpha)} \int_a^b (b - s)^{\alpha-1} \left(p(s)x'(s) + q(s)x(s) - g(s)\right) \mathrm{d}s \right| \\ &\leq \frac{(t_2 - a)^2 - (t_1 - a)^2}{\Gamma(\alpha + 1)}(\mu R + \beta)(b - a)^{\alpha-2}. \end{aligned}$$

It is seen that $|(Qx)(t_2) - (Qx)(t_1)| \to 0$ as $t_2 \to t_1$. Also, we have

$$\begin{aligned} |(Qx)'(t_2) - (Qx)'(t_1)| &= \left| \frac{2(t_2 - a) - 2(t_1 - a)}{(b - a)^2\Gamma(\alpha)} \int_a^b (b - s)^{\alpha-1} \left(p(s)x'(s) + q(s)x(s) - g(s)\right) \mathrm{d}s \right| \\ &\leq \frac{2(t_2 - a) - 2(t_1 - a)}{\Gamma(\alpha + 1)}(\mu R + \beta)(b - a)^{\alpha-2}. \end{aligned}$$

Again, we have that $|(Qx)'(t_2) - (Qx)'(t_1)| \to 0$ as $t_2 \to t_1$. Finally, we observe that

$$|(Qx)''(t_2) - (Qx)''(t_1)| = 0.$$

Thus, we conclude that QB_R is equicontinuous. By Arzelà-Ascoli Theorem, QB_R is compact for each bounded subset $B_R \subset C^2([a, b])$, and thus, Q is compact.

Applying Krasnoselskii's Fixed Point Theorem to the operators P and Q, we conclude that there exists at least one $x \in B_R$ such that $x = Px + Qx$ which is the solution of the FBVP (1.1) and the proof is complete.

Theorem 5. *If the following condition holds*

$$\mu(M_1 + M_2) < 1, \tag{3.4}$$

then the FBVP (1.1) *has a unique solution in* $x \in C^2([a,b])$.

Proof. From Theorem 4, since $\mu(M_1 + M_2) < 1$, the FBVP (1.1) has at least one solution. Let us define the operator $T : C^2([a,b]) \to C^2([a,b])$ by

$$\begin{aligned}(Tx)(t) \;:=\; & \frac{(t-a)^2}{(b-a)^2\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}\left(p(s)x'(s)+q(s)x(s)-g(s)\right)\mathrm{d}s\\ & -\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\left(p(s)x'(s)+q(s)x(s)-g(s)\right)\mathrm{d}s.\end{aligned} \tag{3.5}$$

By the Banach Contraction Principle, we will prove that T has a unique fixed point.

Let $B_R = \{x \in C^2([a,b]) : \|x\|_{C^2} \le R\}$ and choose R such that

$$R \ge \frac{(M_1+M_2)\beta}{1-(M_1+M_2)\mu}.$$

We have

$$\begin{aligned}\|Tx\|_{C^2} \;=\; & \sup_{t\in[a,b]}\left|\frac{(t-a)^2}{(b-a)^2\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}\left(p(s)x'(s)+q(s)x(s)-g(s)\right)\mathrm{d}s\right.\\ & \left.-\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\left(p(s)x'(s)+q(s)x(s)-g(s)\right)\mathrm{d}s\right|\\ & +\sup_{t\in[a,b]}\left|\frac{2(t-a)}{(b-a)^2\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}\left(p(s)x'(s)+q(s)x(s)-g(s)\right)\mathrm{d}s\right.\\ & \left.-\frac{\alpha-1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-2}\left(p(s)x'(s)+q(s)x(s)-g(s)\right)\mathrm{d}s\right|\\ & +\sup_{t\in[a,b]}\left|\frac{2}{(b-a)^2\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}\left(p(s)x'(s)+q(s)x(s)-g(s)\right)\mathrm{d}s\right.\\ & \left.-\frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-3}\left(p(s)x'(s)+q(s)x(s)-g(s)\right)\mathrm{d}s\right|\\ \le\; & (M_1+M_2)(\mu R+\beta).\end{aligned}$$

Thus, $\|Tx\|_{C^2} \le R$, i.e., $TB_R \subset B_R$. Moreover, since p, q, $g \in C^2([a,b])$, we conclude that $Tx \in C^2([a,b])$ for any $x \in C^2([a,b])$, which proves that T maps $C^2([a,b])$ into itself.

Let us prove that T is strictly contractive. Consider $x, y \in C^2([a,b])$. It follows that

$$\begin{aligned}\|Tx-Ty\|_{C^2} \;=\; & \sup_{t\in[a,b]}\left|\frac{(t-a)^2}{(b-a)^2\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}\left(p(s)(x'(s)-y'(s))+q(s)(x(s)-y(s))\right)\mathrm{d}s\right.\\ & \left.-\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\left(p(s)(x'(s)-y'(s))+q(s)(x(s)-y(s))\right)\mathrm{d}s\right|\end{aligned}$$

$$
\begin{aligned}
&+\sup_{t\in[a,b]}\left|\frac{2(t-a)}{(b-a)^2\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}\left(p(s)(x'(s)-y'(s))+q(s)(x(s)-y(s)\right)\mathrm{d}s\right.\\
&\left.-\frac{\alpha-1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-2}\left(p(s)(x'(s)-y'(s))+q(s)(x(s)-y(s))\right)\mathrm{d}s\right|\\
&+\sup_{t\in[a,b]}\left|\frac{2}{(b-a)^2\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}\left(p(s)(x'(s)-y'(s))+q(s)(x(s)-y(s))\right)\mathrm{d}s\right.\\
&\left.-\frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-3}\left(p(s)(x'(s)-y'(s))+q(s)(x(s)-y(s))\right)\mathrm{d}s\right|\\
\leq\ &\frac{1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}\left(|p(s)||x'(s)-y'(s)|+|q(s)||x(s)-y(s)|\right)\mathrm{d}s\\
&+\frac{1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}\left(|p(s)|x'(s)-y'(s)|+|q(s)||x(s)-y(s)|\right)\mathrm{d}s\\
&+\frac{2}{(b-a)\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}\left(|p(s)||x'(s)-y'(s)|+|q(s)||x(s)-y(s)|\right)\mathrm{d}s\\
&+\frac{\alpha-1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-2}\left(|p(s)||x'(s)-y'(s)|+|q(s)||x(s)-y(s)|\right)\mathrm{d}s\\
&+\frac{2}{(b-a)^2\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}\left(|p(s)||x'(s)-y'(s)|+|q(s)||x(s)-y(s)|\right)\mathrm{d}s\\
&+\frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-3}\left(|p(s)||x'(s)-y'(s)|+|q(s)||x(s)-y(s)|\right)\mathrm{d}s\\
\leq\ &\frac{\mu}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}\left(|x(s)-y(s)|+|x'(s)-y'(s)|+|x''(s)-y''(s)|\right)\mathrm{d}s\\
&+\frac{\mu}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}\left(|x(s)-y(s)|+|x'(s)-y'(s)|+|x''(s)-y''(s)|\right)\mathrm{d}s\\
&+\frac{2\mu}{(b-a)\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}\left(|x(s)-y(s)|+|x'(s)-y'(s)|+|x''(s)-y''(s)|\right)\mathrm{d}s\\
&+\frac{\mu(\alpha-1)}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-2}\left(|x(s)-y(s)|+|x'(s)-y'(s)|+|x''(s)-y''(s)|\right)\mathrm{d}s\\
&+\frac{2\mu}{(b-a)^2\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}\left(|x(s)-y(s)|+|x'(s)-y'(s)|+|x''(s)-y''(s)|\right)\mathrm{d}s\\
&+\frac{\mu(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-3}\left(|x(s)-y(s)|+|x'(s)-y'(s)|+|x''(s)-y''(s)|\right)\mathrm{d}s\\
\leq\ &\mu\left(2\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}+2\frac{(b-a)^{\alpha-1}}{\Gamma(\alpha+1)}+\frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)}+2\frac{(b-a)^{\alpha-2}}{\Gamma(\alpha+1)}+\frac{(b-a)^{\alpha-2}}{\Gamma(\alpha-1)}\right)\|x-y\|_{C^2}\\
=\ &\mu(M_1+M_2)\|x-y\|_{C^2}.
\end{aligned}
\tag{3.6}
$$

Since by hypothesis $\mu(M_1 + M_2) < 1$, we conclude that T is strictly contractive.

By Banach Contraction Principle, T has a unique fixed point in $C^2([a, b])$ which is the unique solution of the FBVP (1.1).

4. Ulam-Hyers-Rassias stability analysis

In this section, we analyse the Ulam-Hyers and the Ulam-Hyers-Rassias stabilities of FBVP (1.1). To that purpose, let us first present the definitions of those notions in the sense of our FBVP.

Definition 3. *The FBVP* (1.1) *is Ulam-Hyers stable if there exists a real constant* $k > 0$ *such that, for each* $\epsilon > 0$ *and for each solution* $y \in C^2([a, b])$ *of the inequality problem*

$$\begin{cases} \left|({}^C\mathcal{D}_a^\alpha y)(t) + p(t)y'(t) + q(t)y(t) - g(t)\right| \le \epsilon, \quad t \in [a, b], \\ y'(a) = y(a) = y(b) = 0, \end{cases}$$

there exists a solution $x \in C^2([a, b])$ *of the problem* (1.1) *such that*

$$\|y - x\|_{C^2} \le k\epsilon.$$

Definition 4. *The FBVP* (1.1) *is Ulam-Hyers-Rassias stable with respect to* $\varphi : [a, b] \to \mathbb{R}^+$ *if there exists a real constant* $k_\varphi > 0$ *such that, for each* $\epsilon > 0$ *and for each solution* $y \in C^2([a, b])$ *of the inequality problem*

$$\begin{cases} \left|({}^C\mathcal{D}_a^\alpha y)(t) + p(t)y'(t) + q(t)y(t) - g(t)\right| \le \epsilon\varphi(t), \quad t \in [a, b], \\ y'(a) = y(a) = y(b) = 0, \end{cases}$$

there exists a solution $x \in C^2([a, b])$ *of the problem* (1.1) *with*

$$\|y - x\|_{C^2} \le k_\varphi \epsilon\varphi(t), \quad t \in [a, b].$$

In the next theorem, we present sufficient conditions upon which the FBVP (1.1) is Ulam-Hyers stable.

Theorem 6. *Suppose that* $\mu(M_1 + M_2) < 1$. *Let* $x(t)$ *be the solution of the FBVP* (1.1) *and* $y(t)$ *be such that* $y(a) = y'(a) = y(b) = 0$ *and*

$$\left|({}^C\mathcal{D}_a^\alpha y)(t) + p(t)y'(t) + q(t)y(t) - g(t)\right| \le \epsilon, \quad t \in [a, b], \tag{4.1}$$

where $\epsilon > 0$. *Then, there exists a constant* $k > 0$ *such that*

$$\|y - x\|_{C^2} \le k\epsilon,$$

which means that the FBVP (1.1) *is Ulam-Hyers stable.*

Proof. By Theorems 4 and 5, the solution of the FBVP (1.1) exists and is unique. Let $x(t)$ be that unique solution of the FBVP (1.1) and suppose $y(t)$ satisfies inequality (Eq 4.1). It follows that $y \in C^2([a, b])$ is a solution of inequality (Eq 4.1) if and only if there exists a function $h \in C^2([a, b])$, which depends on y such that

(i) $|h(t)| \le \epsilon$, $t \in [a, b]$, $\epsilon > 0$,
(ii) $h(t) = ({}^C\mathcal{D}_a^\alpha y)(t) + p(t)y'(t) + q(t)y(t) - g(t)$, $t \in [a, b]$,
(iii) $y(a) = y'(a) = y(b) = 0$.

Computing the α-order Riemann-Liouville fractional integral of each member in (ii), according to Proposition 1, we obtain

$$y(t) - y(a) - y'(a)(t-a) - \frac{y''(a)}{2}(t-a)^2 + (I_a^\alpha(py'))(t) + (I_a^\alpha(qy))(t) - (I_a^\alpha(g-h))(t) = 0$$

Since $y(a) = y'(a) = 0$, we have

$$y(t) = d_1(t-a)^2 - \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\left(p(s)y'(s) + q(s)y(s) - g(s) - h(s)\right)\mathrm{d}s$$

where $d_1 = \frac{y''(a)}{2}$.

Moreover, attending that $y(b) = 0$, we have

$$d_1 = \frac{1}{(b-a)^2\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}\left(p(s)y'(s) + q(s)y(s) - g(s) - h(s)\right)\mathrm{d}s$$

and we conclude that

$$\begin{aligned} y(t) &= \frac{(t-a)^2}{(b-a)^2\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}\left(p(s)y'(s) + q(s)y(s) - g(s) - h(s)\right)\mathrm{d}s \\ &\quad -\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\left(p(s)y'(s) + q(s)y(s) - g(s) - h(s)\right)\mathrm{d}s. \end{aligned}$$

Recalling the operator T, defined in (3.5), from (3.6) we already know that under the present conditions T is a contraction and that

$$\|Tx - Ty\|_{C^2} \leq \mu(M_1 + M_2)\|x - y\|_{C^2}.$$

Thus, from Theorem 1, we have

$$\|x - y\|_{C^2} \leq \frac{1}{1 - \mu(M_1 + M_2)}\|Ty - y\|_{C^2}. \tag{4.2}$$

Moreover, we have that

$$\begin{aligned} \|Ty - y\|_{C^2} &= \sup_{t\in[a,b]}|(Ty)(t) - y(t)| + \sup_{t\in[a,b]}|(Ty)'(t) - y'(t)| + \sup_{t\in[a,b]}|(Ty)''(t) - y''(t)| \\ &= \sup_{t\in[a,b]}\left|\frac{(t-a)^2}{(b-a)^2\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}h(s)\mathrm{d}s - \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}h(s)\mathrm{d}s\right| \\ &\quad + \sup_{t\in[a,b]}\left|\frac{2(t-a)}{(b-a)^2\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}h(s)\mathrm{d}s - \frac{\alpha-1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-2}h(s)\mathrm{d}s\right| \\ &\quad + \sup_{t\in[a,b]}\left|\frac{2}{(b-a)^2\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}h(s)\mathrm{d}s - \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-3}h(s)\mathrm{d}s\right| \\ &\leq \frac{1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}|h(s)|\mathrm{d}s + \frac{1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}|h(s)|\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&+\frac{2}{(b-a)\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}|h(s)|\mathrm{d}s+\frac{\alpha-1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-2}|h(s)|\mathrm{d}s\\
&+\frac{2}{(b-a)^2\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}|h(s)|\mathrm{d}s+\frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-3}|h(s)|\mathrm{d}s\\
\leq\;& \epsilon\left(\frac{2(b-a)^{\alpha}}{\Gamma(\alpha+1)}+\frac{2(b-a)^{\alpha-1}}{\Gamma(\alpha+1)}+\frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)}+\frac{2(b-a)^{\alpha-2}}{\Gamma(\alpha+1)}+\frac{(b-a)^{\alpha-1}}{\Gamma(\alpha-1)}\right)\\
=\;& (M_1+M_2)\epsilon.
\end{aligned}
$$

Therefore, taking also (4.2) into account, we obtain

$$
\|x-y\|_{C^2}\leq\frac{M_1+M_2}{1-\mu(M_1+M_2)}\epsilon
$$

and we conclude that the FBVP (1.1) is Ulam-Hyers stable.

In the next theorem, we present sufficient conditions for the FBVP (1.1) to be Ulam-Hyers-Rassias stable.

Theorem 7. *Assume that* $\mu(M_1+M_2)<1$. *Let* $x(t)$ *be the solution of the FBVP* (1.1) *and* $y(t)$ *be such that* $y(a)=y'(a)=y(b)=0$ *and*

$$
\left|({}^{C}\mathcal{D}_a^{\alpha}y)(t)+p(t)y'(t)+q(t)y(t)-g(t)\right|\leq\epsilon\varphi(t),\quad t\in[a,b] \tag{4.3}
$$

where $\epsilon>0$ *and* $\varphi:[a,b]\to\mathbb{R}^+$ *satisfies the property*

$$
(I_a^{\tau}\varphi)(b)\leq\varphi(t),\;\; t\in[a,b],\;\; \tau=\alpha,\alpha-1,\alpha-2,\;\; \alpha\in(2,3). \tag{4.4}
$$

Then, there exists a constant $k_{\varphi}>0$ *such that*

$$
\|y-x\|_{C^2}\leq k_{\varphi}\epsilon\varphi(t),\quad t\in[a,b],
$$

which means that the FBVP (1.1) *is Ulam-Hyers-Rassias stable.*

Proof. By Theorems 4 and 5, the solution of the FBVP (1.1) exists and is unique. Let $x(t)$ be the unique solution of the FBVP (1.1) and suppose that $y(t)$ satisfies inequality (Eq 4.3). It follows that $y\in C^2([a,b])$ is a solution of inequality (Eq 4.3) if and only if there exists a function $f\in C^2([a,b])$ depending on y and such that

(i) $|f(t)|\leq\epsilon\varphi(t)$, $t\in[a,b]$, $\epsilon>0$,
(ii) $f(t)=({}^{C}\mathcal{D}_a^{\alpha}y)(t)+p(t)y'(t)+q(t)y(t)-g(t)$, $t\in[a,b]$,
(iii) $y(a)=y'(a)=y(b)=0$.

Using (ii), we can proceed similarly as in the proof of the previous theorem and obtain

$$
\begin{aligned}
y(t)\;=\;&\frac{(t-a)^2}{(b-a)^2\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}\left(p(s)y'(s)+q(s)y(s)-g(s)-f(s)\right)\mathrm{d}s\\
&-\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\left(p(s)y'(s)+q(s)y(s)-g(s)-f(s)\right)\mathrm{d}s.
\end{aligned}
$$

Recalling the operator T, defined in (3.5), having into account condition (4.4), we have

$$
\begin{aligned}
\|Ty-y\|_{C^2} &= \sup_{t\in[a,b]} |(Ty)(t)-y(t)| + \sup_{t\in[a,b]} |(Ty)'(t)-y'(t)| + \sup_{t\in[a,b]} |(Ty)''(t)-y''(t)| \\
&= \sup_{t\in[a,b]} \left| \frac{(t-a)^2}{(b-a)^2\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1} f(s)\mathrm{d}s - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\mathrm{d}s \right| \\
&\quad + \sup_{t\in[a,b]} \left| \frac{2(t-a)}{(b-a)^2\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1} f(s)\mathrm{d}s - \frac{\alpha-1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-2} f(s)\mathrm{d}s \right| \\
&\quad + \sup_{t\in[a,b]} \left| \frac{2}{(b-a)^2\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1} f(s)\mathrm{d}s - \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-3} f(s)\mathrm{d}s \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1} |f(s)|\mathrm{d}s + \frac{1}{\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1} |f(s)|\mathrm{d}s \\
&\quad + \frac{2}{(b-a)\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1} |f(s)|\mathrm{d}s + \frac{\alpha-1}{\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-2} |f(s)|\mathrm{d}s \\
&\quad + \frac{2}{(b-a)^2\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1} |f(s)|\mathrm{d}s + \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-3} |f(s)|\mathrm{d}s \\
&\leq \frac{1}{\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1} \epsilon\varphi(s)\mathrm{d}s + \frac{1}{\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1} \epsilon\varphi(s)\mathrm{d}s \\
&\quad + \frac{2}{(b-a)\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1} \epsilon\varphi(s)\mathrm{d}s + \frac{\alpha-1}{\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-2} \epsilon\varphi(s)\mathrm{d}s \\
&\quad + \frac{2}{(b-a)^2\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1} \epsilon\varphi(s)\mathrm{d}s + \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-3} \epsilon\varphi(s)\mathrm{d}s \\
&\leq \epsilon\left(\varphi(t) + \varphi(t) + \frac{2}{b-a}\varphi(t) + \varphi(t) + \frac{2}{(b-a)^2}\varphi(t) + \varphi(t)\right),\ t\in[a,b] \\
&\leq \frac{4(b-a)^2 + 2(b-a) + 2}{(b-a)^2} \epsilon\varphi(t),\ t\in[a,b].
\end{aligned}
$$

From the proof of Theorem 5 (cf. (3.6)), we have that the operator T is a contraction with

$$\|Tx - Ty\|_{C^2} \leq \mu(M_1 + M_2)\|x - y\|_{C^2}.$$

Thus, using Banach Contraction Principle (Theorem 1), we obtain that

$$\|x-y\|_{C^2} \leq \frac{\frac{4(b-a)^2+2(b-a)+2}{(b-a)^2}}{1-\mu(M_1+M_2)} \epsilon\varphi(t),\ t\in[a,b].$$

Taking

$$k_\varphi = \frac{\frac{4(b-a)^2+2(b-a)+2}{(b-a)^2}}{1-\mu(M_1+M_2)},$$

we have $\|x-y\|_{C^2} \leq k_\varphi \epsilon\varphi(t)$, $t\in[a,b]$, and so we conclude that the FBVP (1.1) is Ulam-Hyers-Rassias stable.

5. Examples

Consider the following FBVP:

$$\begin{cases} ({}^C\mathcal{D}_0^{\frac{5}{2}}x)(t) + \frac{1}{5}\cos(t)x'(t) + \frac{1}{6}\sin(t)x(t) = t^2, & t \in \left[0, \frac{3}{4}\right] \\ x(0) = x'(0) = x\left(\frac{3}{4}\right) = 0 \end{cases} . \tag{5.1}$$

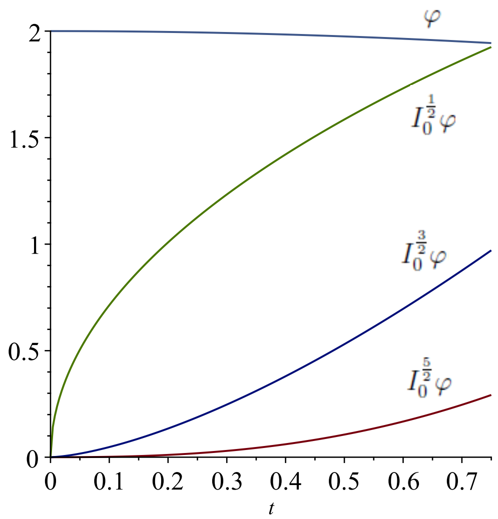

Figure 1. The graphs of $\varphi(t)$, $I_0^{\frac{1}{2}}\varphi(t)$, $I_0^{\frac{3}{2}}\varphi(t)$, $I_0^{\frac{5}{2}}\varphi(t)$, $t \in \left[0, \frac{3}{4}\right]$.

In the notation of (1.1), we have in here $p(t) = \frac{1}{5}\cos(t)$, $q(t) = \frac{1}{6}\sin(t)$, $g(t) = t^2 \in C^2\left(\left[0, \frac{3}{4}\right]\right)$ and $\alpha = \frac{5}{2}$. Moreover, considering the notations (3.1)–(3.3), we realize that

$$\mu = \frac{1}{5}, \quad M_1 < \frac{58}{15\sqrt{\pi}}, \quad M_2 < \frac{8}{3\sqrt{\pi}}.$$

Thus,

$$\mu(M_1 + M_2) < \frac{98}{75\sqrt{\pi}} < 1$$

and we conclude that the FBVP (5.1) has a unique solution and it is Ulam-Hyers stable.

Consider now $\varphi(t) = -0,1t^2 + 2$. For any $t \in \left[0, \frac{3}{4}\right]$, one has

$$I_0^{\frac{5}{2}}\varphi(t) < \varphi(t), \quad I_0^{\frac{3}{2}}\varphi(t) < \varphi(t), \quad I_0^{\frac{1}{2}}\varphi(t) < \varphi(t), \quad t \in \left[0, \frac{3}{4}\right]$$

(see Figure 1).

Therefore, from Theorem 7, we conclude that the FBVP (5.1) is Ulam-Hyers-Rassias stable with respect to φ.

6. Conclusions

Fractional calculus has gained considerable popularity and importance during the last few decades, mainly due to its attractive applications in various areas of science and engineering. In particular, fractional boundary value problems have been used in the fields of physics, biology, chemistry, economics, electromagnetic theory, image and signal processing. In fact, boundary problems involving fractional differential equations model certain situations – such as the study of heredity and memory problems – better than integer-order differential equations. Given the difficulty in obtaining exact explicit solutions for such problems, it becomes important to study their eventual different types of stability, in particular, the Ulam-Hyers and Ulam-Hyers-Rassias stabilities.

In this article, we analyzed a class of fractional boundary value problems involving Caputo's fractional derivative as well as the usual (integer) derivative. Using several *Functional Analysis* techniques (including, for example, Krasnoselskii's Fixed Point Theorem), we obtained sufficient conditions to guarantee the existence of solutions to this class of problems and we also obtained conditions for the uniqueness of these solutions. Finally, we establish – in the form of sufficient conditions – the Ulam-Hyers and Ulam-Hyers-Rassias stabilities. At the end, a concrete example was given to illustrate the obtained theoretical results.

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Conflict of interest

The authors declare there is no conflict of interest.

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