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Research article

The existence and nonexistence of positive solutions for a singular Kirchhoff equation with convection term

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Abstract: This paper considers a singular Kirchhoff equation with convection and a parameter. By defining new sub-supersolutions, we prove a new sub-supersolution theorem. Combining method of sub-supersolution with the comparison principle, for Kirchhoff equation with convection, we get the conclusion about positive solutions when nonlinear term is singular and sign-changing.

Keywords: a singular Kirchhoff equation; nonlinear term; positive solution; sub-supersolution; the comparison principle

1. Introduction

In this work, we study

$$\begin{cases} -a\left(\int_\Omega |\nabla u(x)|^2 dx\right)\triangle u(x) & \\ = \lambda f(x,u) + K(x)g(u) - |\nabla u|^\eta, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

where Ω is a smooth and bounded domain in $\mathbb{R}^N$ $(N \geq 2)$, $a : [0. +\infty) \to (0, +\infty)$ is continuous and increasing with

$$\inf_{t\in[0,+\infty)} a(t) = a(0) = a_0 > 0, \text{ and } \lim_{t\to+\infty} a(t) = +\infty,$$

$K \in C^{0,\gamma}(\overline{\Omega})$, $\lambda > 0$, $0 \leq \eta < 2$.

This work is motivated by [1] where Ghergu and Rădulescu considered

$$\begin{cases} -\triangle u(x) = K(x)g(u) + \lambda f(x,u) - |\nabla u|^a, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

They obtained the existence or nonexistence of solutions. Many other works on the solutions for equations can be found in [2–8] also.

For the case that the nonlinearity is independent on ∇u, many researchers made extensively research in equations of this type, see [9–21] and their references.

But since $h(x,u,\nabla u) = \lambda f(x,u) + K(x)g(u) - |\nabla u|^\eta$ in problem (1.1) depends on gradient, variational methods can not be used to study problem (1.1) in a direct way. According to the works in [1], it is natural to try to use the sub-supersolution approach to study the problem (1.1).

A difficulty is that there is no ready-made sub-supersolution approach for (1.1) although there are some results on the methods of sub-supersolutions for problem (1.1) when nonlinearity h is independent of ∇u or is continuous on $u = 0$, see [22–24].

Our paper will prove the sub-supersolutions theorem for a generalized (2.1) and use the obtained theorem to consider (1.1).

Suppose that the function $f : \overline{\Omega} \times [0,\infty) \to [0,\infty)$ is Hölder continuous, and $f > 0$ on $\overline{\Omega} \times (0,\infty)$. And f satisfies:

$(f1)$ the mapping $s \mapsto \dfrac{f(x,s)}{s}$ with $s \in (0,\infty)$ is decreasing, $\forall x \in \overline{\Omega}$;

$(f2)$ $\lim\limits_{s\to 0} \dfrac{f(x,s)}{s} = +\infty$ and $\lim\limits_{s\to+\infty} \dfrac{f(x,s)}{s} = 0$, uniformly for $x \in \overline{\Omega}$.

$g \in C^{0,\gamma}(0,\infty)$, $g \geq 0$, and decreasing function satisfying

$(g1)$ $\lim\limits_{s\to 0} g(s) = +\infty$;

$(g2)$ $\displaystyle\int_0^1 g(s)ds < +\infty$;

$(g3)$ there are $\alpha \in (0,1)$ and $\theta_0 > 0$, $C > 0$ making $g(s) \leq Cs^{-\alpha}$, $\forall s \in (0,\theta_0)$.

Theorem 1.1. *If* $K(x) > 0$ *in* $\overline{\Omega}$, f *meets* $(f1)-(f2)$, g *meets* $(g1)-(g2)-(g3)$, (1.1) *has at least one solution for all* $\lambda > 0$.

Theorem 1.2. *If* $K(x) < 0$ *in* $\overline{\Omega}$, f *meets* $(f1)-(f2)$, g *meets* $(g1)-(g2)-(g3)$, *there exists* $\lambda^* > 0$ *making* (1.1) *has at least one solution when* $\lambda \geq \lambda^*$, *and there exist* $\lambda_0 > 0$ *enough small such that* (1.1) *has no solution.*

Theorem 1.3. *If* $K(x) < 0$ *in* $\overline{\Omega}$, f *meets* $(f1)-(f2)$, (1.1) *has no solution, if* $\displaystyle\int_0^1 g(s)ds = +\infty$.

This work is organised as follows. In section 2, we give some lemmas and obtain a sub-supersolution theorem for some singular Kirchhoff equation with convection (2.1). In Section 3, we proof the results. Some ideas like [1,22,25–29].

2. The sub-supersolutions approach for problem (2.1)

This section, we discuss

$$\begin{cases} -\triangle u(x) = \dfrac{1}{a(\|u\|^2)} f(x, u(x), \nabla u(x)), \quad x \in \Omega, \\ \quad u|_{\partial\Omega} = 0, \end{cases} \tag{2.1}$$

where $f(x, u, \xi)$ satisfies two conditions:

(F_1) $f(x, u, \xi)$ is continuously differentiable relative to the variables u and ξ and locally Hölder continuous in $\Omega \times (0, +\infty) \times \mathbb{R}^n$;

(F_2) there are $\theta \in (0, 1)$ and $\eta \in [0, 2)$ making there is a corresponding constant $C = C(\Omega; b) > 0$, $\forall b > 0$, such that

$$|f(x; u; \xi)| \leq Cu^{-\theta}\left[1 + |\xi|^{\eta}\right], \forall (x, u, \xi) \in \Omega \times (0, b] \times \mathbb{R}^N.$$

Now consider

$$\begin{cases} |\triangle u| \leq \dfrac{1}{a(0)} |f(x, u, \nabla u)|, x \in \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{2.2}$$

Set

$$\Sigma_R = \left\{ u \in C^2(\Omega) \cap C_0^1(\overline{\Omega}) \text{ satisfies problem (2.2)}, u > 0 | \max_{x \in \overline{\Omega}} |u(x)| \leq R \right\}.$$

Obviously, $0 \in \Sigma_R$ and then Σ_R is not empty for any $R > 0$. For the functions in Σ_R, we have following lemma.

Lemma 2.1. $\forall R > 0$*, there is* $k_0 > 0$ *making*

$$\left(\int_\Omega |\nabla u(x)|^2 dx \right)^{1/2} \leq k_0$$

for all $u \in \Sigma_R$*.*

Proof. Suppose $u \in \Sigma_R$. Multiplying u in both side in (2.2) and integrating on Ω, using Young inequality,

$$\begin{aligned} a(0) \int_\Omega |\nabla u(x)|^2 dx &\leq \int_\Omega |f(y, u(y), \nabla u(y))|\, u(y) dy \\ &\leq C \int_\Omega (u^{1-\theta}(y)) \left[1 + |\nabla u(y)|^{\eta}\right] dy \\ &\leq CR^{1-\theta} \left[|\Omega| + \int_\Omega |\nabla u(y)|^{\eta} dy \right] \\ &\leq CR^{1-\theta} \left[|\Omega| + C_1 + \varepsilon \int_\Omega |\nabla u(y)|^2 dy \right]. \end{aligned}$$

Therefore, there is a $k_0 > 0$ such that

$$\|u\| \leq k_0.$$

The proof is completed. □

Let

$$f^+(x, u, \xi) = \max\{f(x, u, \xi), 0\}$$

and

$$f^{-}(x,u,\xi) = \max\{-f(x,u,\xi),0\}.$$

Then

$$f(x,u,\xi) = f^{+}(x,u,\xi) - f^{-}(x,u,\xi).$$

In the following, we define the supersolution of (2.1) and the corresponding sub-solution.

Definition 2.2. *If the positive function $\overline{u}$ with $\overline{u} \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies*

$$\begin{cases} -\triangle \overline{u}(x) \geq \dfrac{1}{a(0)} f^{+}(x, \overline{u}(x), \nabla \overline{u}(x)), \quad x \text{ in } \Omega, \\ \overline{u}|_{\partial\Omega} = 0, \end{cases}$$

$\overline{u}(x)$ is a upper solution of (2.1).

Suppose $\overline{u}$ is a positive supersolution of (2.1). Since the condition $(F2)$ hold, form Lemma 2.1, for $R = \sup_{x\in\Omega} \overline{u}(x)$, there is $k_0 > 0$ making

$$\|u\| = \sqrt{\left(\int_{\Omega} |\nabla u(x)|^2 dx\right)} \leq k_0$$

for all $u \in \Sigma_R$.

Definition 2.3. *If the positive function $\underline{u}$ with $\underline{u} \in C^{2+\alpha}(\Omega) \cap C^1(\overline{\Omega})$ satisfies $\underline{u}(x) \leq \overline{u}(x)$, $\forall x \in \Omega$ and*

$$\begin{cases} -\triangle \underline{u}(x) \leq & \dfrac{1}{a(k_0^2)} f^{+}(x, \underline{u}(x), \nabla \underline{u}(x)) \\ & - \dfrac{1}{a(0)} f^{-}(x, \underline{u}(x), \nabla \underline{u}(x)), \quad x \text{ in } \Omega, \\ \underline{u}|_{\partial\Omega} = & 0, \end{cases}$$

$\underline{u}(x) > 0$ is a subsolution of (2.1) *corresponding with the supersolution $\overline{u}(x)$.*

Let

$$C^1(\overline{\Omega}) = \{u : \overline{\Omega} \to \mathbb{R} : u(x) \text{ is continuously differentiable on } \overline{\Omega}\}$$

with norm

$$\|u\|_1 = \max\left\{\max_{x\in\overline{\Omega}} |u(x)|, \max_{x\in\overline{\Omega}} |\nabla u(x)|\right\}.$$

Note that $C^1(\overline{\Omega})$ is a Banach space.

We list lemma which will be used later.

Lemma 2.4. *(see [30]) Let $u \in W^{2,p}(\Omega)$ satisfy*

$$|\triangle u(x)| \leq f_0 + K|\nabla u|^2$$

with $u|_{\partial\Omega} = 0$, $|u|_{\infty,\Omega} \leq M \in (0, +\infty)$ and $f_0 \in L^p(\Omega)$. Then there is $k' > 0$, depending u only through M such that

$$|u|_{W^{2,p}(\Omega)} \leq k'.$$

Remark 2.5. *In the above lemma, if* $u|_{\partial\Omega} = \phi(x)$ *with* $\phi \in C^{2+\alpha}(\partial\Omega)$*, we get same conclusion.*

Theorem 2.6. *Set* $\Omega \subseteq \mathbb{R}^N (N \geq 1)$ *be a smooth bounded domain. If* (F_1) *and* (F_2) *hold. Assume* $\overline{u} > 0$ *is a upper solution of* (2.1) *and* $\underline{u} > 0$ *is a lower solution of* (2.1) *corresponding with the supersolution* $\overline{u}$*. Moreover, if there is* $\delta_0 > 0$ *making* $\underline{u}(x) \geq \delta_0 d(x, \partial\Omega)^\gamma$ *with* $0 < \gamma\theta < 1$*. Then* (2.1) *has at least one solution* $u \in C^2(\Omega) \cap C^{1,1-\gamma\theta}(\overline{\Omega})$*,*

$$\underline{u}(x) \leq u(x) \leq \overline{u}(x),$$

$\forall x \in \overline{\Omega}$.

In order to obtain Theorem 2.6, make a sequence of subdomains of Ω with $C^{2+\alpha}$-boundaries, named $\{\Omega_k\}_{k=1}^{\infty}$ such that

$$\Omega_1 \subset\subset \Omega_2 \subset\subset \cdots \subset\subset \Omega_k \subset\subset \Omega_{k+1} \subset\subset \cdots$$

with $\cup_{k=1}^{\infty} \Omega_k = \Omega$. For each k, consider

$$\begin{cases} -\triangle u(x) = \dfrac{1}{a\left(\min\left\{k_0^2, \int_{\Omega_k} |\nabla u(x)|^2 dx\right\}\right)} \\ \qquad\qquad f(x, u(x), \nabla u(x)), \quad x \in \Omega_k, \\ u|_{\partial\Omega_k} = \underline{u}(x) > 0. \end{cases} \tag{2.3}$$

Lemma 2.7. *For each* $k > 0$*,* (2.3) *has a solution* $u_k \in C^1(\overline{\Omega}_k)$ *making*

$$\underline{u}(x) \leq u_k(x) \leq \overline{u}(x), \quad x \in \overline{\Omega}_k.$$

Proof. If u is a solution of problem (2.3) with $\underline{u}(x) \leq u(x) \leq \overline{u}(x)$ on $\overline{\Omega}_k$, we have

$$|\triangle u| \leq C\underline{u}^{-\theta\gamma}(x) \left[1 + |\nabla u|^\eta\right],$$

which together Lemma 5.10 in [30] and the interpolation inequality lemma in [30] infers there is $R_k > 0$ such that

$$\|u\|_1 < R_k.$$

Define $\overline{f} : \overline{\Omega} \times (0, +\infty) \times \mathbb{R}$ as

$$\overline{f}(x, u, \xi) = \begin{cases} f(x, u, \xi), & \text{if } \underline{u}(x) \leq u \leq \overline{u}(x), \\ f(x, \underline{u}(x), \xi) + h_1(x), & \text{if } u < \underline{u}(x), \\ f(x, \overline{u}, \xi) - h_2(x), & \text{if } u > \overline{u}(x), \end{cases}$$

where

$$\begin{cases} h_1(x) = \frac{1}{\underline{u}(x)} \left[|f(x, \underline{u}(x), 0)| + 1\right] \min\left\{\underline{u}(x), \underline{u}(x) - u\right\}, \\ h_2(x) = \frac{1}{\overline{u}(x)} \left[|f(x, \overline{u}(x), 0)| + 1\right] \min\left\{\overline{u}(x), u - \overline{u}(x)\right\}. \end{cases} \tag{2.4}$$

Now consider

$$\begin{cases} -\triangle u(x) = \dfrac{\overline{f}(x, u(x), \nabla u(x))}{a\left(\min\left\{k_0^2, \int_{\Omega_k} |\nabla u(x)|^2 dx\right\}\right)}, x \in \Omega_k, \\ u|_{\partial\Omega_k} = \underline{u}(x). \end{cases} \tag{2.5}$$

First, we prove that the solution of (2.5) is the solution of (2.3).

If u is a solution of (2.5), we will prove that $\underline{u}(x) \le u(x) \le \overline{u}(x)$, $x \in \Omega_k$.

In fact, if there is a $x_0 \in \Omega_k$ with $u(x_0) < \underline{u}(x_0)$, let $A = \{x \in \Omega_k | u(x) < \underline{u}(x)$, there exists a continuous line $\phi : [0,1] \to \Omega_k$, $\phi(0) = x_0$, $\phi(1) = x$ and $u(\phi(t)) < \underline{u}(\phi(t))$ for all $t \in [0,1]\}$. Obviously, $u(x) < \underline{u}(x)$ for all $x \in A$ and $u(x) = \underline{u}(x)$, $\forall x \in \partial A$(note $u(x) = \underline{u}(x)$ for all $x \in \partial\Omega_k$). Now there exists a $x_1 \in A$ such that $u(x_1) - \underline{u}(x_1) = \min_{x\in\overline{A}}(u(x) - \underline{u}(x))$ making $\nabla u(x_1) = \nabla\underline{u}(x_1)$ and

$$
\begin{aligned}
0 &\ge -\triangle\left(u(x_1) - \underline{u}(x_1)\right)\\
&\ge \frac{1}{a\left(\min\left\{k_0^2, \int_{\Omega_k} |\nabla u(x)|^2 dx\right\}\right)}\overline{f}\,(x_1, u(x_1), \nabla u(x_1))\\
&\quad -\frac{1}{a\left(k_0^2\right)} f^+\left(x_1, \underline{u}(x_1), \nabla\underline{u}(x_1)\right) + \frac{1}{a(0)} f^-\left(x_1, \underline{u}(x_1), \nabla\underline{u}(x_1)\right)\\
&= \frac{1}{a\left(\min\left\{k_0^2, \int_{\Omega_k} |\nabla u(x)|^2 dx\right\}\right)} f\left(x_1, \underline{u}(x_1), \nabla u(x_1)\right)\\
&\quad -\frac{1}{a\left(k_0^2\right)} f^+\left(x_1, \underline{u}(x_1), \nabla\underline{u}(x_1)\right) + \frac{1}{a(0)} f^-\left(x_1, \underline{u}(x_1), \nabla\underline{u}(x_1)\right)\\
&\quad +\frac{1}{a\left(\min\left\{k_0^2, \int_{\Omega_k} |\nabla u(x)|^2 dx\right\}\right)} h_1(x_1)\\
&\ge \frac{1}{a\left(k_0^2\right)}\overline{f}^+\left(x_1, \underline{u}(x_1), \nabla\underline{u}(x_1)\right) - \frac{1}{a(0)}\overline{f}^-\left(x_1, \underline{u}(x_1), \nabla\underline{u}(x_1)\right)\\
&\quad -\frac{1}{a\left(k_0^2\right)} f^+\left(x_1, \underline{u}(x_1), \nabla\underline{u}(x_1)\right) + \frac{1}{a(0)} f^-\left(x_1, \underline{u}(x_1), \nabla\underline{u}(x_1)\right)\\
&\quad +\frac{1}{a\left(\min\left\{k_0^2, \int_{\Omega_k} |\nabla u(x)|^2 dx\right\}\right)} h_1(x_1)\\
&= \frac{1}{a\left(\min\left\{k_0^2, \int_{\Omega_k} |\nabla u(x)|^2 dx\right\}\right)} h_1(x_1)\\
&> 0,
\end{aligned}
$$

where h_1 is defined in (2.4). This is contradictory. Thus, $0 < \underline{u}(x) \le u(x)$, $\forall x \in \Omega_k$.

On the other hand, if there is a $x_0 \in \Omega_k$ with $u(x_0) > \overline{u}(x_0)$, let $B = \{x \in \Omega_k | u(x) > \overline{u}(x)$, there is a continuous line $\psi : [0,1] \to \Omega_k$ such that $\psi(0) = x_0$, $\psi(1) = x$ and $u(\psi(t)) > \overline{u}(\psi(t))$, $\forall t \in [0,1]\}$. Obviously, $u(x) > \overline{u}(x)$, $\forall x \in B$ and $u(x) = \overline{u}(x)$, $\forall x \in \partial B$(note $u(x) = \underline{u}(x) \le \overline{\Omega}_k$, $\forall x \in \partial\Omega_k$). Then there is $x_2 \in B$ making $u(x_2) - \overline{u}(x_2) = \max_{x\in\overline{B}}(u(x) - \overline{u}(x))$ such that $\nabla u(x_2) = \nabla\overline{u}(x_2)$ and

$$
\begin{aligned}
0 &\le -\triangle\left(u(x_2) - \overline{u}(x_2)\right)\\
&\le \frac{1}{a\left(\min\left\{k_0^2, \int_{\Omega_k} |\nabla u(x)|^2 dx\right\}\right)}\overline{f}\,(x_2, u(x_2), \nabla u(x_2))\\
&\quad -\frac{1}{a(0)} f^+\left(x_2, \underline{u}(x_2), \nabla\underline{u}(x_2)\right)\\
&= \frac{1}{a\left(\min\left\{k_0^2, \int_{\Omega_k} |\nabla u(x)|^2 dx\right\}\right)} f\,(x_2, \overline{u}(x_2), \nabla\overline{u}(x_2))\\
&\quad -\frac{1}{a(0)} f^+\,(x_2, \overline{u}(x_2), \nabla\overline{u}(x_2))\\
&\qquad -\frac{1}{a\left(\min\left\{k_0^2, \int_{\Omega_k} |\nabla u(x)|^2 dx\right\}\right)} h_2(x_2)\\
&\le \frac{1}{a(0)} f^+\,(x_2, \overline{u}(x_2), \nabla\overline{u}(x_2)) - \frac{1}{a\left(k_0^2\right)} f^-\,(x_2, \overline{u}(x_2), \nabla\overline{u}(x_2))\\
&\quad -\frac{1}{a(0)} f^+\,(x, \overline{u}(x), \nabla\overline{u}(x)) - \frac{1}{a\left(\min\left\{k_0^2, \int_{\Omega_k} |\nabla u(x)|^2 dx\right\}\right)} h_2(x_2)\\
&\le -\frac{1}{a\left(\min\{k_0^2, \int_{\Omega_k} |\nabla u(x)|^2 dx\}\right)} h_2(x_2)\\
&< 0,
\end{aligned}
$$

where h_2 is defined in (2.4). This is contradictory.

Therefore, $0 < \underline{u}(x) \leq u(x) \leq \overline{u}(x)$ for all $x \in \Omega_k$, which implies that u satisfies problem (2.3).

Second, we show that (2.5) has at least one positive solution.

For $u \in C^1(\overline{\Omega}_k)$, define

$$(A_k u)(x) = \frac{1}{a\left(\min\left\{k_0^2, \int_{\Omega_k} |\nabla u(x)|^2 dx\right\}\right)} \cdot \int_{\Omega_k} G_k(x,y)\overline{f}(y, u(y), \nabla u(y))dy, \quad x \in \overline{\Omega}_k,$$

where $G_k(x, y)$ is the Green's function of $-\triangle u(x) = h(x)$, $u_{\partial\Omega_k} = \underline{u}(x)$.

Let

$$E = \left\{u \in C^1(\overline{\Omega}_k) | u = \lambda A_k u, \lambda \in [0, 1]\right\}.$$

By condition (F_2) and (2.4),

$$|\triangle u(x)| \leq \frac{C}{a(0)} \underline{u}^{-\gamma\theta} \left(1 + |\nabla u(x)|^2\right) + h_1(x) + h_2(x),$$

which together with the remark of Lemma 2.4 and the embedding theorem guarantees there is a $C_1 > 0$ such that

$$\|u\|_1 \leq C_1.$$

By Leray-Schauder's fixed point theorem, we have A_k has at least one fixed point u_k in $C^1(\overline{\Omega}_k)$.

Consequently, (2.3) has a solution $u_k > 0$ on Ω_k with $\underline{u}(x) \leq u_k(x) \leq \overline{u}(x)$.

□

Now by the definitions of $\overline{f}$, for each $k \geq 1$, from Theorem 6.2 in [15], we conclude that there is a solution $u_k(x)$ to (2.3) such that

(a) $u_k(x) \in C^{2+\alpha}(\Omega_k) \cap C^2(\overline{\Omega}_k)$;

(b) $\underline{u}(x) \leq u_k(x) \leq \overline{u}(x)$, $x \in \overline{\Omega}_k$.

We extend $u_k(x)$ to the whole domain such that $u_k(x) = \underline{u}(x)$, $\forall x \in \overline{\Omega} \setminus \overline{\Omega}_k$. Then $u_k(x) \in C(\overline{\Omega})$. In this way, we get a sequence of continuous functions $\{u_k(x)\}_{k=1}^{\infty}$ possessing obviously the following properties:

(a) $\underline{u}(x) \leq u_k(x) \leq \overline{u}(x)$, $x \in \overline{\Omega}$;

(b) $-\triangle u_k(x) = \frac{1}{a\left(\min\left\{k^2, \int_{\Omega_k} |\nabla u_k(x)|^2\right\}\right)} f(x, u_k(x), \nabla u_k(x))$, $x \in \Omega_k$ for every $k = 1, 2, \cdots$.

Now we prove the following lemma.

Lemma 2.8. *For each $k = 1, 2, \cdots$, there exists a corresponding constant $C_k > 0$ such that*

$$\|u_j\|_{C^{2+\alpha}(\overline{\Omega}_k)} \leq C_k, \quad \textit{for all } j \geq k + 1. \tag{2.6}$$

Proof. Let k be fixed and take two domains Q_1 and Q_2 such that

$$\Omega_k \subset\subset Q_1 \subset\subset Q_2 \subset\subset \Omega_{k+1}.$$

Then for any $j \geq k+1$ we have

$$-\triangle u_j = \frac{1}{a\left(\min\left\{k_0^2, \int_{\Omega_j} |\nabla u_j|^2 dx\right\}\right)} f(x, u_j(x), \nabla u_j(x)), \quad \text{on } \Omega_{k+1}. \tag{2.7}$$

Denote

$$\overline{f}_j(x) = \frac{1}{a\left(\min\left\{k_0^2, \int_{\Omega_j} |\nabla u_j|^2 dx\right\}\right)} \overline{f}(x, u_j(x), \nabla u_j(x))$$

$(j = k+1, k+2, \cdots)$. Now (2.7) can be rewritten as

$$-\triangle u_j(x) = f_j(x), \text{ on } \Omega_{k+1}. \tag{2.8}$$

First, since $\underline{u}(x) \leq u_j(x) \leq \overline{u}(x)$ on Ω_{k+1} for all $j \geq k+1$, we see that $u_j(x)$ $(j = k+1, k+2, \cdots)$ are uniformly bounded on Ω_{k+1}.

Second, using gradient estimate theorem of Ladyzenskaya and Uraltreva (see [[31], Theorem 3.1]), we know from (2.7) a constant C_1 independent of j such that for any $j \geq k+1$,

$$\max_{x\in Q_2} |\nabla u_j(x)| \leq C_1 \max_{x\in\Omega_{k+1}} u_j(x) \leq C_1 \max_{x\in\overline{\Omega}} \overline{u}(x),$$

which implies that $\nabla u_j(x)$ $(j = k+1, k+2, \cdots)$ are uniformly bounded on Q_2. Therefore, the functions $\overline{f}_j(x)$ $(j = k+1, k+2, \cdots)$ are uniformly bounded on Q_2.

Third, by the interior L^p estimate theorem, we conclude from (2.8) that for any $p > \max\{1, N\}$, there is a corresponding constant C_2 independent of j making for any $j \geq k+1$,

$$\begin{aligned}&||u_j||_{W^{2,p}(Q_1)}\\ &\leq C_2\left(||\overline{f}_j||_{L^p(Q_2)} + ||u_j||_{L^p(Q_2)}\right)\\ &\leq C_2|Q_2|^{\frac{1}{p}}\left(\max_{x\in\overline{Q}_2}\left|\overline{f}_j(x)\right| + \max_{x\in\overline{Q}_2}\left|u_j(x)\right|\right).\end{aligned}$$

Since the last inequality is bounded by a constant independent of j as we have proved, we see that $||u_j||_{W^{2,p}(Q_1)}$ is bounded by a constant independent of j. Now take $p = \frac{N}{1-\alpha}$. Then by applying Sobolev-Morrey embedding inequatlity we conclude that $||u_j||_{C^{1+\alpha}(Q_1)}$ is bounded by a constant independent of j, which furthermore implies that $||f_j||_{C^\alpha(Q_1)}$ is bounded by a similar constant.

Finally, we use the interior Hölder estimate theorem (see [[15], Theorem 6.2] to (2.8) and get another constant C_3 independent of j such that for every $j \geq k+1$

$$||u_j||_{C^{2+\alpha}(\overline{\Omega}_k)} \leq C_3\left(||\overline{f}_j||_{C^\alpha(\overline{Q}_1)} + \max_{x\in\overline{Q}_1} |u_j(x)|\right).$$

From this and the conclusion we have just proved, we get inequality (2.6). □

The proof of Theorem 2.6.

Lemma 2.8 infers there exists a subseuqnece $\{u_{j_l}(x)\}$ of $\{u_j\}$ and $u \in C^2(\Omega)$ such that

$$\|u_{j_l} - u\|_k = \max\left\{ \sum_{1\le s,t\le N} \max_{x\in\overline{\Omega}_k} \left| \frac{\partial^2 u_{j_l}(x)}{\partial x_s \partial x_t}(x) - \frac{\partial^2 u(x)}{\partial x_s \partial x_t}(x) \right|, \right.$$

$$\left. \max_{x\in\overline{\Omega}_k} |\nabla u_{j_l}(x) - \nabla u(x)|, \max_{x\in\overline{\Omega}_k} |u_{j_l}(x) - u(x)| \right\}$$

to 0 as $j_l \to +\infty$ and the corresponding subsequence of $\min\left\{k_0^2, \int_{\Omega_{j_l}} |\nabla u_{j_l}(x)|^2 dx\right\}$ converging to s_0. This implies that $u(x) \in C^2(\Omega)$ and satisfies that

$$\begin{cases} -\triangle u(x) = \dfrac{f(x, u(x), \nabla u(x))}{a(s_0)}, & x \in \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

which implies that

$$u(x) = \frac{\int_\Omega G(x,y) f(y, u(y), \nabla u(y)) dy}{a(s_0)}, \quad x \in \overline{\Omega}.$$

Then

$$|u(x_1) - u(x_2)| \le \frac{1}{a(s_0)} \int_\Omega |G(x_1, y) - G(x_2, y)| \, C d^{-\gamma\theta}(y, \partial\Omega) \left[1 + |\nabla u(x)|^2\right] dy$$

and

$$|\nabla u(x_1) - \nabla u(x_2)| \le \frac{1}{a(s_0)} \cdot \int_\Omega |G_x(x_1, y) - G_x(x_2, y)| \, C d^{-\gamma\theta}(y, \partial\Omega) \left[1 + |\nabla u(x)|^2\right] dy$$

By the standard regularity theory, $u \in C^{1,1-\gamma\theta}(\overline{\Omega})$. Moreover, since $u \in \Sigma_R$, from Lemma 2.1, we know

$$\left(\int_\Omega |\nabla u(x)|^2 dx\right)^{\frac{1}{2}} < k_0.$$

And since

$$\left(\int_\Omega |\nabla u(x)|^2 dx\right)^{\frac{1}{2}} = \lim_{j_l\to+\infty} \left[\int_{\Omega_{j_l}} |\nabla u_{j_l}(x)|^2 dx\right]^{\frac{1}{2}},$$

we have

$$\int_{\Omega_{j_l}} |\nabla u_{j_l}(x)|^2 dx < k_0^2$$

for j_l large enough. And so

$$\min\left\{k_0^2, \int_{\Omega_{j_l}} |\nabla u_{j_l}(x)|^2 dx\right\} = \int_{\Omega_{j_l}} |\nabla u_{j_l}(x)|^2 dx$$

for j_l large enough, which implies that

$$a\left(\min\left\{k_0^2, \int_{\Omega_{j_l}} |\nabla u_{j_l}(x)|^2 dx\right\}\right) = a\left(\int_{\Omega_{j_l}} |\nabla u_{j_l}(x)|^2 dx\right)$$

for j_l large enough.

Consequently,

$$s_0 = \int_\Omega |\nabla u(x)|^2 dx.$$

Then

$$\begin{cases} -\triangle u(x) = \dfrac{f(x, u(x), \nabla u(x))}{a\left(\int_\Omega |\nabla u(x)|^2 dx\right)}, & x \in \Omega, \\ u|_{\partial\Omega} = 0. \square \end{cases}$$

φ_1 is the normalized positive eigenfunction corresponding to the first eigenvalue λ_1 of

$$\begin{cases} -\triangle u(x) = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Lemma 2.9. (see [1]) *Let $F : \overline{\Omega} \times (0, \infty) \to \mathbb{R}$ be a continuous function, and the mapping $s \mapsto \dfrac{F(x, s)}{s}$ is strictly decreasing at each $x \in \Omega$, with $s \in (0, \infty)$. If there are $v, w \in C^2(\Omega) \cap C(\overline{\Omega})$ such that*

(a) $\triangle \omega + F(x, w) \le 0 \le \triangle v + F(x, v)$ *in* Ω*;*
(b) $w, v > 0$ *in* Ω *and* $v \le w$ *on* $\partial\Omega$*;*
(c) $\triangle w \in L^1(\Omega)$ *or* $\triangle v \in L^1(\Omega)$.
Then $v \le \omega$ in Ω.

Lemma 2.10. (see [32]) $\displaystyle\int_\Omega \varphi_1^{-s} < \infty$ *if and only if* $s < 1$.

Lemma 2.11. (see [33]) *The conditions of this lemma are the conditions of the lemma 2.4 in [33]. Then*

$$\begin{cases} -\triangle u(x) = F(x, u), & \text{in } \Omega, \\ u > 0, & \text{on } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

has at least one positive solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$.

3. Proofs of main theorems

3.1. Proof of Theorem 1.1.

Fix $\lambda > 0$.

$$\begin{cases} -\triangle u(x) = \dfrac{1}{a_0}(\lambda f(x, u) + K(x)g(u)), & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

has a solution $\overline{u}_\lambda$.

Let $R = \max_{x\in\overline{\Omega}} \overline{u}(x)$ and define Σ_R as in (2.4). Lemma 2.1 infers there exists a $k_0 > 0$ making

$$\int_\Omega |\nabla u|^2 dx < k_0^2 \tag{3.2}$$

for all $u \in \Sigma_R$.

Let $H : [0,\infty) \to [0,\infty)$ satisfying

$$\begin{cases} H''(t) = g(H(t)), & \forall t > 0, \\ H'(0) = H(0) = 0. \end{cases} \tag{3.3}$$

Equation (3.3) infers H'' is decreasing, while H and H' are nondecreasing on $(0,\infty)$. Then there exist $\xi_t^1,\ \xi_t^2 \in (0,t)$ such that

$$\frac{H(t)}{t} = \frac{H(t) - H(0)}{t - 0} = H'(\xi_t^1) \le H'(t)$$

and

$$\frac{H'(t)}{t} = \frac{H'(t) - H'(0)}{t - 0} = H''(\xi_t^2) \ge H''(t),$$

$\forall t > 0$.

Then

$$H(t) \le tH'(t) \le 2H(t), \quad \forall t > 0.$$

Let

$$\underline{u}_{\lambda_\delta} = \delta H(\varphi_1)$$

where $0 < \delta < 1$. For $a(k_0^2) > 0$ (k_0 is defined in (3.2)), using the fact that g is monotonic, we can conclude

$$\begin{aligned} &-\triangle \underline{u}_{\lambda_\delta} - \frac{1}{a(k_0^2)} K(x) g(\underline{u}_{\lambda_\delta}) + \frac{1}{a_0} |\nabla \underline{u}_{\lambda_\delta}|^\eta \\ &\le -\delta g(H(\varphi_1))|\nabla\varphi_1|^2 + \lambda_1 \delta H'(\varphi_1)\varphi_1 - \frac{1}{a(k_0^2)} K_* g(H(\varphi_1)) \\ &\quad + \frac{1}{a_0}\delta^\eta (H')^\eta(\varphi_1)|\nabla\varphi_1|^\eta \\ &\le -\delta g(H(\varphi_1))|\nabla\varphi_1|^2 + 2\lambda_1 \delta H(\varphi_1) - \frac{1}{a(k_0^2)} K_* g(H(\varphi_1)) \\ &\quad + \frac{1}{a_0}\delta^\eta (H')^\eta(\varphi_1)|\nabla\varphi_1|^\eta, \quad \text{in } \Omega. \end{aligned} \tag{3.4}$$

Let

$$0 < \delta \le \delta_1^* = \min\left\{1, \Big(\frac{a_0 K_* g(H(\|\varphi_1\|_\infty))}{a(k_0^2)(H')^\eta(\|\varphi_1\|_\infty)\||\nabla\varphi_1|\|_\infty^\eta}\Big)^{\frac{1}{\eta}}\right\}$$

such that

$$-\frac{K_* g(H(\|\varphi_1\|_\infty))}{a(k_0^2)} + \frac{\delta^\eta (H')^\eta(\|\varphi_1\|_\infty)\||\nabla\varphi_1|\|_\infty^\eta}{a_0} \le 0, \quad \text{in } \Omega,$$

which together with (3.4) yields

$$\begin{aligned}&-\triangle \underline{u}_{\lambda_\delta}-\frac{1}{a(k_0^2)}K(x)g(\underline{u}_{\lambda_\delta})+\frac{1}{a_0}|\nabla \underline{u}_{\lambda_\delta}|^\eta\\ &\le 2\lambda_1\delta H(\varphi_1)\\ &\le 2\lambda_1\underline{u}_{\lambda_\delta},\quad \text{in } \Omega.\end{aligned} \tag{3.5}$$

Let $0<\delta\le\delta_2^*$ small enough such that

$$\frac{1}{a(k_0^2)}\frac{\lambda f(x,\delta H(\|\varphi_1\|_\infty))}{\delta H(\|\varphi_1\|_\infty)}\ge 2\lambda_1. \tag{3.6}$$

$(f1)$ and (3.6) infer

$$\frac{1}{a(k_0^2)}\frac{\lambda f(x,\underline{u}_{\lambda_\delta})}{\underline{u}_{\lambda_\delta}}\ge\frac{\lambda f(x,\delta H(\|\varphi_1\|_\infty))}{a(k_0^2)\delta H(\|\varphi_1\|_\infty)}\ge 2\lambda_1,\quad \text{in } \Omega.$$

Let us choose $\delta^*=\min\{\delta_1^*,\ \delta_2^*\}$, $\forall\delta\in(0,\delta^*]$. The inequality (3.6)combined (3.5) yields

$$\begin{aligned}&-\triangle \underline{u}_{\lambda_\delta}-\frac{1}{a\left(k_0^2\right)}K(x)g\left(\underline{u}_{\lambda_\delta}\right)+\frac{1}{a_0}\left|\nabla \underline{u}_{\lambda_\delta}\right|^\eta\\ &\le 2\lambda_1\overline{u}_{\lambda_\delta}\\ &\le\frac{1}{a\left(k_0^2\right)}\lambda f\left(x,\underline{u}_{\lambda_\delta}\right),\quad \text{in } \Omega.\end{aligned} \tag{3.7}$$

Equations (3.1) and (3.7) infer $\forall\lambda\ge 0$

$$\begin{cases}-\triangle\overline{u}_\lambda\ge\dfrac{(\lambda f(x,\overline{u}_\lambda)+K(x)g\left(\overline{u}_\lambda\right))}{a_0}, & \text{in } \Omega,\\ \overline{u}_\lambda=0, & \text{on } \partial\Omega\end{cases}$$

and

$$\begin{cases}-\triangle\underline{u}_{\lambda_\delta}\le\dfrac{1}{a\left(k_0^2\right)}\left(\lambda f(x,\underline{u}_{\lambda_\delta})+K(x)g\left(\underline{u}_{\lambda_\delta}\right)\right)\\ \qquad\qquad -\dfrac{1}{a_0}\left|\nabla\underline{u}_{\lambda_\delta}\right|^\eta, & \text{in } \Omega,\\ \underline{u}_{\lambda_\delta}=0, & \text{on } \partial\Omega.\end{cases}$$

Then we have

$$\begin{cases}\triangle\overline{u}_\lambda+\dfrac{\left[\lambda f\left(x,\overline{u}_\lambda\right)+K(x)g\left(\overline{u}_\lambda\right)\right]}{a_0}\le 0, & \text{in } \Omega,\\ \triangle\underline{u}_{\lambda_\delta}+\dfrac{\left[\lambda f\left(x,\underline{u}_{\lambda_\delta}\right)+K(x)g\left(\underline{u}_{\lambda_\delta}\right)\right]}{a_0}\ge 0, & \text{in } \Omega,\\ \overline{u}_\lambda,\underline{u}_{\lambda_\delta}>0, & \text{in } \Omega,\\ \overline{u}_\lambda,\underline{u}_{\lambda_\delta}=0, & \text{on } \partial\Omega,\\ \triangle\overline{u}_\lambda\in L^1(\Omega).\end{cases}$$

From Lemma 2.9 we know $\underline{u}_{\lambda_\delta}\le\overline{u}_\lambda$ in Ω for all $\delta\in(0,\delta^*]$.

Furthermore, from $(g3)$ and the definition of H, we can conclude that

$$\lim_{t\to 0}\frac{H(t)}{t^{\frac{2}{\alpha+1}}}=1.$$

Then we get

$$\lim_{t\to 0}\frac{H(t)}{t^{\gamma}}=+\infty$$

when $\gamma>\frac{2}{\alpha+1}$. It follows that

$$\lim_{x\to\partial\Omega}\frac{H(\varphi_1(x))}{d(x,\partial\Omega)^{\gamma}}=+\infty.$$

Hence, there is a $\delta_0>0$ making

$$\underline{u}_{\lambda_\delta}\geq\delta_0 d\left(x,\partial\Omega\right)^{\gamma}$$

with $0<\gamma\theta<1$.

The Theorem 2.6 guarantees that

$$\begin{cases}-a\left(\int_{\Omega}|\nabla u|^2dx\right)\triangle u=K(x)g\left(u\right)+\lambda f(x,u)-|\nabla u|^{\eta}, & \text{in }\Omega,\\ u>0, & \text{in }\Omega,\\ u=0, & \text{on }\partial\Omega\end{cases}$$

has a solution $u\in H_0^1(\Omega)$ with

$$\underline{u}_{\lambda_\delta}(x)\leq u(x)\leq\overline{u}_{\lambda}(x),\quad\text{in }\Omega.$$

Therefore, (1.1) has at least one positive solution, $\forall\lambda>0$. □

3.2. Proof of Theorem 1.2.

$(f1)$, $(f2)$ and Lemma 2.11 deduce that there is $\overline{u}_{\lambda}\in C^2(\overline{\Omega})$ making

$$\begin{cases}-\triangle\overline{u}_{\lambda}=\dfrac{\lambda f(x,\overline{u}_{\lambda})}{a_0}, & \text{in }\Omega,\\ \overline{u}_{\lambda}>0, & \text{in }\Omega,\\ \overline{u}_{\lambda}=0, & \text{on }\partial\Omega,\end{cases}\tag{3.8}$$

$\forall\lambda>0$.

Let $R=\max_{x\in\overline{\Omega}}\overline{u}(x)$ and define Σ_R as in (2.4). Lemma 2.1 infers there is a $k_0>0$ making

$$\int_{\Omega}|\nabla u|^2dx<k_0^2\tag{3.9}$$

for all $u\in\Sigma_R$.

Let $\underline{u}_\lambda = MH(\varphi_1)$, with $M \geq 1 > 0$ is a constant. Because g is monotonic,

$$\begin{aligned}
&-\triangle \underline{u}_\lambda - \frac{K(x)g\left(\underline{u}_\lambda\right)}{a_0} + \frac{\left|\nabla \underline{u}_\lambda\right|^\eta}{a_0} \\
&\leq \lambda_1 MH'(\varphi_1)\varphi_1 - Mg\left(H(\varphi_1)\right)|\nabla\varphi_1|^2 - \frac{K_* g(H(\varphi_1))}{a_0} \\
&+ \frac{1}{a_0} M^\eta (H')^\eta(\varphi_1)|\nabla\varphi_1|^\eta \\
&\leq \left(-\frac{1}{a_0}K_* - M|\nabla\varphi|^2\right) g(H(\varphi_1))|\nabla\varphi_1|^2 + 2\lambda_1 MH(\varphi_1) \\
&+ \frac{1}{a_0} M^\eta (H')^\eta(\varphi_1)|\nabla\varphi_1|^\eta \quad \text{in } \Omega.
\end{aligned} \tag{3.10}$$

Hopf's maximum principle deduce that there exist δ_0 and $\Sigma \subset \Omega$ making

$$\begin{cases} |\nabla\varphi_1| \geq \delta_0, & \text{in } \Omega \setminus \Sigma, \\ |\varphi_1| \geq \delta_0, & \text{in } \Sigma. \end{cases}$$

On one hand, we consider the case $x \in \Omega \setminus \Sigma$.

Let

$$M \geq M_1 = \max\left\{1, \frac{-K_*}{a_0\delta_0^2}\right\}.$$

Since

$$\lim_{\operatorname{dist}(x,\partial\Omega)\to 0^+} \left(M|\nabla\varphi_1|^\eta + \frac{K_*}{a_0}\right) g\left(H(\varphi_1)\right) = +\infty,$$

if

$$\frac{1}{a_0} M^\eta \left(H'\right)^\eta (\varphi_1)|\nabla\varphi_1|^\eta - \left(M|\nabla\varphi_1|^\eta + \frac{K_*}{a_0}\right) g(H(\varphi_1)) \leq 0 \tag{3.11}$$

in $\Omega \setminus \Sigma$, by letting Σ close enough to the boundary of Ω. The above inequality combined (3.10) yields

$$-\triangle \underline{u}_\lambda - \frac{K(x)g\left(\underline{u}_\lambda\right)}{a_0} + \frac{\left|\nabla \underline{u}_\lambda\right|^\eta}{a_0} \leq 2\lambda_1 \underline{u}_\lambda \quad \text{in } \Omega \setminus \Sigma. \tag{3.12}$$

For $a\left(k_0^2\right) > 0$ (k_0 is defined in (3.9)) and

$$f\left(x, MH(\|\varphi_1\|_\infty)\right) > 0,$$

we can choose

$$\lambda > \lambda_0 = \max\left\{1, \frac{2\lambda_1 Ma\left(k_0^2\right) H(\|\varphi_1\|_\infty)}{\min_{x\in\Omega\setminus\Sigma} f(x, MH(\|\varphi_1\|_\infty))}\right\}$$

making

$$\lambda \frac{1}{a\left(k_0^2\right)} \frac{\min\limits_{x\in\Omega\setminus\Sigma} f(x, MH(\|\varphi_1\|_\infty))}{MH(\|\varphi_1\|_\infty))} \geq 2\lambda_1. \tag{3.13}$$

$(f1)$ and (3.13) decude

$$\frac{1}{a(k_0^2)}\frac{\lambda f\left(x,\underline{u}_\lambda\right)}{\underline{u}_\lambda} \geq \frac{1}{a(k_0^2)}\frac{\lambda f(x,MH(\|\varphi_1\|_\infty))}{MH(\|\varphi_1\|_\infty)} \geq 2\lambda_1, \tag{3.14}$$

in $\Omega \setminus \Sigma$. The last inequality combined (3.12) yields

$$\begin{aligned} & -\triangle\underline{u}_\lambda - \frac{1}{a_0}K(x)g\left(\underline{u}_\lambda\right) + \frac{1}{a_0}|\nabla\underline{u}_\lambda|^\eta \\ & \leq 2\lambda_1\underline{u}_\lambda \\ & \leq \frac{\lambda f\left(x,\underline{u}_\lambda\right)}{a(k_0^2)}, \quad \text{in } \Omega \setminus \Sigma. \end{aligned} \tag{3.15}$$

If $x \in \Sigma$

$$\begin{aligned} & -\triangle\underline{u}_\lambda - \frac{1}{a_0}K(x)g\left(\underline{u}_\lambda\right) + \frac{1}{a_0}\left|\nabla\underline{u}_\lambda\right|^\eta \\ \leq\ & 2\lambda_1 MH\left(\varphi_1\right) - \frac{1}{a_0}K_* g\left(H\left(\varphi_1\right)\right) \\ & + \frac{1}{a_0}M^\eta\left(H'\right)^\eta\left(\varphi_1\right)|\nabla\varphi_1|^\eta, \quad \text{in } \Sigma. \end{aligned}$$

Because $\varphi_1 > 0$ in $\overline{\Sigma}$ and $f > 0$ on $\overline{\Sigma}$, we choose

$$\lambda \geq \lambda_2 = \max\left\{\lambda_0, a^\star\right\}$$

with

$$a^\star = a(k_0^2)\frac{\Phi_1^\star}{\Phi_2^\star}$$

$$\Phi_1^\star = \max_{x\in\overline{\Sigma}}\left\{2\lambda_1 MH(\varphi_1) - \frac{K_* g(H(\varphi_1))}{a_0} + \frac{M^\eta(H')^\eta\left(\varphi_1\right)|\nabla\varphi_1|^\eta}{a_0}\right\}$$

$$\Phi_2^\star = \min_{x\in\overline{\Sigma}} f\left(x, MH\left(\varphi_1\right)\right)$$

such that

$$\begin{aligned} & \frac{\lambda}{a\left(k_0^2\right)}\min_{x\in\overline{\Sigma}} f\left(x, MH\left(\varphi_1\right)\right) \\ \geq & \max_{x\in\overline{\Sigma}}\left(2\lambda_1 MH(\varphi_1) - \frac{1}{a_0}K_* gH\left(\varphi_1\right) + \frac{1}{a_0}M^\eta\left(H'\right)^\eta\left(\varphi_1\right)|\nabla\varphi_1|^\eta\right). \end{aligned}$$

Then

$$\begin{aligned} & -\triangle\underline{u}_\lambda - \frac{K(x)g\left(\underline{u}_\lambda\right)}{a_0} + \frac{1}{a_0}|\nabla\underline{u}_\lambda|^\eta \\ & \leq \frac{\lambda}{a(k_0^2)}\min_{x\in\overline{\Sigma}} f\left(x,\underline{u}_\lambda\right) \\ & \leq \frac{\lambda}{a(k_0^2)} f\left(x,\underline{u}_\lambda\right). \end{aligned} \tag{3.16}$$

It follows from (3.8), (3.15) and (3.16) that for each $\lambda > \lambda^* = \max\{\lambda_1, \lambda_2\}$,

$$\begin{cases} -\triangle\overline{u}_\lambda \geq \dfrac{\lambda f\left(x,\overline{u}_\lambda\right)}{a_0}, & \text{in } \Omega, \\ \overline{u}_\lambda = 0, & \text{on } \partial\Omega \end{cases}$$

and

$$\begin{cases} -\triangle \underline{u}_\lambda \leq \dfrac{\lambda f\left(x, \underline{u}_\lambda\right)}{a\left(k_0^2\right)} + \dfrac{\left(K(x) g\left(\underline{u}_\lambda\right) - |\nabla \underline{u}_\lambda|^\eta\right)}{a_0}, & \text{in } \Omega, \\ \underline{u}_\lambda = 0, & \text{on } \partial\Omega. \end{cases}$$

Furthermore, we obtain

$$\begin{cases} \dfrac{\lambda f\left(x, \overline{u}_\lambda\right)}{a_0} + \triangle \overline{u}_\lambda \leq 0 \leq \dfrac{\lambda f\left(x, \underline{u}_\lambda\right)}{a_0} + \triangle \underline{u}_\lambda, & \text{in } \Omega, \\ \overline{u}_\lambda, \underline{u}_\lambda > 0, & \text{in } \Omega, \\ \overline{u}_\lambda, \underline{u}_\lambda = 0, & \text{on } \partial\Omega, \\ \triangle \overline{u}_\lambda \in L^1(\Omega). \end{cases}$$

Lemma 2.9 infers $\underline{u}_\lambda \leq \overline{u}_\lambda$ in Ω. Then $\underline{u}_\lambda$ and $\overline{u}_\lambda$ are respectively upper and lower solution of the problem (1.1). Moreover, from $(g3)$ and the definition of H, we can conclude that

$$\lim_{t \to 0} \frac{H(t)}{t^{\frac{2}{\alpha+1}}} = 1.$$

Then we have

$$\lim_{t \to 0} \frac{H(t)}{t^\gamma} = +\infty$$

when $\gamma > \frac{2}{\alpha+1}$. It follows that

$$\lim_{x \to \partial\Omega} \frac{H(\varphi_1(x))}{d(x, \partial\Omega)^\gamma} = +\infty,$$

which implies that there is a $\delta_0 > 0$ such that

$$\underline{u}_\lambda \geq \delta_0 d(x, \partial\Omega)^\gamma$$

with $0 < \gamma\theta < 1$ and $0 < \alpha < 1$. By Theorem 2.6, there is a solution $u \in C^1(\overline{\Omega})$ for (1.1), and $\underline{u}_\lambda \leq u \leq \overline{u}_\lambda$ in Ω.

To end the proof, like [1], we have

$$f(x, s) + K(x) g(s) < ms,$$

$\forall (x, s) \in \Omega \times (0, +\infty)$, with

$$m = \max_{x \in \overline{\Omega}} \frac{f(x, c)}{c}.$$

Let

$$\lambda_0 = \min\left\{1, \frac{\lambda_1 a_0}{2m}\right\}.$$

We will prove $(1.1)_\lambda$ has no positive solution as mentioned above for all $\lambda \leq \lambda_0$. Due to

$$f(x, s) + K(x) g(s) < ms,$$

u_0 is a lower solution of

$$\begin{cases} -\triangle u = \dfrac{\lambda m}{a_0} u, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.17}$$

if u_0 is a solution of $(1.1)_\lambda$.

Let k_0 big enough such that $k_0\varphi_1$ is a upper solution for (3.17) and $u_0 \le k_0\varphi_1$ in Ω. Thus, (3.17) has a solution $u \in C^2(\overline{\Omega})$. (3.17) multiply by φ_1 and integrat over Ω,

$$-\int_\Omega \varphi_1 \triangle u dx = \frac{\lambda m}{a_0} \int_\Omega \varphi_1 u dx,$$

that is

$$\lambda_1 \int_\Omega u\varphi_1 dx = \frac{\lambda m}{a_0} \int_\Omega u\varphi_1 dx \le \frac{\lambda_1}{2} \int_\Omega u\varphi_1 dx.$$

Then

$$\int_\Omega u\varphi_1 dx = 0.$$

This is contradictory. Then $(1.1)_\lambda$ has no positive solutions, $\forall \lambda \le \lambda_0$. □

3.3. Proof of Theorem 1.3.

Some ideas is similar to [34] and [1].

Assume that there is $\lambda > 0$ making (1.1) has a solution u_λ. Set

$$b_0 = a\left(\int_\Omega |\nabla u_\lambda|^2 dx\right).$$

$(f1)$, $(f2)$ and Lemma 2.11 deduce that

$$\begin{cases} -\triangle u(x) = \dfrac{\lambda f(x,u)}{a_0}, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

has a positive solution $\overline{u}_\lambda \in C^2(\overline{\Omega})$, $\forall \lambda > 0$. Additionally, there are $C_1,\ C_2 > 0$ satisfying

$$C_1 dist\,(x, \partial\Omega) \le \overline{u}_\lambda(x) \le C_2 dist\,(x, \partial\Omega)\,, \tag{3.18}$$

$\forall x \in \Omega$.

We will consider

$$\begin{cases} -\triangle u - \dfrac{g(u+\varepsilon)}{b_0} K^* = \dfrac{\lambda f(x,u)}{a_0}, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.19}$$

with $K^* = \max_{x\in\overline{\Omega}} K(x) < 0$. Furthermore, we have

$$\begin{cases} \triangle\overline{u}_\lambda + \frac{\lambda f(x,\overline{u}_\lambda)}{a_0} \leq 0 \leq \triangle u_\lambda + \frac{\lambda f(x,u_\lambda)}{a_0}, \quad \text{in } \Omega, \\ \overline{u}_\lambda, u_\lambda > 0, \quad \text{in } \Omega, \\ \overline{u}_\lambda = u_\lambda = 0, \quad \text{on } \partial\Omega, \\ \triangle\overline{u}_\lambda \in L^1(\Omega), \quad (\text{since } \overline{u}_\lambda \in C^2(\overline{\Omega})), \end{cases}$$

Lemma 2.9 infers $u_\lambda \leq \overline{u}_\lambda$ in Ω. We know that u_λ and $\overline{u}_\lambda$ are respectively lower and upper solution of (3.19). Thus, there is a solution $u_\varepsilon \in C^2(\overline{\Omega})$ satisfying

$$u_\lambda \leq u_\varepsilon \leq \overline{u}_\lambda, \quad \text{in } \Omega.$$

Integrating in the problem (3.19),

$$-\int_\Omega \triangle u_\varepsilon dx - K^* \int_\Omega \frac{g\,(u_\varepsilon + \varepsilon)}{b_0} dx = \lambda \int_\Omega \frac{f\,(x, u_\varepsilon)}{a_0} dx.$$

Hence, by the divergence theorem,

$$-\int_{\partial\Omega} \frac{\partial u_\varepsilon}{\partial n} ds - \int_\Omega K^* \frac{g(u_\varepsilon + \varepsilon)}{b_0} dx \leq M, \tag{3.20}$$

with $M > 0$ is a constant. $\frac{\partial u_\varepsilon}{\partial n} \leq 0$ on $\partial\Omega$, and (3.20) infer

$$-\int_\Omega \frac{K^* g(u_\varepsilon + \varepsilon)}{b_0} dx \leq M. \tag{3.21}$$

Because of $u_\varepsilon \leq \overline{u}_\lambda$ in $\overline{\Omega}$, (3.21) infers

$$\int_\Omega g(\overline{u}_\lambda + \varepsilon) dx \leq C$$

for some $C > 0$. Then, we have $\int_\omega g(\overline{u}_\lambda + \varepsilon)dx \leq C$, for any compact subset $\omega \subset \Omega$. When $\varepsilon \to 0^+$, $\int_\omega g(\overline{u}_\lambda)dx \leq C$. Then $\int_\Omega g(\overline{u}_\lambda)dx \leq C$.

However, (3.18) and $\int_0^1 g(s)ds = +\infty$ can conclude

$$\int_\Omega g(\overline{u}_\lambda)dx \geq \int_\Omega g(C_2 dist(x, \partial\Omega))dx = +\infty$$

which contradicts $\int_\Omega g(\overline{u}_\lambda)dx \leq C$. □

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Conflict of interest

The authors declared that they have no conflicts of interest to this work.

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