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Research article

Finite-time flocking with collision-avoiding problem of a modified Cucker-Smale model

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Abstract: To achieve collision-avoiding flocking in finite time, a modified Cucker-Smale model with general inter-driving force is proposed. First, it is proved that the system can achieve conditional collision-avoiding flocking in finite time by imposing appropriate restrictions on the initial states. Moreover, a special case of the inter-driving force is demonstrated. Last, the correctness of the results is verified through numerical simulations.

Keywords: Cucker-Smale model; flocking; finite-time; collision-avoidance; inter-driving force

1. Introduction

Among biological groups such as flocks of birds, herds of beasts, and schools of fish in nature, individuals have limited abilities to perceive, make decisions and act, but they can follow simple rules and cooperate with each other to complete complex behaviors such as migration and nesting and can exhibit an orderly self-organized coordination pattern at the group level. Inspired by this, the concept of flocking has been proposed and widely used in many fields, such as biology [1,2], economics [3,4], sociology [5,6], and engineering [7,8].

For the study of flocking phenomena, many models have been proposed. In 1995, Vicsek et al. proposed a self-propelled particle model that provides inspiration for follow-up research [9]. Each particle in the model adjusts its speed according to the speed of its neighbors to achieve flocking. Based on this, Cucker and Smale proposed a new model in 2007 [10,11]. The main principle of the Cucker-Smale (C-S) model is that each particle adjusts its velocity according to a weighted average of its velocity differences with other individuals. The continuous model can be described as follows:

$$\begin{cases} \dfrac{\mathrm{d}x_i(t)}{\mathrm{d}t} = v_i(t), \quad i = 1, \ldots, N, \quad t > 0, \\ \dfrac{\mathrm{d}v_i(t)}{\mathrm{d}t} = \dfrac{1}{N} \displaystyle\sum_{j=1}^{N} \psi\left(\| x_j(t) - x_i(t) \|\right)\left(v_j(t) - v_i(t)\right), \end{cases} \tag{1.1}$$

where $x_i,\ v_i \in \mathbb{R}^d$ denote the position and velocity of the ith particle. The communication rate ψ is taken as $\psi(r) = K/(1 + r^2)^\beta,\ K > 0,\ \beta \geq 0$ and quantifies the influence between two particles. The results in [10, 11] demonstrate that by taking different values of parameter β, unconditional flocking or conditional flocking, which depends on the initial positions and velocities, can be achieved.

Cucker and Smale established a framework for studying the law of group self-evolution and depicted a general system evolution mechanism. The C-S model has attracted a large number of scholars, who have continued to expand and innovate the model according to actual situations. For example, the collision avoidance results of a UAV formation system [12–14] and the shaping formation of a multi-robot system [15–18]. All these interesting findings require additional bonding forces between different particles in the original C-S system. Therefore, what bonding force conditions allow a system to achieve the set goal without changing the flocking state of the system? To the best of our knowledge, there is no theoretical analysis on this issue.

All the above studies have concluded that time asymptotic flocking occurs, which means that clusters will only form as time tends to infinity. However, in many applications, especially in engineering experiments, it is necessary to implement flocking in a limited time. Based on this, scholars have applied finite-time control technology. Regarding the Vicsek-type system, there are many works on finite-time consensus and flocking [19, 20]. Then, in 2016, Han et al. proposed a continuous non-Lipschitz C-S-type model [21] and carried out a finite-time stability analysis with the condition that the communication rate function has a lower bound. Moreover, in [22, 23], the authors obtained finite-time flocking with collision avoidance.

Motivated by the above works, this paper aims to investigate collision-avoiding finite-time flocking with inter-driving forces. We propose a modified C-S model with a general inter-particle bonding force as follows:

$$\begin{cases} \dfrac{\mathrm{d}x_i(t)}{\mathrm{d}t} = v_i(t), \quad i = 1, \ldots, N, \quad t > 0, \\ \dfrac{\mathrm{d}v_i(t)}{\mathrm{d}t} = \displaystyle\sum_{j=1}^{N} \psi\left(\| x_j(t) - x_i(t) \|\right) \mathrm{sig}\left(v_j(t) - v_i(t)\right)^\theta + F_i(v_1, v_2, \ldots, v_N), \end{cases} \tag{1.2}$$

where $0 < \theta < 1$, $\mathrm{sig}\left(v_j - v_i\right)^\theta = (\mathrm{sign}(v_{j1} - v_{i1})|v_{j1} - v_{i1}|^\theta, \cdots, \mathrm{sign}(v_{jd} - v_{id})|v_{jd} - v_{id}|^\theta)^T$ and sign: $\mathbb{R} \to \{-1, 0, 1\}$ is the sign function. $\psi : [0, \infty) \to [0, \infty)$ is a suitable communication rate between two individuals, and $F_i(v_1, v_2, \ldots, v_N) : \mathbb{R}^d \to \mathbb{R}^d$ is the inter-particle bonding force function. We also impose some restrictions on the initial states, which play important roles in our main results.

In addition, these functions are further assumed to satisfy the following conditions.

Assumption 1.1. *The communication rate ψ is symmetrical, locally Lipschitz continuous and has a lower bound, that is, there exists a constant $\psi^* > 0$ such that*

$$\psi(s) \geq \psi^*. \tag{1.3}$$

Assumption 1.2. *[17] The inter-particle bonding force F_i satisfies the following conditions:*
$(F1)$ $\sum_{i=1}^{N} F_i = 0$, *where* $0 \in \mathbb{R}^d$ *is a zero vector.*
$(F2)$ *There exists a constant* $p > 0$ *such that*

$$\langle F_i, v_i \rangle \leq p \parallel v \parallel^2, \tag{1.4}$$

where $\parallel v \parallel^2 = \sum_{i=1}^{N} \parallel v_i \parallel^2$.

The rest of the paper is organized as follows: In Section 2, we prove the finite-time flocking result by imposing appropriate restrictions on the initial states. In Section 3, we obtain the collision-avoidance result. In Section 4, we give a special case of the inter-driving force and demonstrate our theoretical results by numerical simulations. Finally, the conclusions are given in Section 5.

2. Finite-time flocking

In this section, we present our main result that system (1.2) reaches finite-time flocking. First, we introduce the definition of finite-time flocking with collision-avoidance as follows.

Definition 2.1. *[23] We say system (1.2) reaches finite-time flocking if for any initial condition* $x_i(0), v_i(0)$ *and* $1 \leq i, j \leq N$, *the solutions* $\{x_i, v_i\}(i = 1, \cdots, N)$ *of system (1.2) satisfy*

$$\parallel v_i - v_j \parallel = 0, \forall t \geq T_1, \ \text{and} \ \sup_{0 \leq t \leq \infty} \parallel x_i - x_j \parallel^2 < \infty, \tag{2.1}$$

where $T_1 = \inf\{T : \parallel v_i - v_j \parallel = 0, \forall t \geq T\}$ *is called the convergence time. Moreover, if system (1.2) reaches finite-time flocking and*

$$\min_{i \neq j} \parallel x_i - x_j \parallel > 0, \ t \geq 0, \tag{2.2}$$

then we say that system (1.2) reaches finite-time flocking with collision-avoidance.

We give the following important lemmas to better prove the main results.

Lemma 2.1. *[24] Let* $a_1, a_2, \cdots, a_n > 0$ *and* $0 < r < p$, *then*

$$\left(\sum_{i=1}^{n} a_i^p \right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} a_i^r \right)^{\frac{1}{r}}.$$

Lemma 2.2. *[24] If* $\xi_1, \xi_2, \cdots, \xi_n \geq 0$ *and* $0 < p \leq 1$, *then*

$$\left(\sum_{i=1}^{n} \xi_i \right)^p \leq \sum_{i=1}^{n} \xi_i^p.$$

Lemma 2.3. *[25] Suppose that the function* $V(t)$ *is continuous and positive definite and satisfies the following differential inequality:*

$$\dot{V}(t) \leq -lV^{\alpha}(t) + kV(t), \ \forall t > t_0, \ V^{1-\alpha}(t_0) < \frac{l}{k},$$

where $\alpha \in (0, 1), l$ *and* k *are positive constants. Then,* $V(t) \equiv 0, \forall t \geq t_1$, *where the settling time* t_1 *is given as follows:*

$$t_1 = t_0 + \frac{\ln(1 - \frac{k}{l} V^{1-\alpha}(t_0))}{k(\alpha - 1)}.$$

Next, we give our main conclusion.

Theorem 2.1. *Consider the modified C-S model (1.2). Assume that the communication rate function ψ satisfies Assumption 1.1, the inter-particle bonding force $F_i(v_1, v_2, \ldots, v_N)$ satisfies Assumption 1.2, and the initial states of system (1.2) satisfy*

$$V(0) \leq \frac{1}{2N}\left(\frac{\psi^*}{p}\right)^{\frac{2}{1-\theta}}, \tag{2.3}$$

then, system (1.2) can reach finite-time flocking. The convergence time is estimated by

$$T_1 \leq t_1 = \frac{1}{pN(1-\theta)} \ln\left(\frac{\psi^*}{\psi^* - p(2NV(0))^{\frac{1-\theta}{2}}}\right), \tag{2.4}$$

where $V(0) = \sum_{i=1}^{N} \| v_i(0) \|^2$.

Proof. From the definition of the function $\mathrm{sig}(\cdot)^\theta$, we have

$$\psi\left(\| x_j(t) - x_i(t) \|\right) \mathrm{sig}\left(v_j(t) - v_i(t)\right)^\theta = -\psi\left(\| x_j(t) - x_i(t) \|\right) \mathrm{sig}\left(v_i(t) - v_j(t)\right)^\theta . \tag{2.5}$$

Thus

$$\sum_{i=1}^{N}\sum_{j=1}^{N} \psi\left(\| x_j(t) - x_i(t) \|\right) \mathrm{sig}\left(v_j(t) - v_i(t)\right)^\theta = 0. \tag{2.6}$$

Denote

$$\overline{x}(t) = \frac{1}{N}\sum_{i=1}^{N} x_i(t), \ \overline{v}(t) = \frac{1}{N}\sum_{i=1}^{N} v_i(t). \tag{2.7}$$

We can obtain from system (1.2) that $\mathrm{d}\overline{x} = \overline{v}\mathrm{d}t$, $\mathrm{d}\overline{v} = 0$, that is, $\overline{x}(t) = \overline{x}(0) + \overline{v}(0)t$, $\overline{v}(t) \equiv \overline{v}(0)$.

Without loss of generality, we may assume that $\overline{x}(0) = 0$, $\overline{v}(0) = 0$. Then, it is easy to see that

$$\overline{x}(t) = 0, \ \overline{v}(t) = 0. \tag{2.8}$$

Considering the vectors $v(t) = (v_i(t), v_2(t), \ldots, v_N(t)) \in \mathbb{R}^{d\times N}$ and $x(t) = (x_i(t), x_2(t), \ldots, x_N(t)) \in \mathbb{R}^{d\times N}$, we set positive definite functions

$$\begin{aligned} V(t) &= \sum_{i=1}^{N} \| v_i(t) \|^2, \\ X(t) &= \sum_{i=1}^{N} \| x_i(t) \|^2 . \end{aligned} \tag{2.9}$$

From Eq (2.8), we have

$$\sum_{i,j=1}^{N} \| v_i(t) - v_j(t) \|^2 = 2NV(t), \ \textit{and} \ \sum_{i,j=1}^{N} \| x_i(t) - x_j(t) \|^2 = 2NX(t). \tag{2.10}$$

Therefore, from Definition 2.1, we say that system (1.2) reaches flocking in finite time if for any initial condition $x_i(0), v_i(0)$, the solutions $\{x_i(t), v_i(t)\}(i = 1, \cdots, N)$ to system (1.2) satisfy

$$V(t) = 0, \ \forall t \geq T_1, \ \textit{and} \ \sup_{0\leq t\leq\infty} X(t) < \infty. \tag{2.11}$$

First, we prove the first inequality of condition (2.11). Considering the derivative of $V(t)$ along the second equation of system (1.2), we have

$$\begin{aligned}\frac{\mathrm{d}V(t)}{\mathrm{d}t} &= 2\sum_{i=1}^{N}\langle v_i(t), \dot{v}_i(t)\rangle \\ &= 2\sum_{i=1}^{N}\langle v_i(t), \sum_{j=1}^{N}\psi\left(\| x_j(t)-x_i(t) \|\right)\mathrm{sig}\left(v_j(t)-v_i(t)\right)^{\theta} + F_i(v_1, v_2, \ldots, v_N)\rangle \\ &= 2\sum_{i=1}^{N}\sum_{j=1}^{N}\psi\left(\| x_j(t)-x_i(t) \|\right)\langle v_i(t), \mathrm{sig}\left(v_j(t)-v_i(t)\right)^{\theta}\rangle + 2\sum_{i=1}^{N}\langle v_i(t), F_i(v_1, v_2, \ldots, v_N)\rangle.\end{aligned} \tag{2.12}$$

Using the standard symmetry properties, we obtain that

$$\begin{aligned}&\sum_{i,j=1}^{N}\psi\left(\| x_j(t)-x_i(t) \|\right)\langle v_i(t), \mathrm{sig}\left(v_j(t)-v_i(t)\right)^{\theta}\rangle \\ =&\sum_{i,j=1}^{N}\psi\left(\| x_j(t)-x_i(t) \|\right)\langle v_i(t)-v_j(t), \mathrm{sig}\left(v_j(t)-v_i(t)\right)^{\theta}\rangle \\ &+\sum_{i,j=1}^{N}\psi\left(\| x_j(t)-x_i(t) \|\right)\langle v_j(t), \mathrm{sig}\left(v_j(t)-v_i(t)\right)^{\theta}\rangle \\ =&-\sum_{i,j=1}^{N}\psi\left(\| x_j(t)-x_i(t) \|\right)\langle v_j(t)-v_i(t), \mathrm{sig}\left(v_j(t)-v_i(t)\right)^{\theta}\rangle \\ &-\sum_{i,j=1}^{N}\psi\left(\| x_i(t)-x_j(t) \|\right)\langle v_j(t), \mathrm{sig}\left(v_i(t)-v_j(t)\right)^{\theta}\rangle.\end{aligned} \tag{2.13}$$

Therefore

$$\begin{aligned}&\sum_{i,j=1}^{N}\psi\left(\| x_j(t)-x_i(t) \|\right)\langle v_i(t), \mathrm{sig}\left(v_j(t)-v_i(t)\right)^{\theta}\rangle \\ =&-\frac{1}{2}\sum_{i,j=1}^{N}\psi\left(\| x_j(t)-x_i(t) \|\right)\langle v_j(t)-v_i(t), \mathrm{sig}\left(v_j(t)-v_i(t)\right)^{\theta}\rangle \\ =&-\frac{1}{2}\sum_{i,j=1}^{N}\psi\left(\| x_j(t)-x_i(t) \|\right)\sum_{k=1}^{d}|v_{jk}(t)-v_{ik}(t)|^{\theta+1}.\end{aligned} \tag{2.14}$$

From Lemma 2.1, we have

$$\left(\sum_{k=1}^{d}|v_{jk}-v_{ik}|^{\theta+1}\right)^{\frac{1}{\theta+1}} \geq \left(\sum_{k=1}^{d}|v_{jk}-v_{ik}|^{2}\right)^{\frac{1}{2}} =\| v_j-v_i \| . \tag{2.15}$$

Then

$$\sum_{k=1}^{d}|v_{jk}-v_{ik}|^{\theta+1} \geq\| v_j-v_i \|^{\theta+1} . \tag{2.16}$$

Therefore, we obtain that

$$\begin{aligned}\frac{\mathrm{d}V(t)}{\mathrm{d}t} &= 2\sum_{i=1}^{N}\sum_{j=1}^{N}\psi\left(\| x_j(t)-x_i(t) \|\right)\langle v_i(t), \mathrm{sig}\left(v_j(t)-v_i(t)\right)^{\theta}\rangle + 2\sum_{i=1}^{N}\langle v_i(t), F_i(v_1, v_2, \ldots, v_N)\rangle \\ &= -\sum_{i,j=1}^{N}\psi\left(\| x_j(t)-x_i(t) \|\right)\sum_{k=1}^{d}|v_{jk}(t)-v_{ik}(t)|^{\theta+1} + 2\sum_{i=1}^{N}\langle v_i(t), F_i(v_1, v_2, \ldots, v_N)\rangle \\ &\le -\sum_{i,j=1}^{N}\psi\left(\| x_j(t)-x_i(t) \|\right)\| v_j(t)-v_i(t) \|^{\theta+1} + 2\sum_{i=1}^{N}\langle v_i(t), F_i(v_1, v_2, \ldots, v_N)\rangle.\end{aligned} \tag{2.17}$$

From Assumption 1.1 and condition $(F2)$ of Assumption 1.2, we have

$$\begin{aligned}\frac{\mathrm{d}V(t)}{\mathrm{d}t} &\le -\sum_{i,j=1}^{N}\psi^{*}\| v_j(t)-v_i(t) \|^{\theta+1} + 2pN\sum_{i=1}^{N}\| v_i(t) \|^2 \\ &\le -\psi^{*}\sum_{i,j=1}^{N}\left(\| v_j(t)-v_i(t) \|^2\right)^{\frac{\theta+1}{2}} + 2pN\sum_{i=1}^{N}\| v_i(t) \|^2 .\end{aligned} \tag{2.18}$$

Afterwards, from Lemma 2.2, we have

$$\begin{aligned}\frac{\mathrm{d}V(t)}{\mathrm{d}t} &\le -\psi^{*}\left(\sum_{i,j=1}^{N}\| v_j(t)-v_i(t) \|^2\right)^{\frac{\theta+1}{2}} + 2pN\sum_{i=1}^{N}\| v_i(t) \|^2 \\ &= -\psi^{*}\left(2NV(t)\right)^{\frac{\theta+1}{2}} + 2pNV(t).\end{aligned} \tag{2.19}$$

Furthermore, from Lemma 2.3 and condition (2.3), we can obtain that

$$V(t) \equiv 0, \ t \ge T_1$$

and the convergence time is estimated by

$$T_1 \le \frac{1}{pN(1-\theta)}\ln\left(\frac{\psi^{*}}{\psi^{*}-p(2NV(0))^{\frac{1-\theta}{2}}}\right). \tag{2.20}$$

Next, we show that the second inequality of condition (2.11) holds. Similarly, considering the derivative of $X(t)$ along system (1.2), we have

$$\frac{\mathrm{d}X(t)}{\mathrm{d}t} = 2\sum_{i=1}^{N}\langle x_i(t), v_i(t)\rangle \le 2\sum_{i=1}^{N}\| x_i(t) \|\| v_i(t) \| \le 2X^{\frac{1}{2}}(t)V^{\frac{1}{2}}(t). \tag{2.21}$$

Using the comparison principle of differential equation, we have

$$\begin{aligned}X^{\frac{1}{2}}(t) &\le X^{\frac{1}{2}}(0) + \int_0^t V^{\frac{1}{2}}(s)\mathrm{d}s \\ &\le X^{\frac{1}{2}}(0) + \int_0^{T_1} V^{\frac{1}{2}}(s)\mathrm{d}s.\end{aligned} \tag{2.22}$$

From Lemma 2.3, we have $V(t) \leq V(0)$, so

$$X^{\frac{1}{2}}(t) \leq X^{\frac{1}{2}}(0) + V^{\frac{1}{2}}(0)T_1 < \infty. \tag{2.23}$$

Thus, we have $\sup_{0\leq t\leq\infty} X(t) < \infty$. This completes the proof. □

Remark 2.1. *Theorem 2.1 shows that flocking can be established in finite time, and the convergence time depends on the lower bound of the communication rate, initial states and control parameters θ and p. In fact, condition (2.3) requires small initial velocity fluctuations. This means that to achieve finite-time flocking, the initial velocity differences between particles cannot be too large, and the differences should be smaller when the number of particles is larger, which is realistic. Furthermore, the analytical results show that the group with a smaller lower bound of the communication rate takes longer to reach flocking.*

Remark 2.2. *When $F_i = 0$, the system (1.2) becomes the same model as that in reference [21]. Moreover, regarding the convergence time shown in (2.4), we have*

$$\begin{aligned}\lim_{p\to 0} t_1 &= \lim_{p\to 0} \frac{1}{pN(1-\theta)} \ln\left(\frac{\psi^*}{\psi^* - p(2NV(0))^{\frac{1-\theta}{2}}}\right)\\ &= \frac{1}{N(1-\theta)}\left(\frac{(2NV(0))^{\frac{1-\theta}{2}}}{\psi^* - p(2NV(0))^{\frac{1-\theta}{2}}}\right)\\ &= \frac{(2V(0))^{\frac{1-\theta}{2}}}{\psi^*(1-\theta)} N^{-\frac{1+\theta}{2}},\end{aligned} \tag{2.24}$$

which is consistent with the result in reference [21].

3. Collision-avoidance

Theorem 3.1. *If Theorem 2.1 holds and assume the initial states of system (1.2) satisfy*

$$\min_{i\neq j} \| x_i(0) - x_j(0) \| > 2 \| v(0) \| T_1, \tag{3.1}$$

where T_1 is defined as the inequality (2.4), then system (1.2) reaches finite-time flocking with collision-avoidance.

Proof. Denote

$$\begin{aligned}&V_{ij}(t) = \| v_i(t) - v_j(t) \|,\\ &X_{ij}(t) = \| x_i(t) - x_j(t) \|, \ i \neq j.\end{aligned} \tag{3.2}$$

From definition (2.9), we have $\| v(t) \|^2 = V(t)$. Similar to inequality (2.19), we have

$$\frac{\mathrm{d} \| v(t) \|^2}{\mathrm{d}t} \leq -C_1 \| v(t) \|^{\theta+1} + C_2 \| v(t) \|^2 . \tag{3.3}$$

Then

$$\frac{\mathrm{d} \| v(t) \|}{\mathrm{d}t} \leq -\frac{C_1}{2} \| v(t) \|^{\theta} + \frac{C_2}{2} \| v(t) \| . \tag{3.4}$$

Therefore, from Lemma 2.3 we have $\| v(t) \| \leq \| v(0) \|$, where C_i denotes positive constant whose value depends on N, such as those in inequality (2.19), and may vary from line to line. Then, it is easy to see that

$$V_{ij}(t) = \| v_i(t) - v_j(t) \| \leq 2 \| v(t) \| \leq 2 \| v(0) \| . \tag{3.5}$$

Moreover, a simple calculation shows that

$$\frac{\mathrm{d}X_{ij}^2(t)}{\mathrm{d}t} \leq 2 \| x_i(t) - x_j(t) \| \| v_i(t) - v_j(t) \| = 2X_{ij}(t)V_{ij}(t). \tag{3.6}$$

Then

$$\frac{\mathrm{d}X_{ij}(t)}{\mathrm{d}t} \leq V_{ij}(t). \tag{3.7}$$

Note that $V_{ij}(t) = 0$, for $t \geq T_1$. Thus, we have

$$|X_{ij}(t) - X_{ij}(0)| = \left| \int_0^t \frac{\mathrm{d}X_{ij}(s)}{\mathrm{d}s} \right| \leq \int_0^{T_1} \left| V_{ij}(s) \right| \mathrm{d}s \leq 2 \| v(0) \| T_1. \tag{3.8}$$

Furthermore, we obtain

$$|X_{ij}(t)| \geq |X_{ij}(0)| - |X_{ij}(t) - X_{ij}(0)| \geq \min_{i \neq j} |X_{ij}(0)| - 2 \| v(0) \| T_1 > 0. \tag{3.9}$$

This completes the proof. □

4. A special case

In this section, we give a special form of $F_i(v_1, v_2, \ldots, v_N)$ to explore the convergence time of finite-time flocking, which is described by

$$F_i(v_1, v_2, \ldots, v_N) = k(v_i(t) - \overline{v}(t)), \tag{4.1}$$

where $\overline{v}(t) = \frac{1}{N} \sum_{i=1}^{N} v_i(t)$ and k is a positive constant. Then, we have

$$\sum_{i=1}^{N} F_i(v_1, v_2, \ldots, v_N) = 0 \tag{4.2}$$

and

$$\begin{aligned} \langle F_i, v_i \rangle &= k\langle v_i, v_i \rangle - k\langle \overline{v}, v_i \rangle \\ &\leq \frac{k}{2N} \sum_j \| v_j \|^2 + \frac{3k}{2} \| v_i \|^2 \\ &\leq \frac{k(3N+1)}{2N} \| v \|^2 . \end{aligned} \tag{4.3}$$

Therefore, $F_i(v_1, v_2, \ldots, v_N)$ satisfies Assumption 1.2 and $p = \frac{k(3N+1)}{2N}$. Therefore, according to Theorem 2.1, we can obtain the convergence time.

To illustrate the effectiveness of the above theoretical results, we performed numerical simulations in MATLAB R2019b. To facilitate the simulation and match the actual values, we consider 10 cluster agents in a 2-dimensional space, i.e., $N = 10, d = 2$. The initial positions of the particles are generated by random real numbers in the interval [0, 100], and the initial velocities are generated by random real numbers in the interval [0, 1], which are shown in Table 1.

Table 1. The initial condition.

Agents	Initial position	Initial velocity
1	(3.05409463, 74.40742603)	(0.059618867, 0.681971904)
2	(50.00224355, 47.99221411)	(0.042431137, 0.071445464)
3	(90.47222380, 60.98666484)	(0.521649842, 0.096730025)
4	(61.76663895, 85.94423056)	(0.818148553, 0.817547092)
5	(80.54894245, 57.67215156)	(0.722439592, 0.149865442)
6	(18.29224694, 23.99320105)	(0.659605252, 0.518594942)
7	(88.65119330, 2.867415246)	(0.972974554, 0.648991492)
8	(48.99013885, 16.79271456)	(0.800330575, 0.453797708)
9	(97.86806496, 71.26944716)	(0.432391503, 0.825313795)
10	(50.04716241, 47.10883745)	(0.083469814, 0.133171007)

Next, take $\psi(s) = 0.1$, $\theta = 0.5$, $k = 0.1$. Figure 1(a) displays the evolutions of velocities $\| v_i(t) \|$, and it can be observed that all velocities tend to a constant after approximately $t = 1.53$ s. Moreover, we can obtain the convergence time $t_1 \approx 1.65$ s by simple calculations from inequality (2.4). Furthermore, Figure 1(b,c) displays the evolutions of the minimum distances and maximum distances between agents. The diameter of the group is bounded, and any two agents do not collide during the process of flocking. Therefore, Theorem 2.1 and 3.1 are verified.

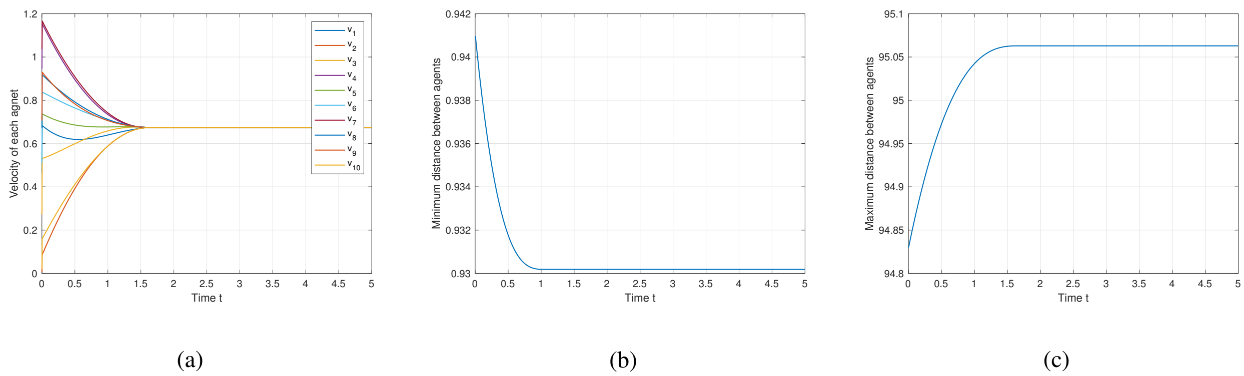
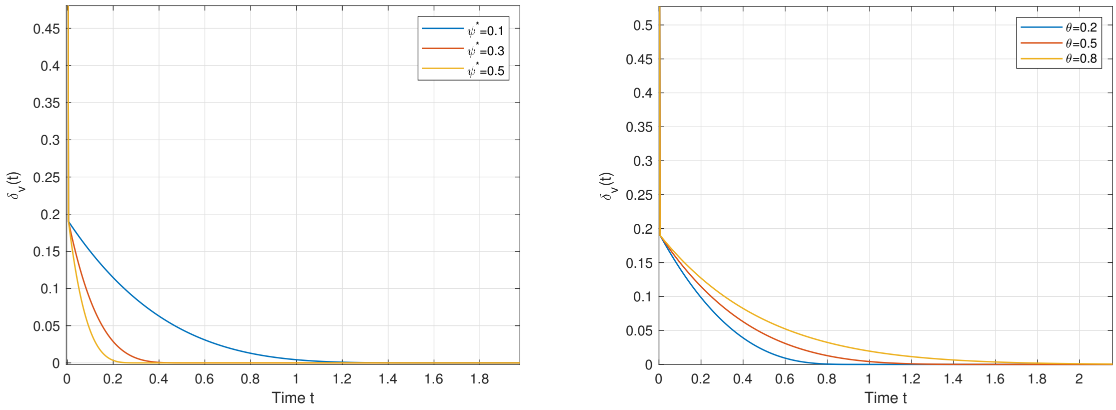


Figure 1. The evolutions of agents. (a) The evolutions of agents' velocities. (b) The evolutions of the minimum distances between agents. (c) The evolutions of the maximum distances between agents.

To further study the convergence time of finite-time flocking, we give a numerical example to analyze the influence of relevant parameters on the convergence speed. Here, $F_i(v_1, v_2, \ldots, v_N) = k(v_i(t) - \overline{v}(t))$, $k = 0.1$, and different values of $\psi(s)$ and θ are applied. $\delta_v(t) = \sum_{i=1}^{N} \| v_i(t) - \overline{v}(t) \|^2$ is used to characterize the transition to ordered flocking behavior. Then,

Figure 2(a) gives the evolutions of the indicator $\delta_v(t)$ with different values of $\psi(s)$ such that $\psi(s) = 0.1,\ 0.3,\ 0.5$, and $\theta = 0.5$ is fixed. This shows that there is a negative correlation between the convergence time T_1 and the lower bound of the communication rate. Figure 2(b) gives the evolutions of the indicator $\delta_v(t)$ with different values of θ such that $\theta = 0.2,\ 0.5,\ 0.8$, and $\psi(s) = 0.1$ is fixed. We observe that there is a positive correlation between the convergence time T_1 and the parameter θ. From inequality (2.4), we obtain that this is consistent with our theoretical analysis.



(a) (b)

Figure 2. The influence of relevant parameters on the convergence speed. (a) The influence of ψ^* on the convergence speed. (b) The influence of θ on the convergence speed.

5. Conclusions

In the practical application of a multi-agent system, we usually add various inter-driving forces to the system so that the system can not only be used in different application scenarios but also form a flocking phenomenon faster. Based on this, this paper presents a C-S model with inter-particle bonding forces to study finite-time flocking. First, we show that the system forms finite-time flocking when the forces between particles satisfy certain conditions and provide the specific estimation expression of the convergence time. Then, we provide sufficient conditions for the system to achieve collision-avoiding results. Finally, we provide a concrete example to illustrate our theoretical results and analyze the influence of relevant parameters on the convergence speed.

For finite-time flocking, to achieve different results in engineering, different forms of inter-driving forces need to be studied. For example, special configurations, such as the line shape and ball shape shown in [16], can be achieved by replacing $F_i(v_1, v_2, \ldots, v_N)$ with $F_i(x_1, x_2, \ldots, x_N)$. This is a very challenging but worthwhile problem and is an important direction for our future research.

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Conflicts of interest

The authors report there are no competing interests to declare.

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