Expected Bayesian estimation for exponential model based on simple step stress with Type-I hybrid censored data

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Abstract: The procedure of selecting the values of hyper-parameters for prior distributions in Bayesian estimate has produced many problems and has drawn the attention of many authors, therefore the expected Bayesian (E-Bayesian) estimation method to overcome these problems. These approaches are used based on the step-stress acceleration model under the Exponential Type-I hybrid censored data in this study. The values of the distribution parameters are derived. To compare the E-Bayesian estimates to the other estimates, a comparative study was conducted using the simulation research. Four different loss functions are used to generate the Bayesian and E-Bayesian estimators. In addition, three alternative hyper-parameter distributions were used in E-Bayesian estimation. Finally, a real-world data example is examined for demonstration and comparative purposes.

Keywords: E-Bayesian estimation; Bayesian estimation; simple step stress; Type-I hybrid censoring; exponential distribution

1. Introduction

In statistical inference, the prior distribution and loss function must be chosen carefully. However, the hyper parameters may influence the prior distribution parameters. We frequently employ the hierarchical Bayesian technique in this case. The concept of hierarchical prior distribution was initially proposed by Lindley and Smith [1]. The hierarchical Bayesian technique requires two steps to complete the prior distribution setting, making it more resilient than the Bayesian method. The method for constructing hierarchical prior distribution was developed by Han [2]. Data analysis has recently employed hierarchical Bayesian techniques; for further information, see Ando and Zellner [3], Han [4], Kzlaslan [5], and Han [6]. For testing data from products with exponential distributions and
the quadratic loss function, from Han [7], the reliability parameter was estimated using E-Bayes and hierarchical Bayes methods. With the help of simulation studies, he proved that the E-Bayesian estimator is both efficient and simple to use. For estimating the dependability parameter of the geometric distribution based on scaled squared loss function in complete samples, Yin and Liu [8] built the E-Bayesian estimation and hierarchical Bayesian estimation algorithms. In terms of calculation complexity, the E-Bayes technique is more stable and convenient than the hierarchical Bayes method, they concluded. For additional information on related studies of the E-Bayesian estimation approach, see Jaheen and Okasha [9], Cai et al. [10], Okasha [11], and Azimi et al. [12]. Because of the rapid development of advanced technology, products and devices are becoming more and more reliable, and product life is increasing. Under normal conditions, obtaining failure information for such highly reliable products is difficult, if not impossible. As a result, accelerated life testing (ALT) is the most common method for obtaining sufficient failure time data in a short period of time. In such test conditions, products are subjected to higher-than-usual levels of stress in order to induce early failures. Failure time data from such accelerated tests are analysed and extrapolated to estimate life characteristics under normal operating conditions. One of the most important types of ALT is the step-stress life testing (SSALT) in which the experimenter can choose one or more stress factors in the experiment, such as temperature, vibration, or humidity that may affect the product’s life. A set of identical experimental units, say \( n \), are examined in an appropriate testing experiment under a starting stress level of \( s_1 \), and then the stress levels are increased to \( s_2, s_3, ..., s_j \) at predetermined times, say \( \tau_1, \tau_2, ..., \tau_j \) respectively. In SSLT if the experiment is performed depending on two stress levels say \( s_0, s_1 \), then this type is reduced to the simple step-stress life testing (SSSSLT).

The loss function, is crucial in Bayesian approaches. The squared error loss function is the most often used loss function in Bayesian inference (SELF). This loss function is symmetrical, meaning that overestimation and underestimation are given equal weight. The following is the definition of the square error loss function (SELF):

\[
L_{BS}(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2, \tag{1.1}
\]

where \( \hat{\theta} \) is an estimator of \( \theta \). The Bayes estimator of \( \theta \) SELF denoted by \( \hat{\theta}_{BS} \) can be obtained as

\[
\hat{\theta}_{BS} = E_{\theta}[\theta]. \tag{1.2}
\]

where \( E_{\theta}[\theta] \) is the expected value is determined with respect to the posterior distribution. Bayesian estimation is derived by using the Degroot loss function (DLF) which is defined by Degroot [13] as follows:

\[
L_{BD}(\hat{\theta}, \theta) = \left( \frac{\theta - \hat{\theta}}{\hat{\theta}} \right)^2, \tag{1.3}
\]

the Bayesian estimator based on DLF is denoted by \( \hat{\theta}_{BD} \) and can be expressed as

\[
\hat{\theta}_{BD} = \frac{E_{\theta}[^2]}{E_{\theta}[\theta]} . \tag{1.4}
\]

The quadratic loss function (QLF) was defined as follows:

\[
L_{BQ}(\hat{\theta}, \theta) = (\theta - \hat{\theta})^2, \tag{1.5}
\]

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we can get bayesian estimation under quadratic loss denoted by $\hat{\theta}_{BQ}$ can be obtained as

$$\hat{\theta}_{BQ} = \frac{E_{\theta}[\theta^{-1}]}{E_{\theta}[\theta^{-2}]}.$$  (1.6)

Under the assumption that the minimal loss occurs at $\hat{\theta} = \theta$, the LINEX loss function (LLF) can be expressed as

$$L_{BL}(\hat{\theta}, \theta) = \exp\left[\nu(\hat{\theta} - \theta)\right] - \nu(\hat{\theta} - \theta) - 1$$  (1.7)

where $\nu \neq 0$. The Bayesian estimator of $\theta$, denoted by $\hat{\theta}_{BL}$ under LLF, the value $\hat{\theta}_{BL}$ which minimizes $E_{\theta}[L_{BL}(\hat{\theta}, \theta)]$ is given by,

$$\hat{\theta}_{BL} = \frac{-1}{\nu} \ln\{E_{\theta}[\exp(-\nu\theta)]\}.$$  (1.8)

where $E_{\theta}[\exp(-\nu\theta)]$ is finite. The maximum likelihood and Bayesian estimation methods are regarded as the inferential features in these investigations. Studying the E-Bayesian estimators and the accompanying properties in the presence of the SSLT model based on Type-I hybrid censoring, however, has not received much attention. Additionally, we present a set of guidelines that we believe applied statisticians and reliability engineers will find extremely useful when selecting the appropriate estimation method to estimate the unknown parameters of the exponential distribution under the SSLT model. Furthermore, a simulation study and analysis of both simulated and real data sets demonstrate that E-Bayesian estimators outperform alternative estimators based on maximum likelihood and Bayesian approaches, encouraging their application in practical contexts. The resulting estimators are obtained based on four different loss functions. by using SEF, DLF, QLF and LLF. This article is organized as follows: Section 2 provides an overview of the step-stress acceleration model depending on the Type-I hybrid censored data. In Section 3, determines the maximum likelihood estimates (ML) of unknown parameters, In Section 4, Bayesian estimation of unknown parameters under different prior distributions and different loss functions are computed. In Section 5, the formulas of E- Bayesian are discussed. Comparison between Bayes and E-Bayes estimates have been made using simulation study in Section 6. A real data set is analyzed in Section 7. Finally, the paper is concluded in Section 8.

### 2. Description of the model

In this section, we assume that the data are drawn from a cumulative exposure model, by applying a simple step-stress technique with Type-I HCS using two stress levels $s_0$ and $s_1$. The lifespan distributions at $s_0$ and $s_1$ are following the exponential distribution with failure rates of 1 and 2, correspondingly. The probability density function (PDF) and cumulative distribution function (CDF) are presented by

$$f_i(y; \lambda_j) = \lambda_j \exp(-\lambda_j y), y \geq 0, \lambda_j > 0, j = 1, 2$$  (2.1)

and

$$F_j(y; \lambda_j) = 1 - \exp(-\lambda_j y), y \geq 0, \lambda_j > 0, j = 1, 2$$  (2.2)

respectively. As a result, the cumulative exposure distribution (CED) $G(y)$ is given as
2) Case 2: Suppose $\tau < Y$

1) Case 1: Suppose $\tau > Y$

3) Case 3: Suppose $\tau = Y$

where $F_j(\cdot)$ is as given in (2.2). The corresponding PDF is:

$$
\begin{align*}
G(y) &= \begin{cases} 
G_1(y) = F_1(y; \lambda_1) & \text{if } 0 < y < \tau, \\
G_2(y) = F_2\left(y - \left(1 - \frac{\lambda_1}{\lambda_2}\right)\tau; \lambda_2\right) & \text{if } \tau \leq y < \infty,
\end{cases}
\end{align*}
$$

(2.3)

Based on the Type-I HCS, we have $n$ units under $s_0$ stress level. At time $\tau$, the stress level is raised to $s_1$, and the life-testing test is finished at $T^*$. Here, $T^* = \min\{Y_{r:n}, T\}$, we will observe the instances below:

- $r \leq n$ and $0 < \tau < T < \infty$ are Predetermined in ahead of time;
- $Y_{1:n} < \cdots < Y_{n:n}$ display the $n$ units’ failure times in order;
- $T$ represents a certain period when the stress level shifts from $s_0$ to $s_1$;
- $Y_{r:n}$ indicates the time at which the $r$th fails;
- $T$ stands for the experiment’s maximum time limit;
- $d$ indicates the number of units that fail prior to time $T$;
- $T^*$ is the random moment at which the life-testing experiment comes to an end;
- $D^*$ stands for the number of components that break before $T$.

Let $m_1$ represent the number of units that fail before time $\tau$, $m_2$ be the number of units that fail after the time $\tau$ and before time $T^*$ at stress level $s_1$, where $T^*$ is termination time of the experiment, it is given by,

$$
\begin{align*}
T^* &= \begin{cases} 
T, & \text{if } T < Y_{r:n}, \\
Y_{r:n}, & \text{if } Y_{r:n} \leq T,
\end{cases}
\end{align*}
$$

(2.5)

Using this notation, we will notice one of the following three cases:

1) Case 1: Suppose $Y_{r:n} \leq \tau < T$, we will observe \{\(y_{1:n} < \cdots < y_{r:n} \leq \tau < T\)\}.

2) Case 2: Suppose $\tau < Y_{r:n} \leq T$, we will observe \{\(y_{1:n} < \cdots < y_{m_1:n} \leq \tau < y_{m_1+1:n} < \cdots < y_{r:n} \leq T\)\}.

3) Case 3: Suppose $T < Y_{r:n}$, we will observe \{\(y_{1:n} < \cdots < y_{m_1:n} \leq \tau < y_{m_1+1:n} < \cdots < y_{m_1+m_2:n} \leq T^* = T\)\}.

We can write the likelihood function of $\lambda_1$ and $\lambda_2$ based on the Type-I hybrid censored sample using (2.3) and (2.4), as follows:

$$
L(\lambda_1, \lambda_2|\mathbf{x}) = \begin{cases} 
\frac{n!}{(n-r)!} \left(\prod_{j=1}^{r} g_1(y_{j:n})\right) \{1 - G_1(Y_{r:n})\}^{n-r}, & \text{Case 1} \\
\frac{n!}{(n-D^*)!} \left(\prod_{j=1}^{m_1} g_1(y_{j:n})\right) \left(\prod_{j=m_1+1}^{D^*} g_2(y_{j:n})\right) \{1 - G_2(T^*)\}^{n-D^*}, & \text{Cases 1 and 2}
\end{cases}
$$

(2.6)

where is the total number of failures and is given by,

$$
D^* = m_1 + m_2 = \begin{cases} 
d, & \text{if } T < Y_{r:n}, \\
r, & \text{if } Y_{r:n} \leq T,
\end{cases}
$$

(2.7)
3. Maximum likelihood estimation

We must maximize the likelihood with regard to \( \lambda_1 \) and \( \lambda_2 \) when computing the ML estimates. Using (2.3), (2.4) and (2.6), then the appropriate likelihood function, which is as follows:

\[
L(\lambda_1, \lambda_2 | x) = \begin{cases} 
\frac{n!}{(n-r)!} \lambda_1^r \exp\left\{-\lambda_1 \sum_{j=1}^{r} y_{j,n} + (n - r) y_{j,n}\right\}, & \text{case 1,} \\
\frac{n!}{(n-D^*)!} \lambda_1^{m_1} \lambda_2^{m_2} \exp\left\{-\lambda_1 W_1(x) - \lambda_2 W_2(x)\right\} & \text{case 2, 3,}
\end{cases}
\]

(3.1)

where

\[
W_1(x) = \sum_{j=1}^{m_1} y_{j,n} + (n - m_1) \tau,
\]

(3.2)

\[
W_2(x) = \begin{cases} 
\sum_{j=m_1+1}^{d} (y_{j,n} - \tau) + (n - d^*) (T^* - \tau), & \text{if } T < Y_{r,n}, \\
\sum_{j=m_1+1}^{r} (y_{j,n} - \tau) + (n - r) (y_{r,n} - \tau) & \text{if } \tau < Y_{r,n} \leq T
\end{cases}
\]

(3.3)

From Eq (3.1), we can deduce the following.

1) In Case 3, when \( m_1 = 0 \) and \( m_2 = 0 \), the MLEs of \( \lambda_1 \) and \( \lambda_2 \) do not exist.

2) In Cases 1 and 3, when \( m_1 \neq 0, m_2 = 0 \), the MLE of \( \lambda_2 \) does not exist, and \( W_1(x) \) is a complete sufficient statistic for \( \lambda_1 \).

3) If \( m_1 = 0, m_2 \neq 0 \) in Cases 2 and 3, the MLE of \( \lambda_1 \) does not exist, and \( W_2(x) \) is a complete sufficient statistic for \( \lambda_2 \).

4) If at least one failure happens before \( \tau \) and between \( \tau \) and \( T \) in Cases 2 and 3, the MLEs of \( \lambda_1 \) and \( \lambda_2 \) do exist, and \( (W_1(x), W_2(x)) \) is a joint complete sufficient statistic for \( (\lambda_1, \lambda_2) \). In this situation, the log-likelihood function of \( \lambda_1 \) and \( \lambda_2 \) is given by,

\[
\log (L(\lambda_1, \lambda_2 | x)) = \log \frac{n!}{(n-D^*)!} + m_1 \log (\lambda_1) + m_2 \log (\lambda_2) - \lambda_1 W_1(x) - \lambda_2 W_2(x)
\]

(3.4)

From (3.4), the MLEs of \( \lambda_1 \) and \( \lambda_2 \) are easily determined as

\[
\tilde{\lambda}_{1 \text{ML}} = \frac{m_1}{W_1(x)}, \quad \tilde{\lambda}_{2 \text{ML}} = \frac{m_2}{W_2(x)}.
\]

(3.5)

(3.6)
4. Bayesian estimation

In this Section, the Bayes estimators for the parameters $\lambda_1$ and $\lambda_2$ using SEF, DLF, QLF and LLF are derived. For creating the Bayesian estimation, we suppose that the parameters $\lambda_1$ and $\lambda_2$ are independently distributed and following gamma distribution. Let $\lambda_1$, $\lambda_2$, have gamma priors with scale parameters $b_i$ and shape parameters $a_i$, $i = 1, 2$. The joint prior density of $\lambda_1$ and $\lambda_2$ can be expressed as follows

$$\pi(\lambda_1, \lambda_2) \propto \prod_{i=1}^{2} \lambda_i^{a_i-1} \exp\left(-b_i\lambda_i\right), b_i, a_i > 0, \text{ for } i = 1, 2. \quad (4.1)$$

The posterior PDF of $\lambda_1$ and $\lambda_2$ is given from (2.6), (4.1), as follows:

$$\pi^*(\lambda_1, \lambda_2|\mathbf{x}) = \frac{1}{I} \prod_{i=1}^{2} \lambda_i^{m_i+a_i-1} \exp\left[-\lambda_i \left(\sum_{i=1}^{2} X_i + b_i\right)\right], \text{ for } i = 1, 2, \quad (4.2)$$

where $I$ is the normalizing constant given as

$$I = \int_{0}^{\infty} \int_{0}^{\infty} \pi^*(\lambda_1, \lambda_2|\mathbf{x}) \, d\lambda_1 \, d\lambda_2 = \prod_{i=1}^{2} \frac{\Gamma(m_i + a_i)}{\left[\sum_{i=1}^{2} X_i + b_i\right]^{(m_i+a_i)}}, \quad (4.3)$$

From (4.2), it is worth noting that the posterior density functions of $\lambda_i$ for $i = 1, 2$ are similar to $\text{gamma}(n_i + a_i, \sum_{i=1}^{2} X_i + b_i)$. Based on the SELF, the Bayes estimators of $\lambda_i$ with $i = 1, 2$ are given by,

$$\hat{\lambda}_{iBS} = E[\lambda_i] = I^{-1} \int_{0}^{\infty} \int_{0}^{\infty} \lambda_i^{m_i+a_i-1} \exp\left[-\lambda_i \left(\sum_{i=1}^{2} X_i + b_i\right)\right] \, d\lambda_1 \, d\lambda_2 = \frac{m_i + a_i}{\sum_{i=1}^{2} X_i + b_i}, \text{ for } i = 1, 2. \quad (4.4)$$

The Bayesian estimate of $\lambda_i$ for $i = 1, 2$ under DLF loss function is given by,

$$\hat{\lambda}_{iBD} = \frac{E[\lambda_i^2]}{E[\lambda_i]} = I^{-1} \int_{0}^{\infty} \int_{0}^{\infty} \lambda_i^{m_i+a_i-1} \exp\left[-\lambda_i \left(\sum_{i=1}^{2} X_i + b_i\right)\right] \, d\lambda_1 \, d\lambda_2 = \frac{m_i + a_i + 1}{\sum_{i=1}^{2} X_i + b_i}, \text{ for } i = 1, 2. \quad (4.5)$$
The Bayesian estimate of $\lambda_i$ for $i = 1, 2$ under QLF is given by,

$$\tilde{\lambda}_{iBQ} = \frac{E[\lambda_i^{-1}]}{E[\lambda_i^{-2}]} = \frac{I^{-1} \int_0^\infty \lambda_i^{-1} \sum_{i=1}^2 \lambda_i^{m_i+a_i-1} \exp[-\lambda_i(W_i(x) + b_i)] d\lambda_1 d\lambda_2}{I^{-1} \int_0^\infty \lambda_i^{-2} \sum_{i=1}^2 \lambda_i^{m_i+a_i-1} \exp[-\lambda_i(W_i(x) + b_i)] d\lambda_1 d\lambda_2}$$

for $i = 1, 2$. The Bayesian estimate of $\lambda_i$ for $i = 1, 2$ under LLF is given by,

$$\tilde{\lambda}_{iBL} = \frac{-1}{\nu} \ln \left[ E [ \exp(-\nu \lambda_i)] \right] = \frac{-1}{\nu} \ln \left\{ I^{-1} \int_0^\infty \int_0^\infty \exp(-\nu \lambda_i) \sum_{i=1}^2 \lambda_i^{m_i+a_i-1} \exp[-\lambda_i(W_i(x) + b_i)] d\lambda_1 d\lambda_2 \right\}$$

for $i = 1, 2$.

5. E-Bayesian estimation method

Here, three different prior distributions of hyper-parameters are investigated in this section to see how they affect the E-Bayesian estimates of $\lambda_i$ for $i = 1, 2$. We select the hyper-parameters $a_i$ and $b_i$ for $i = 1, 2$ to prove that $\pi(\lambda)$ is a decreasing function of $\lambda_i$. The first derivative of $\pi(\lambda_i)$ regarding $\lambda_i$ for $i = 1, 2$ is as follows:

$$\frac{\partial \pi(\lambda_i)}{\partial \lambda_i} \propto \lambda_i^{a_i-1} e^{-b_i\lambda_i} [\lambda_i - 1] - b_i \lambda_i].$$

Thus, for $0 < a_i < 1$ and $b_i > 0$, the prior PDF $\pi(\lambda_i)$ is a decreasing function of $\lambda_i$ for $i = 1, 2$. Suppose that $a_i$ and $b_i$, $i = 1, 2$ are independent with bivariate PDF given by,

$$p(a_i, b_i) = p(a_i)p(b_i), \text{ for } i = 1, 2$$

the E-Bayesian (EB) estimates of the parameter $\lambda_i$ are expectation of the Bayesian estimate of $\lambda_i$ for $i = 1, 2$ and can be obtained as follows:

$$\tilde{\lambda}_{iEB} = E[\lambda_i|x] = \int \lambda_i p(a_i, b_i) da_i db_i.$$
prior distributions affect the estimation of the E-Bayesian of $\lambda_i$ for $i = 1, 2$. We suggest the following prior PDFs

\begin{align}
p_1(a_i, b_i) &= \frac{1}{c_i}, & 0 < a_i < 1, 0 < b_i < c_i, \\
p_2(a_i, b_i) &= \frac{2b_i}{c_i^2}, & 0 < a_i < 1, 0 < b_i < c_i, \\
p_3(a_i, b_i) &= \frac{2(c-b_i)}{c_i}, & 0 < a_i < 1, 0 < b_i < c_i,
\end{align}

(5.4)

For more details, one can refer to Rabie and Li [14–16], and Rabie [17].

5.1. E-Bayesian estimation using SELF

The E-Bayesian estimate of $\lambda_i$ for $i = 1, 2$, under the SEL based on $p_1(a_i, b_i)$, $p_2(a_i, b_i)$, and $p_3(a_i, b_i)$ are computed from (4.4), (5.3) and (5.4), respectively, as follows:

\begin{align}
\hat{\lambda}^1_{i,EBS} &= \int_0^1 \int_0^{c_i} \frac{1}{c_i} \left( \frac{m_i + a_i}{W_i(x) + b_i} \right) \, db \, da_i \\
&= \frac{2m_i + 1}{2c_i} \ln \left( 1 + \frac{c_i}{W_i(x)} \right), \text{ for } i = 1, 2.
\end{align}

(5.5)

\begin{align}
\hat{\lambda}^2_{i,EBS} &= \int_0^1 \int_0^{c_i} \frac{2b_i}{c_i^2} \left( \frac{m_i + a_i}{W_i(x) + b_i} \right) \, db \, da_i \\
&= \frac{2m_i + 1}{c_i} \left[ 1 - \frac{W_i(x)}{c_i} \ln \left( 1 + \frac{c_i}{W_i(x)} \right) \right], \text{ for } i = 1, 2.
\end{align}

(5.6)

\begin{align}
\hat{\lambda}^3_{i,EBS} &= \int_0^1 \int_0^{c_i} \frac{2(c-b_i)}{c_i} \left( \frac{m_i + a_i}{W_i(x) + b_i} \right) \, db \, da_i \\
&= \frac{2m_i + 1}{c_i} \left( 1 + \frac{W_i(x)}{c_i} \ln \left( 1 + \frac{c_i}{W_i(x)} \right) - 1 \right), \text{ for } i = 1, 2.
\end{align}

(5.7)

5.2. E-Bayesian estimation using DLF

Based on $p_1(a_i, b_i)$, $p_2(a_i, b_i)$, and $p_3(a_i, b_i)$, under the DLF, the E-Bayesian estimate of $\lambda_i$ can be derived from (4.5), (5.3) and (5.4), respectively as follows:

\begin{align}
\hat{\lambda}^1_{i,EBD} &= \int_0^1 \int_0^{c_i} \frac{1}{c_i} \left( \frac{m_i + a_i + 1}{W_i(x) + b_i} \right) \, db \, da_i \\
&= \frac{2m_i + 3}{2c_i} \ln \left( 1 + \frac{c_i}{W_i(x)} \right), \text{ for } i = 1, 2.
\end{align}

(5.8)

\begin{align}
\hat{\lambda}^2_{i,EBD} &= \int_0^1 \int_0^{c_i} \frac{2b_i}{c_i^2} \left( \frac{m_i + a_i + 1}{W_i(x) + b_i} \right) \, db \, da_i
\end{align}
estimate of \( p_3 \) using LLF

\[ \frac{2m_i + 3}{c_i} \left[ 1 - \frac{W_i(x)}{c_i} \ln \left( 1 + \frac{c_i}{W_i(x)} \right) \right], \text{ for } i = 1, 2, \tag{5.9} \]

\[ \lambda_{i,EBQ}^3 = \int_0^1 \int_0^{c_i} 2 (c - b_i) \frac{m_i + a_i + 1}{c_i^2} \frac{W_i(x) + b_i}{W_i(x)} db_i da_i \]
\[ = \frac{2m_i + 3}{c_i} \left[ 1 + \frac{W_i(x)}{c_i} \ln \left( 1 + \frac{c_i}{W_i(x)} \right) - 1 \right], \text{ for } i = 1, 2. \tag{5.10} \]

5.3. E-Bayesian estimation using QLF

The E-Bayesian estimate of \( \lambda_i \) for \( i = 1, 2 \), under the QLF based on \( p_1(a_i, b_i) \), \( p_2(a_i, b_i) \), and \( p_3(a_i, b_i) \) are computed from (4.6), (5.3) and (5.4), respectively, as follows:

\[ \lambda_{i,EBQ}^2 = \int_0^1 \int_0^{c_i} \frac{1}{c_i} \frac{m_i + a_i - 2}{W_i(x) + b_i} db_i da_i \]
\[ = \frac{2m_i - 3}{2c_i} \ln \left( 1 + \frac{c_i}{W_i(x)} \right), \text{ for } i = 1, 2, \tag{5.11} \]

\[ \lambda_{i,EBQ}^3 = \int_0^1 \int_0^{c_i} \frac{2b_i}{c_i^2} \frac{m_i + a_i - 2}{W_i(x) + b_i} db_i da_i \]
\[ = \frac{2m_i - 3}{c_i} \left[ 1 - \frac{W_i(x)}{c_i} \ln \left( 1 + \frac{c_i}{W_i(x)} \right) \right], \text{ for } i = 1, 2, \tag{5.12} \]

\[ \lambda_{i,EBQ}^3 = \int_0^1 \int_0^{c_i} \frac{2 (c - b_i)}{c_i^2} \frac{m_i + a_i - 2}{W_i(x) + b_i} db_i da_i \]
\[ = \frac{2m_i - 3}{c_i} \left[ 1 + \frac{W_i(x)}{c_i} \ln \left( 1 + \frac{c_i}{W_i(x)} \right) - 1 \right], \text{ for } i = 1, 2. \tag{5.13} \]

5.4. E-Bayesian estimation using LLF

Also, based on \( p_1(a_i, b_i) \), \( p_2(a_i, b_i) \), and \( p_3(a_i, b_i) \), under the LINEX loss function, the E-Bayesian estimate of \( \lambda_i \), can be derived from (4.8), (5.3) and (5.4), respectively as follows:

\[ \lambda_{i,EBL}^1 = \frac{(m_i + a_i)}{\nu} \int_0^1 \int_0^{c_i} \frac{1}{c_i} \ln \left[ \frac{W_i(x) + b_i + \nu}{W_i(x) + b_i} \right] db_i da_i \]
\[ = \frac{2m_i + 1}{2 \nu c_i} \left\{ (c_i + W_i(x)) \ln \left( 1 + \frac{\nu}{W_i(x) + c_i} \right) + \nu \right. \]
\[ \times \ln \left( 1 + \frac{c_i}{W_i(x) + \nu} \right) - W_i(x) \ln \left( 1 + \frac{\nu}{W_i(x)} \right) \}, \tag{5.14} \]
chosen randomly to be \( \lambda \) where \( \lambda \) is the least mean square error. These values are generated from Gamma distribution as follows:

\[
\Lambda^2_{EBL} = \frac{(m_i + a_i)}{v} \int_0^1 \int_0^{c_i} \frac{2b_i}{c_i^2} \ln \left[ \frac{W_i(x) + b_i + v}{W_i(x) + b_i} \right] db_ida_i
\]

\[
= \frac{2m_i + 1}{2v c_i^2} \left\{ \nu c_i - \nu (2W_i(x) + v) \ln \left( 1 + \frac{c_i}{W_i(x) + v} \right) + \left[ W_i(x) \right]^2 \right. \\
\times \ln \left( 1 + \frac{v}{W_i(x)} \right) - \left( c_i^2 - \left[ W_i(x) \right]^2 \right) \ln \left( 1 + \frac{v}{W_i(x) + c_i} \right) \right\}
\]

\[ (5.15) \]

\[
\Lambda^3_{EBL} = \frac{(m_i + a_i)}{v} \int_0^1 \int_0^{c_i} \frac{2(c_b - b_i)}{c_i^2} \ln \left[ \frac{W_i(x) + b_i + v}{W_i(x) + b_i} \right] db_ida_i
\]

\[
= \frac{2m_i + 1}{2v c_i^2} \left\{ -c_i c_i + W_i(x) (W_i(x) + 2c_i) \ln \left( 1 + \frac{c_i}{W_i(x)} \right) + v (2W_i(x) + c_i) + v \right. \\
\times \ln \left( 1 + \frac{c_i}{W_i(x) + v} \right) + (c_i + [W_i(x)]^2) \ln \left( 1 + \frac{v}{W_i(x) + c_i} \right) \right\}
\]

\[ (5.16) \]

For recent work of Bayesian estimation and loss functions, see for example, Nagy et al. in [18], Nagy and Alrasheedi in [19, 20], and [21], and Raheem et al. in [22].

### 6. Simulation studies

In this section, we provide some simulation results for various choices of \((n, m, \tau, T)\), where, \(n = 50, 80, 100\) and \(m = 30, 64, 80\) with two values of both \(\tau = 0.5, 0.8\) and \(T = 1.6, 2.5\). The values of \(a_i\) and \(b_i\), \(i = 1, 2\) are generated from Eq (5.4). Its chosen to be \((a_1, b_1) = (0.6, 0.7)\) for \(\lambda_1\) with \(c_1 = 0.75\), where \((a_2, b_2) = (0.4, 0.8)\) for \(\lambda_2\) with \(c_2 = 0.85\). For a given values of \(a_i\) and \(b_i\), \(i = 1, 2\), values of \(\lambda_1, \lambda_2\) are generated from Gamma\((a, b)\). By trying and error, the values of parameters have been chosen randomly to be \((\lambda_1, \lambda_2) = (0.85, 0.5)\). In the same way, these values provide short lifetimes and the least mean square error. These values of \(\lambda_1, \lambda_2\) are used to generate Type-I hybrid censored sample from Exponential distribution as follows:

\[
X = \frac{-1}{\lambda_k} (\ln(1 - U)), \quad k = 1, 2,
\]

where, \(U\) is generated from \(U(0, 1)\). All estimators are obtained in an explicit form. The maximum likelihood estimates of \((\lambda_1, \lambda_2)\) are given from Eqs (3.5) and (3.6), respectively. The Bayesian estimates under SELF, DLF, QLF and LLF are obtained from Eqs (4.4), (4.6), (4.5) and (1.8), respectively. The E-Bayesian estimates based on SELF, DLF, QLF and LLF are obtained from Eqs (5.5–5.7), (5.8–5.10), (5.11–5.13) and (5.14–5.16), respectively. All results are listed in Table 1, for \(\lambda_1\) and in Table 2, for \(\lambda_2\). The real data example is performed based on the same procedures and by using the four loss functions and listed the results in Tables 3 and 4.

Figures 1 and 2 were created to demonstrate the differences between the Bayesian and E-Bayesian estimates based on the three prior distributions of the hyperparameters a and b for each loss function in order to examine the pertinent aspects of E-Bayesian estimation. In Figure 1(a)–(d), we compared the
Table 1. The average estimated values (AE) and the mean square error (MSE) for $\lambda_1$ when $\lambda_1 = 0.8571$, $\lambda_2 = 0.5$, $a_1 = 0.6$, $b_1 = 0.7$, $c_1 = 0.75$, $a_2 = 0.4$, $b_2 = 0.8$, $c_2 = 0.85$.

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Table 2. The average estimated values (AE) and the mean square error (MSE) for $\lambda_2$ when $\lambda_1 = 0.8571$, $\lambda_2 = 0.5$, $a_1 = 0.6$, $b_1 = 0.7$, $c_1 = 0.75$, $a_2 = 0.4$, $b_2 = 0.8$, $c_2 = 0.85$.

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Bayesian and E-Bayesian estimates for $\lambda_1$ in case of $\tau = 0.5$ and $T = 1.6$ under the loss functions SEL, DLF, QLF and LLF respectively. Where Figure 2(a)–(d), we compared the Bayesian and E-Bayesian estimates for $\lambda_2$ in case of $\tau = 0.8$ and $T = 2.5$ under the loss functions SEL, DLF, QLF and LLF respectively. In each figure, we have compared the E-Bayesian estimates under the three proposed priors of the hyperparameters $a$ and $b$. From all these graphs we found that: for all proposed loss function and for $j=1,2$, 

1) $\hat{\lambda}_j^B < \hat{\lambda}_j^{EB2} < \hat{\lambda}_j^{EB1} < \hat{\lambda}_j^{EB3}$

2) $\lim_{n \to \infty} \hat{\lambda}_j^{EB2} = \lim_{n \to \infty} \hat{\lambda}_j^{EB1} = \lim_{n \to \infty} \hat{\lambda}_j^{EB3}$

These properties have been discussed in different situations by many authors, see for example Nassar et al. [23].

Figure 1. The Bayesian and E-Bayesian behaviour for the AE of $\lambda_1$ in case of $\tau = 0.5$ and $T = 1.6$.

7. Example of real-life data

In this section, to demonstrate the performance of the offered approaches in the application, we present an example of real-world data. These data were used by Bhaumik et al. [24], representing vinyl chloride data obtained from clean upgradient monitoring wells in mg/l. The exponential distribution has been fitted on these data by Shanker et al. [25], who found that it yields a decent match to the exponential distribution. As shown in the table below, there are 34 observations in this set of data.
Figure 2. The Bayesian and E-Bayesian behaviour for the AE of $\lambda_2$ in case of $\tau = 0.8$ and $T = 2.5$.

We suppose that values of data set represent lifetime of failure observations which follow the exponential distribution. Using a step-stress approach based on Type-I HCS on these data with the same loss functions as before, we obtain estimates of $\lambda_1$ and $\lambda_2$ based on the same used techniques and showed in Tables 3 and 4.

8. Conclusions

We looked at the E-Bayesian estimation of the simple step-stress model under the cumulative exposure model for the exponential distribution with Type-I hybrid censored data in this article. The E-Bayesian estimators are derived by considering the loss functions SEL, DLF, QLF, and LINEX. To the hyperparameters, three different distributions are considered. The average estimates (AE) and mean squared error (MSE) for each of the four loss functions are also calculated. Some E-Bayesian estimator properties are illustrated graphically. A simulation study is carried out to demonstrate the effectiveness of the various estimators. According to the simulation results, E-Bayesian estimates outperform Maximum likelihood and Bayesian estimates. To estimate the parameters of the exponential distribution under the simple step-stress model based on Type-I hybrid censored data, we recommend using the E-Bayesian method. In terms of minimum MSE, E-Bayesian estimators using the prior distribution 3 outperform other estimates. The results of the simulation are confirmed by the analysis of the real data.
Table 3. Real Data the estimated values for $a_1, a_1 = 0.6, b_1 = 0.7, c_1 = 0.75, a_2 = 0.4, b_2 = 0.8, c_2 = 0.85$.

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Table 4. Real Data: the estimated values for $\lambda_2, a_1 = 0.6, b_1 = 0.7, c_1 = 0.75, a_2 = 0.4, b_2 = 0.8, c_2 = 0.85$.

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At the end, we can suggest “the proposed methods in a constant-stress partially accelerated life test model based on a generalized hybrid censoring scheme” as a future work.

Acknowledgements

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Conflicts of interest

The authors declare there is no conflict of interest.

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